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Numerical Computation of Cross-Coupled Algebraic Riccati Equations Related to H_∞-Constrained LQG Control Problem

Hiroaki Mukaidani

Graduate School of Education, Hiroshima University,
1-1-1, Kagamiyama, Higashi-Hiroshima 739-8524 Japan.

Abstract. In this paper, the numerical algorithm for solving the state and output feedback H_∞-constrained LQG control problem is investigated. Although the equations that have to be solved to design the controller consist of the nonlinear cross-coupled algebraic Riccati equations (CAREs), it is newly proven that both the uniqueness and the positive semidefiniteness of the iterative solutions can be guaranteed when disturbance attenuation level γ is sufficiently large. The computational examples are given to demonstrate the efficiency of the proposed algorithm.
keywords: state and output feedback H_∞-constrained LQG control problem, cross-coupled algebraic Riccati equation (CARE), Newton's method, Newton-Kantorovich theorem, reduced-order algorithm.

1 Introduction

Many modern control problems involve solving a set of cross-coupled algebraic Riccati equations (CAREs) (see for example [1, 2, 3]). In [1], an output feedback H_∞-constrained LQG control problem has been formulated. In [2], an H_2-based positive real controller synthesis method has also been investigated. In [3], the global existence of solution to a state feedback mixed H_2/H_∞ control problem has been studied using a dynamic Nash game approach. Although some algorithm for solving the different CAREs have been introduced in these literatures, there is no proof on the convergence of the related algorithm. Moreover, the existence and the uniqueness of the convergence solutions have not been shown.

Up to now, various reliable approaches to the computation of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [8]). One of the approaches is Newton's method [8]. In the past few decades, Newton's method has been applied to the CAREs (see e.g., [9]). However, it is well-known that if the initial conditions are not adequate, the quadratic convergence would not be guaranteed. Moreover, the convergence solutions may not satisfy the required property such as the positive semidefiniteness.

In recent years, Newton's method has been used for various control problems of the singularly perturbed systems (SPS) [4, 5]. It has been shown that Newton's method is very effective and reliable to solve the CAREs of the SPS. However, Newton's method for solving the CAREs related to the H_∞-constrained LQG control problem for the linear state space systems has not been investigated. The reason is that it is difficult to find an appropriate initial guess for the solution for a state space system compared with the SPS. As a result, even if Newton's method is applied to the CAREs related to the H_∞-constrained LQG control problem, the local quadratic convergence may not be guaranteed without the appropriate initial conditions.

In this paper, the state and output feedback H_∞-constrained LQG control problem is investigated from the viewpoint of numerical computation. It should be noted that the state feedback case has not been considered in [1]. The purpose of the paper is to analyze the asymptotic structure of the solution for the CAREs, and to develop the numerical algorithm to solve them. Since the proposed algorithm is based on Newton's method, it is quite different from the existing algorithm [1]. The quadratic local convergence of the algorithm is proved under appropriate initial conditions and a sufficiently large disturbance attenuation level γ. The main idea of this paper is to utilize the theory of the SPS. That is, if the disturbance attenuation level γ is sufficiently large, the newly defined parameter $\varepsilon := \gamma^{-2}$ can be thought as a perturbation. The uniqueness and the asymptotic structure of the solution for the CAREs and the initial conditions are derived by using the implicit function theorem. As a result, the uniqueness and the boundedness of the solution to the CAREs are established. It is worth pointing out that the results obtained are used for the linear state space systems under the large disturbance attenuation level γ. Finally, the computational examples are solved to show the validity of the proposed algorithm.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_p denotes the $p \times p$ identity matrix. vec denotes the column vector of the matrix [7]. $\otimes$ denotes the Kronecker product. $E[\cdot]$ denotes the expectation. $\mathbf{U}_{lm}$ denotes a permutation matrix in Kronecker matrix sense [7] such that $\mathbf{U}_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \mathbf{R}^{l\times m}$.

2 Problem Statement

Consider the following linear system

$$\dot{x}(t) = Ax(t) + D_1 w(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + D_2 w(t), \tag{1b}$$

where, $x(t) \in \mathrm{R}^n$ is the state, $u(t) \in \mathrm{R}^{l_1}$ is the control input, $w(t) \in \mathrm{R}^{l_2}$ is the disturbance, $z(t) \in \mathrm{R}^{k_2}$ is the controlled output. All matrices above are of appropriate dimensions.

The H_∞-constrained LQG control problem addressed in this paper is as follows [1]:

Given the stabilizable and detectable plant (1), determine a dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2a}$$
$$u(t) = C_c x_c(t), \tag{2b}$$

which satisfies the following design criteria:

i. the following closed-loop system is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable.

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{D}w(t), \tag{3a}$$
$$z(t) = \tilde{E}_\infty \tilde{x}(t), \tag{3b}$$

where

$$\tilde{x}(t) := \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix},\ \tilde{A} := \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix},\ \tilde{D} := \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix},\ \tilde{E}_\infty := \begin{bmatrix} E_{1\infty} & 0 \end{bmatrix}.$$

ii. the closed-loop transfer function $H(s) := \tilde{E}_\infty(sI_n - \tilde{A})^{-1}\tilde{D}$ from $w(t)$ to $z(t) := E_{1\infty}x(t) + E_{2\infty}u(t)$ satisfies the constraint

$$\|\tilde{E}_\infty(sI_n - \tilde{A})^{-1}\tilde{D}\|_\infty \le \gamma, \tag{4}$$

where $\gamma > 0$ is a given disturbance attenuation level; and

iii. the performance functional

$$J(A_c,\ B_c,\ C_c) := \lim_{t\to\infty}\frac{1}{t}E\left(\int_0^t [x^T(s)R_1 x(s) + u^T(s)R_2 u(s)]ds\right) \tag{5}$$

is minimized.

Without loss of generality, the following basic assumptions are made [1].

Assumption 1 $D_1 D_2^T = 0$ *is assumed, which effectively implies that plant disturbance and sensor noise are uncorrelated.*

Assumption 2 $(A,\ B,\ C)$ *is assumed to be stabilizable and detectable.*

The following lemma is already known [1].

Lemma 1 *If* $(A_c,\ B_c,\ C_c)$ *solves the auxiliary minimization problem then there exist* Q, P *and* $\hat{Q}$ *such that*

$$A_c := A - Q\bar{\Sigma} - \Sigma P + \gamma^{-2} Q R_{1\infty}, \tag{6a}$$
$$B_c := Q C^T V_2^{-1}, \tag{6b}$$
$$C_c := -R_2^{-1} B^T P, \tag{6c}$$

and such that Q, P *and* $\hat{Q}$ *satisfy*

$$L_1(Q,\ P,\ \hat{Q}) := AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q\bar{\Sigma}Q = 0, \tag{7a}$$
$$L_2(Q,\ P,\ \hat{Q}) := (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1 - P\Sigma P = 0, \tag{7b}$$
$$L_3(Q,\ P,\ \hat{Q}) := (A - \Sigma P + \gamma^{-2} Q R_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \gamma^{-2} Q R_{1\infty})^T + \gamma^{-2}\hat{Q} R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q = 0, \tag{7c}$$

where

$$V_1 = D_1 D_1^T,\ V_2 = D_2 D_2^T,\ R_1 = E_1^T E_1,\ R_2 = E_2^T E_2,$$
$$R_{1\infty} = E_{1\infty}^T E_{1\infty},\ \Sigma = B R_2^{-1} B^T,\ \bar{\Sigma} = C^T V_2^{-1} C.$$

In order to determine a dynamic compensator (2), the CAREs (7) have to be solved. It should be noted that although the algorithm for solving the CAREs (7) has been given in [1], there is no convergence proof. Moreover, it has been stated in [1] that the development of more sophisticated continuation algorithms has been left as future research. Based on this viewpoint, Newton's method seems to be a candidate of an adequate algorithm for solving the CAREs. It is well known that Newton's method is potentially fast and more accurate than the widely used Schur vector method [10] that is superior to the eigenvector approach. The break-even point is between six and eight iterations assuming that an algorithm similar to the one of Bartles-Stewart is used to solve the Lyapunov equation [11]. Hence, Newton's method would result in an appropriate solution for solving the CAREs (7).

