submitted to
Applied Mathematics and Computation

A Numerical Algorithm for Finding Solution of Sign-Indefinite Algebraic Riccati Equations for General Multiparameter Singularly Perturbed Systems

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Abstract. In this paper, a computational algorithm for solving sign-indefinite general multiparameter algebraic Riccati equation (SIGMARE) that arises in the H_∞ filtering problem is investigated. After establishing the asymptotic structure of the solution of the SIGMARE, in order to solve the SIGMARE, Newton's method and two fixed point algorithms are combined. As a result, the new iterative algorithm achieves the quadratic convergence property and succeeds in reducing the computing workspace dramatically. As another important feature, the convergence criteria for small parameters ε_i is derived for the first time. Moreover, it is shown that the uniqueness and positive semidefiniteness of the convergence solutions are guaranteed in the neighborhood of the initial conditions.
keywords: general multiparameter singularly perturbed system (GMSPS), sign-indefinite general multiparameter algebraic Riccati equation (SIGMARE), H_∞ filtering problem, Newton's method, fixed point algorithm.

1 Introduction

Filtering problems for the multiparameter singularly perturbed system (MSPS) have been investigated extensively (see e.g., [3, 11, 12] and reference therein). The multimodeling problems arise in the large scale dynamic systems. For example, the multimodel situation in practice is illustrated by the passenger car model [3]. In order to obtain the optimal solution to the multimodeling problems, the multiparameter algebraic Riccati equation (MARE) needs to be solved. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [4, 5]). One of the approaches is the invariant subspace approach that is based on the Hamiltonian matrix [4]. However, there is no guarantee of symmetry for the computed solution if the ARE is ill-conditioned [4]. It should be noted that it is very difficult to solve the MARE due to high dimension and numerical stiffness [3]. Furthermore, if the ARE has the sign-indefinite quadratic term, it is hard to choose the initial condition of the Klinman algorithm [5].

A popular approach to deal with the MSPS is the two-time-scale design method (see e.g., [1, 2]). However, when the parameters ε_i are not small enough, it is known from [3] that an $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $\mu := [\varepsilon_1 \ \cdots \ \varepsilon_N]$) accuracy is very often not sufficient. More recently, in [11, 12], the recursive algorithms for solving the MARE have been developed. However, there exists the drawback that the recursive algorithm converges only to the approximation solution [12] since the convergence of the recursive algorithm depend on the zero-order solutions. Later, although the application of Newton's method for solving the general multiparameter algebraic Riccati equation (GMARE) has been tackled [14], the considered GMARE needs to have sign-definite quadratic form. Thus, the proposed algorithm is restricted for the practical use. On the other hand, the exact slow-fast decomposition method for solving the MARE has been proposed in [3]. However, these results are restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (see e.g., Assumption 5, [3]). Moreover, the particular MSPS that has two fast subsystems has only been discussed. It is very important to extend to the general MSPS that has N fast subsystems for use of the practical systems.

In this paper, a computational algorithm for solving the sign-indefinite general multiparameter algebraic Riccati equation (SIGMARE) related to the H_∞ filtering problem is developed. It may be noted that the general multiparameter singularly perturbed systems (GMSPS) are considered, as compared to the existing results [3, 11, 12, 13]. Although the Pareto optimal strategy for the GMSPS has been investigated in [14], it should be noted that the ARE has the sign-definite quadratic form. Moreover, it should also be noted that there are no results for the convergence criteria for the small parameter ε_i related to the Newton's method in all existing results.

After showing the uniqueness and boundedness of the solution to such SIGMARE, the asymptotic structure is established. The main contribution of this paper is to propose a new algorithm that combines the Newton's method with two fixed point algorithms. As a result, the new iterative algorithm achieves the quadratic convergence property and succeeds in reducing the computing workspace dramatically. As another important feature, the convergence criteria for small parameters ε_i is derived for the first time. Furthermore, it is not assumed here that the Hamiltonian matrices $T_{ii},\ i = 1,\ \ldots,N$ for the fast subsystems have no eigenvalues in common. Thus, our new results are applicable to more realistic GMSPS. Finally, in order to demonstrate the efficiency and the validity of the algorithm, two computational examples are included.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. $\det L$ denotes the determinant of the square matrix L. I_p denotes the $p \times p$ identity matrix. **block diag** denotes the block diagonal matrix. $\mathrm{vec} M$ denotes the column vector of the matrix M [9]. $\otimes$ denotes Kronecker product. U_{pq} denotes a permutation matrix in Kronecker matrix sense [9] such that $U_{pq}\mathrm{vec}M = \mathrm{vec}M^T,\ M \in \mathbf{R}^{p\times q}$.

2 H_∞ Filtering Problem

Let us consider the linear time-invariant GMSPS

$$\dot{x}_0(t) = A_{00}x_0(t) + \sum_{k=1}^{N} A_{0k}x_k(t) + \sum_{k=1}^{N} D_{0k}w_k(t), \tag{1a}$$

$$\varepsilon_i\dot{x}_i(t) = A_{i0}x_0(t) + A_{ii}x_i(t) + D_{ii}w_i(t),\ i = 1,\ \ldots,N, \tag{1b}$$

with

$$y_i(t) = C_{i0}x_0(t) + C_{ii}x_i(t) + v_i(t),\ i = 1,\ \ldots,N, \tag{2}$$

where $x_i(t) \in \mathbf{R}^{n_i},\ i = 0,\ 1,\ \ldots N$ are state vectors, $y_i(t) \in \mathbf{R}^{p_i},\ i = 1,\ \ldots,N$ are system measurements, $w_i(t) \in \mathbf{R}^{q_i},\ i = 1,\ \ldots,N$ and $v_i(t) \in \mathbf{R}^{r_i},\ i = 1,\ \ldots,N$ are system and measurement disturbances, respectively. All the matrices are constant matrices of appropriate dimensions. It may be noted that the system (1b) is called the fast subsystems with small perturbation parameters ε_i.

$\varepsilon_i,\ \varepsilon_j,\ i,\ j = 1,\ \ldots,N$ are the small positive singular perturbation parameters with the same order of magnitude [1, 2] such that

$$0 < \underline{k}_{ij} \le \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \le \bar{k}_{ij} < \infty. \tag{3}$$

That is, it is assumed that the ratio of ε_i and ε_j is bounded by some positive constants.

In this paper, an H_∞ filter to estimate system states $x_i(t)$ is designed. The states to be estimated are given by a linear combination

$$z_i(t) = G_{i0}x_0(t) + G_{ii}x_i(t),\ i = 1,\ \ldots,N, \tag{4}$$

where $z_i(t) \in \mathbf{R}^{s_i},\ i = 1,\ \ldots,N$.

The estimation problem is to obtain an estimate $\hat{z}_i(t)$ of $z_i(t)$ using the measurements $y_i(t)$ [6]. The measure of the infinite horizon estimation problem is defined as a disturbance attenuation function

$$J = \int_0^\infty \|z(t) - \hat{z}(t)\|_R^2 dt \cdot \left\{ \int_0^\infty (\|w(t)\|_{W^{-1}}^2 + \|v(t)\|^2)dt \right\}^{-1}, \tag{5}$$

where $z(t) = \begin{bmatrix} z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T$, $\hat{z}(t) = \begin{bmatrix} \hat{z}_1^T(t) & \cdots & \hat{z}_N^T(t) \end{bmatrix}^T$, $w(t) = \begin{bmatrix} w_1^T(t) & \cdots & w_N^T(t) \end{bmatrix}^T$ and $v(t) = \begin{bmatrix} v_1^T(t) & \cdots & v_N^T(t) \end{bmatrix}^T$, and where $R \ge 0$ and $W > 0$ are weighting matrices to be chosen by designer. The H_∞ filter is to ensure that the energy gain from the disturbances to the estimation errors $z(t) - \hat{z}(t)$ is less than a prespecified attenuation level γ^2. That is,

$$\sup_{w,\ v} J < \gamma^2. \tag{6}$$

The H_∞ filter of (1) and (2) is given as follows [6]

$$\begin{aligned}\dot{\xi}_0(t) &= A_{00}\xi_0(t) + \sum_{k=1}^{N} A_{0k}\xi_k(t) + \sum_{k=1}^{N} F_{0k}\eta_k(t), && \text{(7a)}\\ \varepsilon_i\dot{\xi}_i(t) &= A_{i0}\xi_0(t) + A_{ii}\xi_i(t) + \sum_{k=1}^{N} F_{ik}\eta_k(t), && \text{(7b)}\\ \eta_i(t) &= y_i(t) - C_{i0}\xi_0(t) - C_{ii}\xi_i(t),\ i = 1,\ \dots, N, && \text{(7c)}\end{aligned}$$

