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Numerical Computation of H_∞ Output Feedback Control for Strongly Coupled Large-Scale Systems

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Abstract. In this paper, H_∞ output feedback control for strongly coupled large-scale systems are discussed. When the positive coupling parameter ε which connect the other subsystems is large, a successive algorithm for solving the algebraic Riccati equations (ARE) is developed for the first time. Since the proposed algorithm is derived using Newton's method, it is noteworthy that the quadratic convergence and uniqueness of the obtained solution are both guaranteed for strongly coupled parameter ε. Moreover, in order to reduce the computation in the resulting Newton's method, the gradient-based iterative (GI) algorithm is combined. As a result, it is shown that the reduced-order computation is attained. Finally, in order to demonstrate the efficiency of the proposed algorithms, computational examples are provided.
keywords: H_∞ output feedback control, strongly-coupled large-scale systems, Newton-Kantorovich theorem, gradient-based iterative algorithm, successive algorithm.

1 Introduction

For large-scale systems, the stability analysis and control and filtering problems have been investigated extensively (see e.g., [4, 5, 6]). In practice, it is known that such systems are represented by multiarea power systems [4], distillation columns [5], and cold rolling [6]. They are widely used to represent system dynamics.

In order to obtain the optimal solution, we must solve the algebraic Riccati equation (ARE) that is parameterized by the positive coupling parameter ε. Various reliable approaches for solving the ARE have been well documented in many literatures (see e.g., [10, 12]). If the dimension of systems are relatively small, these approaches are very useful. However, when these methods are used, the dimension of the computing workspace should be twice that of the ARE. Therefore, the reduced-order computation needs to be considered.

The control problems of weakly coupled large-scale systems have been studied by several researchers (see [5, 13, 14, 15, 16, 17] and references therein). When the positive coupling parameter ε which connect the other subsystems is sufficiently small, the previously used techniques are very efficient. However, as long as the large coupling parameter ε is considered, such methods proposed in these references cannot be applied. This brings a new issue: how to solve ARE for large coupling parameter ε and to guarantee the quadratic convergence and the local uniqueness.

In this paper, H_∞ output feedback control for large-scale systems is investigated. It is assumed that the subsystems of the considered large-scale linear systems are connected by large coupling parameter ε. After establishing the structure of the ARE corresponding to H_∞ control problem by using the implicit function theorem, an algorithm for solving the ARE corresponding this problem related to large coupled large-scale systems is established. In order to guarantee the quadratic convergence and local uniqueness of the obtained solution for any large parameter ε, the successive algorithm that is based on the Newton's method is proposed for the first time. Moreover, the gradient-based iterative (GI) algorithm is combined to attain the reduced-order computation. In order to demonstrate the efficiency of our new algorithms, some computational examples are provided.

Notation: The notations used in this paper are fairly standard. Superscript T denotes matrix transpose. **Tr** denotes sum of the diagonal elements of a matrix. **block diag** denotes the block diagonal matrix. $\lambda_{\max}$ denotes maximum eigenvalue. $\|\cdot\|$ denotes norm of a matrix. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix such that $\|M\|_F^2 := \mathbf{Tr}[M^T M]$. det denotes the determinant of a square matrix. I_p denotes the $p \times p$ identity matrix. vec denotes the column vector of a matrix [1]. $\otimes$ denotes the Kronecker product. $\mathbf{U}_{lm}$ denotes a permutation matrix in Kronecker matrix sense [1] such that $\mathbf{U}_{lm}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \mathbf{R}^{l\times m}$.

2 Output Feedback H_∞ Control Problem for Strongly Coupled Large-Scale Systems

Consider linear time-invariant coupled large-scale systems [5, 13].

$$\dot{x}(t) = A_\varepsilon x(t) + B_{1\varepsilon}w(t) + B_{2\varepsilon}u(t), \tag{1a}$$
$$z(t) = C_1x(t) + D_1u(t), \tag{1b}$$
$$y(t) = C_2x(t) + D_2w(t), \tag{1c}$$

where

$$\begin{aligned}
&x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ w(t) := \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix},\ u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},\ z(t) := \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix},\ y(t) := \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},\\
&A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{bmatrix},\ B_{1\varepsilon} := \begin{bmatrix} B_{111} & \varepsilon B_{112} \\ \varepsilon B_{121} & B_{122} \end{bmatrix},\ B_{2\varepsilon} := \begin{bmatrix} B_{211} & \varepsilon B_{212} \\ \varepsilon B_{221} & B_{222} \end{bmatrix},\\
&C_1 := \begin{bmatrix} \textbf{block diag}\begin{pmatrix} C_{11} & C_{12} \end{pmatrix} \\ 0 \end{bmatrix},\ C_2 := \textbf{block diag}\begin{pmatrix} C_{21} & C_{22} \end{pmatrix},\\
&D_1 := \begin{bmatrix} 0 \\ I_{m_1+m_2} \end{bmatrix},\ D_2 := \begin{bmatrix} I_{q_1+q_2} & 0 \end{bmatrix}.
\end{aligned}$$

$x_i(t) \in \mathbf{R}^{n_i}$ are the state vectors, $w_i(t) \in \mathbf{R}^{l_i}$ are the disturbance, $u_i(t) \in \mathbf{R}^{m_i}$ are the control inputs, $z_i(t) \in \mathbf{R}^{p_i}$ are the controlled output, and $y_i(t) \in \mathbf{R}^{q_i}$ are the measured outputs. ε denotes a relatively large positive coupling parameter that connects the linear system with other subsystems.

Without loss of generality, we now consider H_∞ control problems under the following basic assumption.

Assumption 1 *1. The pair* $(A_\varepsilon,\ B_{1\varepsilon})$ *is stabilizable and* $(C_1,\ A_\varepsilon)$ *is detectable for a given* $\varepsilon \in (0,\ \varepsilon^*]$, $\varepsilon^* > 0$.

2. The pair $(A_\varepsilon,\ B_{2\varepsilon})$ *is stabilizable and* $(C_2,\ A_\varepsilon)$ *is detectable for a given* $\varepsilon \in (0,\ \varepsilon^*]$, $\varepsilon^* > 0$.

3. $D_1^TC_1 = 0$, $B_{1\varepsilon}D_2^T = 0$.

H_∞ optimal control problem for strongly coupled large-scale systems is given below.

[H_∞ **Optimal Control Problem**] Given a stabilizable and detectable plant (1), find all admissible K_ε such that $\|G_\varepsilon(s)\|_\infty < \gamma$, where $G_\varepsilon(s)$ equals the transfer function from w to z.

The following result is well known [8, 9].

Lemma 1 *Under Assumption 1, there exists an admissible controller such that* $\|G_\varepsilon(s)\|_\infty < \gamma$ *if and only if the following three conditions hold.*

i) *The backward ARE*

$$A_\varepsilon^TP_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon(B_{2\varepsilon}B_{2\varepsilon}^T - \gamma^{-2}B_{1\varepsilon}B_{1\varepsilon}^T)P_\varepsilon + C_1^TC_1 = 0, \tag{2}$$

has a unique positive semidefinite stabilizing solution.

ii) *The forward ARE*

$$A_\varepsilon W_\varepsilon + W_\varepsilon A_\varepsilon^T - W_\varepsilon(C_2^TC_2 - \gamma^{-2}C_1^TC_1)W_\varepsilon + B_{1\varepsilon}B_{1\varepsilon}^T = 0, \tag{3}$$

has a unique positive semidefinite stabilizing solution.

iii) $\lambda_{\max}(P_\varepsilon W_\varepsilon) < \gamma^2$.

Moreover, when these conditions hold, one such controller, i.e., a central controller with the free parameter equal to zero, is given by equation (4).

$$u(t) = -B_{2\varepsilon}^T P_\varepsilon \hat{x}(t), \tag{4}$$

where $\dot{\hat{x}}(t) = [A_\varepsilon - (B_{2\varepsilon}B_{2\varepsilon}^T - \gamma^{-2}B_{1\varepsilon}B_{1\varepsilon}^T)P_\varepsilon - ZW_\varepsilon C_2^T C_2]\hat{x} + ZW_\varepsilon C_2^T y(t)$ *with* $Z = (I - \gamma^{-2}W_\varepsilon P_\varepsilon)^{-1}$. *Here,* $\hat{x}(t) = \left[\begin{array}{cc} \hat{x}_1^T(t) & \hat{x}_2^T(t) \end{array}\right]^T \in \mathbf{R}^{n_1+n_2}$ *is the observer state.*

In this section, the asymptotic structure of the solution for the backward ARE (2) is established. The following structure should be assumed.

$$P_\varepsilon = \left[\begin{array}{cc} P_1 & \varepsilon P_{12} \\ \varepsilon P_{12}^T & P_2 \end{array}\right]. \tag{5}$$

It should be noted that the assumption of this structure is also made in [4, 5, 6]. Using the implicit function theorem, let us prove the existence of the implicit functions $P_i = P_i(\varepsilon)$ and $P_{12} = P_{12}(\varepsilon)$ of ε such that

$$P_i = P_i(\varepsilon),\ i = 1,\ 2,\ P_{12} = P_{12}(\varepsilon). \tag{6}$$

In order to simplify the notation, the following matrix is defined.

$$S_\varepsilon := B_{2\varepsilon}B_{2\varepsilon}^T - \gamma^{-2}B_{1\varepsilon}B_{1\varepsilon}^T = \left[\begin{array}{cc} S_1 + \varepsilon^2 S_{11} & \varepsilon(S_{112} + S_{122}) \\ \varepsilon(S_{112} + S_{122})^T & S_2 + \varepsilon^2 S_{22} \end{array}\right],\ S_i = S_i^T,\ S_{ii} = S_{ii}^T,\ i = 1,\ 2,$$
$$Q := C_1^T C_1 = \textbf{block diag}\left(\begin{array}{cc} Q_1 & Q_2 \end{array}\right).$$

