# Home Production and External Effects on Labor-Augmenting Technological Progress*

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## Abstract

In an economy studied in this paper, the ratio of aggregate home consumption to aggregate market consumption, a proxy to the economy-wide resource allocation between market and home sectors, is assumed to have an external effect, either positive or negative, on the rate of labor-augmenting technological progress. It will be shown that a necessary condition for this model to have multiple steady states depends on relative factor intensity between the two sectors. When home production is more labor-intensive, as suggested by many authors, and when people learn more at home than at market, the long-run outcome might depend on both history and (self-fulfilling) expectation.

## 1. Introduction

The qualitative and quantitative importance of household production is widely recognized in macroeconomic modeling. In the field of real business cycle theory, introducing the home sector to a standard model is known to improve the matching of artificial data to actual data in terms of standard deviation and co-variation with output. [c.f., Benhabib, Rogerson, and Wright (1991), Greenwood and Hercowitz (1991), Greenwood, Rogerson, and Wright (1995), McGrattan, Rogerson, and Wright (1997)] In the field of endogenous business cycle theory, the degree of increasing returns required to generate indeterminate equilibria is greatly reduced by introducing the home sector. [c.f., Perli (1998)]

It is also interesting to see how the home sector, through its interaction with the market sector, affects the long-run growth of an economy. Greenwood and Hercowitz (1991) note that "in contrast to physical capital, important components of human capital are produced in the household sector" (p.1211), although the human capital accumulation in their model is an exogenous process. On the other hand, learning-by-doing in the market sector would play an important role in the human capital accumulation as well. Because the resources a society can spend are limited, the appropriate question might be the relative importance of each sector. If people learn more at home, spending too much resource in the market may reduce the speed of human capital accumulation and (per-capita) economic growth, or vice versa. In this paper, I incorporate this consideration into an economic growth model of labor-augmenting technological progress to see the implications of the home sector for the local and global dynamics of the Hamiltonian system. In the model, the ratio of aggregate home consumption to aggregate market consumption, a proxy to the economy-wide resource allocation between the market and home sectors, is assumed to have an external effect, either positive or negative, on the rate of labor-augmenting technological progress. The main findings are summarized as follows. First, a necessary condition for the model to have multiple steady states depends on the relative factor intensity between the market and home technology. If people learn more at home (market) than at market (home), there might be multiple steady states if home (market) technology is more labor-intensive than market (home) technology. The intuitive reason behind this

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observation is explained as follows. A (rational expectation) equilibrium with externality must satisfy two conditions: optimality, and consistency between individual action and aggregate (collective) actions. Consider a case in which people learn more at home than at market. Suppose the economy is on a steady state that is characterized by labor allocation, capital stock, and the ratio of aggregate home consumption to aggregate market consumption, and suppose each individual expects that there is another steady state in which the ratio of aggregate home consumption to aggregate market consumption is higher, hence the rate of labor-augmenting technological progress is higher. In this case, such an expectation can not be self-fulfilling if home technology is less labor-intensive than market technology because each individual will reallocate time more to the market sector and less to the home sector, which contradicts the aggregate consistency condition.

Second, in a parametric model presented in this paper, there could be three steady states. Two are saddle-points, and the remaining one could be a sink of a source. One of the two saddle-points exhibits a high growth rate, and the other tends to be a corner solution in the sense that the rate of technological progress is zero due to a strong external effect. An economy might move to either one of the saddle-points depending on the initial capital stock and technology level. When the initial technology level is low relative to capital stock, the economy might end up with the corner solution. Numerical examples show that the middle steady state could exhibit a Hopf bifurcation with respect to the parameters describing the degree of external effect.

Benhabib et. al. (1991), and Greenwood and Hercowitz (1991) suggest that home sector might be more labor-intensive than market sector. In this case, if people learn more at market than at home, the model predicts an equilibrium path converging to a globally unique saddle-point. On the other hand, if people learn more at home than at market, multiple steady states could emerge, and the outcome might depend on both history and expectation.

This paper is related to the following research. In many optimal growth models, external effects are introduced as the Marshallian externality.[1] Boldrin and Rustichini (1994) showed that in optimal growth models with one state variable (physical capital), while a negative externality generates an indeterminate steady state, a positive externality does not under mild conditions. In the present model, where the external effects are introduced through a mix of consumption goods, it will be shown that both positive and negative externalities could generate an indeterminate steady state depending on technology parameters. Benhabib and Farmer (1994), and Farmer and Guo (1994) showed that an indeterminate steady state could emerge in optimal growth models with a leisure-in-utility function. Perli (1998) introduced home production, and showed by calibration that an indeterminate steady state could emerge. These studies introduce external effects as the Marshallian externality. In the present model, home consumption good might also be interpreted as a leisure which requires both labor and capital inputs, although the source of external effects is different. The "history versus expectation" phenomenon emerges in Krugman (1991), Matsuyama(1991), Futagami and Mino (1995), Gali (1994), and Greiner and Semmler (1996).[2]

This paper is organized as follows. In Section 2, the structure and the assumptions of the optimal growth model are described. Then the first-order conditions are summarized as a system of differential equations with respect to capital and time allocation between the market and home sectors. In Section 3, the steady states of the

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1 See Boldrin and Rustichini (1994), Schmitt-Grohe (1997), Nishimura and Sorger (1999), Benhabib and Farmer (1999) for the survey of this topic.

2 The other related research is as follows. Dynamic general equilibrium models with external effects could have multiple steady states due to the violation of global concavity. Skiba (1978), and Deckert and Nishimura (1983) characterized the solution of optimal growth models with a non-convex technology set. Ethier (1982), Howitt and McAfee (1988), Spear (1991), Boldrin and Rustichini (1994), and Benhabib and Farmer (1994) studied indeterminacy due to external effects. Benhabib and Nishimura (1979, 1998) studied the conditions for indeterminacy in multi-sector optimal growth models. Pelloni and Waldman (1998), and Mino (1999) studied the role of preference structure on the emergence of an indeterminate steady state. Chamley (1993), Bond, Wang and Yip (1996), and Ladron-De Guevara, Ortigueira and Santos (1999) studied the indeterminacy in Uzawa-Lucas type two-sector endogenous growth models. (See Uzawa (1964), and Lucas (1988).)

dynamic system are characterized for two cases: (i) the ratio of aggregate home consumption to aggregate market consumption has a positive external effect on the rate of labor-augmenting technological progress, and (ii) the inverse of the ratio has a positive external effect. The welfare level at each steady state is also compared when there is multiplicity. In Section 4, the structural stability of the dynamic system is analyzed. Section 5 concludes the paper.

## 2. Model

Consider a two-sector model of economic growth driven by labor-augmenting technological progress. The model is described as a continuous-time optimization problem of a representative agent. There are many identical agents in the economy. The population size is normalized to be one. At each moment t, a representative agent is endowed with one unit of labor, and owns $k(t)$ units of capital that has been accumulated through past investment activities. The labor and capital are allocated between two types of production activities. The $I(t) \times 100\%$ of labor and the $z(t) \times 100\%$ of capital is allocated to produce $f[h(t)I(t), z(t)k(t)]$ units of output that is used for market consumption $c_m$ and gross-investment $\dot{k}+\delta k$, where $\delta \in [0, 1)$ is a depreciation rate. The remaining labor and capital is used to produce $g[h(t)(1-I(t)), (1-z(t))k(t)]$ units of home consumption good $c_n$. In both production functions, the labor is measured in efficiency units, $hI$ and $h(1-I)$.