where the filter gain F is obtained from

$$F = X_e C^T = \Phi_e^{-1} X C^T = \Phi_e^{-1}\begin{bmatrix} F_{01} & \cdots & F_{0N} \\ F_{11} & \cdots & F_{1N} \\ \vdots & \ddots & \vdots \\ F_{N1} & \cdots & F_{NN}\end{bmatrix}, \tag{8}$$

and X_e satisfies the SIGMARE

$$A_e X_e + X_e A_e^T - X_e S X_e + U_e = 0, \tag{9}$$

with

$$\begin{aligned}
&\Phi_e := \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix},\ \Pi_e := \textbf{block diag}\begin{pmatrix} \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N}\end{pmatrix},\\
&A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1}A_{f0} & \Pi_e^{-1}A_f \end{bmatrix},\ A_{0f} := \begin{bmatrix} A_{01} & \cdots & A_{0N}\end{bmatrix},\\
&A_{f0} := \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T\end{bmatrix}^T,\ A_f := \textbf{block diag}\begin{pmatrix} A_{11} & \cdots & A_{NN}\end{pmatrix},\\
&C := \begin{bmatrix} C_0 & C_f \end{bmatrix},\ C_0 := \begin{bmatrix} C_{10}^T & \cdots & C_{N0}^T\end{bmatrix}^T,\ C_f := \textbf{block diag}\begin{pmatrix} C_{11} & \cdots & C_{NN}\end{pmatrix},\\
&G := \begin{bmatrix} G_0 & G_f \end{bmatrix},\ G_0 := \begin{bmatrix} G_{10}^T & \cdots & G_{N0}^T\end{bmatrix}^T,\ G_f := \textbf{block diag}\begin{pmatrix} G_{11} & \cdots & G_{NN}\end{pmatrix},\\
&D_e := \begin{bmatrix} D_0 \\ \Pi_e^{-1}D_f \end{bmatrix},\ D_0 := \begin{bmatrix} D_{01} & \cdots & D_{0N}\end{bmatrix},\ D_f := \textbf{block diag}\begin{pmatrix} D_{11} & \cdots & D_{NN}\end{pmatrix},\\
&W := \textbf{block diag}\begin{pmatrix} W_1 & \cdots & W_N\end{pmatrix},\ R := \textbf{block diag}\begin{pmatrix} R_1 & \cdots & R_N\end{pmatrix},\\
&S := C^TC - \gamma^{-2}G^TRG = \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix},\ S_{00} := \sum_{k=1}^{N}(C_{k0}^TC_{k0} - \gamma^{-2}G_{k0}^TR_kG_{k0}),\\
&S_{0f} := \begin{bmatrix} S_{01} & \cdots & S_{0N}\end{bmatrix} = \begin{bmatrix} C_{10}^TC_{11} - \gamma^{-2}G_{10}^TR_1G_{11} & \cdots & C_{N0}^TC_{NN} - \gamma^{-2}G_{N0}^TR_NG_{NN}\end{bmatrix},\\
&S_f := \textbf{block diag}\begin{pmatrix} S_{11} & \cdots & S_{NN}\end{pmatrix}\\
&\quad = \textbf{block diag}\begin{pmatrix} C_{11}^TC_{11} - \gamma^{-2}G_{11}^TR_1G_{11} & \cdots & C_{NN}^TC_{NN} - \gamma^{-2}G_{NN}^TR_NG_{NN}\end{pmatrix},\\
&U_e := D_eWD_e^T = \begin{bmatrix} U_{00} & U_{0f}\Pi_e^{-1} \\ \Pi_e^{-1}U_{0f}^T & \Pi_e^{-1}U_f\Pi_e^{-1}\end{bmatrix},\ U_{00} := \sum_{k=1}^{N} D_{0k}W_kD_{0k}^T,\\
&U_{0f} := \begin{bmatrix} U_{01} & \cdots & U_{0N}\end{bmatrix} = \begin{bmatrix} D_{01}W_1D_{11}^T & \cdots & D_{0N}W_ND_{NN}^T\end{bmatrix},\\
&U_f := \textbf{block diag}\begin{pmatrix} U_{11} & \cdots & U_{NN}\end{pmatrix} = \textbf{block diag}\begin{pmatrix} D_{11}W_1D_{11}^T & \cdots & D_{NN}W_ND_{NN}^T\end{pmatrix}.
\end{aligned}$$

It is noteworthy that the H_∞ filter does not require knowledge of the system and measurement noise intensity matrices as compared with the standard Kalman filter [3, 12]. The difficulty encountered with the H_∞ filter for the GMSPS is that the SIGMARE contains a sign-indefinite quadratic term.

A solution X_e of the SIGMARE (9), if it exists, must contain terms of order ε_i because the matrices A_e and D_e contain the term of ε_i^{-1}-order. Taking this fact into consideration, the solution X_e to the SIGMARE (9) with the following structure has to be found.

$$X_e := \begin{bmatrix} X_{00} & X_{0f} \\ X_{0f}^T & \Pi_e^{-1}X_f \end{bmatrix},\ X_{00} = X_{00}^T,\ \Pi_e^{-1}X_f = X_f^T\Pi_e^{-1},$$

$$X_{0f} := \begin{bmatrix} X_{01}^T \\ \vdots \\ X_{0N}^T \end{bmatrix}^T, \; X_f := \begin{bmatrix} X_{11} & \alpha_{21}X_{12} & \alpha_{31}X_{13} & \cdots & \alpha_{N1}X_{1N} \\ X_{12}^T & X_{22} & \alpha_{32}X_{23} & \cdots & \alpha_{N2}X_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{1(N-1)}^T & X_{2(N-1)}^T & X_{3(N-1)}^T & \cdots & \alpha_{N(N-1)}X_{(N-1)N} \\ X_{1N}^T & X_{2N}^T & X_{3N}^T & \cdots & X_{NN} \end{bmatrix}.$$

In the following analysis, the basic assumption is needed.

Assumption 1 *The Hamiltonian matrices $T_{ii},\ i = 1,\ \dots\ , N$ have not eigenvalues on the imaginary axis, where $T_{ii} := \begin{bmatrix} A_{ii}^T & -S_{ii} \\ -U_{ii} & -A_{ii} \end{bmatrix}$.*

Before solving the SIGMARE (9), the asymptotic structure is investigated. In order to avoid the ill-conditioned caused by the large parameter ε_i^{-1} which is included in the SIGMARE (9), the following useful lemma is introduced [14].

Lemma 1 *The SIGMARE (9) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (10)*

$$F(X) = AX^T + XA^T - XSX^T + U = 0, \tag{10}$$

where $A = \Phi_e A_e$, $U = \Phi_e U_e \Phi_e$ and $X = \Phi_e X_e$.

The GMARE (10) can be partitioned into

$$\begin{aligned} F_{00} &= A_{00}X_{00} + X_{00}A_{00}^T + A_{0f}X_{0f}^T + X_{0f}A_{0f}^T \\ &\quad -X_{00}S_{00}X_{00} - X_{0f}S_fX_{0f}^T - X_{00}S_{0f}X_{0f}^T - X_{0f}S_{0f}^TX_{00} + U_{00} = 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} F_{0f} &= A_{0f}X_f^T + A_{00}X_{0f}\Pi_e + X_{00}A_{f0}^T + X_{0f}A_f^T \\ &\quad -X_{00}S_{00}X_{0f}\Pi_e - X_{0f}S_{0f}^TX_{0f}\Pi_e - X_{00}S_{0f}X_f^T - X_{0f}S_fX_f^T + U_{0f} = 0, \end{aligned} \tag{11b}$$

$$\begin{aligned} F_f &= A_fX_f^T + X_fA_f^T + A_{f0}X_{0f}\Pi_e + \Pi_eX_{0f}^TA_{f0}^T \\ &\quad -X_fS_fX_f^T - \Pi_eX_{0f}^TS_{0f}X_f^T - X_fS_{0f}^TX_{0f}\Pi_e - \Pi_eX_{0f}^TS_{00}X_{0f}\Pi_e + U_f = 0. \end{aligned} \tag{11c}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., [1, 2]), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to 0^+ \\ \varepsilon_i \to 0^+}} \alpha_{ij}. \tag{12}$$

Let $\bar{X}_{00}$, $\bar{X}_{0f}$ and $\bar{X}_f$ be the limiting solutions of the above partitioned ARE (11) as $\varepsilon_i \to 0^+,\ \varepsilon_j \to 0^+,\ i,\ j = 1,\ \cdots,\ N$, then the following equations are obtained.