Substituting the solution P_ε of equation (5) into the ARE (2), the set of algebraic matrix equations (7) is obtained.

$$\begin{aligned}\mathcal{G}_1 = {}& A_{11}^T P_1 + P_1 A_{11} + \varepsilon^2(A_{21}^T P_{12}^T + P_{12}A_{21}) - P_1 S_1 P_1 - \varepsilon^2(P_1 S_{11} P_1 + P_{12}S_{112}^T P_1 \\ & + P_{12}S_{122}^T P_1 + P_1 S_{112} P_{12}^T + P_1 S_{122} P_{12}^T + P_{12} S_2 P_{12}^T + \varepsilon^2 P_{12} S_{22} P_{12}^T) + Q_1 = 0, \end{aligned} \tag{7a}$$
$$\begin{aligned}\mathcal{G}_{12} = {}& A_{11}^T P_{12} + P_1 A_{12} + A_{21}^T P_2 + P_{12}A_{22} - P_1 S_1 P_{12} - P_1 S_{112} P_2 - P_1 S_{122} P_2 - P_{12} S_2 P_2 \\ & + \varepsilon^2(P_1 S_{11} P_{12} + P_{12} S_{112}^T P_{12} + P_{12} S_{122}^T P_{12} + \varepsilon^2 P_{12} S_{22} P_2) = 0, \end{aligned} \tag{7b}$$
$$\begin{aligned}\mathcal{G}_2 = {}& A_{22}^T P_2 + P_2 A_{22} + \varepsilon^2(A_{12}^T P_{12} + P_{12}^T A_{12}) - P_2 S_2 P_2 - \varepsilon^2(P_2 S_{22} P_2 + P_{12}^T S_{112} P_2 \\ & + P_{12}^T S_{122} P_2 + P_2 S_{112}^T P_{12} + P_2 S_{122}^T P_{12} + P_{12}^T S_1 P_{12} + \varepsilon^2 P_{12}^T S_{11} P_{12}) + Q_2 = 0, \end{aligned} \tag{7c}$$

where

$$\mathcal{G}(P_\varepsilon) := \left[\begin{array}{cc} \mathcal{G}_1 & \varepsilon\mathcal{G}_{12} \\ \varepsilon\mathcal{G}_{12}^T & \mathcal{G}_2 \end{array}\right] = A_\varepsilon^T P_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = 0.$$

The 0th order solutions $\bar{P}_i$, $i = 1,\ 2$, and $\bar{P}_{12}$ are defined for $\varepsilon \to +0$ for the algebraic matrix equations. Then, the solutions $\bar{P}_i$ satisfy the ARE (8)

$$A_{ii}^T \bar{P}_i + \bar{P}_i A_{ii} - \bar{P}_i S_i \bar{P}_i + Q_i = 0,\ i = 1,\ 2, \tag{8}$$

where $S_i := B_{1ii}B_{1ii}^T - \gamma^{-2}B_{2ii}B_{2ii}^T$.

The ARE (8) produces a unique positive semidefinite stabilizing solution if γ is sufficiently large.

Let

$\gamma_{P_i} = \inf\{\gamma > 0|$ the ARE (8) has a positive semidefinite stabilizing solution$\}$.

Then, matrix $A_{ii} - S_i\bar{P}_i$ is nonsingular if we choose $\gamma > \gamma_{P_i}$. For the solution P_ε of the ARE (2), the result is given for $\varepsilon \to +0$.

Theorem 1 *It is assumed that the reduced-order ARE (8), which is independent of the perturbation parameter ε, has a positive semidefinite stabilizing solution. If we select a parameter $\gamma > \bar{\gamma}_P = \max\{\gamma_{P_1},\ \gamma_{P_2}\}$, then there exists a small $\bar{\sigma}$ such that for all $\varepsilon \in (0,\ \bar{\sigma})$, the ARE (2) admits a positive semidefinite stabilizing solution P_ε that can be written as equation (9).*

$$P_\varepsilon = \bar{P} + O(\varepsilon) = \textbf{block diag}\left(\begin{array}{cc} \bar{P}_1 & \bar{P}_2 \end{array}\right) + O(\varepsilon). \tag{9}$$

Proof : The proof can be obtained by applying the implicit function theorem [5]. To achieve this, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. It can be shown after some simplification that the Jacobian of the ARE (7) with the limit $\varepsilon \to +0$ is given by equation (10).

$$\mathbf{J} = \left[\begin{array}{ccc} \mathbf{J}_1 & 0 & 0 \\ * & \mathbf{J}_{12} & * \\ 0 & 0 & \mathbf{J}_2 \end{array}\right] \tag{10}$$

with

$$\begin{aligned} \mathbf{J}_i &= (A_{ii} - S_i\bar{P}_i)^T \otimes I_{n_i} + I_{n_i} \otimes (A_{ii} - S_i\bar{P}_i)^T, \\ \mathbf{J}_{12} &= (A_{22} - S_2\bar{P}_2)^T \otimes I_{n_1} + I_{n_2} \otimes (A_{11} - S_1\bar{P}_1)^T. \end{aligned}$$

The Jacobian (10) can be expressed as

$$\det\mathbf{J} = \det\mathbf{J}_1\det\mathbf{J}_{12}\det\mathbf{J}_2. \tag{11}$$

Apparently,$\mathbf{J}_i$ and $\mathbf{J}_{12}$ are nonsingular because the AREs (8) have positive semidefinite stabilizing solutions. Thus, $\det\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\varepsilon = 0$. Therefore, the existence of $\bar{\sigma}$ such that the ARE (2) has an asymptotic structure (9) is directly obtained by applying the implicit function theorem.

The remainder of the proof shows that P_ε is a positive semidefinite stabilizing solution. Using the asymptotic structure (9) for ε, we have equation (12).

$$A_\varepsilon - S_\varepsilon P_\varepsilon = \textbf{block diag}\left(\begin{array}{cc} A_{11} - S_1\bar{P}_1 & A_{22} - S_2\bar{P}_2 \end{array}\right) + O(\varepsilon). \tag{12}$$

The matrices $A_{ii} - S_i\bar{P}_i$ are stable because the ARE (8) has a positive semidefinite stabilizing solution. Therefore, if ε is small, $A_\varepsilon - S_\varepsilon P_\varepsilon$ is also stable. Finally, the positive semidefiniteness can be proved by using the Schur complement [9] for a sufficiently small ε. ■

3 Newton's Method for Solving the ARE

In this section, a new algorithm for solving the ARE (2), which can be calculated with a small dimension and attains quadratic convergence, is proposed.

The Schur vector method [10] is being widely used for solving the ARE (2) because the method has good precision and ensures the algorithm's stability. However, it is well known that in this method, the dimensions of the required workspace for the calculations are twice that of the original full system [10]. On the other hand, in [11], the following comments have been documented.

"Newton's method is potentially fast and more accurate than the widely used Schur vector method. The break-even point is between six and eight iterations assuming that a Bartels-Stewart-like algorithm is used to solve the algebraic Lyapunov equation (ALE)."

Thus, an algorithm that is based on Newton's method and uses the structure of the solution in equation (9) is considered.

Let us consider Newton's method (13).

$$P_\varepsilon^{(n+1)}(A_\varepsilon - S_\varepsilon P_\varepsilon^{(n)}) + (A_\varepsilon - S_\varepsilon P_\varepsilon^{(n)})^T P_\varepsilon^{(n+1)} + P_\varepsilon^{(n)} S_\varepsilon P_\varepsilon^{(n)} + Q_\varepsilon = 0,\ n = 0,\ 1,\ \dots, \tag{13a}$$

$$P_\varepsilon^{(n)} = \left[\begin{array}{cc} P_1^{(n)} & \varepsilon P_{12}^{(n)} \\ \varepsilon P_{12}^{(n)T} & P_2^{(n)} \end{array}\right], \tag{13b}$$

where the initial conditions are chosen as follows.

$$P_{\varepsilon}^{(0)} = \bar{P} = \textbf{block diag} \left(\begin{array}{cc} \bar{P}_1 & \bar{P}_2 \end{array} \right). \tag{14}$$

It should be noted that Newton's method is equivalent to the existing Kleinman algorithm [12].

The algorithm represented by equation (13) has the feature given in the following theorem.