3 Preliminary

Let us consider the following equations that are defined as the parameter $\varepsilon := \gamma^{-2}$.

$$M_1(\varepsilon,\ Q,\ P,\ \hat{Q}) := AQ + QA^T + V_1 + \varepsilon Q R_{1\infty} Q - Q\bar{\Sigma}Q = 0, \tag{8a}$$
$$M_2(\varepsilon,\ Q,\ P,\ \hat{Q}) := (A + \varepsilon[Q + \hat{Q}]R_{1\infty})^T P + P(A + \varepsilon[Q + \hat{Q}]R_{1\infty}) + R_1 - P\Sigma P = 0, \tag{8b}$$
$$M_3(\varepsilon,\ Q,\ P,\ \hat{Q}) := (A - \Sigma P + \varepsilon Q R_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \varepsilon Q R_{1\infty})^T + \varepsilon\hat{Q} R_{1\infty}\hat{Q} + Q\bar{\Sigma}Q = 0. \tag{8c}$$

Setting $\varepsilon = 0$ for the previous equations (8), the following equations hold.

$$M_1(0,\ Q^0,\ P^0,\ \hat{Q}^0) := AQ^0 + Q^0 A^T + V_1 - Q^0\bar{\Sigma}Q^0 = 0, \tag{9a}$$
$$M_2(0,\ Q^0,\ P^0,\ \hat{Q}^0) := A^T P^0 + P^0 A + R_1 - P^0\Sigma P^0 = 0, \tag{9b}$$
$$M_3(0,\ Q^0,\ P^0,\ \hat{Q}^0) := (A - \Sigma P^0)\hat{Q}^0 + \hat{Q}^0(A - \Sigma P^0)^T + Q^0\bar{\Sigma}Q^0 = 0, \tag{9c}$$

where Q^0, P^0 and $\hat{Q}^0$ are zeroth-order solutions of the equations (8).

Using (9), the asymptotic structure of the solutions $Q = Q(\varepsilon)$, $P = P(\varepsilon)$ and $\hat{Q} = \hat{Q}(\varepsilon)$ of the CAREs (8) as $M_k(\varepsilon,\ Q,\ P,\ \hat{Q}) = 0$ is established.

Theorem 1 *Then there exists small* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0,\ \bar{\varepsilon})$, *the CAREs (8) admits the unique solutions* Q, P *and* $\hat{Q}$ *in the neighborhood of* $\varepsilon = 0$, *which can be written as*

$$Q(\varepsilon) = Q^0 + O(\varepsilon), \tag{10a}$$
$$P(\varepsilon) = P^0 + O(\varepsilon), \tag{10b}$$
$$\hat{Q}(\varepsilon) = \hat{Q}^0 + O(\varepsilon). \tag{10c}$$

Proof : It can be done by applying the implicit function theorem to the CAREs (8). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. Taking the partial derivative of the function $M_k(Q,\ P,\ \hat{Q}),\ k = 1,\ 2,\ 3$ with respect to Q, P, $\hat{Q}$ and setting $\varepsilon = 0$ result in (11).

$$\hat{\mathcal{J}}(0,\ Q^0,\ P^0,\ \hat{Q}^0) = \begin{bmatrix} \Psi_1^{(0)} & 0 & 0 \\ 0 & \Psi_2^{(0)} & 0 \\ \Psi_3^{(0)} & \Psi_4^{(0)} & \Psi_2^{(0)} \end{bmatrix}, \tag{11}$$

where

$$\hat{\mathcal{J}}(\varepsilon,\ Q,\ P,\ \hat{Q}) := \begin{bmatrix} \dfrac{\partial \mathrm{vec} M_1}{\partial (\mathrm{vec} Q)^T} & \dfrac{\partial \mathrm{vec} M_1}{\partial (\mathrm{vec} P)^T} & \dfrac{\partial \mathrm{vec} M_1}{\partial (\mathrm{vec} \hat{Q})^T} \\ \dfrac{\partial \mathrm{vec} M_2}{\partial (\mathrm{vec} Q)^T} & \dfrac{\partial \mathrm{vec} M_2}{\partial (\mathrm{vec} P)^T} & \dfrac{\partial \mathrm{vec} M_2}{\partial (\mathrm{vec} \hat{Q})^T} \\ \dfrac{\partial \mathrm{vec} M_3}{\partial (\mathrm{vec} Q)^T} & \dfrac{\partial \mathrm{vec} M_3}{\partial (\mathrm{vec} P)^T} & \dfrac{\partial \mathrm{vec} M_3}{\partial (\mathrm{vec} \hat{Q})^T} \end{bmatrix}, \tag{12}$$

$$\Psi_1^{(0)} := (A - Q^0\bar{\Sigma}) \otimes I_n + I_n \otimes (A - Q^0\bar{\Sigma}),\ \Psi_2^{(0)} := (A - \Sigma P^0)^T \otimes I_n + I_n \otimes (A - \Sigma P^0)^T,$$
$$\Psi_3^{(0)} := I_n \otimes (Q^0\bar{\Sigma}) + (Q^0\bar{\Sigma}) \otimes I_n,\ \Psi_4^{(0)} := -\hat{Q}^0 \otimes \Sigma - \Sigma \otimes \hat{Q}^0.$$

Obviously, $A - Q^0\bar{\Sigma}$ and $A - \Sigma P^0$ are nonsingular because the algebraic Riccati equations (AREs) (9a) and (9b) have the positive semidefinite stabilizing solution under Assumption 2. Thus, $\hat{\mathcal{J}}(\varepsilon,\ Q,\ P,\ \hat{Q})$ is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. ■

4 Newton's Method

In order to obtain the solutions of the CAREs (8), the following new algorithm is given.

$$\begin{aligned} &(A + \gamma^{-2}Q^{(i)}R_{1\infty} - Q^{(i)}\bar{\Sigma})Q^{(i+1)} + Q^{(i+1)}(A + \gamma^{-2}Q^{(i)}R_{1\infty} - Q^{(i)}\bar{\Sigma})^T \\ &\quad + V_1 - \gamma^{-2}Q^{(i)}R_{1\infty}Q^{(i)} + Q^{(i)}\bar{\Sigma}Q^{(i)} = 0, \end{aligned} \tag{13a}$$

$$\begin{aligned} &(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})^T P^{(i+1)} + P^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty}) \\ &\quad + \gamma^{-2}R_{1\infty}Q^{(i+1)}P^{(i)} + \gamma^{-2}P^{(i)}Q^{(i+1)}R_{1\infty} + \gamma^{-2}R_{1\infty}\hat{Q}^{(i+1)}P^{(i)} + \gamma^{-2}P^{(i)}\hat{Q}^{(i+1)}R_{1\infty} \\ &\quad - \gamma^{-2}R_{1\infty}Q^{(i)}P^{(i)} - \gamma^{-2}P^{(i)}Q^{(i)}R_{1\infty} - \gamma^{-2}R_{1\infty}\hat{Q}^{(i)}P^{(i)} - \gamma^{-2}P^{(i)}\hat{Q}^{(i)}R_{1\infty} \\ &\quad + P^{(i)}\Sigma P^{(i)} + R_1 = 0, \end{aligned} \tag{13b}$$

$$\begin{aligned} &(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})\hat{Q}^{(i+1)} + \hat{Q}^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})^T \\ &\quad - \Sigma P^{(i+1)}\hat{Q}^{(i)} - \hat{Q}^{(i)}P^{(i+1)}\Sigma + \gamma^{-2}Q^{(i+1)}R_{1\infty}\hat{Q} + \gamma^{-2}\hat{Q}R_{1\infty}Q^{(i+1)} + Q^{(i)}\bar{\Sigma}Q^{(i+1)} + Q^{(i+1)}\bar{\Sigma}Q^{(i)} \\ &\quad + \Sigma P^{(i)}\hat{Q}^{(i)} + \hat{Q}^{(i)}P^{(i)}\Sigma - Q^{(i)}\bar{\Sigma}Q^{(i)} - \gamma^{-2}Q^{(i)}R_{1\infty}\hat{Q}^{(i)} - \gamma^{-2}\hat{Q}^{(i)}R_{1\infty}Q^{(i)} \\ &\quad - \gamma^{-2}\hat{Q}^{(i)}R_{1\infty}\hat{Q}^{(i)} = 0, \end{aligned} \tag{13c}$$

where $Q^{(0)}$, $P^{(0)}$ and $\hat{Q}^{(0)}$ satisfy the AREs (9) as $Q^{(0)} = Q^0$, $P^{(0)} = P^0$ and $\hat{Q}^{(0)} = \hat{Q}^0$, respectively.