$$\mathcal{A}\bar{X}_{00}^* + \bar{X}_{00}^*\mathcal{A}^T - \bar{X}_{00}^*\mathcal{S}\bar{X}_{00}^* + \mathcal{U} = 0, \tag{13a}$$

$$\bar{X}_{0i}^* = \begin{bmatrix} -\bar{X}_{00}^* & I_{n_0} \end{bmatrix} T_{0i}T_{ii}^{-1} \begin{bmatrix} I_{n_i} \\ \bar{X}_{ii}^* \end{bmatrix}, \tag{13b}$$

$$A_{ii}\bar{X}_{ii}^* + \bar{X}_{ii}^*A_{ii}^T - \bar{X}_{ii}^*S_{ii}\bar{X}_{ii}^* + U_{ii} = 0, \tag{13c}$$

where

$$\begin{bmatrix} \mathcal{A}^T & -\mathcal{S} \\ -\mathcal{U} & -\mathcal{A} \end{bmatrix} := T_{00} - \sum_{k=1}^{N} T_{0k}T_{kk}^{-1}T_{k0}, \tag{14}$$

$$T_{00} := \begin{bmatrix} A_{00}^T & -S_{00} \\ -U_{00} & -A_{00} \end{bmatrix},\ T_{0i} := \begin{bmatrix} A_{i0}^T & -S_{0i} \\ -U_{0i} & -A_{0i} \end{bmatrix},$$

$$T_{i0} := \begin{bmatrix} A_{0i}^T & -S_{0i}^T \\ -U_{0i}^T & -A_{i0} \end{bmatrix},\ T_{ii} := \begin{bmatrix} A_{ii}^T & -S_{ii} \\ -U_{ii} & -A_{ii} \end{bmatrix}.$$

Now, let us define the admissible design parameters [10].

$\gamma_f := \max\{\gamma_{f_1},\ \dots\ ,\gamma_{f_N}\}$, where $\gamma_{f_i} := \inf\{\gamma \mid \gamma \in \Lambda_{f_i}\}$ and $\Lambda_{f_i} := \{\gamma > 0 \mid$ The AREs (13c) $A_{ii}\bar{X}_{ii} + \bar{X}_{ii}A_{ii}^T - \bar{X}_{ii}S_{ii}\bar{X}_{ii} + U_{ii} = 0$ have a positive semidefinite stabilizing solution, respectively.$\}$, $i = 1,\ \dots\ , N$.

Using the similar technique used in [15], it is easy to verify that if a parameter γ is selected such that $\gamma_f := \max\{\gamma_{f_1},\ \dots\ ,\ \gamma_{f_N}\} < \gamma$, then the solution $\bar{X}_f$ has the following form

$$\bar{X}_f^* := \textbf{block diag}\left(\begin{array}{ccc} \bar{X}_{11}^* & \cdots & \bar{X}_{NN}^* \end{array} \right). \tag{15}$$

Moreover, the following set is defined [10].

$\gamma_s := \inf\{\gamma \mid \gamma \in \Lambda_s\}$, where $\Lambda_s := \{0 < \gamma \mid$ The ARE (13a) has a positive semidefinite stabilizing solution.$\}$

As a result, for every $\gamma > \bar{\gamma} = \max\{\gamma_s,\ \gamma_f\}$, the AREs (13a) and (13c) have the positive definite stabilizing solutions. Hence, the limiting behavior of X_e as the parameter $\|\varepsilon\| := \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2} \to +0$ is described by the following lemma.

Lemma 2 *Assume that there exists a positive scalar $\bar{\gamma}$ such that for all $\bar{\gamma} < \gamma$, the AREs (13a) and (13c) have the positive semidefinite stabilizing solutions. Then there exists a small σ^* such that for all $\|\varepsilon\| \in (0,\ \sigma^*)$, for any $\gamma(> \bar{\gamma})$ the GMARE (10) admits a positive semidefinite stabilizing solution $X_e = \Phi_e^{-1}X$ which can be written as*

$$X_e = \Phi_e^{-1}\left[\begin{array}{cc} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \Pi_e\{\bar{X}_{0f}^* + O(\|\mu\|)\}^T & \bar{X}_f^* + O(\|\mu\|) \end{array} \right] = \left[\begin{array}{cc} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \{\bar{X}_{0f}^* + O(\|\mu\|)\}^T & \Pi_e^{-1}\{\bar{X}_f^* + O(\|\mu\|)\} \end{array} \right]. \tag{16}$$

Proof : Since this can be proved by using a technique similar to that used in [15], it is omitted. ■

3 A Numerical Algorithm for Solving the GMARE

In order to solve the GMARE (10) without the ill-conditioned, the following algorithm that is based on the Newton's method is established.

$$(A - X^{(n)}S)X^{(n+1)T} + X^{(n+1)}(A - X^{(n)}S)^T + X^{(n)}SX^{(n)T} + U = 0,\ X^{(0)} = \bar{X},\ n = 0,\ \dots\ , \tag{17}$$

with

$$\bar{X} = \left[\begin{array}{cc} \bar{X}_{00}^* & \bar{X}_{0f}^* \\ \Pi_e\bar{X}_{0f}^* & \bar{X}_f^* \end{array} \right],\ X^{(n)} = \left[\begin{array}{cc} X_{00}^{(n)} & X_{0f}^{(n)} \\ \Pi_e X_{0f}^{(n)T} & X_f^{(n)} \end{array} \right]. \tag{18}$$

The following theorem indicates the convergence of the algorithm (17).

Theorem 1 *Assume that there exists a positive scalar $\bar{\gamma}$ such that for all $\bar{\gamma} < \gamma$, the AREs (13a) and (13c) have the positive semidefinite stabilizing solutions. Under Assumption 1, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$, the Newton's method (17) converges to the exact solution of X with the rate of quadratic convergence, where $X_e = \Phi_e^{-1}X$ is the positive semidefinite stabilizing solution. Moreover, the convergence solution attains a unique solution X_e^* of the SIGMARE (9) in the neighborhood of the initial condition $X^{(0)} = \bar{X}$. In other words, the following condition is satisfied.*

$$\|X^{(n)} - X\| \le \frac{O(\|\mu\|^{2^n})}{2^n\beta\mathcal{L}} = O(\|\mu\|^{2^n}),\ n = 0,\ 1,\ \dots\ , \tag{19}$$

where

$$\mathcal{L} := 2\|S\|,\ \beta := \|[\nabla\mathcal{F}(\bar{\mathcal{X}})]^{-1}\|,\ \eta := \beta\|\mathcal{F}(\bar{\mathcal{X}})\|,\ \theta := \beta\eta\mathcal{L},$$

$$\nabla\mathcal{F}(\mathcal{X}) := \frac{\partial\mathcal{F}(\mathcal{X})}{\partial\mathcal{X}^T},\ \mathcal{F}(\mathcal{X}) := \left[\begin{array}{c} \mathrm{vec}F_{00} \\ \mathrm{vec}F_{0f} \\ \mathrm{vec}F_f \end{array} \right],\ \mathcal{X} := \left[\begin{array}{c} \mathrm{vec}X_{00} \\ \mathrm{vec}X_{0f} \\ \mathrm{vec}X_f \end{array} \right],\ \bar{\mathcal{X}} := \left[\begin{array}{c} \mathrm{vec}\bar{X}_{00}^* \\ \mathrm{vec}\bar{X}_{0f}^* \\ \mathrm{vec}\bar{X}_f^* \end{array} \right].$$

Proof : The proof follows directly by applying Newton-Kantorovich theorem [7] for the GMARE (10). It is easy to verify that function $F(X)$ is differentiable on a certain convex set $\mathcal{D}$. Using the fact that

$$\nabla F(X) := \frac{\partial\mathrm{vec}F(X)}{\partial[\mathrm{vec}\ X]^T} = [(A - XS) \otimes I_{\bar{n}}]U_{\bar{n}\bar{n}} + I_{\bar{n}} \otimes (A - XS) = (I_{\bar{n}^2} + U_{\bar{n}\bar{n}}) \cdot [I_{\bar{n}} \otimes (A - XS)]$$

with $\bar{n} := \sum_{k=0}^{N} n_k$ results in

$$\begin{aligned} & \|\nabla F(X_1) - \nabla F(X_2)\| \le \mathcal{L}\|X_1 - X_2\|, \\ \Rightarrow \quad & \|\nabla \mathcal{F}(\mathcal{X}_1) - \nabla \mathcal{F}(\mathcal{X}_2)\| \le \mathcal{L}\|\mathcal{X}_1 - \mathcal{X}_2\|, \end{aligned}$$

where $\mathcal{L} = 2\|S\|$. Moreover, the following result holds by using the similar technique in [15].

