A Notion of Noncausality in Vector Time Series with Intermediate Components

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1. Introduction

Let $\{y_t\}$ be a vector stochastic process such that y_t is $m\times 1$ observable random vector generated at time t. The vector y_t is partitioned into:

$$y_t=\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n-1,t} \\ y_{n,t} \end{pmatrix} \qquad \begin{matrix} y_{1,t}\in R^{m_1} \\ y_{1,t}\in R^1 \;\; ; i=2,\ \cdots,\ n-1 \\ y_{n,t}\in R^{m_2} \end{matrix} \tag{1}$$

We denote the set of all relevant information from past y_t by $Y_{t-1}^{(-)}$,

$$Y_{t-1}^{(-)}=\{y_{t-s};s\geq 1\} \tag{2.a}$$

similarly the information set from past $y_{j,t}$ by $Y_{j,t-1}^{(-)}$,

$$Y_{j,t-1}^{(-)}=\{y_{j,t-s};s\geq 1\} \quad \text{for} \quad j=1,\ 2,\ \cdots,\ n \tag{2.b}$$

and $Y_{j,t+1}^{(+)}$ denotes the information set from future $y_{j,t}$, i. e.,

$$Y_{j,t+1}^{(+)}=\{y_{j,t+s};s\geq 1\} \quad \text{for} \quad j=1, 2, \cdots, n \tag{2.c}$$

Y_j denotes the information set of all $y_{j,t}$, i. e.,

$$Y_j=\{y_{j,t+s};\text{all integer } s\} \quad \text{for} \quad j=1, 2, \cdots, n \tag{2.d}$$

The projection of χ on Ω (the minimum mean-square linear predictor of χ conditioned on Ω) is denoted by $L(\chi|\Omega)$, where χ is a random vector and Ω is an information set. For example, the projection of $y_{j,t+m}$ on $Y_{t-1}^{(-)}$ is denoted $L(y_{j,t+m}|Y_{t-1}^{(-)})$ for $m\geq 0$ and $j=1, 2, \cdots, n$ and the projction of $y_{n,t}$ on $\{Y_1, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}\}$ is denoted by $L(y_{n,t}|Y_1, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})$. The mean-square error of the projection (the minimum mean-square error of linear predictor) of χ on Ω is denoted by $\sigma^2(\chi|\Omega)$.

The problem we deal with here is on how some predictions of $y_{1,t}$ or future $y_{1,t}$ is deteriolated when all or a part of information from past $y_{n,t}$ is not used. Particularly, our attention is concentrated on a situation such that the deterioration of the prediction of $y_{1,t}$ or future $y_{1,t}$ never results from the absence of information of past $y_{n,t}$. Later, we will provide some conditions for the existence of such a situation. In other words, what we are aiming to research in the present paper are some conditions for a sort of unidirectional stochastic independence from $\{y_{n,t}\}$ to $\{y_{1,t}\}$.

In the system (1) it may be probable that such time series as $\{y_{j,t}\}$; $j=2, \cdots, n-1$ intermediately affect the unidirectional relationship from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ in a sense. In this paper, a existence and a role of $y_{j,t}$; $j=2, \cdots, n-1$, hereafter we call them the intermediate variables, in the causal relationship between $y_{n,t}$ and $y_{1,t}$ are forcussed on. That is, the object of the analysys here is the noncausality in multivariate system with several intermediate variables. But, as seen in later sections, the concept of noncausality in this situation is very complicated and it is extremly difficult to generalize the definition of noncausality in the case of $n=2$, i. e., no intermediate variables (it was originally in Granger[4]) to the case that contains several intermediate variables (i. e., $n \geq 3$). In spite of such a fact, the concept of noncausality in the multivariate case ($n \geq 3$) must be parallel with that in the case of $n=2$. That is, if the noncausality in the case of $n=2$ is stated by a set of properties, that in $n \geq 3$ must be stated by the similar properties.

The present paper is constituted as below. In section 2 we provide some elementary conditions which are required to formulate noncausality in the multivariate case and derive some relationships among these conditions. Next, in section 3, some fundamental definitions for noncausality are formulated based on the conditions given in the previous section and the related properties are deduced. Further, in section 4 we also propose another important concepts and conditions for noncausality. In section 5 it is shown that these concepts and conditions are related to one introduced in Hsiao[5] if stationarity is postulated. Concluding remarks are stated in section 5. All proofs are given in appendix.

2. Some elementary conditions for the existence of noncausality

Consider the following conditions as requirements for the existence of noncausality.

<u>Condition I.</u>

$$L(y_{1,t} | Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, Y_{n,t-1}^{(-)}) =$$

$$L(y_{1,t} | Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) \text{ with probability one} \tag{3}$$

Condition II.

$$L(y_{1,t}|Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}, Y_{n,t-m}^{(-)}) =$$

$$L(y_{1,t}|Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}) \quad m \geq 1, \text{ with probability one} \quad (4)$$

Condition III.

$$L(y_{n,t}|Y_1, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}) =$$

$$L(y_{n,t}|Y_{1,t}^{(-)}, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}) \quad \text{with probability one} (5)$$

Condition IV.

$$L(y_{n,t+m}|Y_1, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}) =$$

$$L(y_{n,t+m}|Y_{1,t}^{(-)}, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})$$

$$m \geq 1, \text{ with probability one} \quad (6)$$

Condition V.

$$L(y_{1,t}|Y_{1,t-s+r-1}^{(-)}, \cdots, Y_{n-1,t-s+r-1}^{(-)}, Y_{n,t-s}^{(-)}) =$$

$$L(y_{1,t}|Y_{1,t-s+r-1}, \cdots, Y_{n-1,t-s+r-1}, Y_{n,t-s-1})$$

$$r \geq 1, s \geq r, \text{ with probability one} \quad (7)$$

Condition VI.

$$L(y_{n,t}|Y_1, Y_{2,t+r-1}^{(-)}, \cdots, Y_{n-1,t+r-1}^{(-)}, Y_{n,t-1}^{(-)}) =$$

$$L(y_{n,t}|Y_{1,t+r-1}^{(-)}, Y_{2,t+r-1}^{(-)}, \cdots, Y_{n-1,t+r-1}^{(-)}, Y_{n,t-1}^{(-)})$$

$$r \geq 1, \text{ with probability one} \quad (8)$$

The implications of Condition I, II and III originate from those in the case of $n=2$. That is, the situation formulated in Condition I corresponds to a multivariate version of the noncausality defined by Granger[4] (Granger-noncausality), that formulated in Condition II is a sort of multivariate version of Pierce and Haugh's[6] noncausality and Condition III corresponds to a multivariate version of the noncausality presented in Sims[7] (Sims-noncausality). From these things we may insist that Condition I, II and III are indispensable for defining the concept of noncausality in the case of $n \geq 3$. Condition IV V and VI may be associated with the concept of noncausality in the m-period ahed prediction; $m \geq 0$. Obviously, Condition IV is just Condition I if $m=0$, Condition V is just Condition II if $r=1$ and Condition VI is just Condition III if $r=1$.