The new algorithm (13) can be constructed by setting $Q^{(i+1)} = Q^{(i)} + \Delta Q^{(i)}$, $P^{(i+1)} = P^{(i)} + \Delta P^{(i)}$ and $\hat{Q}^{(i+1)} = \hat{Q}^{(i)} + \Delta\hat{Q}^{(i)}$, and neglecting $O(\Delta^2)$ term.

Theorem 2 *Suppose that there exist solutions to the CAREs (7). It can be obtained by performing the algorithm (13) which is equal to Newton's method.*

Proof : Taking the vec-operator transformation on both sides of (7) results in

$$\begin{bmatrix} \mathrm{vec} L_1(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_2(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_3(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \end{bmatrix} = \Phi^{(i)} \begin{bmatrix} \mathrm{vec} Q^{(i)} \\ \mathrm{vec} P^{(i)} \\ \mathrm{vec} \hat{Q}^{(i)} \end{bmatrix} + \begin{bmatrix} \mathrm{vec} G^{(i)} \\ \mathrm{vec} H^{(i)} \\ \mathrm{vec} J^{(i)} \end{bmatrix}, \tag{14}$$

where

$$\Phi^{(i)} = \begin{bmatrix} I_n \otimes D^{(i)} + D^{(i)} \otimes I_n & 0 & 0 \\ \Xi^{(i)} & I_n \otimes E^{(i)T} + E^{(i)T} \otimes I_n & \Xi^{(i)} \\ I_n \otimes K^{(i)} + K^{(i)} \otimes I_n & -\hat{Q}^{(i)} \otimes \Sigma - \Sigma \otimes \hat{Q}^{(i)} & I_n \otimes E^{(i)} + E^{(i)} \otimes I_n \end{bmatrix},$$

$$D^{(i)} = A + \gamma^{-2} Q^{(i)} R_{1\infty} - Q^{(i)} \bar{\Sigma},\ E^{(i)} = A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}] R_{1\infty},$$
$$\Xi^{(i)} = P^{(i)} \otimes (\gamma^{-2} R_{1\infty}) + (\gamma^{-2} R_{1\infty}) \otimes P^{(i)},\ K^{(i)} = \gamma^{-2} \hat{Q}^{(i)} R_{1\infty} + Q^{(i)} \bar{\Sigma},$$
$$G^{(i)} = V_1 - \gamma^{-2} Q^{(i)} R_{1\infty} Q^{(i)} + Q^{(i)} \bar{\Sigma} Q^{(i)},$$
$$H^{(i)} = -\gamma^{-2} R_{1\infty} Q^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} Q^{(i)} R_{1\infty} - \gamma^{-2} R_{1\infty} \hat{Q}^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} \hat{Q}^{(i)} R_{1\infty} + P^{(i)} \Sigma P^{(i)} + R_1,$$
$$J^{(i)} = \Sigma P^{(i)} \hat{Q}^{(i)} + \hat{Q}^{(i)} P^{(i)} \Sigma - Q^{(i)} \bar{\Sigma} Q^{(i)} - \gamma^{-2} Q^{(i)} R_{1\infty} \hat{Q}^{(i)} - \gamma^{-2} \hat{Q}^{(i)} R_{1\infty} Q^{(i)} - \gamma^{-2} \hat{Q}^{(i)} R_{1\infty} \hat{Q}^{(i)}.$$

Moreover, taking the vec-operator transformation on both sides of (13) results in

$$\Phi^{(i)} \begin{bmatrix} \mathrm{vec} Q^{(i+1)} \\ \mathrm{vec} P^{(i+1)} \\ \mathrm{vec} \hat{Q}^{(i+1)} \end{bmatrix} + \begin{bmatrix} \mathrm{vec} G^{(i)} \\ \mathrm{vec} H^{(i)} \\ \mathrm{vec} J^{(i)} \end{bmatrix} = 0. \tag{15}$$

Subtracting (14) from (15) and using (12), it is easy to verify that

$$\begin{bmatrix} \mathrm{vec} Q^{(i+1)} \\ \mathrm{vec} P^{(i+1)} \\ \mathrm{vec} \hat{Q}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \mathrm{vec} Q^{(i)} \\ \mathrm{vec} P^{(i)} \\ \mathrm{vec} \hat{Q}^{(i)} \end{bmatrix}$$
$$-\hat{\mathcal{J}}(\varepsilon,\ Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)})^{-1} \times \begin{bmatrix} \mathrm{vec} L_1(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_2(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \\ \mathrm{vec} L_3(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)}) \end{bmatrix}. \tag{16}$$

This is the desired result. ■

Newton's method is well-known and is widely used to find a solution of algebraic nonlinear equations. Its local convergence properties are well understood [6]. Particularly, it is highly expected that the proposed algorithm can converge to the adequate solutions because the initial conditions are close to the exact solutions with the structure of (10) under the sufficiently small parameter $\varepsilon = \gamma^{-2}$. The following theorem indicates the local quadratic convergence and the uniqueness for the convergence solutions.

Theorem 3 *Assume that the conditions of Theorem 1 hold. Then, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, Newton's method (13) converges to the exact solution of Q^*, P^* and $\hat{Q}^*$ with the rate of the quadratic convergence. Moreover, the convergence solutions Q^*, P^* and $\hat{Q}^*$ are unique solution of the CAREs (8) in the neighborhood of the initial conditions $Q^{(0)} = Q^0$, $P^{(0)} = P^0$, $\hat{Q}^{(0)} = \hat{Q}^0$, respectively. That is, the following relations are satisfied.*

$$\|Q^{(i)} - Q^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ , \tag{17a}$$
$$\|P^{(i)} - P^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ , \tag{17b}$$
$$\|\hat{Q}^{(i)} - \hat{Q}^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ . \tag{17c}$$

Proof : The proof of this theorem can be done by using Newton-Kantorovich theorem [6]. It is immediately obtained from the equation (12) that there exists the positive scalar constant $\mathcal{L}(\varepsilon)$ such that for any Q^a, P^a, $\hat{Q}^a$, Q^b, P^b and $\hat{Q}^b$,

$$\|\hat{\mathcal{J}}(\varepsilon,\ Q^a,\ P^a,\ \hat{Q}^a) - \hat{\mathcal{J}}(\varepsilon,\ Q^b,\ P^b,\ \hat{Q}^b)\| \le \mathcal{L}(\varepsilon)\|(Q^a,\ P^a,\ \hat{Q}^a) - (Q^b,\ P^b,\ \hat{Q}^b)\|. \tag{18}$$

Moreover, using (10), we get

$$\hat{\mathcal{J}}(\varepsilon,\ Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)}) = \bar{\mathcal{J}}(0,\ Q^0,\ P^0,\ \hat{Q}^0) + O(\varepsilon). \tag{19}$$