$$\det \nabla \mathcal{F}(\bar{\mathcal{X}}) = \prod_{i=0}^{2} \det J_{ii} + O(\|\mu\|),$$

where

$$\begin{aligned} & J_{00} = \Gamma_0 \otimes I_{n_0} + I_{n_0} \otimes \Gamma_0,\ J_{11} = \Gamma_4 \otimes I_{n_0},\ J_{22} = \Gamma_4 \otimes I_{\hat{n}} + I_{\hat{n}} \otimes \Gamma_4, \\ & \Gamma_0 := \mathcal{A} - \bar{X}_{00}^* \mathcal{S} = \Gamma_1 - \Gamma_2 \Gamma_4^{-1} \Gamma_3, \\ & \Gamma_1 = A_{00} - \bar{X}_{00}^* S_{00} - \bar{X}_{0f}^* S_{0f}^T,\ \Gamma_2 = A_{0f} - \bar{X}_{00}^* S_{0f} - \bar{X}_{0f}^* S_f, \\ & \Gamma_3 = A_{f0} - \bar{X}_f^* S_{0f}^T,\ \Gamma_4 = A_f - \bar{X}_f^* S_f,\ \hat{n} := \sum_{k=1}^{N} n_k. \end{aligned}$$

It is shown that there exists a small $\bar{\sigma}$ such that for sufficiently small parameter $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$, $\nabla \mathcal{F}(\bar{\mathcal{X}})$ is nonsingular because Γ_4 and $\Gamma_0 = \mathcal{A} - \bar{X}_{00}^* \mathcal{S}$ are stable under Assumption 1 and the definition γ_s (see e.g., Theorem 1 [1]). Therefore, there exists β such that $\|[\nabla \mathcal{F}(\bar{\mathcal{X}})]^{-1}\| \equiv \beta$. On the other hand, it can be verified that $\|\mathcal{F}(\bar{\mathcal{X}})\| = O(\|\mu\|)$ because $A\bar{X}^T + \bar{X}A^T - \bar{X}S\bar{X}^T + U = O(\|\mu\|)$. Hence, there exists η such that $\eta = \|[\nabla \mathcal{F}(\bar{\mathcal{X}})]^{-1}\| \cdot \|\mathcal{F}(\bar{\mathcal{X}})\| = O(\|\mu\|)$. Thus, for sufficiently small ε_i, there exists θ such that $\theta \equiv \beta \mathcal{L} \eta < 2^{-1}$ because $\eta = O(\|\mu\|)$. Using Newton-Kantorovich theorem, the strict error estimate is given by (19). Therefore, the proof is completed. ■

It should be noted that no proof exists of whether the proposed algorithm fails to converge for large parameter ε_i. In this paper, the convergence criteria for ε_i is established for the first time. Such a condition is derived from the Newton-Kantorovich theorem [7].

Corollary 1 *If the following inequality holds for any small parameter $\varepsilon_i = \tilde{\varepsilon}_i,\ i = 1,\ \ldots,N$, algorithm (17) guarantees quadratic convergence.*

$$\theta(\tilde{\varepsilon}_1,\ \ldots,\tilde{\varepsilon}_N) := 2\tilde{\beta}\tilde{\eta} \cdot \|S\| < 2^{-1}, \tag{20}$$

where

$$\begin{aligned} & \tilde{\beta} := \beta(\tilde{\varepsilon}_1,\ \ldots,\tilde{\varepsilon}_N),\ \tilde{\eta} = \eta(\tilde{\varepsilon}_1,\ \ldots,\tilde{\varepsilon}_N), \\ & \beta(\varepsilon_1,\ \ldots,\varepsilon_N) := \|[\nabla \mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \ldots,\varepsilon_N)]^{-1}\|, \\ & \eta(\varepsilon_1,\ \ldots,\varepsilon_N) := \beta(\varepsilon_1,\ \ldots,\varepsilon_N) \cdot \|\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \ldots,\varepsilon_N)\|, \\ & \mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \ldots,\varepsilon_N) := \begin{bmatrix} \mathrm{vec} F_{00} \\ \mathrm{vec} F_{0f} \\ \mathrm{vec} F_f \end{bmatrix}. \end{aligned}$$

Proof: Since it is clear that this proof can be derived by applying the Newton-Kantorovich theorem, it has been omitted. ■

4 Reduction Algorithm based on Fixed Point Algorithm

One needs to solve the generalized multiparameter algebraic Lyapunov equation (GMALE) (17) with the dimension $\bar{n} := \sum_{k=0}^{N} n_k$ larger than the dimension $n_i,\ i = 0,\ \ldots,N$ compared with the exact decomposition technique [3]. Thus, in order to reduce the dimension of the workspace, a new algorithm for solving the GMALE (17) which is based on two fixed point algorithms is established. Let us consider the following GMALE (21), in a general form.

$$\Lambda Y^T + Y \Lambda^T + T = 0, \tag{21}$$

where Y is the solution of the GMALE (21). Moreover, Y, Λ and T have the following forms, respectively.

$$\varepsilon := \|\mu\| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2},$$

$$Y := \begin{bmatrix} Y_{00} & Y_{0f} \\ \Pi_e Y_{0f}^T & Y_f \end{bmatrix},\ Y_{00} = Y_{00}^T,\ \Pi_e^{-1} Y_f = Y_f^T \Pi_e^{-1},\ Y_{0f} := \begin{bmatrix} Y_{01}^T \\ \vdots \\ Y_{0N}^T \end{bmatrix}^T,$$

$$\Lambda := \begin{bmatrix} \Lambda_{00} & \Lambda_{0f} \\ \Lambda_{f0} & \Lambda_f \end{bmatrix},\ \Lambda_{0f} := \begin{bmatrix} \Lambda_{01} & \cdots & \Lambda_{0N} \end{bmatrix},\ \Lambda_{f0} := \begin{bmatrix} \Lambda_{10}^T & \cdots & \Lambda_{N0}^T \end{bmatrix}^T,$$

$$\Lambda_f := \begin{bmatrix} \Lambda_{11} & \varepsilon\Lambda_{12} & \cdots & \varepsilon\Lambda_{1N} \\ \varepsilon\Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon\Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Lambda_{N1} & \varepsilon\Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix},$$

$$T := \begin{bmatrix} T_{00} & T_{0f} \\ T_{0f}^T & T_f \end{bmatrix},\ T_{00} = T_{00}^T,\ T_{0f} := \begin{bmatrix} T_{01} & \cdots & T_{0N} \end{bmatrix},\ T_f := \begin{bmatrix} T_{11} & \varepsilon T_{12} & \cdots & \varepsilon T_{1N} \\ \varepsilon T_{12}^T & T_{22} & \cdots & \varepsilon T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon T_{1N}^T & \varepsilon T_{2N}^T & \cdots & T_{NN} \end{bmatrix}.$$

It should be noted that for the GMALE (17),

$$X^{(n+1)} \Rightarrow Y,\ A - X^{(n)} S \Rightarrow \Lambda,\ X^{(n)T} S X^{(n)} + U \Rightarrow T,$$

where $\Rightarrow$ stands for the replacement.

Moreover, taking the asymptotic structure (15) into account, since

$$X_f := \bar{X}_f^* + O(\|\mu\|) = \textbf{block diag} \begin{pmatrix} \bar{X}_{11}^* & \cdots & \bar{X}_{NN}^* \end{pmatrix}^* + O(\|\mu\|),$$

without loss of generality, it can be supposed that Y_f has the following form.

$$X_f^{(n+1)} \Rightarrow Y_f := \begin{bmatrix} Y_{11} & \alpha_{21}\varepsilon Y_{12} & \alpha_{31}\varepsilon Y_{13} & \cdots & \alpha_{N1}\varepsilon Y_{1N} \\ \varepsilon Y_{12}^T & Y_{22} & \alpha_{32}\varepsilon Y_{23} & \cdots & \alpha_{N2}\varepsilon Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon Y_{1(N-1)}^T & \varepsilon Y_{2(N-1)}^T & \varepsilon Y_{3(N-1)}^T & \cdots & \alpha_{N(N-1)}\varepsilon Y_{(N-1)N} \\ \varepsilon Y_{1N}^T & \varepsilon Y_{2N}^T & \varepsilon Y_{3N}^T & \cdots & Y_{NN} \end{bmatrix}. \tag{22}$$

The following condition for the GMALE (21) is assumed.