As stated in the introduction, the rate of labor-augmenting technological progress $\dot{h}/h$ may depend on how people learn at home and at market. Page (1994) compared the growth experiences of the East Asian countries and Latin American countries. Although these two regions had comparable initial states, the subsequent per-capita economic growth turned out to be much higher for the East Asian countries than Latin American countries. Page observes the differences in education system as the key factor. While Latin American countries spent resources more on secondary and higher education, whose return could be internalized through markets, the East Asian countries spent more on primary education. In fact, providing very basic skills such as reading and writing to every individual at an early stage could generate an external effect that enhances economy-wide labor-productivity. (On the other hand, we often observe child-labor in the market place among low-income, stagnant economies.) In this case, economies that spend too much resources in the market sector may experience lower rate of labor-augmenting technological progress than those that spend more in the home sector. On the other hand, learning as a by-product of market production, and network (team) externality on learning at the workplace are stressed in labor economics. [c.f., Arrow (1962), Aoki (1984)] Therefore, each of these two possibilities will be considered. In the following analysis, either the ratio of aggregate market consumption $C_m$ to aggregate home consumption $C_n$ or the inverse of the ratio is assumed to have a positive or negative externality on $\dot{h}/h$. Despite the loss of generality, this assumption enables us to reduce the number of endogenous variables if the utility function is homothetic so that the marginal rate of substitution depends only on the ratio of market consumption to home consumption. Specifically, the external effect is assumed to work in the following functional relationship $\phi(s)=\dot{h}/h$, where $s=C_m/C_n$ or $C_n/C_m$.

Assumption 1: $\phi: R+ \to R+$, $\phi \in C^2$, $\phi(0)=\Phi>0$, $\phi'(s)<0$ on $s \in [0,s)$, and $\phi(s)=0$ on $s \in [s,\infty)$ for some $s \in (0,\infty)$.[3]

In the following, when we say $C_n/C_m$ (or $C_m/C_n$) has a positive external effect on $\dot{h}/h$, it implies that $\dot{h}/h$ is a non-increasing function of $C_m/C_n$ (or $C_n/C_m$), i.e., $\dot{h}/h = \phi(C_m/C_n)$ (or $\dot{h}/h = \phi(C_n/C_m)$).

It is known that one of the key assumptions that characterize the equilibrium dynamics of home production

3 $\phi$ could be negative. In order to have a real-valued steady state, however, $\phi > -\delta-\rho$ must be imposed. See equation (9).

models is high substitutability between the market and home activities. In this paper, however, in order to illuminate the role of external effect on technological progress, I will keep the other aspects of the model as simple as possible. In both sectors, the production functions are assumed to be Cobb-Douglas form: In the market sector, $f(hl, zk)= A(hl)^{\alpha}(zk)^{1-\alpha}$, where $A>0$ is a scale parameter, and $\alpha \in (0, 1)$ is labor's share in the market sector. In the home sector, $g(h(1-l), (1-z)k) = B[h(1-l)]^{\beta}[(1-z)k]^{1-\beta}$, where $B>0$ is a scale parameter and $\beta \in (0, 1)$ is labor's share in the home sector. The instantaneous utility of the representative agent is given by $u(c_m, c_n)=\gamma \ln c_m + \ln c_n$, which is discounted by factor $\rho>0$. $\gamma>0$ is a weight attached to market consumption relative to home consumption. In summary, the representative agent, given $k(0)\equiv k_0$ and $\{s(t);\ t\geq 0\}$ where $s(t)=C_m(t)/C_n(t)$ or $C_n(t)/C_m(t)$, chooses a sequence $\{c_m(t), c_n(t), l(t), z(t);\ t\geq 0\}$ to solve the following problem.

$$\max \int_0^{\infty} u(c_m, c_n)e^{-\rho t}dt$$

subject to

(1) $c_m + \dot{k} + \delta k = A(hl)^{\alpha}(zk)^{1-\alpha}$

(2) $c_n = B[h(1-l)]^{\beta}[(1-z)k]^{1-\beta}$

and $\dot{h}/h=\phi(s(t))$. Define $\hat{k}\equiv k/h$, $\hat{c}_m\equiv c_m/h$, and $\hat{c}_n\equiv c_n/h$. The Maximum Principle is summarized by the following system of differential equations.

$$(3)\quad \dot{\hat{k}}/\hat{k} = A\left[\frac{a}{az+b(1-z)}\right]^{\alpha}\left[\left(1+\frac{\gamma(1-\alpha)}{1-\beta}\right)z-\frac{\gamma(1-\alpha)}{1-\beta}\right](\hat{k})^{-\alpha}-\delta-\phi(s)\equiv F_k(\hat{k},z)$$

$$(4)\quad \dot{z}/z = \frac{1}{\Omega(z)}\left[\left(1+\frac{\gamma(1-\alpha)}{1-\beta}\right)(1-\alpha)A\left[\frac{a}{az+b(1-z)}\right](1-z)(\hat{k})^{-\alpha}-\rho-\alpha\delta-\alpha\phi(s)\right]$$
$$\equiv F_z(\hat{k},z)$$

where $a\equiv \alpha/(1-\alpha)$, $b\equiv \beta/(1-\beta)$, $s=C_m/C_n$ or $C_n/C_m$, and

$$(5)\quad \Omega(z)\equiv\left(\frac{z}{1-z}\right)\left[\frac{(1-\alpha)(\alpha-\beta)(1-z)-\alpha(1-\beta)}{(1-\alpha)(1-\beta)[az+b(1-z)]}\right]$$

It can be shown that $\Omega(z)\leq 0$ for all $z\in[0, 1]$ with $\Omega(z)=0$ only if $z=0$. Define $Q\equiv c_m/c_n$ and $R\equiv 1/Q$. At each moment t, the resource allocation between market sector and home sector requires

(6) $Q(\hat{k},z) = [\gamma A/B][b^{1-\beta}/a^{1-\alpha}](\alpha/\beta)[az+b(1-z)]^{\beta-\alpha}(\hat{k})^{\beta-\alpha}$

(7) $l = az/[az+b(1-z)]$

(3) and (4), with the transversality condition, describe the equilibrium law of motion with respect to $\{\hat{k}(t), z(t)\}$ where s is replaced with $Q(\hat{k}, z)$ or $R(\hat{k}, z)=1/Q(\hat{k}, z)$.

## 3. Steady State and Welfare

In this section, the steady state of the dynamic system (3) and (4) is characterized for the two cases: (i) $C_n/C_m$ has a positive external effect on $\dot{h}/h$, and (ii) $C_m/C_n$ has a positive external effect on $\dot{h}/h$. For each case, a necessary condition for the model to have multiple steady states is shown to be dependent on the relative factor intensity between the market and home sectors. The steady state welfare levels are then compared for multiple steady states.

(i) $C_n/C_m$ has a positive external effect on $\dot{h}/h$: By the aggregate consistency condition and unit population, $C_m/C_n = c_m/c_n = Q$, and the rate of labor-augmenting technological progress is a function of Q, i.e., $\dot{h}/h = \phi(Q)$. Then, by (3), (4), and (6), it can be shown that the steady state is characterized by the following single equation with respect to Q.

$$(8) \qquad Q = H(Q, \Theta)$$

where

$$(9) \qquad H(Q, \Theta) = \left[\frac{\gamma [A(1-\alpha)]^{\beta/\alpha}}{B(1-\beta)}\right] \left(\frac{a}{b}\right)^{\beta} [\phi(Q) + \rho + \delta]^{1-(\beta/\alpha)}$$

and $\Theta$ is a vector of model parameters. The other steady state endogenous variables are expressed as functions of Q:

$$(10) \qquad z(Q) = \left[\frac{(\phi(Q) + \delta)(1-\alpha)}{\phi(Q) + \rho + \delta} + \frac{\gamma(1-\alpha)}{1-\beta}\right] \Big/ \left[\frac{\gamma(1-\alpha)}{1-\beta}\right]$$

$$(11) \qquad \hat{k}(Q) = \left[\frac{a}{az(Q) + b(1-z(Q))}\right]\left[\frac{A(1-\alpha)}{\phi(Q) + \rho + \delta}\right]^{1/\alpha}$$

$$(12) \qquad I(Q) = az(Q)/[az(Q) + b(1-z(Q))]$$

$$(13) \qquad \hat{c}_n(Q) = B[1-I(Q)]^{\beta}[(1-z(Q))\hat{k}(Q)]^{1-\beta}$$

$$(14) \qquad \hat{c}_m(Q) = \hat{c}_n(Q)Q$$

Because $\phi(Q) \in [0, \Phi]$, (10) implies $z(Q) \in (0, 1)$, and $z'(Q) < 0$ for $Q \in [0, \bar{Q})$. In addition, by (12), $I(Q) \in (0, 1)$, and $I'(Q) = (dI/dz) \times z'(Q) < 0$ for $Q \in [0, \bar{Q})$.