Theorem 2 *If the parameter-independent reduced-order ARE (8) has a positive semidefinite stabilizing solution, there exists a small $\tilde{\sigma}$ such that for all $\varepsilon \in (0,\ \tilde{\sigma})$, $0 < \tilde{\sigma} \le \bar{\sigma}$, the iterative algorithm represented by equation (13a) converges to the exact solution of P_{ε} with a rate equal to that of quadratic convergence; here, $P_{\varepsilon}^{(n)}$ is positive semidefinite. Moreover, the convergence solutions equal those of P_{ε} in the ARE (2) in the neighborhood of the initial condition $P_{\varepsilon}^{(0)} = \bar{P}$. Subsequently, we have equation (15).*

$$\|P_{\varepsilon}^{(n)} - P_{\varepsilon}\| = \frac{O(\varepsilon^{2^n})}{\beta\lambda 2^n},\ n = 0,\ 1,\ \ldots\ , \tag{15}$$

where

$$\lambda = 2\|S_{\varepsilon}\| < \infty,\ \beta = \|[\nabla\mathcal{G}(P_{\varepsilon}^{(0)})]^{-1}\|,\ \eta = \beta \cdot \|\mathcal{G}(P_{\varepsilon}^{(0)})\|,\ \theta = \beta\eta\lambda,$$
$$\nabla\mathcal{G}(P_{\varepsilon}) = \frac{\partial \text{vec}\mathcal{G}(P_{\varepsilon})}{\partial(\text{vec}P_{\varepsilon})^T},\ \mathcal{G}(P_{\varepsilon}) = P_{\varepsilon}A_{\varepsilon} + A_{\varepsilon}^T P_{\varepsilon} - P_{\varepsilon}S_{\varepsilon}P_{\varepsilon} + Q.$$

Proof : The proof follows directly by applying the Newton-Kantorovich theorem [2, 3]. Taking the partial derivative of the ARE (2) with respect to P_{ε} yields

$$\nabla\mathcal{G}(P_{\varepsilon}) := \frac{\partial \text{vec}\mathcal{G}(P_{\varepsilon})}{\partial(\text{vec}P_{\varepsilon})^T} = (A_{\varepsilon} - S_{\varepsilon}P_{\varepsilon})^T \otimes I_n + I_n \otimes (A_{\varepsilon} - S_{\varepsilon}P_{\varepsilon})^T. \tag{16}$$

Thus, for any $P_{a\varepsilon}$ and $P_{b\varepsilon} \in \mathbf{R}^{\bar{n}\times\bar{n}}$, $\bar{n} := n_1 + n_2$ the following inequality holds.

$$\|\nabla\mathcal{G}(P_{a\varepsilon}) - \nabla\mathcal{G}(P_{b\varepsilon})\| \le \lambda\|P_{a\varepsilon} - P_{b\varepsilon}\|,$$

where $\lambda = 2\|S_{\varepsilon}\|$.

Moreover, using the stability that is established by equation (12), it is shown that there exists a small $\tilde{\sigma}$ such that for $\varepsilon \in (0,\ \tilde{\sigma})$, $\tilde{\sigma} \le \bar{\sigma}$, $\nabla\mathcal{G}(P_{\varepsilon})$ is nonsingular. Therefore, there exists β such that $\|[\nabla\mathcal{G}(P_{\varepsilon})]^{-1}\| \equiv \beta$. On the other hand, using Theorem 1, it is easy to show that $\|\mathcal{G}(P_{\varepsilon})\| = O(\varepsilon)$. Hence, there exists η such that $\|[\nabla\mathcal{G}(P_{\varepsilon})]^{-1}\| \cdot \|\mathcal{G}(P_{\varepsilon})\| \equiv \eta = O(\varepsilon)$. Thus, for a sufficiently small ε, there exists θ such that $\theta \equiv \beta\lambda\eta < 2^{-1}$ because $\eta = O(\varepsilon)$. Thus, using the Newton-Kantorovich theorem, there exists a small $\tilde{\sigma}$ such that for all $\varepsilon \in (0,\ \tilde{\sigma})$, $\tilde{\sigma} \le \bar{\sigma}$, the iterative algorithm (13a) has quadratic convergence.

Second, the uniqueness of the solution is discussed. Now, let us define $\bar{t}^* \equiv [1 - \sqrt{1-2\theta}]/(\beta\lambda)$. Clearly, $S \equiv \{\ P_{\varepsilon}\ :\ \|P_{\varepsilon} - P_{\varepsilon}^{(0)}\| \le \bar{t}^*\ \}$ is in the convex set D. In the sequel, since $\|P_{\varepsilon} - P_{\varepsilon}^{(0)}\| = O(\varepsilon)$ holds for a small ε, the uniqueness of P_{ε} is guaranteed for subset S by applying the Newton-Kantorovich theorem. ■

Remark 1 *According to the Newton-Kantorovich theorem [2, 3], for any $\varepsilon_0 > 0$, if the following inequality holds (17),*

$$\beta_0\eta_0\lambda_0 := 2\|[\nabla\mathcal{G}_0(P_{\varepsilon}^{(0)})]^{-1}\|^2 \cdot \|S_{0\varepsilon}\| \cdot \|\mathcal{G}_0(P_{\varepsilon}^{(0)})\| < \frac{1}{2}, \tag{17}$$

where

$$\mathcal{G}_0(P_{\varepsilon}^{(0)}) = P_{\varepsilon}^{(0)}A_{0\varepsilon} + A_{0\varepsilon}^T P_{\varepsilon}^{(0)} - P_{\varepsilon}^{(0)}S_{0\varepsilon}P_{\varepsilon}^{(0)} + Q_{0\varepsilon},$$
$$A_{0\varepsilon} := \left[\begin{array}{cc} A_{11} & \varepsilon_0 A_{12} \\ \varepsilon_0 A_{21} & A_{22} \end{array} \right],\ S_{0\varepsilon} := \left[\begin{array}{cc} S_1 + \varepsilon_0^2 S_{11} & \varepsilon_0(S_{112} + S_{122}) \\ \varepsilon_0(S_{112} + S_{122})^T & S_2 + \varepsilon_0^2 S_{22} \end{array} \right],\ Q_{0\varepsilon} := \left[\begin{array}{cc} Q_1 & \varepsilon_0 Q_{12} \\ \varepsilon_0 Q_{12}^T & Q_2 \end{array} \right],$$

the asymptotic structure in equation (9) is also established.

Furthermore, the quadratic convergence is attained. On the other hand, these values of $\bar{\sigma}$ and $\tilde{\sigma}$ are equivalent under the Newton-Kantorovich theorem within the premise of the sufficient condition (17). Hence, when we apply the proposed result to the considered problem, it is sufficient to investigate whether the condition in equation (17) holds or not for any small ε.

3.1 Successive Algorithm

If the coupling parameter ε is not sufficiently small, the initial condition (14) will be not adequate. Hence, in order to achieve a quadratic convergence of the proposed algorithm (13) for any large parameter ε, the successive algorithm is newly proposed. The concept is given below. It is assumed that the step of the successive algorithm is m. Suppose that

$$\varepsilon^{[0]} < \varepsilon^{[1]} < \cdots < \varepsilon^{[m]} < 1.$$

Then, if the following inequality (18) holds, the next approximate solutions $P_{\varepsilon}^{[m+1]} := \lim_{n \to \infty} P_{\varepsilon}^{(n)}$, $m = 1,\ 2,\ ...$ can be computed successively by choosing a new initial condition $P_{\varepsilon}^{(0)} = P_{\varepsilon}^{[m]}$.

$$\theta^{[m]}(\varepsilon^{[m]}) := \beta^{[m]}\eta^{[m]}\lambda^{[m]} < 2^{-1},\ m = 0,\ 1,\ ...\ , \tag{18}$$

where

$$\lambda^{[m]} = 2\|S_{\varepsilon}(\varepsilon^{[m]})\| < \infty,\ S_{\varepsilon} := S_{\varepsilon}(\varepsilon^{[m]}),\ \beta^{[m]} = \|[\nabla\mathcal{G}(P_{\varepsilon}^{[m]})]^{-1}\|,\ \eta^{[m]} = \beta^{[m]} \cdot \|\mathcal{G}(P_{\varepsilon}^{[m]})\|,$$
$$\theta^{[m]}(\varepsilon^{[m]}) = \beta^{[m]}\eta^{[m]}\lambda^{[m]}.$$

This concept employs the fact that when the parameter ε is small, the initial problem is approximated by the reduced-order problem to be a reliable initial solution. The successive algorithm for a relatively large parameter ε is given as follows.

Step 1. Solve (8) for $\bar{P}_{ii}$. When $m = 0$, choose as the initial conditions (14).

Step 2. For sufficiently small $\varepsilon^{[0]}$, compute $\beta^{[0]}$ and $\eta^{[0]}$.

Step 3. If the following inequality holds, solve (13) for $P_{\varepsilon}^{[1]}$ under the condition $P_{\varepsilon}^{(0)} = P_{\varepsilon}^{[0]}$.

$$\theta^{[0]} = \beta^{[0]}\eta^{[0]}\lambda^{[0]} < 2^{-1}. \tag{19}$$

If the inequality (19) does not hold, decrease $\varepsilon^{[0]}$ and go to Step 2.

Step 4. Select $\varepsilon^{[1]}$ such that $\varepsilon^{[0]} < \varepsilon^{[1]}$. Let $m = 1$ and check for inequality (18). If this inequality holds, solve (13) for $P_{\varepsilon}^{[2]}$ under the condition that $P_{\varepsilon}^{(0)} = P_{\varepsilon}^{[1]}$.

Step 5. Select $\varepsilon^{[m]}$ such that $\varepsilon^{[0]} < \varepsilon^{[1]} < \cdots < \varepsilon^{[m]}$. If the inequality (18) holds, increment $m \to m+1$ and solve (13) for $P_{\varepsilon}^{[m+1]} := \lim_{n \to \infty} P_{\varepsilon}^{(n)}$ under the condition that $P_{\varepsilon}^{(0)} = P_{\varepsilon}^{[m]}$.

Step 6. Repeat Step 5 until the desired magnitude of ε is attained. If the desired ε is achieved, stop. Otherwise, declare that no control law exists for $\varepsilon^{[m]}$.

The main result for the above algorithm is stated as follows.