Assumption 2 $\Lambda_{ii},\ i = 1,\ \dots\ , N$ *and* $\Lambda_0 := \Lambda_{00} - \sum_{k=1}^{N} \Lambda_{0k}\Lambda_{kk}^{-1}\Lambda_{k0}$ *are stable.*

The GMALE (21) can be changed as follows by partitioning.

$$\Lambda_{00} Y_{00} + Y_{00}\Lambda_{00}^T + \Lambda_{0f} Y_{0f}^T + Y_{0f}\Lambda_{0f}^T + T_{00} = 0, \tag{23a}$$

$$\Lambda_{00} Y_{0f}\Pi_e + \Lambda_{0f} Y_f^T + Y_{00}\Lambda_{f0}^T + Y_{0f}\Lambda_f^T + T_{0f}, \tag{23b}$$

$$\Lambda_f Y_f^T + Y_f \Lambda_f^T + \Lambda_{f0} Y_{0f}\Pi_e + \Pi_e Y_{0f}^T \Lambda_{f0}^T + T_f = 0. \tag{23c}$$

The following algorithm (24) for solving the GMALE (23) is given.

$$\Lambda_f Y_f^{(l+1)T} + Y_f^{(l+1)}\Lambda_f^T + \Lambda_{f0} Y_{0f}^{(l)}\Pi_e + \Pi_e Y_{0f}^{(l)T}\Lambda_{f0}^T + T_f = 0,\ l = 0,\ 1,\ \dots\ , \tag{24a}$$

$$\begin{aligned} &\Lambda_0 Y_{00}^{(l+1)} + Y_{00}^{(l+1)}\Lambda_0^T - \Lambda_{0f}\Lambda_f^{-1}(\Lambda_{00} Y_{0f}^{(l)}\Pi_e + T_{0f})^T - (\Lambda_{00} Y_{0f}^{(l)}\Pi_e + T_{0f})\Lambda_f^{-T}\Lambda_{0f}^T \\ &\quad + \Lambda_{0f}\Lambda_f^{-1}(\Lambda_{f0} Y_{0f}^{(l)}\Pi_e + \Pi_e Y_{0f}^{(l)T}\Lambda_{f0}^T + T_f)\Lambda_f^{-T}\Lambda_{0f}^T + T_{00} = 0,\ l = 0,\ 1,\ \dots\ , \end{aligned} \tag{24b}$$

$$Y_{0f}^{(l+1)} = -(\Lambda_{0f} Y_f^{(l+1)T} + Y_{00}^{(l+1)}\Lambda_{f0}^T + \Lambda_{00} Y_{0f}^{(l)}\Pi_e + T_{0f})\Lambda_f^{-T},\ l = 0,\ 1,\ \dots\ , \tag{24c}$$

where

$$
\begin{aligned}
&Y_{0f}^{(0)} = \bar{Y}_{0f},\ \bar{Y}_{0f} = -(\Lambda_{0f}\bar{Y}_f^T + \bar{Y}_{00}\Lambda_{f0}^T + T_{0f})\bar{\Lambda}_f^{-T},\\
&\bar{Y}_f := \textbf{block diag}\begin{pmatrix} \bar{Y}_{11} & \cdots & \bar{Y}_{NN} \end{pmatrix},\\
&\Lambda_{jj}\bar{Y}_{jj} + \bar{Y}_{jj}\Lambda_{jj}^T + T_{jj} = 0,\ j = 1,\ \dots ,N,\\
&\bar{\Lambda}_0\bar{Y}_{00} + \bar{Y}_{00}\bar{\Lambda}_0^T - \Lambda_{0f}\bar{\Lambda}_f^{-1}T_{0f}^T - T_{0f}\bar{\Lambda}_f^{-T}\Lambda_{0f}^T + \Lambda_{0f}\bar{\Lambda}_f^{-1}\bar{T}_f\bar{\Lambda}_f^{-T}\Lambda_{0f}^T + T_{00} = 0,\\
&\Lambda_0 = \Lambda_{00} - \Lambda_{0f}\Lambda_f^{-1}\Lambda_{f0},\ \bar{\Lambda}_0 = \Lambda_{00} - \Lambda_{0f}\bar{\Lambda}_f^{-1}\Lambda_{f0},\\
&\bar{\Lambda}_f := \textbf{block diag}\begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{NN} \end{pmatrix},\ \bar{T}_f := \textbf{block diag}\begin{pmatrix} T_{11} & \cdots & T_{NN} \end{pmatrix}.
\end{aligned}
$$

The following theorem indicates the convergence of the algorithm (24).

Theorem 2 *Under Assumption 2, the fixed point algorithm (24) converges to the exact solutions* Y_{00}, Y_{f0} *and* Y_f *with the rate of convergence of* $O(\|\mu\|^{l+1})$, *that is*

$$\|Y_f^{(l)} - Y_f\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \dots , \tag{25a}$$

$$\|Y_{00}^{(l)} - Y_{00}\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \dots , \tag{25b}$$

$$\|Y_{0f}^{(l)} - Y_{0f}\| = O(\|\mu\|^{l+1}),\ l = 0,\ 1,\ \dots . \tag{25c}$$

Proof: The proof of Theorem 2 can be done by using the mathematical induction. It is easy to verify that the first order approximations Y_{00}, Y_{0f} and Y_f corresponding to the small parameter ε_i are $\bar{Y}_{00}$, $\bar{Y}_{0f}$ and $\bar{Y}_f$, respectively. It follows from these equations that

$$\|Y_f^{(0)} - Y_f\| = \|\bar{Y}_f - Y_f\| = O(\|\mu\|), \tag{26a}$$

$$\|Y_{00}^{(0)} - Y_{00}\| = \|\bar{Y}_{00} - Y_{00}\| = O(\|\mu\|), \tag{26b}$$

$$\|Y_{0f}^{(0)} - Y_{0f}\| = \|\bar{Y}_{0f} - Y_{0f}\| = O(\|\mu\|). \tag{26c}$$

When $l = h,\ h \ge 1$, it is assumed that

$$\|Y_f^{(h)} - Y_f\| = O(\|\mu\|^{h+1}), \tag{27a}$$

$$\|Y_{00}^{(h)} - Y_{00}\| = O(\|\mu\|^{h+1}), \tag{27b}$$

$$\|Y_{0f}^{(h)} - Y_{0f}\| = O(\|\mu\|^{h+1}). \tag{27c}$$

Subtracting the partitioned algebraic matrix equation (23) from (24) and setting $k = h$, the following equations hold under the above assumptions (27).

$$\Lambda_f(Y_f^{(h+1)} - Y_f)^T + (Y_f^{(h+1)} - Y_f)\Lambda_f^T = O(\|\mu\|^{h+2}), \tag{28a}$$

$$\Lambda_0(Y_{00}^{(h+1)} - Y_{00}) + (Y_{00}^{(h+1)} - Y_{00})\Lambda_0^T = O(\|\mu\|^{h+2}), \tag{28b}$$

$$Y_{0f}^{(h+1)} - Y_{0f} = -[\Lambda_{0f}(Y_f^{(h+1)} - Y_f)^T + (Y_{00}^{(h+1)} - Y_{00})\Lambda_{f0}^T + O(\|\mu\|^{h+2})]\Lambda_f^{-T}. \tag{28c}$$

After the cancellation takes place, since Λ_0 and $\bar{\Lambda}_f = \textbf{block diag}\begin{pmatrix} \Lambda_{11} & \cdots & \Lambda_{NN} \end{pmatrix}$ are stable from Assumption 2, the following relations hold using the result of [8].

$$\|Y_f^{(h)} - Y_f\| = O(\|\mu\|^{h+2}), \tag{29a}$$

$$\|Y_{00}^{(h)} - Y_{00}\| = O(\|\mu\|^{h+2}), \tag{29b}$$

$$\|Y_{0f}^{(h)} - Y_{0f}\| = O(\|\mu\|^{h+2}). \tag{29c}$$

Consequently, the error equations (25) hold for all $k \in \mathbf{N}$. This completes the proof of Theorem 2. ■

When the ALE (24a) is solved, very large computational dimension $\hat{n} := \sum_{k=1}^{N} n_k$ is needed. Thus, the reduction of the dimension of the computing workspace must be needed. Therefore, the new algorithm for solving the ALE (24a) which is based on the other fixed point algorithm is established. Let us consider the following ALE (30), in a general form.

$$\Psi_e Z_e^T + Z_e \Psi_e^T + V_e = 0, \tag{30}$$

where Z_e is the solution of the ALE (30). Moreover, Z_e, Ψ_e and V_e have the following forms, respectively.