In order to provide some important relationships among these conditions we make the following asuumption.

<u>Assumption 1. (Regularity Condition)</u>

$$\operatorname*{Plim}_{k\to\infty} L(y_{j,t} | V_{t-1}^{(j)},\ W_{t-k}) = L(y_{j,t} | V_{t-1}^{(j)}) \tag{9}$$

where $V_{t-1}^{(j)}$ is a subset of $\{Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, Y_{j+1,t}^{(-)}, \cdots, Y_{n,t}^{(-)}\}$ (provided that $Y_{0,t}^{(-)} \equiv Y_{n+1,t}^{(-)} \equiv \{\phi\}$) ;$j=1, 2, \cdots, n$ and W_{t-k} is a subset of $Y_{t-k}^{(-)}$.

Under this assumption the following result is derived immediately.

<u>Proposition 1.</u>

(i) Condition II is equivalent to Condition III.

(ii) Condition IV, V and VI are equivalent.

The proof is given in appendix. It has been showed in many papers that Granger-noncausality is equivalent to Sims-noncausality. (See, for example, Sims[7], Chamberlain[1] and Florens and Mouchart[2, 3].) That is, in the case of $n=2$, Condition I is equivalent to Condition III (under Assumption 1.). Further, in the case of $n=2$ it can be shown that the following relationship holds.

Proposition 2. Condition I is equivalent to Condition IV in the case of $n=2$.

The proof is given in appendix. This proposition implies that in the case of $n=2$ all conditions from I to VI are equivalent. But, in the case of $n \geq 3$, the equivalence of Condition I and III does not necessarily hold and Condition IV does not always follow from Condition I. In view of the consistency with the case of $n=2$, we should define the noncausality so that all conditions holds for the case of $n \geq 3$ as well. That is, Condition IV should be imposed as the minimum requirement for the noncausality since all conditions from I to VI follow from Condition IV.

3. Some fundamental definitions and properties of noncausality

Before providing different definitions of the noncausality, we make the following assumption.

Assumption 2.

$\sigma^2(y_{1,h,t} | Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$, $\sigma^2(y_{1,h,t} | Y_{1,t-1}^{(-)})$; $h=1, \cdots, m_1$, $\sigma^2(y_{j,t} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$ and $\sigma^2(y_{j,t} | Y_{j,t-1}^{(-)})$; $j=1, \cdots, n-1$ are not zero and are bounded, where $y_{1,t} = (y_{1,1,t}, \cdots, y_{1,m_1,t})'$.

Consider the following condition.

$$L(y_{1,t+m} | Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, Y_{n,h,t-1}^{(-)}) =$$

$$L(y_{1,t+m} \mid Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})$$

$$h=1, \cdots, m_2, \ m \geq 0, \text{ with probability one} \qquad (6)'$$

where $y_{n,t}=(y_{n,1,t}, \cdots, y_{n,m_2,t})'$ and $Y^{(-)}_{n,h,t-1}=\{y_{n,h,t-s}; s\geq 1\}$. From (A.9) in the proof of Proposition 1. (ii), that (6) and (6)′ are equivalent is immediately established (since it is obvious that (6)′ is equivalent to (A.9)). The equivalency implies that the analysis is not affected at all even if we replace $y_{n,t}$ with $y_{n,h,t}$ $(1\geq h\geq m_2)$ in the case of $m_2\geq 2$. Therefore, with no loss of generality, we can assume $m_2=1$ in the subsequent discussion.

It may be natural that the concept of noncausality in the case of $n\geq 3$ is defined to make the operation of the intermediate variables explicit. In that viewpoint, the definitions of noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ must be stated so that a sort of noncausality from $\{y_{n,t}\}$ to $\{y_{j,t}\}$ or from $\{y_{j,t}\}$ to $\{y_{1,t}\}$ $(j=2, \cdots, n-1)$ can be sufficiently counted in addition to Condition I. The subsequent analysis is meaningless if $n=2$. Therfore we suppose $n\geq 3$ below.

Definition 1. (Type I Noncausality)

We say that $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type I's sence when the following condition (11) holds in addition to Condition I.

$$\sum_{j=2}^{n-1} E\{\{L(y_{1,t+m} \mid Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j,t}, Y^{(-)}_{j+1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})$$

$$-L(y_{1,t+m} \mid Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots, Y^{(-)}_{n-1,t-1})\}$$

$$\{L(y_{n,t-s} \mid Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j,t}, Y^{(-)}_{j+1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})$$

$$-L(y_{n,t-s} \mid Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots, Y^{(-)}_{n-1,t-1})\}\}=0$$

$$m\geq 1, \ s\geq 1 \qquad (11)$$

The relationship between Type I noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ and Condition IV is stated as follows:

Proposition 3. $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type I's sence if and only if Condition IV holds.

The proof is given in appendix. It should be noted that $L(\cdot \mid Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) \equiv L(\cdot \mid Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)})$ when $j=n-1$ in (11). Consider (11). $L(y_{1,t+m} \mid Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) - L(y_{1,t+m} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$ may be interpreted as one related to a sort of noncausality from $\{y_{j,t}\}$ to $\{y_{1,t}\}$ $(n-1 \geq j \geq 2)$. We can see easily that the expression is parallel to Condition II when $t+m$ is replaced by t. Similarly, $L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) - L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$ can be related to the concept of noncausality from $\{y_{n,t}\}$ to $\{y_{j,t}\}$ and it recalls Condition I, which is checked as follows:

From $L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) - L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) = L[y_{n,t-s} \mid y_{j,t} - L(y_{j,t} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t}^{(-)})]$ $(j=2, \cdots, n-1)$,

$$L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) - L(y_{n,t-s} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) = 0$$

$(s \geq 1)$ with probability one (12)

is equivalent to

$$L[y_{n,t-s} \mid y_{j,t} - L(y_{j,t} \mid Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) = 0$$

$(s \geq 1)$ with probability one (12-a)

Similarly, from the fundamental property of the projection, we can see easily that (12-a) is equivalent to

$$E\{y_{n,t-s}[y_{j,t}-L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})]\}=0 \quad (s\geq 1) \tag{12-b}$$

From $E\{y_{n,t-s}[y_{j,t}-L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})]\}=$ $E\{[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})]y_{j,t}\}$,

(12-b) is equivalent to

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})]\}=0 \quad (s\geq 1) \tag{12-c}$$

By the same arguments with the above one, it is shown that (12-c) is equivalent to

$$L[y_{j,t}|y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\cdots,Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\cdots,Y^{(-)}_{n-1,t-1})\,;s\geq 1]=0 \quad \text{with probability one} \tag{12-d}$$

where $L[y_{j,t}|y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t})\,;s\geq 1]$ denotes the projection of $y_{j,t}$ on $\{y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t})\,;s=1,\ 2,\ \cdots\}$. From $L[y_{j,t}|y_{n,t-s}-L(y_{n,t-s}|$ $Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})\,;s\geq 1]=L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},$ $Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1},\ Y^{(-)}_{n,t-1})-L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,$ $Y^{(-)}_{n,t-1})\,;j=2,\ \cdots,\ n-1$, it is also shown that (12-d) is equivalent to

$$L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1},\ Y^{(-)}_{n,t-1})$$
$$-L(y_{j,t}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})=0$$

$(j=2, \cdots, n-1)$ with probability one (12-e)

Thus, for $j=2, \cdots, n-1$, the equivalence of (12) and (12-e) is established.