Theorem 3 *Assume that the conditions of Theorem 2 hold and ARE (2) have a solution. Suppose that for $\varepsilon^{[m]}$, $P_{\varepsilon}^{[m]}$, the Jacobian $\mathbf{J} := \mathbf{J}(\varepsilon^{[m]},\ P_{\varepsilon}^{[m]})$ is nonsingular at $\varepsilon = \varepsilon^{[m]}$. Then, there exists a small $\hat{\sigma}^*$ such that for all $\varepsilon \in (0,\ \hat{\sigma}^*)$, Newton's method (13) converges to exact solutions P_{ε}^* with a quadratic convergence rate. Moreover, for each step m, convergence solutions P_{ε}^* is the unique solution for ARE (2) in the neighborhood of the initial conditions $P_{\varepsilon}^{(0)} = P_{\varepsilon}^{[m]}$. In other words, the following relations are satisfied.*

$$\|P_{\varepsilon}^{(n)} - P_{\varepsilon}^*\| = \frac{(2\theta^{[m]})^{2^n}}{\beta^{[m]}\lambda^{[m]}2^n},\ n = 0,\ 1,\ ...\ , \tag{20}$$

where $0 < 2\theta^{[m]} < 1$.

Proof : The proof is similar to that of Theorem 2. In other words, since the proof of Theorem 3 can also be derived by using the Newton-Kantorovich theorem, it is omitted. ■

It should be noted that the bound of ε can be obtained successively with regard to the sufficient condition.

4 Iterative Algorithm for Solving ALE

It is possible to solve the algorithm in equation (13a) by using a linear equation because such an algorithm is based on the ALE. However, this method results in an increase in the workspace dimensions for the numerical computation when the dimensions of the matrices P_1, P_{12}, and P_2 increase. That is, let us consider the following linear equation (21).

$$(13) \Leftrightarrow \mathcal{A}^{(n)} \begin{bmatrix} \mathrm{vec}P_1^{(n+1)} \\ \mathrm{vec}P_{12}^{(n+1)} \\ \mathrm{vec}P_2^{(n+1)} \end{bmatrix} = - \begin{bmatrix} \mathrm{vec}\bar{Q}_1 \\ \mathrm{vec}\bar{Q}_{12} \\ \mathrm{vec}\bar{Q}_2 \end{bmatrix}, \tag{21}$$

where

$$\mathcal{A}^{(n)} := \begin{bmatrix} \bar{A}_1^T \otimes I_{n_1} + I_{n_1} \otimes \bar{A}_1^T & \varepsilon^2(\bar{A}_{21}^T \otimes I_{n_1} + I_{n_1} \otimes \bar{A}_{21}^T) & 0 \\ \bar{A}_{12}^T \otimes I_{n_2} & \bar{A}_2^T \otimes I_{n_1} + I_{n_2} \otimes \bar{A}_1^T & I_{n_1} \otimes \bar{A}_{21}^T \\ 0 & \varepsilon^2(\bar{A}_{21}^T \otimes I_{n_1} + (I_{n_1} \otimes \bar{A}_{21}^T)\mathbf{U}_{n_1 n_2}) & (\bar{A}_{12} \otimes I_{n_2})\mathbf{U}_{n_1 n_2} + I_{n_2} \otimes \bar{A}_{12} \end{bmatrix},$$

$$A_\varepsilon - S_\varepsilon P_\varepsilon^{(n)} := \begin{bmatrix} \bar{A}_1 & \varepsilon \bar{A}_{12} \\ \varepsilon \bar{A}_{21} & \bar{A}_2 \end{bmatrix}, \ P_\varepsilon^{(n)} S_\varepsilon P_\varepsilon^{(n)} + Q_\varepsilon := \begin{bmatrix} \bar{Q}_1 & \varepsilon \bar{Q}_{12} \\ \varepsilon \bar{Q}_{12}^T & \bar{Q}_2 \end{bmatrix}.$$

In this case, if the dimensions n_1 and n_2 are large, the dimension of $\mathcal{A}^{(n)}$ would be quite large because the Kronecker products are used.

4.1 Fixed Point Algorithm

In order to reduce the computational dimension, the fixed point algorithm for solving the ALE (13a) has been formulated in [13, 18]. This algorithm and the convergence proof that is different from the existing results [13, 18] is summarized. Let us consider the following ALE (22), in a general form.

$$E_\varepsilon^T X_\varepsilon + X_\varepsilon E_\varepsilon + H_\varepsilon = 0, \tag{22}$$

where $E_i \in \mathbf{R}^{n_i \times n_i}$ is stable and $X_i = X_i^T \geq 0 \in \mathbf{R}^{n_i \times n_i}$, $H_i = H_i^T \in \mathbf{R}^{n_i \times n_i}$, $i = 1,\ 2$.

$$E_\varepsilon := \begin{bmatrix} E_1 & \varepsilon E_{12} \\ \varepsilon E_{21} & E_2 \end{bmatrix}, \ H_\varepsilon := \begin{bmatrix} H_1 & \varepsilon H_{12} \\ \varepsilon H_{12}^T & H_2 \end{bmatrix}, \ X_\varepsilon = \begin{bmatrix} X_1 & \varepsilon X_{12} \\ \varepsilon X_{12}^T & X_2 \end{bmatrix}.$$

It should be noted that for the ALE (22),

$$P_\varepsilon^{(n+1)} \Rightarrow X_\varepsilon, \ A_\varepsilon - S_\varepsilon P_\varepsilon^{(n)} \Rightarrow E_\varepsilon, \ P_\varepsilon^{(n)} S_\varepsilon P_\varepsilon^{(n)} + Q_\varepsilon \Rightarrow H_\varepsilon.$$

where $\Rightarrow$ stands for the replacement.

Substituting X_ε into the ALE (22), we have the following set of three linear equations (23).

$$E_1^T X_1 + X_1 E_1 + \varepsilon^2(E_{21}^T X_{12}^T + X_{12} E_{21}) + H_1 = 0, \tag{23a}$$
$$E_1^T X_{12} + X_1 E_{12} + E_{21}^T X_2 + X_{12} E_2 + H_{12} = 0, \tag{23b}$$
$$E_2^T X_2 + X_2 E_2 + \varepsilon^2(E_{12}^T X_{12} + X_{12}^T E_{12}) + H_2 = 0. \tag{23c}$$

By considering the form of equation (23), we propose the following algorithm in equation (24) to solve the ALE (22).

$$E_1^T X_1^{(k+1)} + X_1^{(k+1)} E_1 + \varepsilon^2(E_{21}^T X_{12}^{(k)T} + X_{12}^{(k)} E_{21}) + H_1 = 0, \tag{24a}$$
$$E_2^T X_2^{(k+1)} + X_2^{(k+1)} E_2 + \varepsilon^2(E_{12}^T X_{12}^{(k)} + X_{12}^{(k)T} E_{12}) + H_2 = 0, \tag{24b}$$
$$E_1^T X_{12}^{(k+1)} + X_{12}^{(k+1)} E_2 + X_1^{(k+1)} E_{12} + E_{21}^T X_2^{(k+1)} + H_{12} = 0, \tag{24c}$$
$$X_i^{(0)} = 0, \ i = 1,\ 2, X_{12} = 0, \ k = 0,\ 1,\ \ldots .$$

The following theorem indicates the convergence of the algorithm in equation (24).

Theorem 4 *If $E_i,\ i=1,\ 2$ is stable, there exists a small $\hat{\sigma}$ such that for all $\varepsilon \in (0,\ \hat{\sigma}),\ 0 < \hat{\sigma}$, the iterative algorithm in equation (24) converges to the exact solutions of X_i and X_{21} with a rate equal to that of linear convergence. Subsequently, we obtain equation (25).*

$$\|X_i^{(k)} - X_i\| = O(\varepsilon^{2k}),\ i=1,\ 2, \tag{25a}$$
$$\|X_{12}^{(k)} - X_{12}\| = O(\varepsilon^{2k}),\ k=1,\ 2,\ \ldots\ . \tag{25b}$$

Proof : The proof follows directly by applying the fixed point theorem [7]. First, it is easy to verify that the algorithms in equations (24) and (26) are identical.

$$\begin{aligned}
X_1^{(k+1)} &:= \mathcal{Z}_1(X_{12}^{(k)}) \\
&= \varepsilon^2 \int_0^\infty \exp(E_1^T s)(E_{21}^T X_{12}^{(k)T} + X_{12}^{(k)} E_{21}) \exp(E_1 s) ds + \int_0^\infty \exp(E_1^T s) H_1 \exp(E_1 s) ds, & (26a) \\
X_2^{(k+1)} &:= \mathcal{Z}_2(X_{12}^{(k)}) \\
&= \varepsilon^2 \int_0^\infty \exp(E_2^T s)(E_{12}^T X_{12}^{(k)} + X_{12}^{(k)T} E_{12}) \exp(E_2 s) ds + \int_0^\infty \exp(E_2^T s) H_2 \exp(E_2 s) ds, & (26b) \\
X_{12}^{(k+1)} &:= \mathcal{Z}_3(X_1^{(k+1)},\ X_2^{(k+1)}) = \mathcal{Z}_3(X_{12}^{(k)}). & (26c)
\end{aligned}$$

Thus, taking into account the stability of E_1 with regard to equation (26a), there exist $m_1 > 0$ and $\phi_1 > 0$ such that $\|\exp(E_1^T s)\| \le m_1 \exp(-\phi_1 s)$ [19]. As a result, for any X_{12}^a and X_{12}^b

$$\|\mathcal{Z}_1(X_{12}^a) - \mathcal{Z}_1(X_{12}^b)\| \le 2\varepsilon^2 \|E_{21}\| \cdot \|X_{12}^a - X_{12}^b\| \int_0^\infty m_1^2 \exp(-2\phi_1 s) ds := \varepsilon^2 \mathcal{M}_1 \|X_{12}^a - X_{12}^b\|,$$

there exist $\mathcal{M}_1$ and the parameter $\varepsilon = \varepsilon_1$ such that $\varepsilon^2 \mathcal{M}_1 < 1$. Using a technique similar to that given above, there exist $\mathcal{M}_i$ and the parameter $\varepsilon = \varepsilon_i,\ i=2,\ 3$ such that $\varepsilon^2 \mathcal{M}_i < 1$.