$$Z_e := \begin{bmatrix} Z_{11} & \alpha_{21}\varepsilon Z_{12} & \alpha_{31}\varepsilon Z_{13} & \cdots & \alpha_{N1}\varepsilon Z_{1N} \\ \varepsilon Z_{12}^T & Z_{22} & \alpha_{32}\varepsilon Z_{23} & \cdots & \alpha_{N2}\varepsilon Z_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon Z_{1(N-1)}^T & \varepsilon Z_{2(N-1)}^T & \varepsilon Z_{3(N-1)}^T & \cdots & \alpha_{N(N-1)}\varepsilon Z_{(N-1)N} \\ \varepsilon Z_{1N}^T & \varepsilon Z_{2N}^T & \varepsilon Z_{3N}^T & \cdots & Z_{NN} \end{bmatrix},$$

$$\Psi_e := \begin{bmatrix} \Psi_{11} & \varepsilon\Psi_{12} & \cdots & \varepsilon\Psi_{1N} \\ \varepsilon\Psi_{21} & \Psi_{22} & \cdots & \varepsilon\Psi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Psi_{N1} & \varepsilon\Psi_{N2} & \cdots & \Psi_{NN} \end{bmatrix}, \ V_e := \begin{bmatrix} V_{11} & \varepsilon V_{12} & \cdots & \varepsilon V_{1N} \\ \varepsilon V_{12}^T & V_{22} & \cdots & \varepsilon V_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon V_{1N}^T & \varepsilon V_{2N}^T & \cdots & V_{NN} \end{bmatrix}.$$

It should be noted that for the ALE (24a),

$$Y_f^{(l+1)} \Rightarrow Z_e, \ \Lambda_f \Rightarrow \Psi_e, \ \Lambda_{f0} Y_{0f}^{(l)} \Pi_e + \Pi_e Y_{0f}^{(l)T} \Lambda_{f0}^T + T_f \Rightarrow V_e$$

where $\Rightarrow$ stands for the replacement. Furthermore, the ALE (30) is a part of the ALE (21). Namely, the ALE (30) stands for the ALE (24a).

Without loss of generality, the following condition for the ALE (30) is also assumed.

Assumption 3 $\Psi_{ii}, \ i = 1, \ \ldots \ , N$ *are stable.*

The following algorithms (31) for solving the ALE (30) are newly given.

$$\Psi_{11} Z_{11}^{(m+1)} + Z_{11}^{(m+1)} \Psi_{11}^T + \varepsilon^2 \sum_{k=2}^{N} \alpha_{k1} (\Psi_{1k} Z_{1k}^{(m)T} + Z_{1k}^{(m)} \Psi_{1k}^T) + V_{11} = 0, \tag{31a}$$

$$\vdots$$

$$\begin{aligned} &\Psi_{ii} Z_{ii}^{(m+1)} + Z_{ii}^{(m+1)} \Psi_{ii}^T + \varepsilon^2 \sum_{k=1}^{i-1} (\Psi_{ik} Z_{ki}^{(m)} + Z_{ki}^{(m)T} \Psi_{ik}^T) \\ &\quad + \varepsilon^2 \sum_{k=i+1}^{N} \alpha_{ki} (\Psi_{ik} Z_{ik}^{(m)T} + Z_{ik}^{(m)} \Psi_{ik}^T) + V_{ii} = 0, \ i = 2, \ \ldots \ , N-1, \end{aligned} \tag{31b}$$

$$\vdots$$

$$\Psi_{NN} Z_{NN}^{(m+1)} + Z_{NN}^{(m+1)} \Psi_{NN}^T + \varepsilon^2 \sum_{k=1}^{N-1} (\Psi_{Nk} Z_{kN}^{(m)} + Z_{kN}^{(m)T} \Psi_{Nk}^T) + V_{NN} = 0, \tag{31c}$$

$$\begin{aligned} &\Psi_{11} Z_{12}^{(m+1)} + \alpha_{21} Z_{12}^{(m+1)} \Psi_{22}^T + Z_{11}^{(m+1)} \Psi_{21}^T + \Psi_{12} Z_{22}^{(m+1)} \\ &\quad + \varepsilon \sum_{k=3}^{N} (\alpha_{k2} \Psi_{1k} Z_{2k}^{(m)T} + \alpha_{k1} Z_{1k}^{(m)} \Psi_{2k}^T) + V_{12} = 0, \end{aligned} \tag{31d}$$

$$\vdots$$

$$\begin{aligned} &\Psi_{(N-1)(N-1)} Z_{(N-1)N}^{(m+1)} + \alpha_{N(N-1)} Z_{(N-1)N}^{(m+1)} \Psi_{NN}^T + Z_{(N-1)(N-1)}^{(m+1)} \Psi_{N(N-1)}^T + \Psi_{(N-1)N} Z_{NN}^{(m+1)} \\ &\quad + \varepsilon \sum_{k=1}^{N-2} (\Psi_{(N-1)k} Z_{kN}^{(m)} + Z_{k(N-1)}^{(m)T} \Psi_{Nk}^T) + V_{(N-1)N} = 0, \end{aligned} \tag{31e}$$

$m = 0, \ 1, \ \cdots,$

Table 1. Convergence solution $X_e := \begin{bmatrix} X_{00} & X_{0f} \\ X_{0f}^T & \Pi_e^{-1} X_f \end{bmatrix}$.

$$X_{00} = \begin{bmatrix} 1.5086 & -4.3532e-01 & -1.7459e-02 & 4.5398e-02 & 5.2563e-01 \\ -4.3532e-01 & 1.5161 & 4.6813e-02 & -1.5798e-02 & -5.1866e-01 \\ -1.7459e-02 & 4.6813e-02 & 7.8337e-03 & 7.7249e-04 & 1.7988e-03 \\ 4.5398e-02 & -1.5798e-02 & 7.7249e-04 & 7.7619e-03 & -1.5713e-03 \\ 5.2563e-01 & -5.1866e-01 & 1.7988e-03 & -1.5713e-03 & 2.4501 \end{bmatrix}$$

$$X_{0f} = \begin{bmatrix} -1.3497e-02 & 4.3835e-02 & -1.9364e-01 & -1.5877e-01 \\ -2.8435e-01 & -2.6978e-01 & -1.1308e-01 & -6.5749e-02 \\ 8.8320e-03 & -1.8211e-02 & -6.7232e-03 & -6.3278e-03 \\ -2.9107e-03 & -2.8025e-03 & 1.1808e-02 & -1.4738e-02 \\ 3.8083e-02 & -2.7966e-02 & -9.4401e-03 & 5.0702e-02 \end{bmatrix}$$

$$\Pi_e^{-1} X_f = \begin{bmatrix} 1.4174 & 1.4870 & 4.8844e-02 & 3.5509e-02 \\ 1.4870 & 4.8399 & 3.9452e-02 & 3.1745e-02 \\ 4.8844e-02 & 3.9452e-02 & 2.7847 & 2.9315 \\ 3.5509e-02 & 3.1745e-02 & 2.9315 & 9.5689 \end{bmatrix}$$

where

$$Z_{ii}^{(0)} = \bar{Z}_{ii},\ Z_{ij}^{(0)} = \bar{Z}_{ij},\ i < j,$$
$$\Psi_{ii}\bar{Z}_{ii} + \bar{Z}_{ii}\Psi_{ii}^T + V_{ii} = 0,\ \Psi_{ii}\bar{Z}_{ij} + \alpha_{ji}\bar{Z}_{ij}\Psi_{jj}^T + \bar{Z}_{ii}\Psi_{ji}^T + \Psi_{ij}\bar{Z}_{jj} + V_{ij} = 0.$$

The following theorem indicates the convergence of the fixed point algorithm (31).

Theorem 3 *Under Assumption 3, the fixed point algorithm (31) converges to the exact solution Z_{ij} with the rate of*

$$\|Z_{ii}^{(m)} - Z_{ii}\| = O(\varepsilon^{m+2}),\ m = 1,\ \dots, \tag{32a}$$
$$\|Z_{ij}^{(m)} - Z_{ij}\| = O(\varepsilon^{m+1}),\ i < j,\ m = 1,\ \dots\ . \tag{32b}$$

Proof : Since the proof of Theorem 3 can be also done by using mathematical induction and the fixed point theorem, it is omitted. ■

An algorithm which solves the SIGMARE (9) with the small positive parameters ε_i is given below.

Step 1. *Solve the AREs (13) that are given as the initial conditions of the Newton's method (17).*

Step 2. *In order to carry out the Newton's method (17), apply the new proposed algorithm (24).*

Step 3. *In order to reduce the dimension of the workspace for solving the ALE (24a), apply the new proposed algorithm (31).*

Step 4. *Solve the solutions $Y_f^{(l+1)}$ and $Y_{00}^{(l+1)}$ of the ALE (24a) and (24b), respectively and compute $Y_{f0}^{(l+1)}$ using the relation of (24c). As a result, the sequence of solution of the Newton's method (17) is obtained.*

Step 5. *If the new combined algorithm converges, go to Step 6. Otherwise, increment $n \to n+1$ and go to Step 2.*

Step 6. *Calculate the solution X_e of the SIGMARE (9) by using (16).*

5 Computational Example

In order to demonstrate the efficiency of the proposed algorithm, the computational examples are given.