In the following definition the role of the intermediate variables, which implies a sort of noncausality from $\{y_{n,t}\}$ to $\{y_{j,t}\}$ or from $\{y_{k,t}\}$ to $\{y_{1,t}\}$ $(j, k=2, \cdots, n-1)$ is stated more clearly.

Definition 2. (Type II Noncausality)

We say that $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense when either (i) or (ii) given below holds for all $j=2, \cdots, n-1$ in addition to Condition I.

(i) $$L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) = L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$$

$m \geq 1$, with probablity one (13)

(ii) $$L(y_{j,t} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, Y_{n,t-1}^{(-)}) = L(y_{j,t} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$$

with probability one (14)

For the existence of Type II noncausality a condition wihich is stronger than Condition IV is required.

Condition VII.

In addition to (3),

$$L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, Y_{n,t-1}^{(-)}) = L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$$

$m \geq 1, j=2, \cdots, n-1$, with probability one (15)

The relationship between Type II noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ and Condition VII is stated as follows:

Proposition 4. $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense if and only if Condition VII holds.

The proof is given in appendix.

4. Another concepts for noncausality and related properties

As a definition of noncausality, a concept which is more restricted than that in Definition 2. may be adopted.

Definition 3. (Type III Noncausality)

We say that $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense when either (i) or (ii) given below holds in addition to Condition I

(i) $L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) =$

$L(y_{1,t+m} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$

$m \geq 1$, for all $j=2, \cdots, n-1$, with probablity one (16)

(ii) $L(y_{j,t} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, Y_{n,t-1}^{(-)}) =$

$L(y_{j,t} | Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})$

for all $j=2, \cdots, n-1$, with probability one (17)

The situation formulated in either (16) or (17) implies that every intermediate variable takes the same pattern on the contribution to noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$. That is, one intermediate variable contributes to the noncausality in the same way as the other. Thus this definition states that all intermediate variables in the system (1) must be handled as a bundle of the variables in a sense.

For the existence of Type III noncausality, we need to extend and strengthen such a condition as Condition VII. For that, let Θ_t denote the set which consists of the $(n-2)$-dimensional vectors expressed by any invertible linear transformation of the intermediate vector, $(y_{2,t}, \cdots, y_{n-1,t})'$, i. e.,

$$\theta_t = \{(z_{2,t}, \cdots, z_{n-1,t})' = (y_{2,t}, \cdots, y_{n-1,t})' A_t ; A_t \in \nabla\} \quad (18)$$

where ∇ denotes the set whose elements is a $(n-2)\times(n-2)$ non-sigular matrix. Further, we adopt the following notations.

$$z_{1,h,t} \equiv y_{1,h,t} \ (h=1, \cdots, m_1), \quad z_{1,t} \equiv y_{1,h} \quad z_{n,t} \equiv y_{n,t} \quad (19)$$

And we denote the set of all relavant information from past $z_{j,t}$ by $Z^{(-)}_{j,t-1}$, i. e.,

$$Z^{(-)}_{j,t-1} = \{z_{j,t-s} ; s \geq 1\} \quad j=1, 2, \cdots, n \quad (20)$$

<u>Condition VIII.</u>

$$L(z_{1,t} | Z^{(-)}_{1,t-1}, \cdots, Z^{(-)}_{n-1,t-1}, Z^{(-)}_{n,t-1}) =$$

$$L(z_{1,t} | Z_{1,t-1}, \cdots, Z_{n-1,t-1}) \text{ with probability one} \quad (21)$$

$$L(z_{1,t+m} | Z^{(-)}_{1,t}, \cdots, Z^{(-)}_{j-1,t}, Z^{(-)}_{j,t-1}, \cdots, Z^{(-)}_{n-1,t-1}, Z^{(-)}_{n,t-1}) =$$

$$L(z_{1,t+m} | Z^{(-)}_{1,t}, \cdots, Z^{(-)}_{j-1,t}, Z^{(-)}_{j,t-1}, \cdots, Z^{(-)}_{n-1,t-1})$$

$$\text{with probability one} \quad (22)$$

for all $m \geq 1$, for all $j=2, \cdots, n-1$, for all $(z_{2,t}, \cdots, z_{n-1,t})' \in \Theta_t$.

The relationship between Type III noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ and Condition VIII is stated as follows:

Proposition 5. $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense if and only if Condition VIII holds.

The proof is given in appendix. Condition VIII is meaningful but extremely restricted. For the existence of Type III noncausality, more pragmatic conditions may be needed.

Condition VIII′

$$L(y_{1,t+m}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n,t-1}^{(-)}, y_{j,t}+by_{j+1,t})=$$

$$L(y_{1,t+m}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}, y_{j,t}+by_{j+1,t})$$

$$m\geq 1, \text{ for a real number } b, \text{ with probability one} \qquad (23)$$

Proposition 6. $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense if and only if Condition VIII′ holds in addition to Condition VII.

We can easily confirm that the concepts of noncausality presented in the paper is getting weaker in order of Type I, II and III. That is, $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type I's sense if $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense and $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense if $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense. Further, with an additional restriction, the following result of the eauivalence between the definitions of noncausality is derived.

Proposition 7. Suppose $n=3$. Then, the three concepts of noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ (i. e., Type I, II and III noncausality) are equivalent.

5. Notion of noncausality in stationary process

Suppose that $\{y_t\}$ is second order stationary, i. e.,

$$E\{y_t y'_{t+s}\} = R(s) \quad \text{for all integer } s \tag{24}$$

Then, (13) and (14) can be written as follows:

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j,t-m}, Y^{(-)}_{j+1,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1}) =$$

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j-1,t-m}, Y^{(-)}_{j,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})$$

$$h=1, \cdots, m_1, m \geq 1, \tag{13)'}$$

$$\sigma^2(y_{j,t} | Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots, Y^{(-)}_{n-1,t-1}, Y^{(-)}_{n,t-1}) =$$

$$\sigma^2(y_{j,t} | Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots, Y^{(-)}_{n-1,t-1}) \tag{14)'}$$

Similarly, (16) and (17) is written as

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j,t-m}, Y^{(-)}_{j+1,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1}) =$$

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j-1,t-m}, Y^{(-)}_{j,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})$$

$$h=1, \cdots, m_1, m \geq 1, \text{ for all } j=2, \cdots, n-1 \tag{16)'}$$

$$\sigma^2(y_{j,t} | Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots, Y^{(-)}_{n-1,t-1}, Y^{(-)}_{n,t-1})$$

$$= \sigma^2(y_{1,h,t} | Y_{1,t}, \cdots, Y_{j-1,t}, Y_{j,t-1}, \cdots, Y_{n-1,t-1})$$

$$\text{for all } j=2, \cdots, n-1 \tag{17)'}$$

Also, we introduce the following expressions in connection with one given above.

