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Set optimization theory and its applications

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***Abstract*—We study an optimization problem which is called a set optimization problem. We investigate the dual space of an ordered vector space in which the set optimization problem is embedded, and characterize the dual problem.**

I. Introduction and Preliminaries

Let X, Y be normed spaces, S a nonempty convex subset of X, K a closed convex cone in Y and assume that $\mathrm{int}K \neq \emptyset$ and $K \cap (-K) = \{\theta_Y\}$, where θ_Y is the null vector of Y, and F a map from S to 2^Y. In this paper, we consider the following minimization problem:

$$\text{(P)} \qquad \begin{array}{ll} \text{minimize} & F(x) \\ \text{subject to} & x \in S. \end{array}$$

There are two types criteria of solutions for the problem with respect to the convex cone K. One is vector optimization (VP); the notions of solutions is based on comparisons between vectors. In this paper, two binary relations $\leq_K$ and $<_K$ are defined on Y: for any a, $b \in Y$,

- $a \leq_K b$ if $a + K \ni b$, and
- $a <_K b$ if $a + \mathrm{int}K \ni b$.

Note that the latter condition $a + \mathrm{int}K \ni b$ is equivalent to $a + K \supset b + rV$ for some $r > 0$, where $V = \{y \in Y \mid \|y\| \leq 1\}$. For vector optimization (VP), minimal and weak minimal solutions are defined as follows. An element $x_0 \in S$ is said to be

- a minimal solution of (VP) if $F(x_0)$ has a minimal element of $\cup_{x \in S} F(x)$ with respect to K, that is, there exists $y_0 \in F(x_0)$ such that
$$F(x_0) \cap (y_0 - K) = \{y_0\}, \tag{1}$$
- a weak minimal solution of (VP) if $F(x_0)$ has a weak minimal element of $\cup_{x \in S} F(x)$ with respect to K, that is, there exists $y_0 \in F(x_0)$ such that
$$F(x_0) \cap (y_0 - \mathrm{int}K) = \emptyset. \tag{2}$$

Also, condition (1) is equivalent to there are no (x, y) satisfying

$$x \in S, y \in F(x), y \leq_K y_0, \text{ and } y_0 \not\leq_K y, \tag{3}$$

and condition (2) is equivalent to there are no (x, y) satisfying

$$x \in S, y \in F(x), \text{ and } y <_K y_0. \tag{4}$$

The other is set optimization (SP); the notions of solutions is based on comparisons between sets. Six binary relations on the family of sets are introduced in [1], we use the following two relations in this paper. For nonempty set A, $B \subset Y$,

- $A \leq_K^l B$ if $\mathrm{cl}(A + K) \supset B$, and
- $A <_K^l B$ if $\exists r > 0$ such that $A + K \supset B + rV$,

where addition and scalar multiplication is defined as follows:

$$A + B = \{a + b \mid a \in A, b \in B\},$$
$$\lambda A = \{\lambda a \mid a \in A, \lambda \in \mathbb{R}\}.$$

Clearly these relations are generalizations of the binary relations on Y. We define minimal and weak minimal solutions for (SP). An element $x_0 \in S$ is said to be

- a minimal solution of (SP) if there is no x satisfying
$$x \in S, F(x) \leq_K^l F(x_0), \text{ and } F(x_0) \not\leq_K^l F(x), \tag{5}$$
- a weak minimal solution of (SP) if there is no x satisfying
$$x \in S \text{ and } F(x) <_K^l F(x_0). \tag{6}$$

We can see that (3) and (5) are analogous, also (4) and (6).

In general, one of useful methods to solve an optimization problem is duality. When we study dual problems of (VP), the dual space Y^* of Y is an important role. However when we study dual problems of (SP), it is difficult to consider vector structure of a family of sets, but also dual space of the family.

In this paper, we study dual problems of set optimization problem (SP), by using methodology of dual spaces. The layout of the paper is as follows. In Section 2, we show previous results, a way of construction of an ordered vector space in which set optimization (SP) is embedded. In Section 3, we establish new results concerned with the dual space of the ordered vector space and characterize the dual problems of set optimization problem (SP).

II. The embedding space

In this section, we define a range of the objective function F, and show a way of certain ordered vector space in which set optimization problem (SP) is embedded. All results of the section, see [2].

Let $C_0(Y)$, the range of F, be the family of all nonempty K-convex and K-bounded subsets of Y, where a subset $A \subset Y$ is said to be K-convex if $A + K$ is convex, and A is said to be K-bounded if there exists $y \in Y$ such that $y + K \supset A$.

