

Efficient Numerical Procedures for Solving Closed-Loop Stackelberg Strategies with Small Singular Perturbation Parameter

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Abstract. In this paper, the computation of the linear closed-loop Stackelberg strategies with small singular perturbation parameter that characterizes singularly perturbed systems (SPS) are studied. The attention is focused on a new numerical algorithm for solving a set of cross-coupled algebraic Lyapunov and Riccati equations (CALRE). It is proven that the new algorithm guarantees the local quadratic convergence. A numerical example is solved to show a reduction of the average CPU time compared with the existing algorithm.

keywords: Stackelberg strategies, cross-coupled algebraic Lyapunov and Riccati equations, singularly perturbed systems, Newton's method.

1 Introduction

The linear Stackelberg strategies and their applications have been investigated intensively in many studies (see e.g., [1, 2, 3] and reference therein). There exist three different types of Stackelberg strategies : (a) open-loop strategies, (b) closed-loop strategies, and (c) feedback strategies. Particularly, the linear closed-loop Stackelberg strategies in sequential decision-making problems have been studied in [1]. It is well-known that, in order to obtain the closed-loop Stackelberg strategies, it is necessary to solve a set of cross-coupled algebraic Lyapunov and Riccati equations (CALREs). Although a numerical algorithm for solving the CALRE has been introduced in [1], there is no proof on the convergence of the algorithm. Moreover, it is easy to verify that the convergence speed of the algorithm is very slow through the simulation.

When an integrated controlling system includes the singular perturbations that are known as the small time constants, masses, capacitances, and similar parasitic parameters, such dynamic systems are called singularly perturbed systems (SPSs). In general, the SPS arise in large scale dynamic systems [5]. The control problems of the SPS have been investigated extensively (see e.g., [5, 6, 11] and reference therein). In order to obtain the optimal solution, the algebraic Riccati equation (MARE), which is parameterized by the small positive parameter ε such as the singular perturbation parameter needs to be solved. Various reliable approaches to

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the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures. One of the approaches is the invariant subspace approach based on the Hamiltonian matrix [4]. However, such an approach is not adequate to the SPS since the dimension of the required workspace to carry out the calculations for the Hamiltonian matrix is twice the dimension of the original full-system. As another disadvantage, there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill-conditioned [4]. Particularly, the existence of the small singular perturbation parameter results in the ill-conditioned.

The linear Stackelberg strategies of the SPS have been studied by using composite controller design [7, 8]. It is well known that the composite design is very useful when the parameter in the systems represents a small perturbation, whose value is not known exactly. However, the resulting composite Stackelberg strategies guarantee only a near optimality. Therefore, as long as the value of the small perturbation parameter ε is known, much effort should be made towards finding the exact strategies without the ill-conditioning.

The recursive algorithm for solving the CALRE of the SPS has been developed [9]. It has been shown that the recursive algorithm is very effective to solve the CALRE when the system matrices are functions of a small perturbation parameter ε . However, the overall convergence of the algorithm is not guaranteed because the existing algorithm [1] has been used. Moreover, even if the recursive algorithm converges, it only converges to the 0-order approximation solution, and the convergence speed is very slow. Furthermore, since in the existing algorithm [1, 9], the step size related to the cost function needs to be updated at each iteration, a lot of computation time for algebraic manipulations must be needed.

In this paper, the linear closed-loop Stackelberg strategies of the SPS are considered. After defining a set of the generalized cross-coupled algebraic Lyapunov and Riccati equations (GALRE), the uniqueness and boundedness of the solution to the GALRE and their asymptotic structure are studied. A new numerical algorithm for solving the GALRE is proposed. Since the proposed numerical computation is based on Newton's method, the local quadratic convergence is guaranteed. Moreover, unlike the existing algorithm [1, 9], there is no need to update the parameter because there is no design parameter in the new algorithm. Therefore, the computation can be done directly. As another important feature, the cost performance degradation using the high-order approximate strategy that is based on the iterative solutions is exactly proved for the first time compared with the existing result [7]. The simulation results show that the proposed algorithm succeeds in improving the convergence rate and reducing the CPU time.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of M . $\text{vec}M$ denotes an ordered stack of the columns of M [10]. \otimes denotes Kronecker product. U_{lm} denotes a permutation matrix in Kronecker matrix sense [10] such that $U_{lm}\text{vec}M = \text{vec}M^T$, ($M \in \mathbf{R}^{l \times m}$). $E[\cdot]$ denotes the expectation. The trace of M is denoted by $\text{Trace } M$.