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{n-1,t-m})$$

$$= \sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-m}, Y^{(-)}_{2,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})$$

$$h=1, \cdots, m_1, m \geq 1 \tag{25}$$

$$\sigma^2(y_{1,h,t} | Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1}, \cdots, Y^{(-)}_{n-1,t-1}) = \sigma^2(y_{1,h,t} | Y^{(-)}_{1,h,t-1})$$

$$h=1, \cdots, m_1 \tag{26}$$

$$\sigma^2(y_{j,t} | Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n,t-1}) = \sigma^2(y_{j,t} | Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})$$

$$j=2, \cdots, n-1 \tag{27}$$

Then the relationships between these expressions are stated as follows:

Proposition 8.

(i) Either (16)′ or (17)′ holds if and only if either (25) or (17)′ holds.

(ii) Either (25) or (17)′ holds if and only if either (26) or (17)′ holds.

(iii) Suppose that Condition I holds. Then, either (26) or (17)′ holds if and only if either (26) or (27) holds.

We note that it is unnecessary for the proof of this proposition to suppose second order stationarity of $\{y_t\}$. This proposition implies that we can provide another representation of the definition of Type III noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$. That is, $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense when any one of (16)′, (17)′, (25), (26) and (27) holds in addition to Condition I. Further, this suggests that Type III noncausality from $\{y_{n,t}\}$ to $\{y_{1,t}\}$ can be also defined as follows:

Definition 3′. (Type III Noncausality or Hsiao's No Causality)

We say that $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense (or Hsiao's[5] sense) when either (26) or (27) holds in addition to Condition I.

This definition is adopted from Hsiao[5]. That is, Definition 3′ is corresponding to 'no causality' defined by Hsiao[5]. (Strictly speaking, Definition 3′ is just a multivariate generalization of one defined by Hsiao[5] for the case of $n=3$.) It should by noted that (26) implies that $\{(y_{2,t}, \cdots, y_{n-1,t})'\}$ does not cause $\{y_{1,t}\}$ in Granger's[4] sense.

Similarly, Condition I and (27) imply that $\{y_{n,t}\}$ does not cause $\{(y_{1,t}, \cdots, y_{n-1,t})'\}$ in Granger's[4] sense. Further, Condition I and (26) implies that $\{(y_{2,t}, \cdots, y_{n,t})'\}$ does not cause $\{y_{1,t}\}$ in Granger's[4] sense. On the other hand, (25) implies that $\{(y_{2,t}, \cdots, y_{n-1,t})'\}$ does not cause $\{y_{1,t}\}$ in Pierce and Haugh's[6] sense. (We note that these noncausality concepts are essentially for the case of $n=2$.)

The operation of the intermediate variables may be not necessarily restricted to noncausarity from $\{y_{n,t}\}$ to $\{y_{i,t}\}$ or $\{y_{j,t}\}$ to $\{y_{1,t}\}$ $(i, j=2, \cdots, n-1)$ but contain noncausality between the intermediate variables, i. e., such one as noncausality from $\{y_{j,t}\}$ to $\{y_{i,t}\}$ $(i \neq j; i, j=2, \cdots, n-1)$ as well. This point has never been mentioned in the above three definitions. As a definiton of noncausality, Definition 1 and 2 is simply insufficient as has been mentioned. On the other hand, reversely, Definition 3 may be too restricted because the interaction among the intermediate variables is ignored. Thus we finally come to the following definition.

Definition 4. (Type IV Noncausality)

We say that $\{y_{n,t}\}$ does not cause $\{y_{1,t}\}$ in Type IV's sense when any one of (16)′, (17)′ and (27) holds in addition to Condition I.

6. Concluding Remarks

In this paper the noncausality between two time series in multivariate system with several intermediate variables has been analyzed. It was emphasized that the concepts of noncausality in this situation are much more complicated than that defined under the system with no intermediate variables and that the derivation of some meaningful conditions for the existence is extremely difficult. We also pointed out the possibility to formulate several concepts on noncausality under the system which contains several intermediate variables. The causality concepts presented in the paper are very general and valuable in the sense that these contain one defined in Hsiao[5] as a special case and have some closed relationships with the definitions by Granger[4] or Pierce and Haugh[6] etc. for the case of no intermediate variables. Although some properties and conditions on non-causality derived in the paper may appear to be unuseful under general situation (dealt with in the paper), we can easily see some pragmatic implications these possess when the system partitioned into three or four subsystems is considered. From that pointview, owr results will provide a definite guide for the empirical research to investigate some causal relationships among several macroeconmic variables.

Appendix: Proofs of Propositions

Proof of Proposition 1. From the fundamental properties of projection, we have

$$L(y_{1,t}|Y_{1,t-m}^{(-)}, \cdots, Y_{n,t-m}^{(-)}) - L(y_{1,t}|Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}, Y_{n,t-m-1}^{(-)})$$

$$= L[y_{1,t}|y_{n,t-m} - L(y_{n,t-m}|Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}, Y_{n,t-m-1}^{(-)})$$

$$(m \geq 1) \qquad \text{(A.1)}$$

$$L(y_{n,t}|Y_1, Y_{2,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}) -$$

$$L(y_{n,t}|Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})$$

$$= L(y_{n,t}|y_{1,t+s} - L(y_{1,t+s}|Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}); s \geq 1) \quad \text{(A.2)}$$

From (A.1) and (A.2), it follows that (4) is equivalent to

$$L[y_{1,t}|y_{n,t-m} - L(y_{n,t-m}|Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}, Y_{n,t-m-1}^{(-)})] = 0$$

$m \geq 1$, with probability one (A.3)

Similarly, (5) is equivalent to

$$L[y_{n,t} | y_{1,t+s} - L(y_{1,t+s} | Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)}) ; s \geq 1] = 0$$

with probability one (A.4)

Further, (A.3) is written as

$$E\{y_{1,t}[y_{n,t-m} - L(y_{n,t-m} | Y_{1,t-m}^{(-)}, \cdots, Y_{n-1,t-m}^{(-)}, Y_{n,t-m-1}^{(-)})]'\} = 0$$

$m \geq 1$ (A.5)

(A.4) is also written as

$$E\{y_{n,t}[y_{1,t+s} - L(y_{1,t+s} | Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})]'\} = 0$$

$s \geq 1$ (A.6)

Since $E\{y_{n,t}[y_{1,t+s} - L(y_{1,t+s} | Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})]'\} = E\{[y_{n,t} - L(y_{n,t} | Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})] y_{1,t+s}'\} ; s \geq 1$,

(A.6) must be equivalent to

$$E\{y_{1,t+s}[y_{n,t} - L(y_{n,t} | Y_{1,t}^{(-)}, \cdots, Y_{n-1,t}^{(-)}, Y_{n,t-1}^{(-)})]' = 0 \quad s \geq 1 \quad \text{(A.7)}$$

Hence we deive the equivalence of (A.5) and (A.6), therefore, the equivalence of (A.3) and (A.6), which also implies that (4) and (5) are equivalent. Thus the eqyivalence of Condition II and III is established.