Hence, it follows that $\hat{\mathcal{J}}(\varepsilon,\ Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)})$ is nonsingular under $\det \hat{\mathcal{J}}(0,\ Q^0,\ P^0,\ \hat{Q}^0) \neq 0$ for sufficiently small ε. Therefore, there exists β such that $\beta = \|\hat{\mathcal{J}}(\varepsilon,\ Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)})^{-1}\|$. On the other hand, since $L_k(Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)}) = O(\varepsilon)$, there exists η such that $\eta = \|[\hat{\mathcal{J}}(\varepsilon,\ Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)})]^{-1}\| \cdot \|L_k(Q^{(0)},\ P^{(0)},\ \hat{Q}^{(0)})\| = O(\varepsilon)$. Thus, there exists θ such that $\theta = \beta\eta\mathcal{L}(\varepsilon) < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, using the Newton-Kantorovich theorem, we can show that Q^*, P^* and $\hat{Q}^*$ are the unique solution in the subset. Moreover, the error estimate is given by (17). ■

It may be noted that the Newton's method (13) is well defined. That is, the Lyapunov equations in (13) are solvable in each step due to the following reason. For the sufficiently large parameter γ, the following equation holds.

$$\Phi^{(i)} \to \Psi^{(i)} := \begin{bmatrix} \Psi_1^{(i)} & 0 & 0 \\ 0 & \Psi_2^{(i)} & 0 \\ \Psi_3^{(i)} & \Psi_4^{(i)} & \Psi_2^{(i)} \end{bmatrix}, \tag{20}$$

where

$$\Psi_1^{(i)} := (A - Q^{(i)}\bar{\Sigma}) \otimes I_n + I_n \otimes (A - Q^{(i)}\bar{\Sigma}),\ \Psi_2^{(i)} := (A - \Sigma P^{(i)})^T \otimes I_n + I_n \otimes (A - \Sigma P^{(i)})^T,$$
$$\Psi_3^{(i)} := I_n \otimes (Q^{(i)}\bar{\Sigma}) + (Q^{(i)}\bar{\Sigma}) \otimes I_n,\ \Psi_4^{(i)} := -\hat{Q}^{(i)} \otimes \Sigma - \Sigma \otimes \hat{Q}^{(i)},$$
$$A - Q^{(i)}\bar{\Sigma} = A - Q^0\bar{\Sigma} + O(\varepsilon),\ A - \Sigma P^{(i)} = A - \Sigma P^0 + O(\varepsilon).$$

The matrices $A - Q^0\bar{\Sigma}$ and $A - \Sigma P^0$ are stable because the AREs (9a) and (9b) have the positive semidefinite stabilizing solutions. Thus, if the parameter $\varepsilon = \gamma^{-2}$ is small, $A - Q^{(i)}\bar{\Sigma}$ and $A - \Sigma P^{(i)}$ are also stable. Finally, the Newton's method (13) is well defined for each step.

The algorithm is now summarized.

Step 1. Calculate Q^0, P^0 and $\hat{Q}^0$ by using the initial conditions (9).

Step 2. Compute the solutions $Q^{(i+1)}$, $P^{(i+1)}$ and $\hat{Q}^{(i+1)}$ by using the following linear equation.

$$\begin{bmatrix} \mathrm{vec}Q^{(i+1)} \\ \mathrm{vec}P^{(i+1)} \\ \mathrm{vec}\hat{Q}^{(i+1)} \end{bmatrix} = -[\Phi^{(i)}]^{-1} \begin{bmatrix} \mathrm{vec}G^{(i)} \\ \mathrm{vec}H^{(i)} \\ \mathrm{vec}J^{(i)} \end{bmatrix}. \tag{21}$$

Step 3. If $i \geq 1$ check for

$$\mathcal{E}(\varepsilon) := \sum_{k=1}^{3} \|L_k(Q^{(i)},\ P^{(i)},\ \hat{Q}^{(i)})\| < \phi \tag{22}$$

for a given convergence criterion $\phi > 0$.

Step 4. If convergence is not achieved in Step 3, increment $i \to i+1$ and go to Step 2. Otherwise, stop Newton iterations (13) and compute the controller (2) with the relations (6).

4.1 Reduced-order Computations for Newton's Method

Ones need to solve the linear equation (21) with quite large dimension $3n^2 \times 3n^2$. Thus, in order to avoid this drawback, a computation method to solve these linear equations is established.

Let us consider the following differential equations.

$$\dot{\mathbf{Q}} = D^{(i)}\mathbf{Q} + \mathbf{Q}D^{(i)T} + G^{(i)}, \tag{23a}$$
$$\dot{\mathbf{P}} = \gamma^{-2}R_{1\infty}\mathbf{Q}P^{(i)} + \gamma^{-2}P^{(i)}\mathbf{Q}R_{1\infty} + E^{(i)T}\mathbf{P} + \mathbf{P}E^{(i)} + \gamma^{-2}R_{1\infty}\hat{\mathbf{Q}}P^{(i)} + \gamma^{-2}P^{(i)}\hat{\mathbf{Q}}R_{1\infty} + H^{(i)}, \tag{23b}$$
$$\dot{\hat{\mathbf{Q}}} = K^{(i)}\mathbf{Q} + \mathbf{Q}K^{(i)T} - \Sigma\mathbf{P}\hat{Q}^{(i)} - \hat{Q}^{(i)}\mathbf{P}\Sigma + E^{(i)}\hat{\mathbf{Q}} + \hat{\mathbf{Q}}E^{(i)T} + J^{(i)}, \tag{23c}$$

where

$$\mathbf{Q} := \mathbf{Q}(t),\ \mathbf{P} := \mathbf{P}(t),\ \hat{\mathbf{Q}} := \hat{\mathbf{Q}}(t),\ \mathbf{Q}(0) = I_n,\ \mathbf{P}(0) = I_n,\ \hat{\mathbf{Q}}(0) = I_n.$$

It is important to note that the stability of the differential equations (23) is guaranteed because the matrix (20) is stable for sufficiently large γ. Thus, the solutions of (23) tend to some finite values as $t \to \infty$. Finally, the required solutions of the linear algebraic equations (13) can be obtained as the convergence solutions. It should be noted that a fourth order Runge-Kutta method is used to integrate the differential equations (23).

In this case, since the required workspace for the matrix calculus is $3n \times 3n$, the proposed computation method is very attractive in the sense that it is easy to implement. For example, in the next numerical example, when the dimension $n = 8$ the proposed algorithm requires 192×192 dimensions, while the algorithm (21) requires 24×24 dimensions for the matrix calculation. It is concluded that such example results in a 15.625% reduction of the workspace compared with the existing result.

5 Extension to General Case of CAREs

In this section the extension of Lemma 1 is considered. That is, the compensator that has the fixed dimension n_c which may be less than the plant order n is established.