$$\begin{aligned}
\|\mathcal{Z}_2(X_{12}^a) - \mathcal{Z}_2(X_{12}^b)\| &= \varepsilon^2 \mathcal{M}_2 \|X_{12}^a - X_{12}^b\|, \\
\|\mathcal{Z}_3(X_{12}^a) - \mathcal{Z}_3(X_{12}^b)\| &= \varepsilon^2 \mathcal{M}_3 \|X_{12}^a - X_{12}^b\|.
\end{aligned}$$

If the convergence factor is chosen to satisfy

$$0 < \varepsilon < \min\left\{ \frac{1}{\sqrt{\mathcal{M}_1}},\ \frac{1}{\sqrt{\mathcal{M}_2}},\ \frac{1}{\sqrt{\mathcal{M}_3}} \right\} \tag{27}$$

then the algorithm in equation (24) attains linear convergence for the fixed point theorem. ■

When the coupling parameter ε is sufficiently small, the fixed point algorithm (24) is very efficient. However, as long as the coupling parameter ε is large, such algorithm cannot be applied. Hence, in order to guarantee the convergence for large parameter ε, we formulate a new algorithm for solving the ALE (22) that is based on the GI algorithm.

4.2 Gradient-Based Iterative (GI) Algorithm

Let us consider the following cross-coupled algebraic Lyapunov equation (CALE) (28), in a general form.

$$E_1^T X + X E_1 + F_{21}^T Y^T + Y F_{21} + H_1 = 0, \tag{28a}$$
$$E_1^T Y + X E_{12} + E_{21}^T Z + Y E_2 + H_{12} = 0, \tag{28b}$$
$$E_2^T Z + Z E_2 + F_{12}^T Y + Y^T F_{12} + H_2 = 0. \tag{28c}$$

It should be noted that for the CALE (28),

$$X_1 \Rightarrow X,\ X_{12} \Rightarrow Y,\ X_2 \Rightarrow Z,\ \varepsilon^2 E_{12} \Rightarrow F_{12},\ \varepsilon^2 E_{21} \Rightarrow F_{21}.$$

In order to solve the CALE (28), the GI algorithm is given below.

$$\left.\begin{aligned} X_1(k+1) &= X(k) - \mu E_1 L_1(k) \\ X_2(k+1) &= X(k) - \mu L_1(k) E_1^T - \mu L_2(k) E_{12}^T \end{aligned}\right\}, \tag{29a}$$

$$\left.\begin{aligned} Z_1(k+1) &= Z(k) - \mu L_3(k) E_2^T \\ Z_2(k+1) &= Z(k) - \mu E_2 L_3(k) - \mu E_{21} L_2(k) \end{aligned}\right\}, \tag{29b}$$

$$\left.\begin{aligned} Y_1(k+1) &= Y(k) - \mu L_1(k) F_{21}^T - \mu L_2(k) E_2^T \\ Y_2(k+1) &= Y(k) - \mu E_1 L_2(k) - \mu F_{12} L_3(k) \end{aligned}\right\}, \tag{29c}$$

$$\left.\begin{aligned} Y_3^T(k+1) &= Y^T(k) - \mu F_{21} L_1(k) \\ Y_4^T(k+1) &= Y^T(k) - \mu L_3(k) F_{12}^T \end{aligned}\right\}, \tag{29d}$$

$$\begin{aligned} X(k+1) &= \frac{X_1(k)+X_2(k)}{2},\ Z(k+1) = \frac{Z_1(k)+Z_2(k)}{2}, \\ Y(k+1) &= \frac{Y_1(k)+Y_2(k)}{2},\ Y^T(k+1) = \frac{Y_3^T(k)+Y_4^T(k)}{2},\ k = 0,\ 1,\ \dots\ , \end{aligned} \tag{29e}$$

where

$$\begin{aligned} & X(0) = 0,\ Y(0) = 0,\ Z(0) = 0, \\ & L_1(k) := E_1^T X(k) + X(k) E_1 + F_{21}^T Y^T(k) + Y(k) F_{21} + H_1, \\ & L_2(k) := E_1^T Y(k) + X(k) E_{12} + E_{21}^T Z(k) + Y(k) E_2 + H_{12}, \\ & L_3(k) := E_2^T Z(k) + Z(k) E_2 + F_{12}^T Y(k) + Y^T(k) F_{12} + H_2. \end{aligned}$$

Theorem 5 *If the CALE (28) has the unique solutions X^*, Y^* and Z^*, then the iterative solutions $X(k)$, $Y(k)$ and $Z(k)$ that are given by the GI algorithm (29) converge to X^*, Y^* and Z^*. That is, the following conditions are satisfied.*

$$\lim_{k\to\infty} X(k) = X^*, \tag{30a}$$
$$\lim_{k\to\infty} Y(k) = Y^*, \tag{30b}$$
$$\lim_{k\to\infty} Z(k) = Z^*. \tag{30c}$$

In order to prove Theorem 5, the following well-known result will be used.

Lemma 2 *Let us consider the CALE (31).*

$$E_1^T \hat{X} + \hat{X} E_1 + F_{21}^T \hat{Y}^T + \hat{Y} F_{21} = 0, \tag{31a}$$
$$E_1^T \hat{Y} + \hat{X} E_{12} + E_{21}^T \hat{Z} + \hat{Y} E_2 = 0, \tag{31b}$$
$$E_2^T \hat{Z} + \hat{Z} E_2 + F_{12}^T \hat{Y} + \hat{Y}^T F_{12} = 0. \tag{31c}$$

The CALE (31) has unique solutions $\hat{X} = 0$, $\hat{Y} = 0$ and $\hat{Z} = 0$ if and only if the following equation holds.

$$\det \begin{bmatrix} E_1^T \otimes I_{n_1} + I_{n_1} \otimes E_1^T & F_{21}^T \otimes I_{n_1} + (I_{n_1} \otimes F_{21}^T)\mathbf{U}_{n_2 n_1} & 0 \\ E_{12}^T \otimes I_{n_1} & I_{n_2} \otimes E_1^T + E_2^T \otimes I_{n_1} & I_{n_2} \otimes E_{21}^T \\ 0 & (F_{12}^T \otimes I_{n_2})\mathbf{U}_{n_2 n_1} + I_{n_2} \otimes F_{12}^T & E_2^T \otimes I_{n_2} + I_{n_2} \otimes E_2^T \end{bmatrix} \neq 0. \tag{32}$$

Proof : Define the error matrices

$$\begin{aligned} & \tilde{X}(k) := X(k) - X^*,\ \tilde{X}_i(k) := X_i(k) - X^*,\ \tilde{Z}(k) := Z(k) - Z^*,\ \tilde{Z}_i(k) := Z_i(k) - Z^*,\ i = 1,\ 2, \\ & \tilde{Y}(k) := Y(k) - Y^*,\ \tilde{Y}_i(k) := Y_i(k) - Y^*,\ i = 1,\ 2, \\ & \tilde{Y}^T(k) := Y^T(k) - Y^{*T},\ \tilde{Y}_i^T(k) := Y_i^T(k) - Y^{*T},\ i = 3,\ 4. \end{aligned}$$

Using the CALE (28) and GI algorithm (29), it is not difficult to obtain

$$\left.\begin{array}{l}\tilde{X}_1(k+1)=\tilde{X}(k)-\mu E_1\tilde{L}_1(k)\\ \tilde{X}_2(k+1)=\tilde{X}(k)-\mu\tilde{L}_1(k)E_1^T-\mu\tilde{L}_2(k)E_{12}^T\end{array}\right\}, \tag{33a}$$

$$\left.\begin{array}{l}\tilde{Z}_1(k+1)=\tilde{Z}(k)-\mu\tilde{L}_3(k)E_2^T\\ \tilde{Z}_2(k+1)=\tilde{Z}(k)-\mu E_2\tilde{L}_3(k)-\mu E_{21}\tilde{L}_2(k)\end{array}\right\}, \tag{33b}$$

$$\left.\begin{array}{l}\tilde{Y}_1(k+1)=\tilde{Y}(k)-\mu\tilde{L}_1(k)F_{21}^T-\mu\tilde{L}_2(k)E_2^T\\ \tilde{Y}_2(k+1)=\tilde{Y}(k)-\mu E_1\tilde{L}_2(k)-\mu F_{12}\tilde{L}_3(k)\end{array}\right\}, \tag{33c}$$

$$\left.\begin{array}{l}\tilde{Y}_3^T(k+1)=\tilde{Y}^T(k)-\mu F_{21}\tilde{L}_1(k)\\ \tilde{Y}_4^T(k+1)=\tilde{Y}^T(k)-\mu\tilde{L}_3(k)F_{12}^T\end{array}\right\}, \tag{33d}$$

where

$$\begin{aligned}\tilde{L}_1(k)&:=E_1^T\tilde{X}(k)+\tilde{X}(k)E_1+F_{21}^T\tilde{Y}^T(k)+\tilde{Y}(k)F_{21},\\ \tilde{L}_2(k)&:=E_1^T\tilde{Y}(k)+\tilde{X}(k)E_{12}+E_{21}^T\tilde{Z}(k)+\tilde{Y}(k)E_2,\\ \tilde{L}_3(k)&:=E_2^T\tilde{Z}(k)+\tilde{Z}(k)E_2+F_{12}^T\tilde{Y}(k)+\tilde{Y}^T(k)F_{12}.\end{aligned}$$