At first we define a binary relation $\equiv$ on $C_0(Y)$ with respect to the convex cone K. For each (A, B), $(C, D) \in C_0(Y)^2$,

$$(A, B) \equiv (C, D) \overset{\text{def}}{\iff} A + D + K = B + C + K.$$

Then $\equiv$ is an equivalence relation on $C_0(Y)^2$, and we write the quotient set of $C_0(Y)$ by $\equiv$ as $\mathcal{V}$, that is

$$\mathcal{V} = \{[A, B] \mid (A, B) \in C_0(Y)^2\},$$

where $[A, B]$ is the equivalence class of $(A, B) \in C_0(Y)^2$,

$$[A, B] = \{(C, D) \in C_0(Y)^2 \mid (A, B) \equiv (C, D)\}.$$

Next we introduce a vector structure and a norm in $\mathcal{V}$. Define addition and scalar multiplication on $\mathcal{V}$ by

$$[A, B] + [C, D] = [A + C, B + D],$$

$$\lambda \cdot [A, B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0 \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{V}, +, \cdot)$ is a vector space over $\mathbb{R}$. Also for a given base W of K^+, K^+ is the positive polar cone of K by

$$K^+ = \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0,\ \forall y \in K\},$$

define real-valued function $\|\cdot\|$ on $\mathcal{V}$ as

$$\|[A, B]\| = \sup_{y^* \in W} |\inf \langle y^*, A\rangle - \inf \langle y^*, B\rangle|$$

for all $[A, B] \in \mathcal{V}$. Then $\|\cdot\|$ is a norm on $\mathcal{V}$. Also we can check that

$$\mu(K) = \{[A, B] \in \mathcal{V} \mid B \leq_K^l A\}$$

is a closed solid pointed convex cone in $\mathcal{V}$, and define the positive polar cone of $\mu(K)$ as

$$\mu(K)^+ = \{T \in \mathcal{V}^* \mid \langle T, [A, B]\rangle \geq 0, \forall [A, B] \in \mu(K)\}.$$

Now, we have a duality result of (SP).

Theorem 1. Assume that F is a function from S to $C_0(Y)$. If

$$\bigcup_{x \in S} [F(x), 0] + \mu(K)$$

is convex in $\mathcal{V}$, then for any $x_0 \in S$, the following are equivalent:

1) x_0 is a weak minimal solution of (SP),
2) $\exists T \in \mu(K)^+$ such that $T \neq 0$ and

$$T([F(x), F(x_0)]) \geq 0,\ \forall x \in S.$$

In general, this kind of theorem is useful to calculate solutions. In this case, however, there is a difficulty because to observe the dual space $\mathcal{V}^*$ of $\mathcal{V}$ is not easy. In the next section, we show an idea to treat the dual space $\mathcal{V}^*$.

III. Main Results

Let $\mathcal{B}al(Y)$ be the family of all nonempty balanced subsets of Y, where a subset $A \subset Y$ is balanced if $\lambda A \subset A$ when $|\lambda| \leq 1$. Since $\mathcal{B}al(Y)$ is closed under the addition of sets and the scalar multiplication, we can define a notion of linearly independent on the family $\mathcal{B}al(Y)$. If $A_1, A_2, \ldots, A_n \in \mathcal{B}al(Y)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, then

$$\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n$$

is said to be the linear combination of these sets with these scalars as coefficients. A subfamily $\mathcal{A}$ of $\mathcal{B}al(Y)$ is said to be linearly independent if none of them can be written as a linear combination of finitely many other sets in $\mathcal{A}$. Now we have the following lemma.

Lemma 1. Let $\{A_1, A_2, \ldots, A_l\} \subset \mathcal{B}al(Y) \cap C_0(Y)$ be linearly independent. Then

$$\left\{\left[x + \sum_{i=1}^{l} \lambda_i A_i,\ y + \sum_{i=1}^{l} \mu_i A_i\right] \,\middle|\, \lambda_i, \mu_i \geq 0\right\}^* = Y^* \times \mathbb{R}^l.$$

By using the lemma, we have a duality result of (SP).

Theorem 2. Let $\{A_1, A_2, \ldots, A_l\} \subset \mathcal{B}al(Y) \cap C_0(Y)$ be linearly independent, and assume that F is a function from S to

$$\{\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_l A_l \mid \lambda_1, \ldots, \lambda_l \geq 0\},$$

that is, there exist function f from S to Y and nonnegative-valued functions $g_1, g_2, \ldots, g_l$ defined on S such that

$$F(x) = f(x) + \sum_{i=1}^{l} g_i(x) A_i$$

for all $x \in S$. If

$$\bigcup_{x \in S} [F(x), 0] + \mu(K)$$

is convex, then for any $x_0 \in S$, the following are equivalent:

1) x_0 is a weak minimal solution of (SP),
2) there exists nonzero $(y_0^*, \mu_1, \mu_2, \ldots, \mu_l) \in K^+ \times [0, \infty)^l$ such that

$$\left\langle y_0^*, f(x_0)\right\rangle + \sum_{i=1}^{l} \mu_i g_i(x_0) \leq \left\langle y_0^*, f(x)\right\rangle + \sum_{i=1}^{l} \mu_i g_i(x)$$

holds for each $x \in S$.

References

[1] D. Kuroiwa, T. Tanaka, and T.X.D. Ha, On cone convexity of set-valued maps. Nonlinear Anal. 30 (1997), no. 3, 1487–1496.

[2] D. Kuroiwa, On derivatives of set-valued maps and optimality conditions for set optimization. J. Nonlinear Convex Anal. 10 (2009), no. 1, 41–50.