5.1 Example 1

The system matrices of the GMSPS (1) are given as follows.

$$A_{00} = \begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix},\ A_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix},\ A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix},\ A_{11} = A_{22} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix},$$

$$A_{12} = A_{21} = O_{2\times 2},\ D_{01} = D_{02} = O_{5\times 1},\ D_{11} = D_{22} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C = G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ W = I_9,\ R = 20.$$

Table 2. Error per iterations.

n	$\|F(X^{(n)})\|$
0	$7.7301e-03$
1	$1.1110e-03$
2	$6.6036e-07$
3	$5.9791e-13$

Table 3. Error $\|F(X)\|$.

$\varepsilon_1 = \varepsilon_2$	Proposed algorithm	MATLAB
$1.0e-02$	$7.7810e-13$	$7.2625e-13$
$1.0e-03$	$2.2080e-13$	$6.0496e-12$
$1.0e-04$	$1.8019e-14$	$2.0564e-10$
$1.0e-05$	$2.1075e-14$	$2.2429e-08$

It should be noted that the technique proposed in [3] to the MSPS can not be applied because the Hamiltonian matrices $T_{ii},\ i = 1,\ 2$ have eigenvalues in common. The small parameters are chosen as $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.005$. For every boundary value $\gamma > \bar{\gamma} = \max\{\gamma_s,\ \ \gamma_f\} = 4.4722$, the ARE (13a) and the AREs (13c) have the positive definite stabilizing solution, where $\gamma_{f_1} = 3.6515$, $\gamma_{f_2} = 3.6515$ and $\gamma_s = 4.4722$.

When $\gamma = 5$, a solution of the SIGMARE (9) is given in Table 1. It can be verified that the algorithm (17) converges to the exact solution with accuracy of $\|F(X^{(n)})\| < 10^{-12}$ after three iterations. In order to verify the exactitude of the solution, the remainder per iteration is computed by substituting $X^{(n)}$ into the GMARE (10). In Table 2, the results of the error $\|F(X^{(n)})\|$ per iterations are given. It can be seen that the initial guess (18) for the algorithm (17) is quite good and the proposed algorithm (17) has quadratic convergence.

In order to compare the solution X_e that is computed by using the function `are` of MATLAB with the solution that is obtained through the algorithm (17), the remainder of the errors are given in Table 3. From Table 3, it is shown that the resulting algorithm of this paper is very useful for the small parameters ε_i. Moreover, it appears that the proposed algorithm is superior than the function `are` of MATLAB that is based on the Schur method [4].

From the viewpoint of this example, it should be noted that when the fixed-point algorithm is applied, even if the number of fast subsystems is greater than two, the computing workspace required for the filter gain is the same as the dimension of the fast subsystems. In other words, even if the GMSPS (1) are composed of N two-dimensional fast subsystems, the required workspace is two for the algorithm (31).

5.2 Example 2

Consider the system (1) with

$$\Phi_e := \begin{bmatrix} 1 & 0 \\ 0 & \Pi_e \end{bmatrix},\ \Pi_e := \textbf{block diag} \left(\begin{matrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{matrix} \right),$$

$$A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1} A_{f0} & \Pi_e^{-1} A_f \end{bmatrix},\ A_{0f} := \begin{bmatrix} A_{01} & A_{02} & A_{03} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

$$A_{f0} := \begin{bmatrix} A_{10}^T & A_{20}^T & A_{30}^T \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T,$$

$$A_f := \textbf{block diag} \left(\begin{matrix} A_{11} & A_{22} & A_{33} \end{matrix} \right) = \textbf{block diag} \left(\begin{matrix} -1 & -1 & -1 \end{matrix} \right),$$

$$D_e := \begin{bmatrix} D_0 \\ \Pi_e^{-1} D_f \end{bmatrix},\ D_0 := \begin{bmatrix} D_{01} & D_{02} & D_{03} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

Table 4. Error per iterations.

n	$\|F(X^{(n)})\|$
0	$9.7210e-02$
1	$1.2914e-03$
2	$5.6044e-07$
3	$5.6265e-11$
4	$3.5975e-15$

Table 5. Convergence solution.

$$X_e = \begin{bmatrix} 1.4520 & 2.1707 & 3.2454 & 4.3079 \\ 2.1707 & 4.3214e+02 & 2.7135 & 3.6477 \\ 3.2454 & 2.7135 & 2.2011e+02 & 6.7781 \\ 4.3079 & 3.6477 & 6.7781 & 1.5314e+02 \end{bmatrix}$$

Table 6. Convergence criteria.

	ε_1=ε_2=ε_3=$1.0e-02$	ε_1=ε_2=ε_3=$1.0e-03$	ε_1=ε_2=ε_3=$1.0e-04$	ε_1=ε_2=ε_3=$1.0e-05$
θ	$3.2184e-01$	$3.3568e-02$	$3.3711e-03$	$3.3726e-04$

Table 7. Solutions and errors.

$$X^{\mathrm{MAT}} = \begin{bmatrix} 1.4214 & 2.1542 & 3.2305 & 4.3060 \\ 2.1551e-07 & 4.3037e-01 & 3.0881e-06 & 1.2496e-05 \\ 6.4630e-07 & -3.2858e-05 & 4.3050e-01 & 6.0356e-05 \\ 1.2914e-06 & 2.7288e-08 & 8.8993e-06 & 4.3050e-01 \end{bmatrix}$$

$$X^{\mathrm{NEW}} = \begin{bmatrix} 1.4225 & 2.1566 & 3.2319 & 4.3072 \\ 2.1566e-07 & 4.3050e-01 & 2.7156e-07 & 3.6647e-07 \\ 6.4639e-07 & 5.4312e-07 & 4.3050e-01 & 1.3678e-06 \\ 1.2922e-06 & 1.0994e-06 & 2.0517e-06 & 4.3050e-01 \end{bmatrix}$$

$\|F(X^{\mathrm{MAT}})\|$	$\|F(X^{\mathrm{NEW}})\|$
$1.1312e-02$	$3.7343e-15$

$$D_f := \textbf{block diag}\begin{pmatrix} D_{11} & D_{22} & D_{33} \end{pmatrix} = \textbf{block diag}\begin{pmatrix} -1 & -1 & -1 \end{pmatrix},$$

$$C = G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ W = I_4, \ R = 1, \ \gamma = 2.$$

In order to evaluate the convergence criteria, the values of θ of (20) should be calculated. First, the partitioned equation of the GMARE (10) can be obtained as follows.

$$F(X) = AX^T + XA^T - XSX^T + U = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ F_2 & F_5 & F_6 & F_7 \\ F_3 & F_6 & F_8 & F_9 \\ F_4 & F_7 & F_9 & F_{10} \end{bmatrix}, \ X = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_5 & \alpha_{21}p_6 & \alpha_{31}p_7 \\ p_3 & p_6 & p_8 & \alpha_{32}p_9 \\ p_4 & p_7 & p_9 & p_{10} \end{bmatrix},$$

where $F_i := F_i(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)$, $i = 1,\ \ldots, 10$, $\Gamma := 1 - \gamma^{-2}$,

$$\begin{aligned}
F_1(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_2^2 + p_3^2 + p_4^2)\Gamma + 2(p_1 + p_2 + p_3 + p_4) + 3, \\
F_2(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_2p_5 + \alpha_{21}p_3p_6 + \alpha_{31}p_4p_7)\Gamma + p_1 + (\varepsilon_1 - 1)p_2 + p_5 + \alpha_{21}p_6 + \alpha_{31}p_7 + 1, \\
F_3(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_2p_6 + p_3p_8 + \alpha_{32}p_4p_9)\Gamma + 2p_1 + (\varepsilon_2 - 1)p_3 + p_6 + p_8 + \alpha_{32}p_9 + 1, \\
F_4(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_2p_7 + p_3p_9 + p_4p_{10})\Gamma + 3p_1 + (\varepsilon_3 - 1)p_4 + p_7 + p_9 + p_{10} + 1, \\
F_5(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -[p_5^2 + (\alpha_{21}p_6)^2 + (\alpha_{31}p_7)^2]\Gamma + 2(\varepsilon_1p_2 - p_5) + 1, \\
F_6(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_5p_6 + \alpha_{21}p_6p_8 + \alpha_{31}\alpha_{32}p_7p_9)\Gamma + 2\varepsilon_1p_2 + \varepsilon_2p_3 - \alpha_{21}p_6 - p_6, \\
F_7(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_5p_7 + \alpha_{21}p_6p_9 + \alpha_{31}p_7p_{10})\Gamma + 3\varepsilon_1p_2 + \varepsilon_3p_4 - \alpha_{31}p_7 - p_7, \\
F_8(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -[p_6^2 + p_8^2 + (\alpha_{32}p_9)^2]\Gamma + 4\varepsilon_2p_3 - 2p_8 + 1,
\end{aligned}$$

$$
\begin{aligned}
F_9(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_6p_7+p_8p_9+\alpha_{32}p_9p_{10})\Gamma+3\varepsilon_2p_3+2\varepsilon_3p_4-p_9\alpha_{32}-p_9,\\
F_{10}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) &= -(p_7^2+p_9^2+p_{10}^2)\Gamma+6\varepsilon_3p_4-2p_{10}+1.
\end{aligned}
$$