By Assumption 1, $H(0; \Theta) > 0$, $H(Q; \Theta)$ is a positive constant for $Q \in [\bar{Q}, \infty)$, and

$$(15) \qquad \frac{\partial}{\partial Q} H(Q, \Theta) \begin{pmatrix} > \\ < \end{pmatrix} 0 \quad \text{az} \quad \alpha \begin{pmatrix} < \\ > \end{pmatrix} \beta$$

for $Q \in [0, \bar{Q})$. When $\alpha = \beta$, $H(Q; \Theta)$ is a constant at $\gamma A/B$. By a straightforward calculation, the sign of the second-order derivative $\partial^2 H/\partial Q^2$ is shown to be dependent on the elasticity of $\phi$ and $\phi'$ with respect to Q as follows.

$$(16) \qquad \frac{\partial^2}{\partial Q^2} H(Q, \Theta) = H(Q, \Theta) \left[\frac{(1-\beta/\alpha)\phi'}{Q(\phi + \rho + \delta)}\right] \left[\left(\frac{Q\phi''}{\phi'}\right) - \left(\frac{\beta}{\alpha}\right)\left(\frac{Q\phi'}{\phi + \rho + \delta}\right)\right]$$

These observations imply (8) has a unique solution $Q^*$ when $\alpha \geqq \beta$. Therefore, the dynamic system (3) and (4) has a unique steady state $(\hat{k}(Q^*), z(Q^*))$ when the market sector is more labor-intensive than the home sector. The steady state growth rate $\phi(Q^*)$ may be positive if $Q^*$ is an interior solution of (8), or zero if $Q^*$ is a corner solution $Q^* = H(Q; \Theta) > Q$.

Figure 1-a

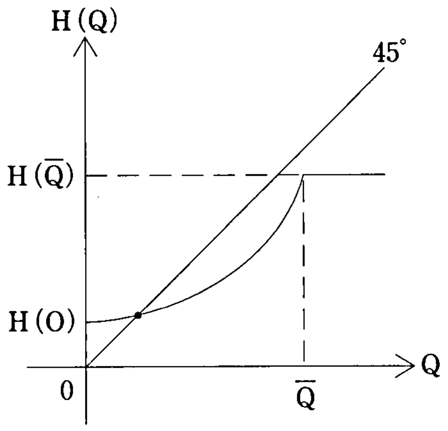



Figure 1-b

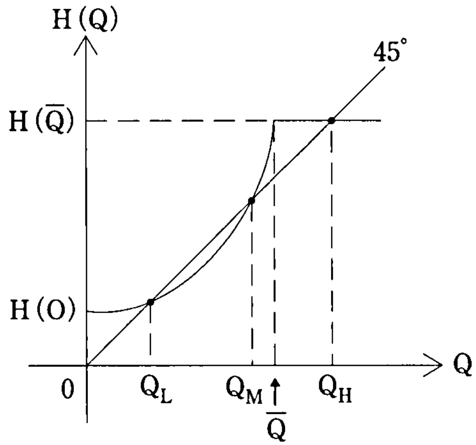



Figure 1-c

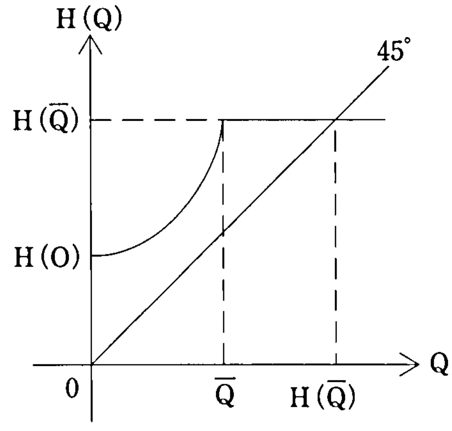



Figure 2-a

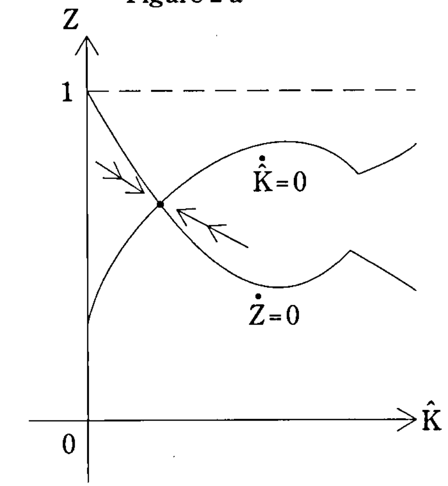



Figure 2-b

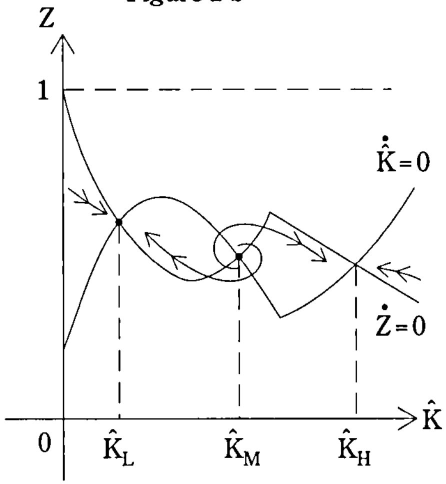
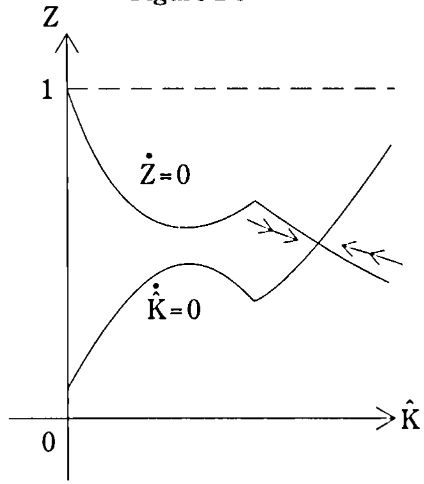



Figure 2-c



When $\alpha < \beta$, (8) might have multiple solutions. For example, if $\phi''(Q) \leqq 0$, (16) implies $\partial^2 H / \partial Q^2 > 0$. In this case, there could be as many as three steady states, two interior and one corner.[4] Figure 1-a depicts a case in which (8) has a unique interior solution, while Figure 1-c depicts a case in which (8) has a unique corner solution $Q = H(\overline{Q};\ \Theta) > \overline{Q}$. In Figure 1-b, (8) has two interior solutions, $Q_L$ and $Q_M$, and one corner solution $Q_H = H(\overline{Q};\ \Theta) > \overline{Q}$ such that $Q_L < Q_M < Q_H$. The corresponding growth rates satisfy $\phi(Q_L) > \phi(Q_M) > \phi(Q_H) = 0$. We summarize these observations by the following theorem.

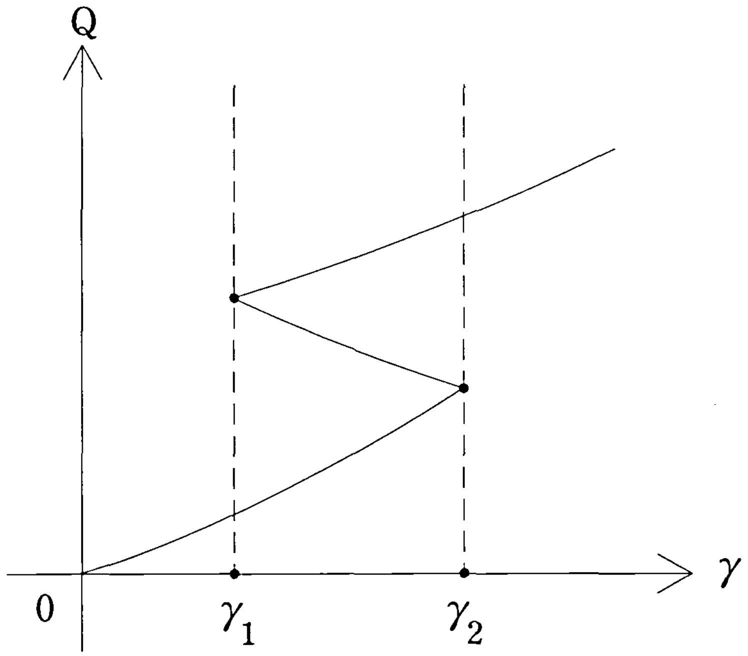
Theorem 1: Suppose the ratio of aggregate home consumption to aggregate market consumption $C_n/C_m$ has a positive external effect on the rate of labor-augmenting technological progress $\dot{h}/h$. If the market sector is at least as labor-intensive as the home sector, i.e., $\alpha \geqq \beta$, the dynamic system (3) and (4) has a unique steady state.