Lemma 2 *[1] Let $n_c \le n$, suppose there exist Q, P, $\hat{Q}$ and $\hat{P}$ satisfying*

$$H_1(Q,\ P,\ \hat{Q},\ \hat{P}) := AQ + QA^T + V_1 + \gamma^{-2} Q R_{1\infty} Q - Q\bar{\Sigma}Q + \Lambda Q \bar{\Sigma} Q \Lambda^T = 0, \tag{24a}$$

$$H_2(Q,\ P,\ \hat{Q},\ \hat{P}) := (A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty})^T P + P(A + \gamma^{-2}[Q + \hat{Q}]R_{1\infty}) + R_1 - P\Sigma P + \Lambda^T P \Sigma P \Lambda = 0, \tag{24b}$$

$$H_3(Q,\ P,\ \hat{Q},\ \hat{P}) := (A - \Sigma P + \gamma^{-2} Q R_{1\infty})\hat{Q} + \hat{Q}(A - \Sigma P + \gamma^{-2} Q R_{1\infty})^T + \gamma^{-2}\hat{Q} R_{1\infty} \hat{Q} + Q\bar{\Sigma}Q - \Lambda Q \bar{\Sigma} Q \Lambda^T = 0, \tag{24c}$$

$$H_4(Q,\ P,\ \hat{Q},\ \hat{P}) := (A - Q\bar{\Sigma} + \gamma^{-2} Q R_{1\infty})^T \hat{P} + \hat{P}(A - Q\bar{\Sigma} + \gamma^{-2} Q R_{1\infty}) + P\Sigma P - \Lambda P \Sigma P \Lambda^T = 0, \tag{24d}$$

$$\mathrm{rank}\hat{Q} = \mathrm{rank}\hat{P} = \mathrm{rank}\hat{Q}\hat{P} = n_c. \tag{24e}$$

Then $(A_c,\ B_c,\ C_c)$ be given by

$$A_c := \Gamma(A - Q\bar{\Sigma} - \Sigma P + \gamma^{-2} Q R_{1\infty}) G^T, \tag{25a}$$

$$B_c := \Gamma Q C^T V_2^{-1}, \tag{25b}$$

$$C_c := -R_2^{-1} B^T P G^T, \tag{25c}$$

where $\hat{Q}\hat{P} = G^T M \Gamma$, $\Gamma G^T = I_{n_c}$, $\Lambda = I_n - G^T\Gamma$, $G,\ \Gamma \in \mathrm{R}^{n_c \times n}$, $M \in \mathrm{R}^{n_c \times n_c}$.

Moreover, the reduced-order dynamic output controller (2) with (25) satisfy the conditions i, ii and iii.

It should be noted that if $\mathrm{rank}\hat{Q}\hat{P} = n_c$ holds, it has been shown from [1] that there exist the matrices G, Γ, and M.

In order to obtain the solutions of the CAREs (24), Newton's method is given as follows.

$$(A + \gamma^{-2} Q^{(i)} R_{1\infty} - Q^{(i)}\bar{\Sigma}) Q^{(i+1)} + Q^{(i+1)}(A + \gamma^{-2} Q^{(i)} R_{1\infty} - Q^{(i)}\bar{\Sigma})^T + \Lambda[Q^{(i+1)}\bar{\Sigma} + \bar{\Sigma}Q^{(i+1)}]\Lambda + V_1 - \gamma^{-2} Q^{(i)} R_{1\infty} Q^{(i)} + Q^{(i)}\bar{\Sigma}Q^{(i)} - \Lambda Q^{(i)}\bar{\Sigma}Q^{(i)}\Lambda = 0, \tag{26a}$$

$$(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})^T P^{(i+1)} + P^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty}) + \Lambda[P^{(i+1)}\Sigma + \Sigma P^{(i+1)}]\Lambda + \gamma^{-2} R_{1\infty} Q^{(i+1)} P^{(i)} + \gamma^{-2} P^{(i)} Q^{(i+1)} R_{1\infty} + \gamma^{-2} R_{1\infty} \hat{Q}^{(i+1)} P^{(i)} + \gamma^{-2} P^{(i)} \hat{Q}^{(i+1)} R_{1\infty} - \gamma^{-2} R_{1\infty} Q^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} Q^{(i)} R_{1\infty} - \gamma^{-2} R_{1\infty} \hat{Q}^{(i)} P^{(i)} - \gamma^{-2} P^{(i)} \hat{Q}^{(i)} R_{1\infty} + P^{(i)}\Sigma P^{(i)} + R_1 - \Lambda P^{(i)}\Sigma P^{(i)}\Lambda = 0, \tag{26b}$$

$$(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})\hat{Q}^{(i+1)} + \hat{Q}^{(i+1)}(A - \Sigma P^{(i)} + \gamma^{-2}[Q^{(i)} + \hat{Q}^{(i)}]R_{1\infty})^T - \Sigma P^{(i+1)}\hat{Q}^{(i)} - \hat{Q}^{(i)}P^{(i+1)}\Sigma + \gamma^{-2} Q^{(i+1)} R_{1\infty}\hat{Q} + \gamma^{-2}\hat{Q} R_{1\infty} Q^{(i+1)} - \Lambda[Q^{(i+1)}\bar{\Sigma} + \bar{\Sigma}Q^{(i+1)}]\Lambda + Q^{(i)}\bar{\Sigma}Q^{(i+1)} + Q^{(i+1)}\bar{\Sigma}Q^{(i)} + \Sigma P^{(i)}\hat{Q}^{(i)} + \hat{Q}^{(i)}P^{(i)}\Sigma - Q^{(i)}\bar{\Sigma}Q^{(i)} - \gamma^{-2} Q^{(i)} R_{1\infty}\hat{Q}^{(i)} - \gamma^{-2}\hat{Q}^{(i)} R_{1\infty} Q^{(i)} - \gamma^{-2}\hat{Q}^{(i)} R_{1\infty}\hat{Q}^{(i)} + \Lambda Q^{(i)}\bar{\Sigma}Q^{(i)}\Lambda = 0, \tag{26c}$$

$$(A - Q^{(i)}\bar{\Sigma} + \gamma^{-2} Q^{(i)} R_{1\infty})^T \hat{P}^{(i+1)} + \hat{P}^{(i+1)}(A - Q^{(i)}\bar{\Sigma} + \gamma^{-2} Q^{(i)} R_{1\infty}) - \bar{\Sigma}Q^{(i+1)}\hat{P}^{(i)} - \hat{P}^{(i)}Q^{(i+1)}\bar{\Sigma} + \gamma^{-2} R_{1\infty} Q^{(i+1)}\hat{P}^{(i)} + \gamma^{-2}\hat{P}^{(i)} Q^{(i+1)} R_{1\infty} + P^{(i)}\Sigma P^{(i+1)} + P^{(i+1)}\Sigma P^{(i)} - \Lambda[P^{(i+1)}\Sigma + \Sigma P^{(i+1)}]\Lambda + \bar{\Sigma}Q^{(i)}\hat{P}^{(i)} + \hat{P}^{(i)}Q^{(i)}\bar{\Sigma} - \gamma^{-2} R_{1\infty} Q^{(i)}\hat{P}^{(i)} - \gamma^{-2}\hat{P}^{(i)} Q^{(i)} R_{1\infty} - P^{(i)}\Sigma P^{(i)} + \Lambda P^{(i)}\Sigma P^{(i)}\Lambda = 0, \tag{26d}$$

where $Q^{(0)}$, $P^{(0)}$, $\hat{Q}^{(0)}$ and $\hat{P}^{(0)}$ satisfies the following AREs, respectively.

$$Q^{(0)} = Q^+,\ P^{(0)} = P^+,\ \hat{Q}^{(0)} = \hat{Q}^+,\ \hat{P}^{(0)} = \hat{P}^+,$$

$$AQ^+ + Q^+A^T + V_1 - Q^+\bar{\Sigma}Q^+ + \Lambda Q^+\bar{\Sigma}Q^+\Lambda^T = 0, \tag{27a}$$

$$A^TP^+ + P^+A + R_1 - P^+\Sigma P^+ + \Lambda^TP^+\Sigma P^+\Lambda = 0, \tag{27b}$$

$$(A - \Sigma P^+)\hat{Q}^+ + \hat{Q}^+(A - \Sigma P^+)^T + Q^+\bar{\Sigma}Q^+ - \Lambda Q^+\bar{\Sigma}Q^+\Lambda^T = 0, \tag{27c}$$

$$(A - Q^+\hat{\Sigma})^T\hat{P}^+ + \hat{P}^+(A - Q^+\hat{\Sigma}) + P^+\Sigma P^+ - \Lambda^TP^+\Sigma P^+\Lambda = 0. \tag{27d}$$

Remark 1 *It should be noted that the AREs (27a) and (27b) are not the ordinary ones. However, these AREs can be solved by applying Newton's method. For example, Newton's method for solving the ARE (27a) is given below.*

$$(A - Q^{+(i)}\bar{\Sigma})Q^{+(i+1)} + \Lambda Q^{+(i)}\bar{\Sigma}Q^{+(i+1)}\Lambda^T + Q^{+(i+1)}(A - Q^{+(i)}\bar{\Sigma})^T + \Lambda Q^{+(i+1)}\bar{\Sigma}Q^{+(i)}\Lambda^T$$
$$+V_1 + Q^{+(i)}\bar{\Sigma}Q^{+(i)} - \Lambda Q^{+(i)}\bar{\Sigma}Q^{+(i)}\Lambda^T = 0,$$

where $Q^{+(0)}$ is the positive semidefinite solutions such that $A - Q^{+(0)}\bar{\Sigma}$ is stable.