(ii) By the similar arguments with that of (i), we can establish the following results:

(6) is equivalent to

$$L[y_{1,t+m} | y_{n,t-s} - L(y_{n,t-s} | Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)}) ; s \geq 1] = 0$$

$m \geq 1$, with probablilty one (A.8)

(7) is equivalent to

$$L[y_{1,t} | y_{n,t-s} - L(y_{n,t-s} | Y^{(-)}_{1,t-s+r-1}, \cdots, Y^{(-)}_{n-1,t-s+r-1}, Y^{(-)}_{n,t-s-1})] = 0 \quad r \geq 1,\ s \geq r, \text{ with probability one} \tag{A.9}$$

(8) is equivalently to

$$L[y_{n,t} | y_{1,t+s} - L(y_{1,t+s} | Y^{(-)}_{1,t+r-1}, \cdots, Y^{(-)}_{n-1,t+r-1}, Y^{(-)}_{n,t-1}) ; s \geq 1] = 0 \quad r \geq 1, \text{ with probability one} \tag{A.10}$$

(A.8) is equivalent to

$$E\{y_{1,t+m}[y_{n,t-s} - L(y_{n,t-s} | Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})]'\} = 0 \quad m \geq 0,\ s \geq 1 \tag{A.11}$$

(A.9) is equivalent to

$$E\{y_{1,t}[y_{n,t-s} - L(y_{n,t-s} | Y^{(-)}_{1,t-s+r-1}, \cdots, Y^{(-)}_{n-1,t-s+r-1}, Y^{(-)}_{n,t-s-1})]'\} = 0 \quad r \geq 1,\ s \geq r \tag{A.12}$$

(A.10) is equivalent to

$$E\{y_{n,t}[y_{1,t+s} - L(y_{1,t+s} | Y^{(-)}_{1,t+r-1}, \cdots, Y^{(-)}_{n-1,t+r-1}, Y^{(-)}_{n,t-1})]'\} = 0 \quad r \geq 1,\ s \geq r \tag{A.13}$$

(A.13) is equivalent to

$$E\{y_{1,t+s}[y_{n,t} - L(y_{n,t} | Y^{(-)}_{1,t+r-1}, \cdots, Y^{(-)}_{n-1,t+r-1}, Y^{(-)}_{n,t-1})]'\} = 0 \quad r \geq 1,\ s \geq r \tag{A.14}$$

We can see that (A.12) is the same with (A.14) when t in (A.12) is replaced by $t+s$. Hence (A.12) is equivalent to (A.13), therfore, (7) and (8) are equivalent. *Thus the equivalence of Condition V and VI is established.*

From the fundamental property,

$$y_{n,t-m-s}-L(y_{n,t-m-s}|Y^{(-)}_{1,t-m-1},\cdots,Y^{(-)}_{n-1,t-m-1},Y^{(-)}_{n,t-m-s-1})=$$
$$y_{n,t-m-s}-L(y_{n,t-m-s}|Y^{(-)}_{1,t-m-1},\cdots,Y^{(-)}_{n-1,t-m-1})$$
$$-L[y_{n,t-m-s}|y_{n,t-m-s-r}-L(y_{n,t-m-s-r}|Y^{(-)}_{1,t-m-1},\cdots,Y^{(-)}_{n-1,t-m-1})$$
$$;r\geq 1]\quad m\geq 0,\ s\geq 1 \tag{A.15}$$

Auumption 1. and (A.15) imply that (A.11) is equivalent to

$$E\{y_{1,t}[y_{n,t-m-s}-L(y_{n,t-m-s}|Y^{(-)}_{1,t-m-1},\ \cdots,$$
$$Y^{(-)}_{n-1,t-m-1},\ Y^{(-)}_{n,t-m-s-1})]'\}=0\quad m\geq 0,\ s\geq 1 \tag{A.16}$$

By letting $s-r=m'$ in (A.12), we have

$$E\{y_{1,t}[y_{n,t-m'-r}-L(y_{n,t-m'-r}|Y^{(-)}_{1,t-m'-1},\ \cdots,\ Y^{(-)}_{n-1,t-m'-1},$$
$$Y^{(-)}_{n,t-m'-r-1})]'\}=0\quad m\geq 0,\ r\geq 1 \tag{A.12$'$}$$

Reversely, by letting $m'+r=s$ in (A.12)$'$, we have (A.12). The equivalence of (A.12) and (A.16) follows from that (A.12)$'$ is just (A.16). Hence (A.12) is equivalent to (A.11). *Thus the equivalence of Condition V and IV is prooved.* Q. E. D.

Proof of Proposition 2. It suffices to show that Condition IV holds if Condition I holds. By arguments similar to the proof of Proposition 1, it is shown that Condition I in the case of $n=2$ is equivalent to

$$E\{y_{1,t}[y_{2,t-s}-L(y_{2,t-s}|Y^{(-)}_{1,t-1})]'\}=0\quad s\geq 1, \tag{A.17}$$

From the fundamental property of projection, (A.17) implies

$$E\{y_{2,t-m-s}[y_{1,t-r}-L(y_{1,t-r}|Y^{(-)}_{1,t-r-1})]'\}=0$$

$$m\geq 1,\ s\geq 1,\ m\geq r\geq 1 \qquad \text{(A.18)}$$

therfore,

$$L[y_{2,t-m-s}|y_{1,t-r}-L(y_{1,t-r}|Y^{(-)}_{1,t-r-1})]=0$$

$$m\geq 1,\ s\geq 1,\ m\geq r\geq 1, \text{ with probability one} \qquad \text{(A.18)}'$$

From $y_{2,t-m-s}-L(y_{2,t-m-s}|Y^{(-)}_{1,t-m-1})=y_{2,t-m-s}-L(y_{2,t-m-s}|Y^{(-)}_{1,t-1})+\sum_{r=1}^{m} L[y_{2,t-m-s}|y_{1,t-r}-L(y_{1,t-r}|Y^{(-)}_{1,t-r-1})$ $(m,\ s\geq 1)$ and (A.18)

$$y_{2,t-m-s}-L(y_{2,t-m-s}|Y^{(-)}_{1,t-m-1})=y_{2,t-m-s}-L(y_{2,t-m-s}|Y^{(-)}_{1,t-1})$$

$$m\geq 1,\ s\geq 1,\ m\geq r\geq 1, \text{ with probability one} \qquad \text{(A.19)}$$

(A.17) and (A.19) implies

$$E\{y_{1,t+m}[y_{2,t-s}-L(y_{2,t-s}|Y^{(-)}_{1,t-1})]'\}=0 \quad m\geq 1,\ s\geq 1, \qquad \text{(A.20)}$$

(A.17) and (A.20) is equivalent to Condition IV. Q. E. D.