Therefore, $\alpha < \beta$ is a necessary condition for the model to have multiple steady states. A change in each parameter $\theta \in \Theta \equiv \{\rho, \delta, \gamma, A, B, \alpha, \beta\}$ will cause a shift in the graph of $H(Q;\ \Theta)$. When $\alpha < \beta$, the effects are summarized as follows.

$$(17) \qquad H(Q, \overset{-}{\rho}, \overset{-}{\delta}, \overset{+}{\gamma}, \overset{+}{A}, \overset{-}{B}, \overset{(*)}{\alpha}, \overset{(*)}{\beta})$$

The "+" ("−") sign above the parameters implies an upward (a downward) shift. For example, an increase in $\gamma$, the relative weight attached to $c_m$ relative to cn in the utility function, causes $H(Q;\Theta)$ to shift upward.

## Figure 3: Solution to $Q = H(Q;\gamma)$




4 If $\phi''>0$, (8) might have three interior solutions. See footnote 5.

Figure 3 depicts the relationship between $\gamma$ and Q that is a solution to Q=H(Q; $\Theta$). There are two critical values of $\gamma$, denoted as $\gamma_1$ and $\gamma_2$ such that $\gamma_1 < \gamma_2$. For $\gamma \in (0, \gamma_1)$, Q=H(Q; $\Theta$) has a unique interior solution that is depicted by Figure 1-a. At $\gamma = \gamma_1$, two additional solutions begin to emerge (Figure 1-b). Beyond $\gamma = \gamma_2$, two interior solutions disappear, and Q=H(Q; $\Theta$) once again has a unique solution that might be a corner solution $Q = H(\bar{Q};\Theta) > \bar{Q}$(Figure 1-c). The effect of a change in $\alpha$ or $\beta$ is somewhat ambiguous because it affects not only the position but also the curvature of H(Q; $\Theta$). The following theorem can be verified by differentiating H(Q; $\Theta$) with respect to $\alpha$ or $\beta$.

Theorem 2:

(i) If $\frac{A(1-\alpha)}{\phi(Q)+\rho+\delta} \leqq (b/a)^a < 1$, then $\partial H/\partial\alpha > 0$ and $\partial H/\partial\beta \leqq 0$.

(ii) If $(b/a)^a < \frac{A(1-\alpha)}{\phi(Q)+\rho+\delta} \leqq 1$, then $\partial H/\partial\alpha \geqq 0$ and $\partial H/\partial\beta > 0$.

(iii) If $(b/a)^a < 1 < \frac{A(1-\alpha)}{\phi(Q)+\rho+\delta}$, then $\partial H/\partial\alpha < 0$ and $\partial H/\partial\beta > 0$.

The lifetime utility of a representative agent, evaluated at steady state, is expressed as a function of Q as follows.

$$(18)\qquad \int_0^\infty u(c_m, c_n)e^{-\rho t}dt = \rho^{-1}[(1+\gamma)\mathrm{In}\hat{c}_m(Q) - \mathrm{In}Q] + \rho^{-2}(1+\gamma)\phi(Q)$$

When there are three steady states $Q_L < Q_M < Q_H$, a definite conclusion about utility comparison is difficult to draw in general. When Q is large, the first-term on the right-hand side of (18), which represent the level effect, may also be large if $\hat{c}_m$ is increasing in Q. On the other hand, the larger Q implies a smaller growth rate $\phi$(Q) of the labor-augmenting technological progress. This growth effect is captured by the second term on the right-hand side of (18). Therefore, when $\gamma$ is small, i.e., a representative agent greatly prefers $c_m$ over $c_n$, and when $\rho$ is large so that the growth effect is heavily discounted, the welfare loss due to the external effect of large Q may not be significant. As (18) shows, however, the growth effect is much larger than the level effect because the former works by the order of $\rho^{-2}$ while the latter works by the order of $\rho^{-1}$. A numerical example is provided in Table 1. In the example, the functional form of $\phi$(Q) is given by

$$(19)\qquad \phi(Q) = \begin{cases} \psi(\bar{Q}-Q)^{\varepsilon}, & Q\in[0,\bar{Q}) \\ 0, & Q\in[\bar{Q},\infty] \end{cases}$$

where $\psi > 0$ and $\varepsilon > 0$. When $\alpha < \beta$, it can be shown that $\partial H/\partial\psi < 0$, $\partial H/\partial\bar{Q} < 0$, and $\partial H/\partial\varepsilon \geqq (<)$ as $\bar{Q}-Q \leqq (>)\ 1$. The parameter values are specified as $\rho$=0.042, $\delta$=0.078, $\gamma$=1, A=1.8, B=1, $\psi$=1, $\varepsilon$=0.5, $\bar{Q}$=2, $\alpha$=0.7, and $\beta$=0.87. The values for $\{\rho, \delta, \alpha, \beta\}$ are the same as those in the benchmark model of Greenwood and Hercowitz (1991). (8) has three solutions: $Q_L$=1.496, $Q_M$=1.98, and $Q_H$=H($\bar{Q}$; $\Theta$)=2.39. As expected, as Q gets larger, the level effect improves wile the growth effect deteriorates the lifetime utility. At $\rho$=0.042, the latter effect dominates the former. Obviously, the growth effect is zero at the corner steady state $Q_H$=2.39>$\bar{Q}$=2.[5]

5 This parametric model can have three interior steady states when $\varepsilon$ is large. By (19), for $Q\in[0,\bar{Q})$, $\phi''(Q) \leqq (>)\ 0$ as $\varepsilon \leqq (>)\ 1$. Therefore, by (16), $H''(Q) > 0$ if $\alpha < \beta$ and $\varepsilon \leqq 1$ for $Q\in[0,\bar{Q})$. In this case, if there are three steady states, two are interior and the remaining one is a corner. On the other hand, for $\rho$=0.01, $\delta$=0.05, $\gamma$=1, A=1.15, B=1, $\psi$=1, $\bar{Q}$=2, $\alpha$=0.7, $\beta$=0.87, and $\varepsilon$=5, (8) has three interior solutions: $Q_L$=0.699, $Q_M$=1.26, and $Q_H$=1.58. In this case H(Q; $\Theta$) is convex at small Q, and concave at large Q.

Table 1 : Steady State Values of Endogenous Variables and Utility Level

| | Q | $\hat{k}$ | z | $\hat{c}_m$ | $\hat{c}_n$ | Level Effect | Growth Effect | Utility |
|---|---|---|---|---|---|---|---|---|
| $Q_L$ | 1.4964 | 0.3857 | 0.7839 | 0.5322 | 0.3557 | -39.6322 | 804.597 | 764.965 |
| $Q_M$ | 1.9798 | 1.9755 | 0.7738 | 0.9004 | 0.4548 | -21.2564 | 160.982 | 139.725 |
| $Q_H$ | 2.3932 | 5.8939 | 0.7566 | 1.3241 | 0.5533 | -7.40965 | 0 | -7.40965 |

•$C_n/C_m$ is assumed to have a positive external effect on the rate of labor-augmenting technological progress $\dot{h}/h$.
•The steady state values of endogenous variables are calculated from (10) ~ (14).
•The steady state utility level is calculated from (18). The level effect is the first term on the right-hand side of (18), and the growth effect is the second term on the right-hand side of (18).
•The parameter values are $\rho=0.042$, $\delta=0.078$, $\gamma=1$, A=1.8, B=1, $\bar{Q}=2$, $\psi=1$, $\varepsilon=0.5$, $\alpha=0.7$, and $\beta=0.87$.