On the other hand, the algebraic Lyapunov equations (ALEs) can also be solved by substituting the solutions Q^+ and P^+ into the ALEs (27c) and (27d).

Theorem 4 *Suppose that there exist solutions to the CAREs (24). It can be obtained by performing the algorithm (26).*

Proof : Since the proof can be done by using the similar manner used by Theorem 2, it is omitted. ■

The following theorem is easily seen in view of Newton's method.

Theorem 5 *Assume that the conditions of Theorem 1 hold. Then, there exists a small ρ^* such that for all $\varepsilon \in (0,\ \rho^*)$, Newton's method (26) converges to the exact solution of Q^*, P^*, $\hat{Q}^*$ and $\hat{P}^*$ with the rate of the quadratic convergence. Moreover, these convergence solutions are unique solution of the CAREs (24) in the neighborhood of the initial conditions.*

$$\|Q^{(i)} - Q^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ , \tag{28a}$$

$$\|P^{(i)} - P^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ , \tag{28b}$$

$$\|\hat{Q}^{(i)} - \hat{Q}^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ , \tag{28c}$$

$$\|\hat{P}^{(i)} - \hat{P}^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \dots\ . \tag{28d}$$

Proof : It is clear that this theorem can also be proved by using Newton-Kantorovich theorem [6]. That is, the CAREs are smooth and there exists the partial derivative of the function $H_k(Q,\ P,\ \hat{Q},\ \hat{P}) = 0$. Thus, since the similar manner used in the proof of Theorem 3 results in (28), it is omitted. ■

It should be noted that the following asymptotic structure (29) can be obtained by setting $i = 0$ without implicit function theorem.

$$Q(\varepsilon) = Q^+ + O(\varepsilon), \tag{29a}$$

$$P(\varepsilon) = P^+ + O(\varepsilon), \tag{29b}$$

$$\hat{Q}(\varepsilon) = \hat{Q}^+ + O(\varepsilon), \tag{29c}$$

$$\hat{P}(\varepsilon) = \hat{P}^+ + O(\varepsilon). \tag{29d}$$

6 State Feedback H_∞-Constrained LQG Control Problem

In this section, the state feedback H_∞-constrained LQG control problem is investigated. Thus far, it should be noted that the state feedback case has not been considered in [1]. Moreover, since the implementation of the controller is easy whenever the state information is available, it appears that it is important to study this control problem.

Consider the following linear system

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{30a}$$
$$z_i(t) = C_i x(t) + D_i u(t),\ i = 0,\ 1, \tag{30b}$$

where, $x(t) \in \mathrm{R}^n$ is the state, $u(t) \in \mathrm{R}^{l_1}$ is the control input, $w(t) \in \mathrm{R}^{l_2}$ is the disturbance with $E[w(t)w^T(t)] = I_{l_2}$, $z_i(t) \in \mathrm{R}^{k_i}$, $i = 1,\ 2$ is the controlled output. All matrices above are of appropriate dimensions. In the following analysis, the basic assumption is needed.

Assumption 3 *The triple* $(A,\ B_i,\ C_i)$, $i = 0,\ 1$ *is stabilizable and detectable.*

Moreover, without loss of generality, the following assumption is made in the LQG control setting.

Assumption 4 $C_0^T C_0 := R_1 \geq 0$, $C_0^T D_0 = 0$, $D_0^T D_0 := R_2 > 0$.

The state feedback H_∞-constrained LQG control problem addressed in this paper is as follows:
Given the stabilizable and detectable plant (30), determine a state feedback gain K

$$u(t) = Kx(t) \tag{31}$$

which satisfies the following design criteria:

i. the following closed-loop system is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable.

$$\dot{x}(t) = \tilde{A}x(t) + B_1 w(t), \tag{32a}$$
$$z_i(t) = \tilde{C}_i x(t),\ i = 0,\ 1, \tag{32b}$$

where $\tilde{A} = A + B_2 K$ and $\tilde{C}_i = C_i + D_i K$, $i = 0,\ 1$.

ii. the closed-loop transfer function $H(s) := \tilde{C}_1 (sI_n - \tilde{A})^{-1} B_1$ from $w(t)$ to $z_1(t) := C_1 x(t) + D_1 u(t)$ satisfies the constraint

$$\|\tilde{C}_1 (sI_n - \tilde{A})^{-1} B_1\|_\infty < \gamma, \tag{33}$$

where $\gamma > 0$ is a given disturbance attenuation level; and

iii. the following performance index is minimized

$$J(K) = \lim_{t\to\infty} E[x^T (C_0 + D_0 K)^T (C_0 + D_0 K)x] = \lim_{t\to\infty} E[x^T (R_1 + K^T R_2 K)x] = \mathbf{Tr}[\tilde{Q}\tilde{R}], \tag{34}$$

where $\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + B_1 B_1^T = 0$, $\tilde{R} := R_1 + K^T R_2 K = \tilde{C}_0^T \tilde{C}_0$, $\tilde{Q} := \lim_{t\to\infty} E[x(t)x(t)^T]$.

As an important implication of [1], the following theorem can be established.

Theorem 6 *Suppose there exists a* $Q > 0$ *such that* $\tilde{A}^T + Q\tilde{R}_\infty$ *is stable, thereby satisfying ARE (35).*

$$\tilde{A}Q + Q\tilde{A}^T + Q\tilde{R}_\infty Q + B_1 B_1^T = 0, \tag{35}$$

where $\tilde{R}_\infty = \gamma^{-2}\tilde{C}_1^T \tilde{C}_1$.

Then, $\tilde{A}$ *is asymptotically stable. Moreover,* $\|H(s)\|_\infty < \gamma$ *and* $\tilde{Q} \leq Q$.

Furthermore, in order for K *to exist such that the upper bound* $J(K) = \mathbf{Tr}[\tilde{Q}\tilde{R}] \leq \mathbf{Tr}[Q\tilde{R}]$ *becomes as small as possible, it is necessary that there exist* P *and* K *that satisfy (36a) and (36b).*

$$(\tilde{A} + Q\tilde{R}_\infty)^T P + P(\tilde{A} + Q\tilde{R}_\infty) + \tilde{R} = 0, \tag{36a}$$
$$R_2 K + B_2^T P + \gamma^{-2} D_1^T \tilde{C}_1 QP = 0, \tag{36b}$$

where $\tilde{C}_i = C_i + D_i K$, $i = 0,\ 1$.