Taking Frobenius norm of both sides of (33a) results in

$$\begin{aligned}&\|\tilde{X}_1(k+1)\|_F^2=\|\tilde{X}(k)-\mu E_1\tilde{L}_1(k)\|_F^2=\left\|\tilde{X}_1(k)-\mu\begin{bmatrix}E_1 & 0\end{bmatrix}\begin{bmatrix}\tilde{L}_1(k) & \tilde{L}_2(k)\\ 0 & \tilde{L}_3(k)\end{bmatrix}\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}\right\|_F^2\\ &\le\|\tilde{X}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{X}(k)E_1\tilde{L}_1(k)\right]+n_1\mu^2\|E_1\|_F^2\mathcal{L}(k),\\ &\quad\|\tilde{X}_2(k+1)\|_F^2=\|\tilde{X}(k)-\mu\tilde{L}_1(k)E_1^T-\mu\tilde{L}_2(k)E_{12}^T\|_F^2\\ &=\left\|\tilde{X}_1(k)-\mu\begin{bmatrix}I_{n_1} & 0\end{bmatrix}\begin{bmatrix}\tilde{L}_1(k) & \tilde{L}_2(k)\\ 0 & \tilde{L}_3(k)\end{bmatrix}\begin{bmatrix}E_1^T\\ E_{12}^T\end{bmatrix}\right\|_F^2\\ &\le\|\tilde{X}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{X}(k)\left(\tilde{L}_1(k)E_1^T+\tilde{L}_2(k)E_{12}^T\right)\right]+n_1\mu^2\left(\|E_1\|_F^2+\|E_{12}\|_F^2\right)\mathcal{L}(k),\end{aligned}$$

where $\mathcal{L}(k):=\|\tilde{L}_1(k)\|_F^2+\|\tilde{L}_2(k)\|_F^2+\|\tilde{L}_3(k)\|_F^2$.

Using the similar technique, the following inequalities hold.

$$\left.\begin{array}{l}\|\tilde{X}_1(k+1)\|_F^2\le\|\tilde{X}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{X}(k)E_1\tilde{L}_1(k)\right]+n_1\mu^2\|E_1\|_F^2\mathcal{L}(k)\\ \|\tilde{X}_2(k+1)\|_F^2\le\|\tilde{X}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{X}(k)\left(\tilde{L}_1(k)E_1^T+\tilde{L}_2(k)E_{12}^T\right)\right]+n_1\mu^2\left(\|E_1\|_F^2+\|E_{12}\|_F^2\right)\mathcal{L}(k)\end{array}\right\}, \tag{34a}$$

$$\left.\begin{array}{l}\|\tilde{Z}_1(k+1)\|_F^2\le\|\tilde{Z}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Z}(k)\tilde{L}_3(k)E_2^T\right]+n_2\mu^2\|E_2\|_F^2\mathcal{L}(k)\\ \|\tilde{Z}_2(k+1)\|_F^2\le\|\tilde{Z}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Z}(k)\left(E_2\tilde{L}_3(k)+E_{21}\tilde{L}_2(k)\right)\right]+n_2\mu^2\left(\|E_2\|_F^2+\|E_{21}\|_F^2\right)\mathcal{L}(k)\end{array}\right\}, \tag{34b}$$

$$\left.\begin{array}{l}\|\tilde{Y}_1(k+1)\|_F^2\le\|\tilde{Y}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Y}(k)\left(\tilde{L}_1(k)F_{21}^T+\tilde{L}_2(k)E_2^T\right)\right]+n_1\mu^2\left(\|E_2\|_F^2+\|F_{21}\|_F^2\right)\mathcal{L}(k)\\ \|\tilde{Y}_2(k+1)\|_F^2\le\|\tilde{Y}(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Y}(k)\left(E_1\tilde{L}_2(k)+F_{12}\tilde{L}_3(k)\right)\right]+n_2\mu^2\left(\|E_1\|_F^2+\|F_{12}\|_F^2\right)\mathcal{L}(k)\end{array}\right\}, \tag{34c}$$

$$\left.\begin{array}{l}\|\tilde{Y}_3^T(k+1)\|_F^2\le\|\tilde{Y}^T(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Y}^T(k)F_{21}\tilde{L}_1(k)\right]+n_1\mu^2\|F_{21}\|_F^2\mathcal{L}(k)\\ \|\tilde{Y}_4^T(k+1)\|_F^2\le\|\tilde{Y}^T(k)\|_F^2-2\mu\mathbf{Tr}\left[\tilde{Y}^T(k)\tilde{L}_3(k)F_{12}^T\right]+n_2\mu^2\|F_{12}\|_F^2\mathcal{L}(k)\end{array}\right\}. \tag{34d}$$

Hence, the following results hold.

$$\begin{aligned}&\|\tilde{X}(k+1)\|_F^2=\frac{\|\tilde{X}_1(k+1)+\tilde{X}_2(k+1)\|_F^2}{4}\le\frac{\|\tilde{X}_1(k+1)\|_F^2+\|\tilde{X}_2(k+1)\|_F^2}{2}\\ &=\|\tilde{X}(k)\|_F^2-\mu\mathbf{Tr}\left[\tilde{X}(k)\{E_1\tilde{L}_1(k)+\tilde{L}_1(k)E_1^T+\tilde{L}_2(k)E_{12}^T\}\right]+\frac{n_1\mu^2}{2}\left(2\|E_1\|_F^2+\|E_{12}\|_F^2\right)\mathcal{L}(k), \qquad (35a)\\ &\|\tilde{Z}(k+1)\|_F^2\end{aligned}$$

$$\leq \|\tilde{Z}(k)\|_F^2 - \mu\mathbf{Tr}\left[\tilde{Z}(k)\{\tilde{L}_3(k)E_2^T + E_2\tilde{L}_3(k) + E_{21}\tilde{L}_2(k)\}\right] + \frac{n_2\mu^2}{2}\left(2\|E_2\|_F^2 + \|E_{21}\|_F^2\right)\mathcal{L}(k), \tag{35b}$$

$$\|\tilde{Y}(k+1)\|_F^2$$
$$= \|\tilde{Y}(k)\|_F^2 - \mu\mathbf{Tr}\left[\tilde{Y}(k)\{\tilde{L}_1(k)F_{21}^T + E_1\tilde{L}_2(k) + \tilde{L}_2(k)E_2^T + F_{12}\tilde{L}_3(k)\}\right]$$
$$+\frac{\mu^2}{2}\left(n_2\|E_1\|_F^2 + n_1\|E_2\|_F^2 + n_2\|F_{12}\|_F^2 + n_1\|F_{21}\|_F^2\right)\mathcal{L}(k), \tag{35c}$$

$$\|\tilde{Y}^T(k+1)\|_F^2$$
$$\leq \|\tilde{Y}^T(k)\|_F^2 - \mu\mathbf{Tr}\left[\tilde{Y}^T(k)\{F_{21}\tilde{L}_1(k) + \tilde{L}_3(k)F_{12}^T\}\right] + \frac{\mu^2}{2}\left(n_1\|F_{21}\|_F^2 + n_2\|F_{12}\|_F^2\right)\mathcal{L}(k). \tag{35d}$$

Thus, summing the above four inequalities, the following result is satisfied.

$$\begin{aligned}&\|\tilde{X}(k+1)\|_F^2 + 2\|\tilde{Y}(k+1)\|_F^2 + \|\tilde{Z}(k+1)\|_F^2\\ &\leq \|\tilde{X}(k)\|_F^2 + 2\|\tilde{Y}(k)\|_F^2 + \|\tilde{Z}(k)\|_F^2 - \mu\mathcal{L}(k) + \mu^2\mathcal{L}(k)\Lambda,\end{aligned} \tag{36}$$

where

$$\Lambda := \left(n_1 + \frac{n_2}{2}\right)\|E_1\|_F^2 + \left(\frac{n_1}{2} + n_2\right)\|E_2\|_F^2 + \frac{n_1}{2}\|E_{12}\|_F^2 + \frac{n_2}{2}\|E_{21}\|_F^2 + n_1\|F_{21}\|_F^2 + n_2\|F_{12}\|_F^2.$$

Moreover, summing the inequality (36) from zero to N results in

$$\begin{aligned}&\|\tilde{X}(N+1)\|_F^2 + 2\|\tilde{Y}(N+1)\|_F^2 + \|\tilde{Z}(N+1)\|_F^2\\ &\leq \|\tilde{X}(0)\|_F^2 + 2\|\tilde{Y}(0)\|_F^2 + \|\tilde{Z}(0)\|_F^2 - \mu(1-\Lambda\mu)\sum_{k=0}^{N}\mathcal{L}(k).\end{aligned} \tag{37}$$

If the convergence factor is chosen to satisfy

$$0 < \mu < \Lambda^{-1} \tag{38}$$

then

$$\sum_{k=0}^{N}\mathcal{L}(k) < \sum_{k=0}^{\infty}\left(\|\tilde{L}_1(k)\|_F^2 + \|\tilde{L}_2(k)\|_F^2 + \|\tilde{L}_3(k)\|_F^2\right) < \infty. \tag{39}$$

This implies that as $k \to \infty$, $\tilde{L}_i(k) \to 0$, $i = 1,\ 2,\ 3$. According to Lemma 2, $\tilde{X}(k) \to 0$, $\tilde{Y}(k) \to 0$ and $\tilde{Z}(k) \to 0$ as $k \to \infty$. This completes the proof of Theorem 5. ■

4.3 The Dual ARE

In this section, a numerical algorithm for solving the dual ARE (3) of the ARE (2) is given as an important algorithm.