Proof of Proposition 3. By tedious calculation, we derive

$$\sum_{j=2}^{n-1} E\Big(\{L(y_{1,t+m}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j,t},\ Y^{(-)}_{j+1,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})$$

$$-L(y_{1,t+m}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})\}$$

$$\{L(y_{n,t-s}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j,t},\ Y^{(-)}_{j+1,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})$$

$$-L(y_{n,t-s}|Y^{(-)}_{1,t},\ \cdots,\ Y^{(-)}_{j-1,t},\ Y^{(-)}_{j,t-1},\ \cdots,\ Y^{(-)}_{n-1,t-1})\}\Big)$$

$$=E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t-1},\cdots,Y^{(-)}_{n-1,t-1})]\}$$

$$-E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\cdots,Y^{(-)}_{n-1,t})]\}$$

$$m\geq 1,\ s\geq 1 \tag{A.21}$$

(if part) (11) follows readily from (A.11) and (A.21). Note that Condition IV implies (A.11) and that Condition I is contained in Condition IV.

(only if part) (11) and (A.21) imply

$$E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t-1},\cdots,Y^{(-)}_{n-1,t-1})]\}$$

$$=E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\cdots,Y^{(-)}_{n-1,t})]\}$$

$$m\geq 1,\ s\geq 1 \tag{A.22}$$

Further,

$$E\{y_{1,t+2}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\cdots,Y^{(-)}_{n-1,t})]\}=0,\quad s\geq 0 \tag{A.23}$$

From (A.22) and (A.23)

$$E\{y_{1,t+3}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t},\cdots,Y^{(-)}_{n-1,t})]\}=0,\quad s\geq 0 \tag{A.24}$$

From (A.22) and (A.24), we have

$$E\{y_{1,t+3}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t-1},\cdots,Y^{(-)}_{n-1,t-1})]\}=0$$
$$s\geq 1 \tag{A.25}$$

Thus, inductively, we derive (A.11). (A.11) is equivalent to Condition IV. Q. E. D.

Proof of Proposition 4. By the arguments similar to the proof of Proposition 1, it is shown that (13), (14) and (15) are equivalent to (A.26), (A.27) and (A.28), respectively.

$$E\{y_{1,t+m}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\}=0$$

$$m\geq 1 \tag{A.26}$$

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,$$

$$Y_{n-1,t-1}^{(-)})]\}=0 \quad s\geq 1 \tag{A.27}$$

$$E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,$$

$$Y_{n-1,t-1}^{(-)})]\}=0 \quad m\geq 1,\ s\geq 1,\ j=2,\cdots,n-1 \tag{A.28}$$

Therfore, it suffices to show that either (A.26) or (A.27) holds for all $j=2, \cdots, n-1$ if and only if (A.28) holds when Condition I holds (since Condition VII contains Condition I).

(if part) Using the same arguments with that in the proof of Proposition 1 or 2, from (A.28) we can derive

$$E\{y_{1,t+m}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)},\cdots,Y_{j,t-1}^{(-)},Y_{j+1,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\}=0$$

$$m\geq 1,\ s\geq 1,\ j=2,\cdots,n-1 \tag{A.29}$$

From (A.28) and (A.29), we have

$$E\{y_{1,t+m}L[y_{n,t-s}|y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\}=0$$

$$m\geq 1,\ s\geq 1,\ j=2,\cdots,n-1 \tag{A.30}$$

Under Assumption 2, (A.30) implies that

$$E\Big(y_{1,h,t+m}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)$$

$$\Big(\sigma^2(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})\Big)^{-1}$$

$$E\Big([y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]y_{n,t-s}\Big)=0$$

$$h=1,\cdots,m_1,\ m\geq 1,\ s\geq 1,\ j=2,\cdots,n-1 \tag{A.31}$$

Noting that $E\{[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]y_{n,t-s}\}$ $=E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]\}$, we can see that (A.31) is equivalent to

either (i) $E\{y_{1,h,t+m}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]\}=0$

$$h=1, \cdots, m_1, \ m\geq 1 \qquad \text{(A.26)}'$$

or (ii) $E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]\}=0$

$$s\geq 1 \qquad \text{(A.27)}'$$

for each $j=2, \cdots, n-1$. Thus the result is established.

(only if part) If either (A.26) or (A.27) holds for all $j=2, \cdots, n-1$, we have

$$E\{y_{1,t}L[y_{n,t-m-s}|y_{j,t-m}-L(y_{j,t-m}|Y_{1,t-m}^{(-)}, \cdots, Y_{j-1,t-m}^{(-)}, Y_{j,t-m-1}^{(-)}, \cdots, Y_{n-1,t-m-1}^{(-)})]\}=0$$

$$m\geq 1, \ s\geq 1, \ j=2, \cdots, n-1 \qquad \text{(A.32)}$$

Noting that $y_{n,t-m-s}-L(y_{n,t-m-s}|Y_{1,t-m}^{(-)}, \cdots, Y_{j-1,t-m}^{(-)}, Y_{j,t-m-1}^{(-)}, \cdots, Y_{n-1,t-m-1}^{(-)})=y_{n,t-m-s}-L(y_{n,t-m-s}|Y_{1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})+\sum_{r=1}^{m-1}\sum_{i=1}^{n-1}L[y_{n,t-m-s}|y_{1,t-r}-L(y_{i,t-r}|Y_{1,t-r}^{(-)}, \cdots, Y_{i-r,t-r}^{(-)}, Y_{i,t-r-1}^{(-)}, \cdots, Y_{n-1,t-r-1})]+\sum_{i=j}^{n-1}Ly_{n,t-m-s}|y_{1,t-m}-L(y_{i,t-m}|Y_{1,t-m}^{(-)}, \cdots, Y_{i-1,t-m}^{(-)}, Y_{i,t-m-1}^{(-)}, \cdots, Y_{n-1,t-m-1}^{(-)})]$, we derive (A.28) from (A.32) and Condition I. Q. E. D.

<u>Proof of Proposition 5.</u> It is easily shown that the following (A.33), (A.34) and (A.35) are equivalent to (16), (17) and (22), respectively.

$$E\{y_{1,t+m}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]\}=0$$

$$m \geq 1,\ j=2, \cdots, n-1 \qquad \text{(A.33)}$$

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots,$$

$$Y_{n-1,t-1}^{(-)})]\}=0 \quad s \geq 1,\ j=2, \cdots, n-1 \qquad \text{(A.34)}$$

$$E\{z_{1,t+m}[z_{n,t-s}-L(z_{n,t-s}|Z_{1,t}^{(-)}, \cdots, Z_{j-1,t}^{(-)}, Z_{j,t-1}^{(-)}, \cdots,$$

$$Z_{n-1,t-1}^{(-)})]\}=0$$

$$m \geq 1,\ s \geq 1,\ j=2, \cdots, n-1,\ (z_{2,t}, \cdots, z_{n-1,t})' \in \Theta_t, \qquad \text{(A.35)}$$

Then, it suffices to show that either (A.33) or (A.34) holds if and only if (A.35) holds. It should be noted that (A.28) is a special case of (A.35).