It should be noted that since it is very difficult to solve the optimization problem of the cost $\mathbf{Tr}[\tilde{Q}\tilde{R}]$, the cost bound $\mathbf{Tr}[Q\tilde{R}]$ should be minimized by using $\tilde{Q} \le Q$.
Proof : The first part of this theorem can be directly obtained by applying the bounded real lemma. The second part can be proved by tracing the existing result of [12]. To prove $\tilde{Q} \le Q$, subtracting $\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + B_1 B_1^T = 0$ from (35) results in

$$\tilde{A}(Q - \tilde{Q}) + (Q - \tilde{Q})\tilde{A}^T + Q\tilde{R}_\infty Q = 0.$$

Since $\tilde{A}$ is stable, it is easy to obtain the following equation.

$$Q - \tilde{Q} = \int_0^\infty \exp[\tilde{A}t] Q\tilde{R}_\infty Q \exp[\tilde{A}^T t] dt \ge 0.$$

For the rest of this proof, to determine a matrix gain K such that the bound $\mathbf{Tr}[Q\tilde{R}]$ is minimized subject to the constraint (36), the Lagrange multiplier approach can be used. That is, let us consider the Hamiltonian $\mathcal{H}$

$$\mathcal{H}(Q,\ P,\ K) = \mathbf{Tr}\big[Q\tilde{R} + (\tilde{A}Q + Q\tilde{A}^T + Q\tilde{R}_\infty Q + B_1 B_1^T)P\big], \tag{37}$$

where $P \in R^{n\times n}$ is a symmetric matrix of Lagrange multipliers. Necessary conditions for a K to be optimal can be found by setting $\dfrac{\partial \mathcal{H}}{\partial Q}$ and $\dfrac{\partial \mathcal{H}}{\partial K}$ equal to zero, and solving the resulting equations (36) simultaneously for K. ■

6.1 Asymptotic Structure of the CAREs

Let us consider the following CAREs that are defined as the parameter $\varepsilon := \gamma^{-2}$.

$$N_1(\varepsilon,\ Q,\ P,\ K) := \tilde{A}Q + Q\tilde{A}^T + \varepsilon Q\tilde{C}_1^T\tilde{C}_1 Q + B_1 B_1^T = 0, \tag{38a}$$
$$N_2(\varepsilon,\ Q,\ P,\ K) := (\tilde{A} + \varepsilon Q\tilde{C}_1^T\tilde{C}_1)^T P + P(\tilde{A} + \varepsilon Q\tilde{C}_1^T\tilde{C}_1) + \tilde{R} = 0, \tag{38b}$$
$$N_3(\varepsilon,\ Q,\ P,\ K) := D_0^T\tilde{C}_0 + [B_2^T + \varepsilon D_1^T\tilde{C}_1 Q]P = 0, \tag{38c}$$

Setting $\varepsilon = 0$ for the previous CAREs (38), the following equations hold.

$$N_1(0,\ Q^0,\ P^0,\ K^0) = (A + B_2K^0)Q^0 + Q^0(A + B_2K^0)^T + B_1B_1^T = 0, \tag{39a}$$
$$N_2(0,\ Q^0,\ P^0,\ K^0) = (A + B_2K^0)^T P^0 + P^0(A + B_2K^0) + R_1 + K^{0T}R_2K^0 = 0, \tag{39b}$$
$$N_3(0,\ Q^0,\ P^0,\ K^0) = D_0^T\tilde{C}_0 + B_2^T P^0 = R_2K^0 + B_2^T P^0 = 0, \tag{39c}$$

where Q^0, P^0 and K^0 are zeroth-order solutions of the CAREs (38).

Using (39), the asymptotic structure of the solutions $Q = Q(\varepsilon)$, $P = P(\varepsilon)$ and $K = K(\varepsilon)$ of the CAREs (38) as $N_k(\varepsilon,\ Q,\ P,\ K) = 0$ is established.

Theorem 7 *Then there exists small $\tilde{\varepsilon} > 0$ such that for all $\varepsilon \in (0,\ \tilde{\varepsilon})$, the CAREs (38) admit the unique solutions Q, P and K in the neighborhood of $\varepsilon = 0$, which can be written since*

$$Q(\varepsilon) = Q^0 + O(\varepsilon), \tag{40a}$$
$$P(\varepsilon) = P^0 + O(\varepsilon), \tag{40b}$$
$$K(\varepsilon) = K^0 + O(\varepsilon). \tag{40c}$$

Proof: It can be done by applying the implicit function theorem to the CAREs (38). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. Taking the partial derivative of the function $N_k(Q,\ P,\ K),\ k = 1,\ 2,\ 3$ with respect to Q, P, K and setting $\varepsilon = 0$ results in (41).

$$\hat{\mathcal{J}}(0,\ Q^0,\ P^0,\ K^0) = \begin{bmatrix} \Psi_{11}^{(0)} & 0 & \Psi_{13}^{(0)} \\ 0 & \Psi_{22}^{(0)} & 0 \\ 0 & \Psi_{32}^{(0)} & \Psi_{33}^{(0)} \end{bmatrix}, \tag{41}$$

where

$$\hat{\mathcal{J}}(\varepsilon,\ Q,\ P,\ K) := \begin{bmatrix} \dfrac{\partial \text{vec} N_1}{\partial (\text{vec} Q)^T} & \dfrac{\partial \text{vec} N_1}{\partial (\text{vec} P)^T} & \dfrac{\partial \text{vec} N_1}{\partial (\text{vec} K)^T} \\ \dfrac{\partial \text{vec} N_2}{\partial (\text{vec} Q)^T} & \dfrac{\partial \text{vec} N_2}{\partial (\text{vec} P)^T} & \dfrac{\partial \text{vec} N_2}{\partial (\text{vec} K)^T} \\ \dfrac{\partial \text{vec} N_3}{\partial (\text{vec} Q)^T} & \dfrac{\partial \text{vec} N_3}{\partial (\text{vec} P)^T} & \dfrac{\partial \text{vec} N_3}{\partial (\text{vec} K)^T} \end{bmatrix}, \tag{42}$$

$$\Psi_{11}^{(0)} := (A + B_2 K^0) \otimes I_n + I_n \otimes (A + B_2 K^0),\ \Psi_{13}^{(0)} := Q_0 \otimes B_2 + [B_2 \otimes Q^0] \mathbf{U}_{mn},$$
$$\Psi_{22}^{(0)} := (A + B_2 K^0)^T \otimes I_n + I_n \otimes (A + B_2 K^0)^T,\ \Psi_{32}^{(0)} := I_n \otimes B_2^T,\ \Psi_{33}^{(0)} := I_n \otimes R_2.$$

It should be noted that since $R_2 > 0$ under Assumption 4, $K^0 = -R_2^{-1} B_2^T P^0$. Then the algebraic equation (39b) can be changed as $A^T P^0 + P^0 A - P^0 S_2 P^0 + R_1 = 0$, where $S_2 := B_2 R_2^{-1} B_2^T$. Hence, under Assumption 3, there exists the unique positive semi-definite stabilizing solution P^0. Furthermore, $A + B_2 K^0 = A - B_2 R_2^{-1} B_2 P^0$ is stable. Thus, since $\det \hat{\mathcal{J}}(0,\ Q^0,\ P^0,\ K^0) = \Pi_{i=1}^3 \det \Psi_{ii}^{(0)}$, $\hat{\mathcal{J}}(\varepsilon,\ Q,\ P,\ K)$ is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 7 is obtained directly by using the implicit function theorem. ■

In order to obtain the solutions of the CAREs (38), the following new algorithm which is based on Newton's method is given.