Let us consider the dual ARE (40).

$$A_\varepsilon W_\varepsilon + W_\varepsilon A_\varepsilon^T - W_\varepsilon U W_\varepsilon + T_\varepsilon = 0, \tag{40}$$

where

$$W_\varepsilon = \begin{bmatrix} W_1 & \varepsilon W_{12} \\ \varepsilon W_{12}^T & W_2 \end{bmatrix},\ W_i = W_i^T \geq 0,$$
$$U := C_2^T C_2 - \gamma^{-2}C_1^T C_1 = \mathbf{block\ diag}\begin{pmatrix} U_1 & U_2 \end{pmatrix},$$
$$T_\varepsilon := B_{1\varepsilon}B_{1\varepsilon}^T = \begin{bmatrix} T_1 + \varepsilon^2 T_{11} & \varepsilon(T_{112} + T_{122}) \\ \varepsilon(T_{112} + T_{122})^T & T_2 + \varepsilon^2 T_{22} \end{bmatrix},\ T_i = T_i^T,\ T_{ii} = T_{ii}^T.$$

The 0th order solutions $\bar{W}_i,\ i=1,\ 2$ and $\bar{W}_{12}$ are also defined as $\varepsilon \to +0$ for the dual ARE.

$$A_i\bar{W}_i + \bar{W}_iA_i^T - \bar{W}_iU_i\bar{W}_i + T_i = 0. \tag{41}$$

where $U_i := C_{2i}C_{2i}^T - \gamma^{-2}C_{1i}C_{1i}^T$.

The ARE (41) will produce a unique positive semidefinite stabilizing solution if γ is sufficiently large.

Let

$\gamma_{W_i} = \inf\{\gamma > 0|$ the ARE (41) has a positive semidefinite stabilizing solution$\}$.

Then, matrix $A_{ii} - \bar{W}_iU_i$ is nonsingular if we choose $\gamma > \gamma_{W_i}$.

Theorem 6 *It is assumed that the reduced-order ARE (41), which is independent of the perturbation parameter ε, has a positive semidefinite stabilizing solution. If we select a parameter $\gamma > \bar{\gamma}_W = \max\{\gamma_{W_1},\ \gamma_{W_2}\}$, then there exists a small $\bar{\rho}$ such that for all $\varepsilon \in (0,\ \bar{\rho})$, the ARE (41) admits a positive semidefinite stabilizing solution W_ε that can be written as equation (42).*

$$W_\varepsilon = \bar{W} + O(\varepsilon) = \textbf{block diag}\begin{pmatrix} \bar{W}_1 & \bar{W}_2 \end{pmatrix} + O(\varepsilon). \tag{42}$$

Proof : Since this can be proved by using a technique similar to that used in Theorem 1, it is omitted. ■

In order to solve the ARE (40), we can apply Newton's method by using the asymptotic structure in equation (42).

Theorem 7 *Let us consider Newton's method (43).*

$$(A_\varepsilon - W_\varepsilon^{(n)}U)W_\varepsilon^{(n+1)} + W_\varepsilon^{(n+1)}(A_\varepsilon - W_\varepsilon^{(n)}U)^T + W_\varepsilon^{(n)}UW_\varepsilon^{(n)} + T_\varepsilon = 0,\ n = 0,\ 1,\ \ldots, \tag{43a}$$

$$W_\varepsilon^{(n)} = \begin{bmatrix} W_1^{(n)} & \varepsilon W_{12}^{(n)} \\ \varepsilon W_{12}^{(n)T} & W_2^{(n)} \end{bmatrix}, \tag{43b}$$

$$W_\varepsilon^{(0)} = \textbf{block diag}\begin{pmatrix} \bar{W}_1 & \bar{W}_2 \end{pmatrix}. \tag{43c}$$

If the parameter-independent reduced-order ARE (41) has a positive semidefinite stabilizing solution, there exists a small $\tilde{\rho}$ such that for all $\varepsilon \in (0,\ \tilde{\rho})$, $0 < \tilde{\rho} \le \bar{\rho}$, the iterative algorithm in equation (43a) converges to the exact solution W_ε with a rate equal to that of quadratic convergence. Subsequently, we obtain equation (44).

$$\|W_\varepsilon^{(n)} - W_\varepsilon\| = \frac{O(\varepsilon^{2^n})}{\delta\xi 2^n},\ n = 0,\ 1,\ \ldots, \tag{44}$$

where

$$\xi = 2\|U\| < \infty,\ \delta = \|[\nabla\mathcal{F}(W_\varepsilon^{(0)})]^{-1}\|,\ \psi = \delta \cdot \|\mathcal{F}(W_\varepsilon^{(0)})\|,\ \phi = \delta\psi\xi,$$
$$\nabla\mathcal{F}(W_\varepsilon) = \frac{\partial \mathrm{vec}\mathcal{F}(W_\varepsilon)}{\partial(\mathrm{vec}W_\varepsilon)^T},\ \mathcal{F}(W_\varepsilon) = A_\varepsilon W_\varepsilon + W_\varepsilon A_\varepsilon^T - W_\varepsilon UW_\varepsilon + T_\varepsilon.$$

Proof : Since this can be proved by using a method similar to that of Theorem 2 that is based on the Newton-Kantorovich theorem [2, 3], it is omitted. ■

It should be noted that the computation of the algorithm in equation (43a) can be done by substituting it into equation (24) or (29) as follows.

$$W_\varepsilon \Rightarrow X_\varepsilon,\ (A_\varepsilon - W_\varepsilon^{(n)}U)^T \Rightarrow E_\varepsilon,\ W_\varepsilon^{(n)}UW_\varepsilon^{(n)} + T_\varepsilon \Rightarrow H_\varepsilon. \tag{45}$$

Finally, we succeed in establishing the numerical algorithm with the reduced-order computations for solving the dual ARE of strongly coupled systems.

5 Computational Examples

In order to verify the efficiency of the proposed algorithms, some computational examples are provided.

5.1 Example 1

It should be noted that since Newton's method (43) can be employed by using steps similar to those of the proposed algorithm in equation (13), it is omitted. The system matrices of the large-scale systems (1) are given as follows as a modification of [4].

$$A_{11} = \begin{bmatrix} 0 & 1 & -0.266 & -0.009 \\ -2.75 & -2.78 & -1.36 & -0.037 \\ 0 & 0 & 0 & 1 \\ -4.95 & 0 & -55.5 & -0.039 \end{bmatrix},\ \varepsilon A_{12} = \begin{bmatrix} 0.0024 & 0 & -0.087 & 0.002 \\ -0.185 & 0 & 1.11 & -0.011 \\ 0 & 0 & 0 & 0 \\ 0.222 & 0 & 8.17 & 0.004 \end{bmatrix},$$

$$\varepsilon A_{21} = \begin{bmatrix} 0.021 & 0 & 0.121 & 0.003 \\ -1.1 & 0 & -1.62 & -0.015 \\ 0 & 0 & 0 & 0 \\ -2.43 & 0 & 1.37 & -0.034 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.21 & 1 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0 & 0 & 0 & 1 \\ -3.1 & 0 & -56 & 0.032 \end{bmatrix},$$

$$B_{111} = \begin{bmatrix} 0 \\ 36.1 \\ 0 \\ 0 \end{bmatrix},\ B_{122} = \begin{bmatrix} 0 \\ 78.9 \\ 0 \\ 0 \end{bmatrix},\ B_{112}=B_{121}=0,\ B_{211} = \begin{bmatrix} 0 \\ 3.5 \\ 0 \\ 0 \end{bmatrix},\ B_{222} = \begin{bmatrix} 0 \\ 4.2 \\ 0 \\ 0 \end{bmatrix},\ B_{212}=B_{221}=0,$$

$$C_1 = \begin{bmatrix} \sqrt{0.5} I_8 \\ 0_{2\times 8} \end{bmatrix},\ D_1 = \begin{bmatrix} 0_{8\times 2} \\ I_2 \end{bmatrix}.$$

The two basic quantities for the system are $\gamma_{P_1} = 9.7396 \times 10^{-2}$ and $\gamma_{P_2} = 5.3678 \times 10^{-2}$. Thus, for every boundary value $\gamma > \bar{\gamma}_P = \max\{\gamma_{P_1},\ \gamma_{P_2}\} = 9.7396 \times 10^{-2}$, the ARE (2) has a positive semidefinite stabilizing solution for a sufficiently small ε. On the other hand, using MATLAB, the minimum value of γ^* such that there exists a dynamic feedback controller is $\gamma^* = 9.7396 \times 10^{-2}$ for $\varepsilon = 10^{-3}$.

In order to verify the exactitude of the solution, we calculate the error per iteration in Table 1, where $\gamma = 10$, and the convergence condition is given by $\|\mathcal{G}(P_\varepsilon^{(n)})\| < 10^{-10}$. Hence, it can be observed from Table 1 that the algorithm in equation (13a) attains quadratic convergence.