(if part) Using Proposition 4, from (A.35), it is shown that either (A.26) or (A.27) must hold for all $j=2, \cdots, n-1$. And, from (A.35), we also have

$$E\{y_{1,t+m}L[y_{n,t-s}|y_{j,t}+by_{j+1,t}-L(y_{j,t}+by_{j+1,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)},$$

$$Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]\}=0 \quad m \geq 1,\ s \geq 1,\ j=2, \cdots, n-1 \ \text{(A.36)}$$

where b is a real number. Noting that $y_{j,t}+by_{j+1,t}-L(y_{j,t}+by_{j+1,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})=y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})+b\Big(y_{j+1,t}-L(y_{j+1,t}|Y_{1,t}^{(-)}, \cdots, Y_{j,t}^{(-)}, Y_{j+1,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})\Big) +bL[y_{j+1,t}|y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)}, \cdots, Y_{j-1,t}^{(-)}, Y_{j,t-1}^{(-)}, \cdots, Y_{n-1,t-1}^{(-)})]$, it is shown that (A.36) holds if and only if either (A.37) or (A.38) below holds.

$$E\Big(y_{1,t+m}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)$$
$$+bE\Big(y_{1,t+m}[y_{j+1,t}-L(y_{j+1,t}|Y_{1,t}^{(-)},\cdots,Y_{j,t}^{(-)},Y_{j+1,t-1}^{(-)},\cdots,$$
$$Y_{n-1,t-1}^{(-)})]\Big)+bE\Big(y_{1,t+m}L[y_{j+1,t}|y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},$$
$$Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)=0 \quad m\geq 1 \qquad \text{(A.37)}$$

$$E\Big(y_{n,t-s}[y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)$$
$$+bE\Big(y_{n,t-s}[y_{j+1,t}-L(y_{j+1,t}|Y_{1,t}^{(-)},\cdots,Y_{j,t}^{(-)},Y_{j+1,t-1}^{(-)},\cdots,$$
$$Y_{n-1,t-1}^{(-)})]\Big)+bE\Big(y_{n,t-s}L[y_{j+1,t}|y_{j,t}-L(y_{j,t}|Y_{1,t}^{(-)},\cdots,Y_{j-1,t}^{(-)},$$
$$Y_{j,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)=0 \quad s\geq 1 \qquad \text{(A.38)}$$

The result for either (A.33) or (A.34) is easily derived using the results for either (A.26) or (A.27) and either (A.37) or (A.38).

(only if part) Using the arguments similar to the proof of Proposition 4, it is shown that (A.35) holds if and only if either (A.39) or (A.40) holds.

$$E\{z_{1,t+m}[z_{j,t}-L(z_{j,t}|Z_{1,t}^{(-)},\cdots,Z_{j-1,t}^{(-)},Z_{j,t-1}^{(-)},\cdots,Z_{n-1,t-1}^{(-)})]\}=0$$
$$m\geq 1,\ j=2,\ \cdots,\ n-1 \qquad \text{(A.39)}$$

$$E\{z_{n,t-s}[z_{j,t}-L(z_{j,t}|Z_{1,t}^{(-)},\cdots,Z_{j-1,t}^{(-)},Z_{j,t-1}^{(-)},\cdots,Z_{n-1,t-1}^{(-)})]\}=0$$
$$s\geq 1,\ j=2,\ \cdots,\ n-1 \qquad \text{(A.40)}$$

Noting that there exists a $(a_{2,j,t},\cdots,a_{n-1,j,t})'\in R^{n-2}$ such that $z_{j,t}=\sum_{r=2}^{n-1} a_{r,j,t}y_{r,t};j=2,\ \cdots,\ n-2$ and $z_{j,t}-L(z_{j,t}|Z_{1,t}^{(-)},\ \cdots,\ Z_{j-1,t}^{(-)},\ Z_{j,t-1}^{(-)},\ \cdots,$ $Z_{n-1,t-1}^{(-)})=\sum_{r=2}^{n-1} a_{r,j,t}\Big(y_{r,t}-L(y_{r,t}|Y_{1,t}^{(-)},\cdots,Y_{r-1,t}^{(-)},Y_{r,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})]\Big)$ $-L\Big(z_{j,t}\Big|\sum_{r=2}^{n-1} a_{r,j,t}\{y_{r,t}-L(y_{r,t}|Y_{1,t}^{(-)},\cdots,Y_{r-1,t}^{(-)},Y_{r,t-1}^{(-)},\cdots,Y_{n-1,t-1}^{(-)})+\sum_{q=2}^{r-1}$

$L[y_{r,t}|y_{q,t}-L(y_{q,t}|Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{q-1,t}, Y^{(-)}_{q,t-1}, \cdots, Y^{(-)}_{n-1,t-1})]\})$, it is immediately established that either (A.39) or (A.40) holds if and only if either (A.33) or (A.34) holds. Q. E. E.

Proof of Proposition 6. The result is obtained in the proof of Proposition 5. Q. E. D.

Proof of Proposition 7. It suffices to show that

(i) $\{y_{3,t}\}$ does not cause $\{y_{1,t}\}$ in Type III's sense if $\{y_{3,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense

(ii) $\{y_{3,t}\}$ does not cause $\{y_{1,t}\}$ in Type II's sense if $\{y_{3,t}\}$ does not cause $\{y_{1,t}\}$ in Type I's sense.

In the case of $n=3$ (the case of only one intermediate variable), Definition 2 and 3 turn to the same, which implies that (i) holds. From Proposition 3, Condition IV is a necessary and sufficient condition for Type I noncausality. In the case of $n=3$, Condition IV is written as

$$\sigma^2(y_{1,h,t+m}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1}, Y^{(-)}_{3,t-1})=\sigma^2(y_{1,h,t+m}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1})$$

$$h=1, \cdots, m_1, m\geq 0, \tag{A.41}$$

From Proposition 4, Condition VII is a necessary and sufficient condition for the Type II noncausality. When $n=3$, Condition VII is written as

$$\sigma^2(y_{1,h,t}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1}, Y^{(-)}_{3,t-1})=\sigma^2(y_{1,h,t}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1})$$

$$h=1, \cdots, m_1 \tag{A.42}$$

$$\sigma^2(y_{1,h,t+m}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1}, Y^{(-)}_{3,t-1})=\sigma^2(y_{1,h,t+m}|Y^{(-)}_{1,t-1}, Y^{(-)}_{2,t-1})$$

$$h=1, \cdots, m_1, m\geq 1, \tag{A.43}$$

$$\sigma^2(y_{1,h,t+m} | Y_{1,t}^{(-)}, Y_{2,t-1}^{(-)}, Y_{3,t-1}^{(-)}) = \sigma^2(y_{1,h,t+m} | Y_{1,t}^{(-)}, Y_{2,t-1}^{(-)})$$