Table 2 : Steady State Values of Endogenous Variables and Utility Level

| | R | $\hat{k}$ | z | $\hat{c}_m$ | $\hat{c}_n$ | Level Effect | Growth Effect | Utility |
|---|---|---|---|---|---|---|---|---|
| $R_L$ | 1.3081 | 0.67 | 0.3109 | 0.3494 | 0.457 | -27.4063 | 570.0 | 542.594 |
| $R_M$ | 1.9793 | 5.5274 | 0.2758 | 0.5778 | 1.1437 | -1.3746 | 34.4175 | 33.0429 |
| $R_H$ | 2.1432 | 8.3837 | 0.2596 | 0.6518 | 1.3946 | 4.3921 | 0 | 4.3921 |

• $C_m/C_n$ is assumed to have a positive external effect on the rate of labor-augmenting technological progress $\dot{h}/h$.
•The steady state values of endogenous variables are calculated from (10) ~ (13) and $\hat{C}_m=\hat{C}_n/R$.
•The steady state utility level is calculated from (23). The level effect is the first term on the right-hand side of (23), and the growth effect is the second term on the right-hand side of (23).
•The parameter values are $\rho=0.042$, $\delta=0.078$, $\gamma=0.35$, A=2, B=1, $\bar{R}=2$, $\psi=1$, $\varepsilon=0.8$, $\alpha=0.8$, and $\beta=0.6$.

(ii) $C_m/C_n$ has a positive external effect on $\dot{h}/h$ : In this case, the rate of labor-augmenting technological progress is a function of $R=c_n/c_m$, i.e., $\dot{h}/h=\phi(R)$. It can be shown that the steady state is also characterized by the following single equation with respect to R.

$$(20) \qquad R = F(R, \Theta)$$

where

$$(21) \qquad F(R, \Theta) = \left[\frac{B(1-\beta)}{\gamma\,[A(1-\alpha)]^{\beta/\alpha}}\right]\left(\frac{b}{a}\right)^{\beta}[\phi(R)+\rho+\delta]^{(\beta/\alpha)-1}$$

Notice that $F(R;\ \Theta)=1/H(R;\ \Theta)$. The other steady state variables are determined by (10) − (13) where Q is replaced with R, and $\hat{c}_m(R)=\hat{c}_n(R)/R$. Obviously, $\partial F/\partial R > (<)\ 0$ as $\alpha > (<)\ \beta$ for $R\in[0,\bar{R})$. When $\alpha=\beta$, $F(R;\ \Theta)$ is a constant at $B/(\gamma A)$. Therefore, the statement of Theorem 1 is reversed, i.e., the dynamic system (3) and (4) has a unique steady state when $\alpha \leqq \beta$. when $\alpha > \beta$, multiple steady states might emerge depending on the parameter values.

When $\alpha > \beta$, the effects of the changes in parameter values on the graph of $F(R;\ \Theta)$ are summarized as follows.

$$(22) \qquad F(Q\,;\ \bar{\rho}, \bar{\delta}, \bar{\gamma}, \bar{A}, \overset{+}{B}, \overset{(*)}{\alpha}, \overset{(*)}{\beta})$$

As before, the "+" ("−") sign above parameters implies an upward (a downward) shift. The effect of the change in $\alpha$ or $\beta$ is described by conditions similar to Theorem 2, except the order of inequality signs are reversed. When the growth rate $\phi$ as a function of R is given by (19), it can be shown that $\partial F/\partial \psi<0$, $\partial F/\partial \bar{R}<0$, and $\partial F/\partial \epsilon \geqq (<) 0$ as $\bar{R}-R \leqq (>) 1$.

The lifetime utility of a representative agent, evaluated at steady state, is expressed as a function of R as follows.

$$(23) \qquad \int_0^{\infty} u(c_m,c_n)e^{-\rho t}dt=\rho^{-1}[(1+\gamma)\ln\hat{c}_n(R)-\gamma\ln R]+\rho^{-2}(1+\gamma)\phi(R)$$

As before, the first term on the right-hand side is the level effect, and the second term is the growth effect. When $\alpha>\beta$, it can be shown that $\hat{c}_n$ is an increasing function of R.[6] Therefore, when there are three steady states, a definite conclusion is difficult to draw in general. A numerical example is provided in Table 2. In the example, $\phi$ (R) is given by (19), and the parameter values are specified as $\rho=0.042$, $\delta=0.078$, $\gamma=0.35$, A=2, B=1, $\psi=1$, $\epsilon=0.8$, $\bar{R}=2$, $\alpha=0.8$, and $\beta=0.6$. (20) has two interior solutions, $R_L=1.308$ and $R_M=1.979$, and a corner solution $R_H=F(\bar{R};\Theta)=2.143$. As R gets larger, the level effect improves while the growth effect deteriorates the lifetime utility. The latter effect, however, dominates the former at $\rho=0.042$.

## 4. Structural Stability and Bifurcation

In this section, the structural stability of the dynamic system (3) and (4) is analyzed under the two assumptions: (i) $C_n/C_m$ has a positive external effect on $\dot{h}/h$, and (ii) $C_m/C_n$ has a positive external effect on $\dot{h}/h$. In general, when a dynamic system has multiple steady states, the local stability of each steady state alternates its property. In the following, attention is paid to the possibility that the equilibrium path is indeterminate. Given the initial capital stock $\hat{k}(0)$, the equilibrium path is said to be indeterminate if there are multiple z(0) such that $\{\hat{k}(t), z(t); t\geqq 0\}$ satisfies the system of differential equations, (3) and (4), and the transversality condition. The local multiplicity of z(0) at a steady state is uncountable if the dynamic system has a locally stable manifold of dimension two at the steady state. On the other hand, the multiplicity of z(0) may be countable if the dynamic system has a stable manifold of dimension zero and an "overlap" [c.f., Krugman (1991), Matsuyama (1991)] at one of the multiple steady states. For example, a steady state might have expanding spiral paths converging toward other steady states. In this case, for an initial capital stock in some neighborhood of the steady state, there are countablly many z(0) such that $\{\hat{k}(t), z(t); t\geqq 0\}$ satisfies (3), (4), and the transversality condition.

The stability of each steady state could be analyzed by phase-diagrams. The following theorem will be used in the analysis. (The proof is given in Appendix A.)

Theorem 3: If $\dot{h}/h=\phi$ is an exogenously given constant, the dynamic system (3) and (4) has a unique steady state that is a global saddle-point in $(\hat{k}, z)\in R_+\times[0, 1]$.

The nonlinear system (3) and (4) is linearized at a steady state $(\hat{k}_s, z_s)$ as follows.

$$(24) \qquad \begin{bmatrix}\dot{\hat{k}}\\ \dot{z}\end{bmatrix}=\Gamma\begin{bmatrix}\hat{k}-\hat{k}_s\\ z-z_s\end{bmatrix},\ \Gamma\equiv\begin{bmatrix}\partial F_k(\hat{k}_s,z_s)/\partial\hat{k} & \partial F_k(\hat{k}_s,z_s)/\partial z\\ \partial F_z(\hat{k}_s,z_s)/\partial\hat{k} & \partial F_z(\hat{k}_s,z_s)/\partial z\end{bmatrix}$$

Denote the characteristic equation of $\Gamma$ by

---

6 For $R\in[0,\bar{R})$, $z'(R)<0$ and $I'(R)=(dI/dz)\,z'(R)<0$. When $\alpha>\beta$, it can be shown that $k'(R)>0$ and $\hat{c}_n'(R)>0$ for $R\in[0,\bar{R})$.

(25) $P(\zeta) \equiv \zeta^2 - Tr(\Gamma)\zeta + Det(\Gamma) = 0$

where Tr($\Gamma$) is the trace of $\Gamma$, and Det($\Gamma$) is the determinant of $\Gamma$.