5.2 Example 2

Consider the system (1) with

$$A_\varepsilon = \left[\begin{array}{cc|c} 0 & 1 & \varepsilon \\ -1 & -1 & 0 \\ \hline 0 & \varepsilon & 0 \end{array}\right],\ B_{1\varepsilon} = B_{2\varepsilon} = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \hline \varepsilon & 1 \end{array}\right],\ C_1 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],\ D_1 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right].$$

It is easy to verify that for every boundary value $\gamma > 1$, the ARE (2) has a positive semidefinite stabilizing solution for a sufficiently small ε. In order to evaluate the convergence criteria, the values of $\theta^{[m]}$ of (18) should be calculated. First, the partitioned equation of the ARE (2) can be obtained as follows.

$$\mathcal{G}(P_\varepsilon) = A_\varepsilon^T P_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = \begin{bmatrix} g_1 & g_2 & g_3 \\ g_2 & g_4 & g_5 \\ g_3 & g_5 & g_6 \end{bmatrix},\ P_\varepsilon = \begin{bmatrix} p_1 & p_2 & \varepsilon p_3 \\ p_2 & p_4 & \varepsilon p_5 \\ \varepsilon p_3 & \varepsilon p_5 & p_6 \end{bmatrix},$$

where $g_i := g_i(p_1,\ \dots,p_6,\ \varepsilon)$, $i = 1,\ \dots,6$, $\alpha := (1+\varepsilon^2)(1-\gamma^{-2})$,

$$\begin{aligned}
g_1(p_1,\ \dots,p_6,\ \varepsilon) &= -2p_2 - \varepsilon^2 \alpha p_3^2 + 1, \\
g_2(p_1,\ \dots,p_6,\ \varepsilon) &= p_1 - p_2 + \varepsilon^2 p_3 - p_4 - \varepsilon^2 \alpha p_3 p_5, \\
g_3(p_1,\ \dots,p_6,\ \varepsilon) &= p_1 - p_5 - \alpha p_3 p_6, \\
g_4(p_1,\ \dots,p_6,\ \varepsilon) &= 2p_2 - 2p_4 + 2\varepsilon^2 p_5 - \varepsilon^2 \alpha p_5^2, \\
g_5(p_1,\ \dots,p_6,\ \varepsilon) &= p_2 + p_3 - p_5 + p_6 - \alpha p_5 p_6, \\
g_6(p_1,\ \dots,p_6,\ \varepsilon) &= 2\varepsilon^2 p_3 - \alpha p_6^2 + 1.
\end{aligned}$$

Therefore, the related equations (16) are given below.

$$\tilde{\mathcal{G}}(\mathcal{P}_\varepsilon,\ \varepsilon) := \left[\begin{array}{cccccc} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \end{array}\right]^T,\ \mathcal{P}_\varepsilon^T := \left[\begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \end{array}\right],$$

$$\frac{\partial \tilde{\mathcal{G}}(\mathcal{P}_\varepsilon,\ \varepsilon)}{\partial \mathcal{P}_\varepsilon} = \nabla \tilde{\mathcal{G}}(\mathcal{P}_\varepsilon,\ \varepsilon) := \left[\begin{array}{cccccc} 0 & -2 & -2\varepsilon^2\alpha p_3 & 0 & 0 & 0 \\ 1 & -1 & \varepsilon^2(1-\alpha p_5) & -1 & -\varepsilon^2\alpha p_3 & 0 \\ 1 & 0 & -\alpha p_6 & 0 & -1 & -\alpha p_3 \\ 0 & 2 & 0 & -2 & 2\varepsilon^2(1-\alpha p_5) & 0 \\ 0 & 1 & 1 & 0 & -1-\alpha p_6 & 1-\alpha p_5 \\ 0 & 0 & 2\varepsilon^2 & 0 & 0 & -2\alpha p_6 \end{array}\right].$$

Therefore, it results in the following equation.

$$\det \nabla \tilde{\mathcal{G}}(\mathcal{P}_\varepsilon,\ \varepsilon^{[m]}) = 8\bar{\alpha}\bar{p}_6[(\bar{\alpha}\bar{p}_6)^2 + \bar{\alpha}\bar{p}_6 + 1] + O(\varepsilon),\ \bar{\alpha} := 1 - \gamma^{-2},\ \bar{p}_6 := \frac{1}{\sqrt{\bar{\alpha}}}.$$

It should be noted that $\bar{\alpha}\bar{p}_6 \neq 0$. Thus, there exists $\beta^{[m]}$ for sufficiently small ε such that

$$\beta^{[m]} = \|[\nabla \tilde{\mathcal{G}}(\mathcal{P}_\varepsilon,\ \varepsilon^{[m]})]^{-1}\|.$$

In order to verify the validity of the proposed successive algorithm in the previous section, iterations are carried out. Since for small parameter $\varepsilon^{[0]} = 0.001$ at $m = 0$, $\theta^{[0]} = 0.01230027 < 0.5$ holds, the quadratic convergence can be verified without the simulation. Thus, for sufficiently small ε, it can be concluded that the uniqueness of the solution is guaranteed if inequality (19) holds. The simulation results using the successive algorithm are given for $\varepsilon = 0.001 \sim 0.1201070$. The convergence criteria for all m is given as Table 2. It is observed that since for all m, inequality (18) ($\theta^{[m]} < 2^{-1}$) is satisfied, Newton's method (13) exhibits a quadratic convergence. Moreover, the uniqueness of the convergence solutions are guaranteed at the neighborhood of each $\varepsilon = \varepsilon^{[m]}$. In other words, if the parameter ε begins from $\varepsilon^{[0]} = 0.001$, the initial conditions $P_\varepsilon^{(0)} = \bar{P}$ of (14) satisfy inequality (19). For the next step, if $\varepsilon^{[m]}$ is chosen such that $\varepsilon^{[0]} < \varepsilon^{[1]} < \cdots < \varepsilon^{[m]}$, for all m, inequality (18) also holds. Therefore, when solution $P_\varepsilon^{(n)}$ are solved by using Newton's method (13), quadratic convergence is achieved. In fact, for all m, this useful phenomenon has been observed. Moreover, the local uniqueness would be achieved at the neighborhood of each $\varepsilon = \varepsilon^{[m]}$. It should be noted that the required solution is obtained by repeating the successive algorithm recursively until the bound of $\varepsilon = 0.1201070$. In this case, it may be also noted that the error estimations (20) are satisfied.

5.3 Example 3

Consider the system (22) with

$$\begin{aligned} &E_\varepsilon := -2I_{n_1+n_2} + \varepsilon\mathbf{rand}(n_1+n_2),\ E_1 \in \mathbf{R}^{n_1\times n_1},\ E_2 \in \mathbf{R}^{n_2\times n_2}, \\ &H_\varepsilon = 2I_{n_1+n_2} + \varepsilon M^T M,\ M = \mathbf{rand}(1,\ n_1+n_2),\ H_1 \in \mathbf{R}^{n_1\times n_1},\ H_2 \in \mathbf{R}^{n_2\times n_2}, \\ &n_1 = 20,\ n_2 = 10, \end{aligned}$$

where $\mathbf{rand}(m,\ n)$ denotes a scalar value drawn from a uniform distribution on the unit interval with m-by-n matrix of the same dimension.

The convergence of the GI algorithm in equation (29) is demonstrated as $n_1 = 20$ and $n_2 = 10$. The small parameter of the GI algorithm (29) is chosen as $\mu = 10^{-2}$. For various large ε, Table 3 shows the required iteration for the GI algorithm (29) versus the fixed point algorithm (24), where the convergence condition is given by $\|\mathcal{L}(X_\varepsilon^{(k)})\| = \|E_\varepsilon^T X_\varepsilon^{(k)} + X_\varepsilon^{(k)} E + H_\varepsilon\| < 10^{-10}$. It should be noted that "NC" stands for no convergence. From Table 3, it can be verified that the GI algorithm can succeed in obtaining the solution for a large ε, while the fixed point algorithm (24) cannot. Furthermore, even if the large-scale systems (22) are composed of two four-dimensional subsystems, the required workspace is four. This feature is very useful from the practical viewpoint.

Table 1. $\|\mathcal{G}(P_\varepsilon^{(n)})\|$

$n \setminus \varepsilon$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$
0	6.3881×10^{-2}	6.3881×10^{-3}	6.3881×10^{-4}	6.3881×10^{-5}	6.3883×10^{-6}
1	3.5601×10^{-4}	3.5458×10^{-6}	3.5444×10^{-8}	3.4186×10^{-10}	2.7829×10^{-11}
2	3.5601×10^{-4}	2.2232×10^{-11}	2.0435×10^{-11}	2.1110×10^{-11}	
3	2.5761×10^{-11}				

Table 2. Convergence criteria.

m	$\varepsilon^{[m]}$	$\theta^{[m]} < 2^{-1}$
$m = 0 \sim 392$	$\varepsilon^{[m]} = 0.001 + m \times 0.0001$	ALL O.K.
$m = 392 \sim 4396$	$\varepsilon^{[m]} = 0.0402 + (m - 392) \times 0.00001$	ALL O.K.
$m = 4396 \sim 44263$	$\varepsilon^{[m]} = 0.08024 + (m - 4396) \times 0.000001$	ALL O.K.

Table 3. Required Iterations

Algorithm $\setminus \varepsilon$	$\varepsilon = 0.1$	$\varepsilon = 0.3$	$\varepsilon = 0.5$	$\varepsilon = 0.6$
GI algorithm	15313	434650	11943	6256
Fixed point algorithm	24	NC	NC	NC

6 Conclusion

A numerical algorithm for solving the ARE that is related to H_∞ output feedback control problem for strongly coupled large-scale systems has been investigated. In order to solve ARE, the successive algorithm that is based on Newton's method has been derived. As a result, it has been shown that the quadratic convergence is guaranteed under the appropriate initial condition. Moreover, the local uniqueness of the solutions has been proved for any parameter ε. Then, in order to reduce the dimension of matrix calculation, a GI algorithm has been combined with Newton's method instead of a fixed point algorithm. Thus, the dimension of the computation for the algebraic manipulation will be same as that of the solutions. Finally, the numerical example used yielded excellent results.

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