$$\begin{aligned} 0 &= [A + B_2 K^{(i)} + \varepsilon Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)}] Q^{(i+1)} + Q^{(i+1)} [(A + B_2 K^{(i)} + \varepsilon Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)}]^T \\ &\quad + B_2 K^{(i+1)} Q^{(i)} + Q^{(i)} K^{(i+1)T} B_2^T + \varepsilon Q^{(i)} \tilde{C}_1^{(i)T} D_1 K^{(i+1)} Q^{(i)} + \varepsilon Q^{(i)} K^{(i+1)T} D_1^T \tilde{C}_1^{(i)} Q^{(i)} \\ &\quad + \varepsilon Q^{(i)} [W - K^{(i)T} D_1^T D_1 K^{(i)}] Q^{(i)} - 2\varepsilon Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)} Q^{(i)} - B_2 K^{(i)} Q^{(i)} - Q^{(i)} K^{(i)T} B_2^T + B_1^T B_1, \end{aligned} \tag{43a}$$

$$\begin{aligned} 0 &= [A + B_2 K^{(i)} + \varepsilon Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)}]^T P^{(i+1)} + P^{(i+1)} [(A + B_2 K^{(i)} + \varepsilon Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)}] \\ &\quad + \varepsilon \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)} Q^{(i+1)} P^{(i)} + \varepsilon P^{(i)} Q^{(i+1)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)} + \varepsilon P^{(i)} Q^{(i)} \tilde{C}_1^{(i)T} D_1 K^{(i+1)} + \varepsilon P^{(i)} Q^{(i)} K^{(i+1)T} D_1^T \tilde{C}_1^{(i)} \\ &\quad + \varepsilon K^{(i+1)T} D_1^T \tilde{C}_1^{(i)} Q^{(i)} P^{(i)} + \varepsilon \tilde{C}_1^{(i)T} D_1 K^{(i+1)} Q^{(i)} P^{(i)} \\ &\quad + K^{(i+1)T} (R_2 K^{(i)} + B_2 P^{(i)}) + (R_2 K^{(i)} + B_2 P^{(i)})^T K^{(i+1)} \\ &\quad + \varepsilon P^{(i)} Q^{(i)} [W - K^{(i)T} D_1^T D_1 K^{(i)}] + \varepsilon [W - K^{(i)T} D_1^T D_1 K^{(i)}]^T Q^{(i)} P^{(i)} \\ &\quad - 2\varepsilon P^{(i)} Q^{(i)} \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)} - 2\varepsilon \tilde{C}_1^{(i)T} \tilde{C}_1^{(i)} Q^{(i)} P^{(i)} + R_1 - K^{(i)T} R_2 K^{(i)} - K^{(i)T} B_2^T P^{(i)} - P^{(i)} B_2 K^{(i)}, \end{aligned} \tag{43b}$$

$$\begin{aligned} 0 &= (R_2 + \varepsilon D_1^T D_1) K^{(i+1)} + \varepsilon D_1^T \tilde{C}_1^{(i)} Q^{(i+1)} P^{(i)} + [B_2^T + \varepsilon D_1^T \tilde{C}_1^{(i)} Q^{(i)}] P^{(i+1)} \\ &\quad + \varepsilon D_1^T C_1 Q^{(i)} P^{(i)} - 2\varepsilon D_1^T \tilde{C}_1^{(i)} Q^{(i)} P^{(i)}, \end{aligned} \tag{43c}$$

where $W := C_1^T C_1$. $Q^{(0)}$, $P^{(0)}$ and $K^{(0)}$ satisfy the following algebraic equations as $Q^{(0)} = Q^0$, $P^{(0)} = P^0$ and $K^{(0)} = K^0$, respectively.

$$(A - S_2 P^0) Q^0 + Q^0 (A - S_2 P^0)^T + B_1 B_1^T = 0, \tag{44a}$$
$$A^T P^0 + P^0 A - P^0 S_2 P^0 + R_1 = 0, \tag{44b}$$
$$K^0 = -R_2^{-1} B_2^T P^0. \tag{44c}$$

The following theorem indicates the local quadratic convergence and the uniqueness for the convergence solutions.

Theorem 8 *There exists a small $\tilde{\rho}^*$ such that for all the $\varepsilon \in (0,\ \tilde{\rho}^*)$, Newton's method (43) converges to exact solutions Q^*, P^* and K^* with a quadratic convergence rate. Moreover, the convergence solutions Q^*, P^* and K^* are the unique solutions of the CAREs (38) in the neighborhood of the initial conditions $Q^{(0)} = Q^0$, $P^{(0)} = P^0$, $K^{(0)} = K^0$, respectively. In other words, the following relations are satisfied.*

$$\|Q^{(i)} - Q^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \ldots\ , \tag{45a}$$
$$\|P^{(i)} - P^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \ldots\ , \tag{45b}$$
$$\|K^{(i)} - K^*\| \le O(\varepsilon^{2^i}),\ i = 0,\ 1,\ \ldots\ . \tag{45c}$$

Proof : Since this proof can be done by using the similar steps of the proof of Theorem 3, it is omitted. ■

Although Theorem 7 and 8 are the straightforward extensions of the output feedback H_∞-constrained LQG control problem, it is noteworthy that the local uniqueness and the quadratic convergence are both achieved by applying the Newton-Kantorovich theorem. Moreover, it may be noted that there are no results for solving the state feedback H_∞-constrained LQG control problem

7 Computational Example

In order to demonstrate the efficiency of the proposed algorithm, the computational examples are given.

7.1 Example 1: Dynamic Output Feedback Case

The system matrices are given below [1].

$$A = \begin{bmatrix} -0.161 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.004 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5822 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -9.9835 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.4073 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3.982 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.0713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_1 = E_{1\infty} = 10^{-3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T,$$

$$D_1 = \begin{bmatrix} B & O_{8\times 1} \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In order to verify the exactitude of the solution, the remainder per iteration is computed by substituting $Q^{(i)}$, $P^{(i)}$ and $\hat{Q}^{(i)}$ into the CAREs (7). Table 1 shows the error per iterations. In the case of $\gamma = 100$, it should be noted that the algorithm (13) converges to the exact solution with accuracy of $\mathcal{E}(\varepsilon) < 1.0e-9$ after three iterations. Hence, the initial conditions (9) are quite good under the large parameter γ. Moreover, it can be seen from Table 1 that the algorithm (13) attains the quadratic convergence.

The required iterations of the proposed algorithm (13) versus the existing algorithm [1] are presented in Table 2. It can be seen from Table 2 that the proposed algorithm have relatively small number of iterations than the existing algorithm [1]. Hence, the resulting algorithm of this paper is very reliable.

Table 1. Errors per iterations.

k	$\mathcal{E}(\varepsilon)$
1	$9.4536e-02$
2	$2.0622e-07$
3	$9.6526e-10$

Table 2. Number of iterations.

γ	Newton's Method	Existing Method [1]
1.2	6	23
5.2	3	9

It should be noted that the considered example is ill-conditioning because the matrices E_1 and $E_{1\infty}$ have the 10^{-3}-order. Thus, the solutions cannot be obtained under the use of the algorithms (21) only. In this case, since the inverse matrix calculation of the algorithm (21) is not needed, the algorithm (23) for solving the equation (13) seems to be very useful.

7.2 Example 2: State Feedback Case

Consider the system (30) with

$$B_1 = D_1,\ B_2 = B,$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ D_0 = D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It should be noted that the element of some matrices have same elements of the given matrices in the previous example 1.

In order to verify the exactitude of the solution, the remainder per iteration is computed by substituting $Q^{(i)}$, $P^{(i)}$ and $K^{(i)}$ into the CAREs (38). Table 3 shows the error per iterations. In the case of $\gamma = 1000$, it should be noted that the algorithm (43) converges to the exact solution with accuracy of $\mathcal{F}(\varepsilon) < 1.0e - 10$ after three iterations, where

$$\mathcal{F}(\varepsilon) := \sum_{k=1}^{3} \|N_k(Q^{(i)},\ P^{(i)},\ K^{(i)})\|$$

. Hence, it can be seen from Table 3 that the algorithm (43) attains the quadratic convergence.

Table 3.

k	$\mathcal{E}(\varepsilon)$
0	$1.3430e - 04$
1	$9.9533e - 09$
2	$6.1823e - 12$

8 Conclusion

The numerical algorithm for solving the state and output feedback H_∞-constrained LQG control problem has been tackled. The main contribution of this paper is to provide the new algorithms for solving the CAREs. Finally, the convergence proof of the algorithms has been shown for the first time. Since the new algorithms are based on Newton's method, the quadratic local convergence is guaranteed for a large parameter γ. As a result, the CAREs can be solved quickly. Moreover, it has been proven that the uniqueness and boundedness for the solutions of the CAREs are also guaranteed. Finally, the numerical examples have shown excellent results that the proposed algorithm has succeeded in reducing the computational workspace and the quadratic convergence has been attained.

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