(i) $C_n/C_m$ has a positive external effect on $\dot{h}/h$: As before, three cases are considered depending on the relative size of $\alpha$ and $\beta$. When $\alpha = \beta$, Q is constant at $\gamma$ A/B on the transition path ($\hat{k}$(t), z(t)) satisfying (3) and (4). Therefore, by Theorem 3, the system has a unique steady state that is a global saddle-point. When $\alpha > \beta$, $\dot{\hat{k}}=0$ and $\dot{z}=0$ loci in ($\hat{k}$, z)$\in R_+ \times [0, 1]$ plane are constructed as follows. In (3), (4), and (6), az+b(1-z) is bounded between [b, a] for all z$\in$[0, 1]. In (6), when $\hat{k}$ is very small, Q($\hat{k}$, z) is very large. Therefore, $\phi$(Q)=0 at small $\hat{k}$. For the same reason, $\phi$(Q) approaches $\Phi$ as $\hat{k}$ gets larger. These observations imply that at $\hat{k}$=0, z = [$\gamma$(1$-\alpha$)/(1$-\beta$)] / [1+ $\gamma$(1$-\alpha$)/(1$-\beta$)] for $\dot{\hat{k}}$ to be zero, and z=1 for $\dot{z}$ to be zero. On the other hand, at large $\hat{k}$, z must be large on $\dot{\hat{k}}$=0 locus, and z must be small on $\dot{z}$=0 locus. In the steady state analysis, we saw that Q=H(Q; $\Theta$) has a unique solution when $\alpha > \beta$. This implies that $\dot{\hat{k}}$=0 and $\dot{z}$=0 loci have a unique intersection, and $\dot{\hat{k}}$=0 locus must cut $\dot{z}$=0 locus from below at the intersection in ($\hat{k}$, z)-plane. An inspection of the vector field in the phase-diagram indicates that the steady state is a global saddle-point. Strictly speaking, each locus may consist of two graphs: one with ($\hat{k}$, z) such that Q($\hat{k}$, z)$\in$[0, $\bar{Q}$), and the other with ($\hat{k}$, z) such that Q($\hat{k}$, z)$\in$[$\bar{Q}$, $\infty$). As mentioned before, the latter emerges at small $\hat{k}$. The corner solution Q=H($\bar{Q}$; $\Theta$)$>\bar{Q}$ with zero growth rate occurs if $\dot{\hat{k}}$=0 and $\dot{z}$=0 loci have an intersection at the part of graph where Q($\hat{k}$, z)$\in$[$\bar{Q}$, $\infty$).

When $\alpha < \beta$, we saw in the steady state analysis that Q=H(Q; $\Theta$) could have three solutions depending on the parameter values. Therefore, $\dot{\hat{k}}$=0 and $\dot{z}$=0 loci cannot be monotone at the same time. As before, in (3), (4) and (6), az+b(1-z) is bounded between [a, b] for all z$\in$[0, 1]. In (6), $\hat{k}$=0 implies Q($\hat{k}$, z)=0 and $\phi$(0)$\hat{k}$=$\Phi$. On the other hand, when $\hat{k}$ is very large, Q($\hat{k}$, z) is very large. Therefore, $\phi$(Q)=0 at large $\hat{k}$. These observations imply that at $\dot{\hat{k}}$=0, z = [$\gamma$(1$-\alpha$)/(1$-\beta$)] / [1+ $\gamma$(1$-\alpha$)/(1$-\beta$)] for $\hat{k}$ to be zero, and z=1 for $\dot{\hat{k}}$ to be zero. At large $z$, $\dot{\hat{k}}$=0 and $\dot{\hat{k}}$=0 loci are reduced to those of the exogenous growth model with growth rate $\phi$(Q)=0. Figure 4-a depicts one possible shape of $\dot{\hat{k}}$=0 locus. The locus is a mix of two graphs: one with ($\bar{Q}$, z) such that Q($\hat{k}$, z)$\in$[0, $\bar{Q}$) (bold line), and the other with ($\hat{k}$, z) such that Q($\hat{k}$, z)$\in$[$\bar{Q}$, $\infty$) (broken line). Similarly, Figure 4-b depicts one possible shape of $\dot{z}$=0 locus. Again, the locus is a mix of two graphs: one with Q($\hat{k}$, z)$\in$[0, $\bar{Q}$), and the other with Q($\hat{k}$, z)$\in$[$\bar{Q}$, $\infty$).

Figure 4-a: Locus of $\dot{\hat{k}}$=0

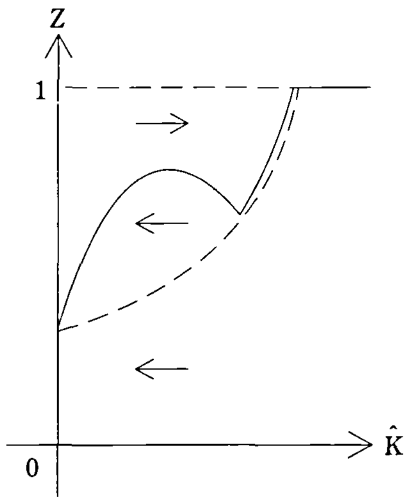
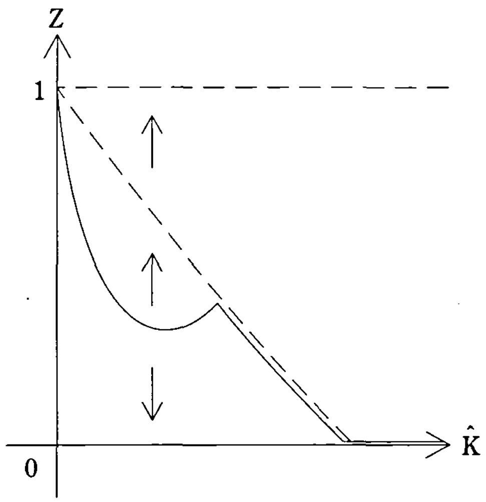
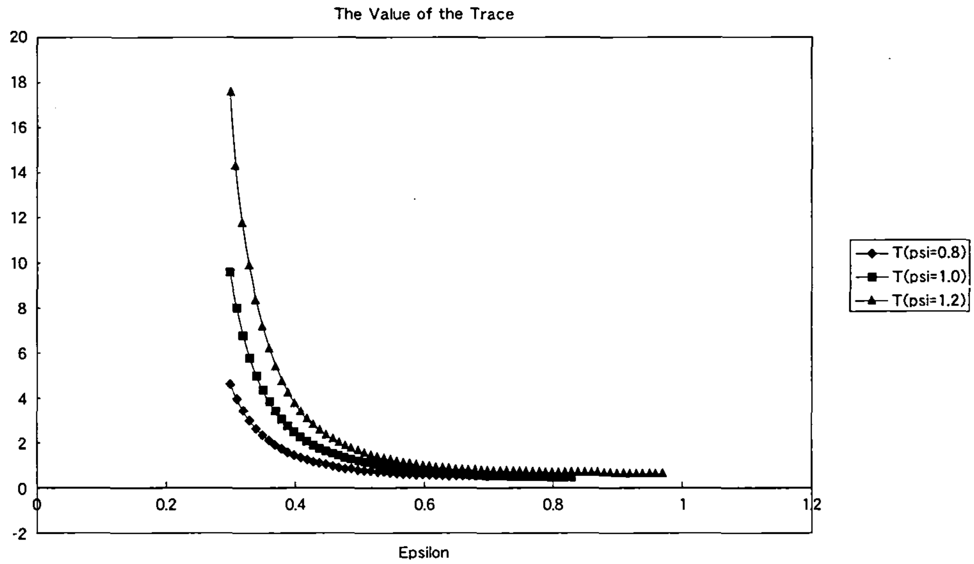



Figure 4-b: Locus of $\dot{z}$=0



In the steady state analysis for $\alpha < \beta$, we saw that three situations emerge with respect to the solution for Q=H(Q; Θ). They were depicted in Figures 1-a, 1-b, and 1-c. Figures 2-a, 2-b, and 2-c are the phase-diagrams corresponding to each one of these three situations. In Figure 2-a, there is one interior steady state that is a global saddle-point. In Figure 2-b, there are three steady states. The left one ($\hat{k}(Q_L)$, $z(Q_L)$) and the right one ($\hat{k}(Q_H)$, $z(Q_H)$) are saddle-points. An inspection of the vector field reveals that the middle one ($\hat{k}(Q_M)$, $z(Q_M)$) cannot be a saddle-point. It could be either a source or a sink (or a center) depending on the parameter values. A sufficient condition for the equilibrium to be locally indeterminate at ($\hat{k}(Q_M)$, $z(Q_M)$) is $\mathrm{Tr}(\Gamma)<0$ because the signs of the real parts of characteristic roots must be the same.