$$h=1, \cdots, m_1, \ m \geq 1, \qquad \text{(A.44)}$$

Since (A.41) is equivalent to (A.42) and (A.43), it suffices to derive (A.44) from (A.42) and (A.43). From the fundamental property of projection, it is shown that (A.44) is equivalent to

$$E\Big(y_{1,t+m}\{y_{3,t-s} - L(y_{3,t-s} | Y_{1,t-1}^{(-)}, Y_{2,t-1}^{(-)})$$

$$- L[y_{3,t-s} | y_{1,t} - L(y_{1,t} | Y_{1,t-1}^{(-)}, Y_{2,t-1}^{(-)})]\}\Big) = 0$$

$$m \geq 1, \ s \geq 1 \qquad \text{(A.44)}'$$

Note that (A.42) implies

$$L[y_{3,t-s} | y_{1,t} - L(y_{1,t} | Y_{1,t-1}^{(-)}, Y_{2,t-1}^{(-)})] = 0$$

$$s \geq 1, \text{ with probability one} \qquad \text{(A.45)}$$

(A.43) and (A.45) imply (A.44)′. Thus (ii) is proved. Q. E. D.

Proof of Proposition 8. (i) (ii) It suffices to show that (16)′, (25) and (26) are equivalent. (25) implies that $\{(y_{2,t}, \cdots, y_{n-1,t})'\}$ does not cause $\{y_{1,t}\}$ in Pierce and Haugh's sense, or equivalently, Sims' sense (see Proposition 1 (i)). (26) implies that $\{(y_{2,t}, \cdots, y_{n-1,t})'\}$ does not cause $\{y_{1,t}\}$ in Granger's sense. (25) or (26) is corresponding to the noncausality in the case of $n=2$ since $\{(y_{2,t}, \cdots, y_{n-1,t})'\}$ can be regarded as $\{y_{n,t}\}$. The equivalence of Granger-noncausality and Sims-noncausality has been proved in many papers. See, for example, Sims[7], Chamberlain[1] and Florens and Mourchart[2, 3]. Thus the equivalence of (25) and (26) is proved. On the other hand, (16)′ is equivalent to

$$L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j,t-m}, Y^{(-)}_{j+1,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})=$$

$$L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j-1,t-m}, Y^{(-)}_{j,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})$$

$$m\geq 1,\ j=2,\ \cdots,\ n-1, \text{ with probability one} \qquad (16)''$$

and (25) is equivalent to

$$L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{n-1,t-1})=L(y_{1,t}|Y^{(-)}_{1,t-m}, Y^{(-)}_{2,t-m-1}, \cdots,$$

$$Y^{(-)}_{n-1,t-m-1}) \quad m\geq 1, \text{ with probability one} \qquad (25)'$$

Noting $L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{n-1,t-m})-L(y_{1,t}|Y^{(-)}_{1,t-m}, Y^{(-)}_{2,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})=\sum_{j=2}^{n-1}\Big(L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j,t-m}, Y^{(-)}_{j+1,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})-L(y_{1,t}|Y^{(-)}_{1,t-m}, \cdots, Y^{(-)}_{j-1,t-m}, Y^{(-)}_{j,t-m-1}, \cdots, Y^{(-)}_{n-1,t-m-1})\Big);m\geq 1$, (16)″ implies (25)′. Thus the equivalence of (16)′ and (25) is proved.

(iii) It suffices to show that (3) and (17)′ are equivalent to (3) and (27). (3) is equivalent to

$$E\{y_{1,t}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})]\}=0, \quad s\geq 1, \qquad (3)'$$

(17)′ is equivalent to

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t}, \cdots, Y^{(-)}_{j-1,t}, Y^{(-)}_{j,t-1}, \cdots,$$

$$Y^{(-)}_{n-1,t-1})]\}=0, \quad s\geq 1,\ j=2,\ \cdots,\ n-1 \qquad (17)''$$

and (27) is equivalent to

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y^{(-)}_{1,t-1}, \cdots, Y^{(-)}_{n-1,t-1})]\}=0$$

$$s\geq 1,\ j=2,\ \cdots,\ n-1 \qquad (27)'$$

From the fundamental property of projection,

$$E\{y_{j,t}[y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)},\ \cdots,\ Y_{j-1,t}^{(-)},\ Y_{j,t-1}^{(-)},\ \cdots,$$

$$Y_{n-1,t-1}^{(-)})]\}=E\Big(y_{j,t}\{y_{n,t-s}-L(y_{n,t-s}|Y_{1,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})$$

$$-L[y_{n,t-s}|y_{r,t}-L(y_{r,t}|Y_{1,t-1},\ \cdots,\ Y_{n-1,t-1});r=1,\ \cdots,$$

$$j-1]\}\Big)=0 \quad s\geq 1,\ j=2,\ \cdots,\ n-1 \qquad \text{(A.46)}$$

Further, (3)′ and (27)′ imply

$$L[y_{n,t-s}|y_{r,t}-L(y_{r,t}|Y_{1,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)});r=1,\ \cdots,\ j-1]=0$$

$$s\geq 1,\ j=2,\ \cdots,\ n-1,\ \text{with probability one} \qquad \text{(A.47)}$$

(17)′ follows from (27)′, (A.46) and (A.47)

On the other hand, we always have

$$E\Big(y_{j,t}\{y_{n,t-s}-L(y_{n,t-s}|Y_{1,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})\}\Big)=$$

$$E\Big(y_{j,t}\{y_{n,t-s}-L(y_{n,t-s}|Y_{1,t}^{(-)},\ \cdots,\ Y_{j-1,t}^{(-)},\ Y_{j,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})$$

$$+\sum_{r=1}^{j-1}\{L(y_{n,t-s}|Y_{1,t}^{(-)},\ \cdots,\ Y_{r,t}^{(-)},\ Y_{r+1,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})$$

$$-L(y_{n,t-s}|Y_{1,t},\ \cdots,\ Y_{r-1,t}^{(-)},\ Y_{r,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})\}\Big)$$

$$s\geq 1,\ j=2,\ \cdots,\ n-1 \qquad \text{(A.48)}$$

Also, (3)′ and (17)″ imply

$$L[y_{n,t-s}|y_{r,t}-L(y_{r,t}|Y_{1,t}^{(-)},\ \cdots,\ Y_{r-1,t}^{(-)},\ Y_{r,t-1}^{(-)},\ \cdots,\ Y_{n-1,t-1}^{(-)})]=0$$

$$s\geq 1,\ j=2,\ \cdots,\ n-1,\ r=1,\ \cdots,$$

$$j-1,\ \text{with probability one} \qquad \text{(A.49)}$$

(27)′ follows from (17)″, (A.48) and (A.49). Thus the equivalence of (17)′ and (27) is established. Q. E. D.

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