Therefore, the related equations (20) are given below.

$$
\begin{aligned}
&\beta(\varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) := \|[\nabla\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)]^{-1}\|,\\
&\eta(\varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) := \beta(\varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)\cdot\|\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)\|,\\
&\nabla\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) := \frac{\partial\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)}{\partial\mathcal{X}^T},\\
&\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3) := \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10}\end{bmatrix}^T,\\
&\mathcal{X}^T := \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10}\end{bmatrix},\\
&\nabla\mathcal{F}(\mathcal{X},\ \varepsilon_1,\ \varepsilon_2\ ,\varepsilon_3)\\
:={}&\begin{bmatrix}
2 & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 & 0 & 0 & 0 & 0\\
1 & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} & \Xi_{27} & 0 & 0 & 0\\
2 & -p_6\Gamma & \Xi_{33} & \Xi_{34} & 0 & \Xi_{36} & 0 & \Xi_{38} & \Xi_{39} & 0\\
3 & -p_7\Gamma & \Xi_{43} & \Xi_{44} & 0 & 0 & \Xi_{47} & 0 & \Xi_{49} & \Xi_{410}\\
0 & 2\varepsilon_1 & 0 & 0 & \Xi_{55} & \Xi_{56} & \Xi_{57} & 0 & 0 & 0\\
0 & 2\varepsilon_1 & \varepsilon_2 & 0 & \Xi_{65} & \Xi_{66} & \Xi_{67} & \Xi_{68} & \Xi_{69} & 0\\
0 & 3\varepsilon_1 & 0 & \varepsilon_3 & \Xi_{75} & \Xi_{76} & \Xi_{77} & 0 & \Xi_{79} & \Xi_{710}\\
0 & 0 & 4\varepsilon_2 & 0 & 0 & \Xi_{86} & 0 & \Xi_{87} & \Xi_{89} & 0\\
0 & 0 & 3\varepsilon_2 & 2\varepsilon_3 & 0 & \Xi_{96} & \Xi_{97} & -p_9\Gamma & \Xi_{99} & \Xi_{910}\\
0 & 0 & 0 & 6\varepsilon_3 & 0 & 0 & \Xi_{107} & 0 & \Xi_{109} & \Xi_{1010}
\end{bmatrix},\\
&\Xi_{12}=-2p_2\Gamma+2,\ \Xi_{13}=-2p_3\Gamma+2,\ \Xi_{14}=-2p_4\Gamma+2,\ \Xi_{22}=-p_5\Gamma+\varepsilon_1-1,\\
&\Xi_{23}=-\alpha_{21}p_6\Gamma,\ \Xi_{24}=-\alpha_{31}p_7\Gamma,\ \Xi_{25}=-p_2\Gamma+1,\ \Xi_{26}=(-p_3\Gamma+1)\alpha_{21},\ \Xi_{27}=(-p_4\Gamma+1)\alpha_{31},\\
&\Xi_{33}=-p_8\Gamma+\varepsilon_2-1,\ \Xi_{34}=-\alpha_{32}p_9\Gamma,\ \Xi_{36}=-p_2\Gamma+1,\ \Xi_{38}=-p_3\Gamma+1,\ \Xi_{39}=(-p_4\Gamma+1)\alpha_{32},\\
&\Xi_{43}=-p_9\Gamma,\ \Xi_{44}=-p_{10}\Gamma+\varepsilon_3-1,\ \Xi_{47}=-p_2\Gamma+1,\ \Xi_{49}=-p_3\Gamma+1,\ \Xi_{410}=-p_4\Gamma+1,\\
&\Xi_{55}=-2p_5\Gamma-2,\ \Xi_{56}=-2\alpha_{21}^2p_6\Gamma,\ \Xi_{57}=-2\alpha_{31}^2p_7\Gamma,\ \Xi_{65}=-p_6\Gamma,\\
&\Xi_{66}=-p_5\Gamma-\alpha_{21}p_8\Gamma-1-\alpha_{21},\ \Xi_{67}=-\alpha_{31}\alpha_{32}p_9\Gamma,\ \Xi_{68}=-\alpha_{21}p_6\Gamma,\\
&\Xi_{69}=-\alpha_{31}\alpha_{32}p_7\Gamma,\ \Xi_{75}=-p_7\Gamma,\ \Xi_{76}=-\alpha_{21}p_9\Gamma,\ \Xi_{77}=-p_5\Gamma-\alpha_{31}p_{10}\Gamma-1-\alpha_{31},\\
&\Xi_{79}=-\alpha_{21}p_6\Gamma,\ \Xi_{710}=-\alpha_{31}p_7\Gamma,\ \Xi_{86}=-2p_6\Gamma,\ \Xi_{87}=-2p_8\Gamma-2,\ \Xi_{89}=-2\alpha_{32}^2p_9\Gamma,\\
&\Xi_{96}=-p_7\alpha_{21}\Gamma,\ \Xi_{97}=-p_6\alpha_{31}\Gamma,\ \Xi_{99}=-p_8\Gamma-\alpha_{32}p_{10}\Gamma-1-\alpha_{32},\\
&\Xi_{910}=-\alpha_{32}p_9\Gamma,\ \Xi_{107}=-2\alpha_{21}p_7\Gamma,\ \Xi_{109}=-2p_9\Gamma,\ \Xi_{1010}=-2p_{10}\Gamma-2.
\end{aligned}
$$

Using the above results, the value of the convergence criteria (20) of the Newton's method is

$$\theta := \theta(0.001,\ 0.002,\ 0.003) = 6.9774e-02 < 2^{-1}$$

for $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.002$, $\varepsilon_3 = 0.003$. Hence, since the inequality (20) holds, the asymptotic structure of the solutions with uniqueness, positive semidefiniteness and quadratic convergence is attained. In fact, it can also be verified that the algorithm (17) converges to the exact solution with quadratic convergence after four iterations. The remainder per iteration is given in Table 4. Moreover, a solution of the SIGMARE (9) is given in Table 5. On the other hand, Table 6 shows the values of θ for various values of ε_i. It is verified for the first time that for various ε_i, since the convergence criteria is satisfied, the quadratic convergence is guaranteed for sufficiently small ε_i.

It should be noted that convergence criteria (20) is a conservative condition. Hence, even if such a condition is not satisfied, a required solution that attains quadratic convergence might exist.

Finally, in order to compare the solution X^{MAT} that is computed by using the function `are` of MATLAB with the solution X^{NEW} that is obtained through the algorithm (17), the remainder of the errors and these solutions are given in Table 7 under the conditions that the sufficiently small perturbation parameters are chosen as $\varepsilon_1 = 1.0e-07$, $\varepsilon_2 = 2.0e-7$, and $\varepsilon_3 = 3.0e-07$. From Table 7, it can also be verified that as compared with the previous example, the resulting algorithm of this paper is very reliable for the small parameters ε_i. Moreover, the computational dimension that carries out the Newton's iterations is scalar. Thus, taking into account the fact that the Schur method [4] needs eight dimensions, it appears that the proposed algorithm is very attractive.

6 Conclusion

In this paper, a new iterative algorithm for solving the SIGMARE that has sign-indefinite quadratic form has been proposed. The proposed algorithm consist of the Newton's method and two fixed point algorithms. As a result, it has been proven that the solution of the SIGMARE converges to a positive semi-definite stabilizing solution with the rate of convergence of $O(\|\mu\|^{2^n})$. Moreover, the reduction of the computational work space can be attained even if the GMSPS has many fast subsystems as compared with the existing results [11, 12]. As another important features, the assumption that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common is not needed. This advantage admits the using of the proposed algorithm to the GMSPS. Moreover, the convergence criteria of the Newton's method for the GMSPS has been derived for the first time. Finally, the numerical examples have shown excellent results that the proposed algorithm has succeeded in reducing the computational workspace and the quadratic convergence has been attained under the condition that the convergence criteria was satisfied.

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