Figure 5 : The Value of the Trace $\mathrm{Tr}(\Gamma)$ evaluated at ($\hat{k}(Q_M)$, $z(Q_M)$)




*The parameter values are $\rho$=0.042, $\delta$=0.078, $\gamma$=1, A=1.8, B=1, $\overline{Q}$=2, $\alpha$=0.7, $\beta$=0.87

Using the parameter values of Table 1 as a bench mark, $\mathrm{Tr}(\Gamma)$ is calculated for different combinations of parameters $\{\varepsilon, \psi\}$ describing $\phi(Q)$ function (19). In Figure 5, the horizontal axis measures ( and the vertical axis measures $\mathrm{Tr}(\Gamma)$ evaluated at ($\hat{k}(Q_M)$, $z(Q_M)$). The three curves in the figure correspond to $\psi$= 0.8, 1.0, and 1.2. As the figure shows, it is difficult to find a parameter configuration that generates local indeterminacy. Even if $\mathrm{Tr}(\Gamma)$ decreases as $\varepsilon$ gets larger, the middle steady state $Q_M$ disappears at some $\varepsilon$, and the dynamic system has a unique corner steady state Q=H($\overline{Q}$; Θ) > $\overline{Q}$ for ( beyond this value.[7] The global indeterminacy, however, is not precluded. For example, when $\psi$=1.0, the characteristic roots $\zeta_1$ and $\zeta_2$ of P ($\zeta$) = 0 (equation (25)) evaluated at ($\hat{k}(Q_M)$, $z(Q_M)$) are complex with a positive real part for $\varepsilon \in (0.42, 0.82)$. (These characteristic roots are real, positive and distinct for $\varepsilon \in (0, 0.42)$. ) Therefore, there might be expanding spirals emanating from ($\hat{k}(Q_M)$, $z(Q_M)$) which are equilibrium paths converging toward each saddle point, ($\hat{k}(Q_L)$, $z(Q_L)$) or ($\hat{k}(Q_H)$, $z(Q_H)$).[8] In the situation depicted by Figure 2-b, the long-run

7 In this parametric example, there are three steady states when $\psi$=0.8 and $\varepsilon \in (0, 0.69)$, $\psi$=1.0 and $\varepsilon \in (0, 0.84)$, and $\psi$=1.2 and $\varepsilon \in (0, 0.98)$.

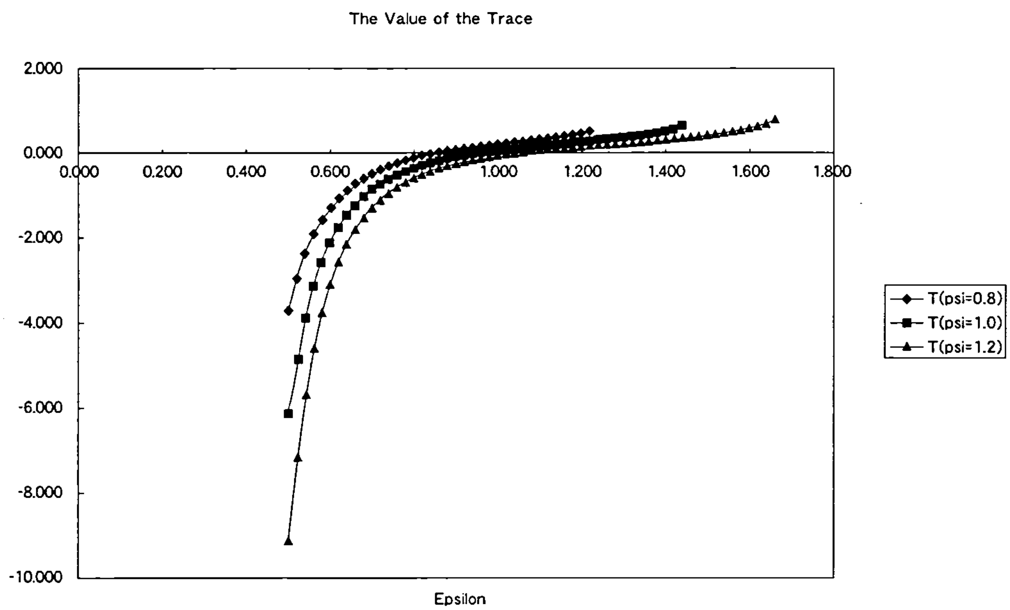
8 Under the parameter values of Table 1, the characteristic roots at ($\hat{k}(Q_L)$, $z(Q_L)$) are -1.9 and 1.75, and those at ($\hat{k}(Q_H)$, $z(Q_H)$) are -0.047 and 0.165.

outcome of the economy might depend on history (the initial capital stock $\hat{k}(0)$) as well as self-fulfilling expectation. If $\hat{k}(0)$ is small, then a representative agent is able to choose a unique $z(0)$ such that $(\hat{k}(t), z(t))$ converges to $(\hat{k}(R_L), z(R_L))$. If $\hat{k}(0)$ is large, although there is a unique $z(0)$ satisfying the dynamic system (3) and (4), $(\hat{k}(t), z(t))$ will converge to the Pareto-inferior steady state $(\hat{k}(Q_H), z(Q_H))$ with no growth. If $\hat{k}(0)$ is near $\hat{k}(Q_M)$, there might be multiple $z(0)$ that satisfy the law of motion. Depending on the expectation held by the agents, the economy could end up in either one of the two saddle-points.

In Figure 2-c, there is one corner steady state $(\hat{k}(Q), z(Q))$, where $Q=H(\overline{Q}; \Theta) > \overline{Q}$, which is a global saddle point. This corner solution with zero growth rate $\phi(Q)=0$ occurs if $\dot{\hat{k}}=0$ and $\dot{z}=0$ loci have a unique intersection at the part of graph where $Q(\hat{k}, z) \in [\overline{Q}, \infty)$.

(ii) $C_m/C_n$ has a positive external effect on $\dot{h}/h$: As we saw in Section 3, the steady state is a solution of $R=F(R; \Theta)$ where $F(R; \Theta)=1/H(R; \Theta)$. Therefore, three cases emerge depending on the relative size of $\alpha$ and $\beta$, and the stability of the dynamic system (3) and (4) can be analyzed in the same way as the previous case (i). When $\alpha=\beta$, $R(\hat{k}, z)$ is a constant at $B/(\gamma A)$ on the path $\{\hat{k}(t), z(t)\}$ satisfying (3) and (4). Therefore, by Theorem 3, the dynamic system has a unique steady state that is a global saddle-point. When $\alpha < \beta$, the system is shown to have a unique steady state that is a global saddle-point in phase-diagram. The growth rate $\phi(R)$ could be zero in the steady state. When $\alpha > \beta$, $R=F(R; \Theta)$ might have three solutions $R_L$, $R_M$, and $R_H$, such that $R_L < R_M < R_H$, depending on the parameter values, and there are three steady states corresponding to each R. It can be shown that $(\hat{k}(R_L), z(R_L))$ and $(\hat{k}(R_H), z(R_H))$ are saddle-points, and $(\hat{k}(R_M), z(R_M))$ could be a source or a sink. Unlike case (i), however, local indeterminacy at the middle steady state might emerge.

Figure 6 : The Value of the Trace $Tr(\Gamma)$ evaluated at $(\hat{k}(R_M), z(R_M))$




*The parameter values are $\rho=0.042$, $\delta=0.078$, $\gamma=0.35$, A=2, B=1, $\overline{R}=2$, $\alpha=0.8$, $\beta=0.6$

Figure 6 shows the graph of $Tr(\Gamma)$ evaluated at $(\hat{k}(R_M), z(R_M))$ for different combinations of parameters $\{\varepsilon, \psi\}$ describing $\phi(R)$ function. The other parameter values are the same as Table 2. In the figure, the horizontal axis measures $\varepsilon$, and the vertical axis measures $Tr(\Gamma)$. The three curves in the figure corresponds to $\psi$=0.8, 1.0, and 1.2.[9] As the figure shows, $Tr(\Gamma)$ is negative at small $\varepsilon$. That is, each characteristic root of $\Gamma$ has a negative real part. In addition, changes in the parameter values might cause bifurcation in the structural stability at the middle steady state. For example, when $\psi$=1.0, the characteristic roots $\zeta_1$ and $\zeta_2$ of $P(\zeta) = 0$ (equation (25)), evaluated at $(\hat{k}(R_M), z(R_M))$ for each different value of $\varepsilon$, vary as follows.

* $\varepsilon \in (0, 0.66)$: $\zeta_1$ and $\zeta_2$ are real, negative and distinct.
* $\varepsilon \in (0.67, 0.96)$: $\zeta_1$ and $\zeta_2$ are complex with a negative real part.
* $\varepsilon \in (0.97, 1.38)$: $\zeta_1$ and $\zeta_2$ are complex with a positive real part.
* $\varepsilon \in (1.39, 1.45)$: $\zeta_1$ and $\zeta_2$ are real, positive and distinct.
* $\varepsilon \in (1.46, \infty)$: The middle steady state $(\hat{k}(R_M), z(R_M))$ disappears, and the dynamic system (3) and (4) has a unique corner steady state. (See Figures 1-c and 2-c.)

In this example, $\varepsilon$=0.966 is a Hopf (subcritical) bifurcation point. (See Wiggins (1990), pp. 270-278 for the definition.) When $\varepsilon < 0.966$, there is an open neighborhood $K_\varepsilon$ of $\hat{k}(R_M)$ such that for each $\hat{k}(0) \in K_\varepsilon$, there is an open set $Z_\varepsilon$ such that $\{\hat{k}(t), z(t); t \geqq 0\}$ with any $z(0) \in Z_\varepsilon$ is an equilibrium path.

## 5. Conclusion

In this paper, we introduced external effects on the rate of labor-augmenting technological progress through a mix of market consumption and home consumption as a proxy for the relative magnitude of resources spent on the market and home sectors. Then we analyzed the model with respect to multiplicity and structural stability of equilibrium path. It turned out that the necessary condition for the model to have multiple steady states depends on the relative factor intensity between the market and home sectors. When the ratio of home consumption to market consumption has a positive external effect on the rate of technological progress, multiple steady states might emerge if the home sector is more labor-intensive than the market sector. The opposite statement holds when the ratio of market consumption to home consumption has a positive external effect. In reality, the home sector seems more labor-intensive than the market sector, though direct observation is not available. By matching the model-generated moments to actual data, Benhabib et. al. (1991) obtain the share of labor to be 0.92 in the home sector and 0.64 in the market sector, while Greenwood and Hercowitz (1991) suggest 0.87 in the home sector and 0.7 in the market sector. Therefore, there might be multiple steady states if people learn more at home than at market. In the model, there could be three steady states: two saddle-points and a remaining one being a source or a sink. It is difficult, however, to find a parameter configuration that makes the middle steady state locally indeterminate. Therefore, even if there is a chance of multiplicity, what we observe about the state of the economy might be either one of the two saddle-points.

The relationship between the parameter values and the structural stability of the dynamic system (3) and (4) has important welfare implications. Consider the case in which the ratio of home consumption to market consumption has a positive external effect on the rate of labor-augmenting technological progress. As we saw, when $\alpha < \beta$ and there are three steady states, $Q_L < Q_M < Q_H$, $Q_L$ is Pareto superior to $Q_M$, and $Q_M$ is Pareto superior to $Q_H$ under a plausible parameter configuration. For example, in a situation depicted by Figure 2-b, the economy might end up at the Pareto inferior corner steady state $(\hat{k}(Q_H), z(Q_H))$ with zero growth rate $\phi(Q_H) = 0$ when: (i) $\hat{k}(0)$ is so large that there is a unique $z(0)$ such that $(\hat{k}(t), z(t))$ satisfying (3) and (4) converges to the zero growth steady state, or (ii) $\hat{k}(0)$ is near $\hat{k}(Q_M)$ so that the economy could converge either to the interior steady state with positive growth or to the corner steady state with zero growth depending on the agents' expectation. In these cases, a social planner may confiscate and destroy initial capital stock so that $\hat{k}(0)$ becomes

9 In this example, there are three steady states when $\psi$=0.8 and $\varepsilon \in (0, 1.23)$, $\psi$=1.0 and $\varepsilon \in (0, 1.45)$, and $\psi$=1.2 and $\varepsilon \in (0, 1.67)$.

small enough for the economy to pick up a unique z(0) such that $(\hat{k}(t), z(t))$ converges to the Pareto superior interior steady state. In case (ii), the planner need not destroy $\hat{k}(0)$ to achieve the Pareto superior outcome if it can induce each agent to pick up a desirable z(0) by coordinating the expectation. In addition, the planner might be able to eliminate the Pareto inferior steady state if it has policy tools to affect the parameter values. For example, as we saw in the bifurcation analysis, a decrease in A or an increase in B causes $H(Q; \Theta)$ to shift downward. Accordingly, the phase-diagram in $(\hat{k}, z)$-plane might change from Figure 2-b to Figure 2-a. Therefore, taxing the market production and subsidizing the home production would eliminate the Pareto inferior steady state, and leave the interior steady state with positive growth as a globally unique saddle-point. Such an intervention might be regarded as a Pigouvian tax-subsidy policy because the source of welfare loss is an insufficient home consumption relative to market consumption.

In this paper, in order to illuminate the role of external effects on technological progress through consumption mix, other aspects of the model are kept as simple as possible. For this reason, I employed Cobb-Douglas production functions for both sectors, and a logarithmic utility function. It is known, however, that high substitutability between the market and home activities is one of the key assumptions that enable real business cycle models with home production to perform better than those without home production.[10] Therefore, it will be interesting to incorporate this external effect into models with general CES production functions and a CES utility function. The calibration and simulation of such models might give us further insights about the mechanism of multiplicity, indeterminacy and the roles played by history and expectation.

Appendix A: The Proof of Theorem 3

(Theorem 3: When $\dot{h}/h=\phi$ is an exogenously given constant, the dynamic system (3) and (4) has a unique steady state that is a global saddle-point in $(\hat{k}, z)\in R_{+}\times[0, 1]$.)

Proof: In the steady state of the system $(\hat{k}(Q), z(Q))$ given by (10) and (11), $\phi\geqq 0$ is a given constant. By (10), the steady state value of z is uniquely determined. In (11), $\hat{k}$ is a monotone function of z. ( $\partial\hat{k}/\partial z\gtreqless 0$ as $\alpha\lesseqgtr\beta$ ) Therefore, the steady state value of $\hat{k}$ is also uniquely determined. This implies that $\dot{\hat{k}}=0$ and $\dot{z}=0$ loci has a unique intersection in $(\hat{k}, z)\in R_{+}\times[0, 1]$ plane. From (3), $\dot{\hat{k}}=0$ locus is given by

$$\text{(A1)}\quad \hat{k}=\left\{\left(\frac{A}{\delta+\phi}\right)\left[\left(1+\frac{\gamma(1-\alpha)}{1-\beta}\right)z-\left(\frac{\gamma(1-\alpha)}{1-\beta}\right)\right]\right\}^{1/\alpha}\left[\frac{a}{az+b(1-z)}\right].$$

By (A1), $\hat{k}=0$ when $z=[(\gamma(1-\alpha)/(1-\beta)]/[1+\gamma(1-\alpha)/(1-\beta)]$, and $\hat{k}=[A/(\delta+\phi)]^{1/\alpha}$ when z= 1. From (4), $\dot{z}=0$ locus is given by

$$\text{(A2)}\quad \hat{k}=\left\{\left(1+\frac{\gamma(1-\alpha)}{1-\beta}\right)\left(\frac{A(1-\alpha)}{\rho+\alpha(\delta+\phi)}\right)(1-z)\right]^{1/\alpha}\left[\frac{a}{az+b(1-z)}\right].$$

By (A2), $\hat{k}=0$ when z=1, and

10 McGrattan et. al. (1997) applied MLE on linearized stochastic real business cycle models with home production to estimate unobservable parameters. Then they imposed a restriction that the home sector production function is Cobb-Douglas form. They found that the restrictions is binding.

$$\text{(A3)} \quad \hat{k} = \left\{\left(1+\frac{\gamma\,(1-\alpha)}{1-\beta}\right)\left(\frac{A(1-\alpha)}{\rho+\alpha\,(\delta+\phi)}\right)\right]^{1/\alpha}\left[\frac{a}{b}\right].$$

when z=0. Based on these observations, the unique steady state of the system is shown to be a global saddle-point in phase-diagram.

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