2 Problem Statement

Consider a linear time-invariant SPS [7, 9]

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u_1 + B_{12}u_2, \quad (1a)$$

$$\varepsilon \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2 \quad (1b)$$

with $x_i(0) = x_i^0$ and the quadratic cost functions

$$J_i = \frac{1}{2} \int_0^\infty [z^T Q_i z + u_i^T R_{ii} u_i + u_j^T R_{ij} u_j] dt, \quad (2)$$

and

$$R_{ii} > 0, R_{ij} \geq 0, i = 1, 2,$$

$$Q_i = \begin{bmatrix} C_{i1}^T C_{i1} & C_{i1}^T C_{i2} \\ C_{i2}^T C_{i1} & C_{i2}^T C_{i2} \end{bmatrix}, z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $x_i \in \mathbf{R}^{n_i}$, $i = 1, 2$ are the state vector, $u_i \in \mathbf{R}^{m_i}$, $i = 1, 2$ are the control input. All the matrices are constant matrices of appropriate dimensions. ε is the small positive singular parameter. It is supposed that the small parameter is exactly known.

Let us introduce the partitioned matrices

$$A_\varepsilon = \Phi_\varepsilon^{-1} A, B_{i\varepsilon} = \Phi_\varepsilon^{-1} B_i,$$

$$S_{i\varepsilon} = B_{i\varepsilon} R_{11}^{-1} R_{i1} R_{11}^{-1} B_{i\varepsilon}^T = \Phi_\varepsilon^{-1} S_i \Phi_\varepsilon^{-1},$$

$$\Phi_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix},$$

$$S_i = B_1 R_{11}^{-1} R_{i1} R_{11}^{-1} B_1^T = \begin{bmatrix} S_{i11} & S_{i12} \\ S_{i12}^T & S_{i22} \end{bmatrix}, i = 1, 2.$$

It is assumed that the decision-maker denoted by Player 2 is the leader, and Player 1 is the follower. Under the assumption that both players employ closed-loop strategies $u_i := u_i(z, t)$, a strategy set (u_1^*, u_2^*) is called a Stackelberg strategy if the following conditions hold.

$$J_2(u_1^*, u_2^*) \leq J_2(u_1^0(u_2), u_2), \forall u_2 \in \mathbf{R}^{m_2}, \quad (3)$$

where

$$J_1(u_1^0(u_2), u_2) = \min_{u_1} J_1(u_1, u_2), \quad (4)$$

and

$$u_1^* = u_1^0(u_2^*). \quad (5)$$

Closed-loop Stackelberg strategies of the linear quadratic problems for the SPS have been studied in [7, 9]. According to these studies, it is well-known that the closed-loop Stackelberg strategies have the following form.

$$u_i(z, t) = -F_i z = -F_i z(t). \quad (6)$$

It is shown that the gain F_i is dependent on the initial state of the systems $z(0)$. To eliminate this dependence on $z(0)$, it is assumed that $E[z(0)] = 0$, $E[z(0)z^T(0)] = I_n$, where $n := n_1 + n_2$. With this viewpoint the performance criterion is modified as $\bar{J}_i(u_1, u_2) = E[J_i(u_1, u_2)]$, $i = 1, 2$.

The gain F_2 in the leader's strategy is obtained by solving the following CALRE.

$$A_{c\varepsilon}^T M_{1\varepsilon} + M_{1\varepsilon} A_{c\varepsilon} + M_{1\varepsilon} S_{1\varepsilon} M_{1\varepsilon} + F_2^T R_{12} F_2 + Q_1 = 0, \quad (7a)$$

$$A_{c\varepsilon}^T M_{2\varepsilon} + M_{2\varepsilon} A_{c\varepsilon} + M_{1\varepsilon} S_{2\varepsilon} M_{1\varepsilon} + F_2^T R_{22} F_2 + Q_2 = 0, \quad (7b)$$

$$N_1 A_{c\varepsilon}^T + A_{c\varepsilon} N_1 - S_{1\varepsilon} M_{2\varepsilon} N_2 - N_2 M_{2\varepsilon} S_{1\varepsilon} + S_{2\varepsilon} M_{1\varepsilon} N_2 + N_2 M_{1\varepsilon} S_{2\varepsilon} = 0, \quad (7c)$$

$$N_2 A_{c\varepsilon}^T + A_{c\varepsilon} N_2 + I_n = 0, \quad (7d)$$

$$R_{12} F_2 N_1 + R_{22} F_2 N_2 - B_{2\varepsilon}^T (M_{1\varepsilon} N_1 + M_{2\varepsilon} N_2) = 0, \quad (7e)$$

where

$$F_1 := R_{11}^{-1} B_{1\varepsilon}^T M_{1\varepsilon}, \quad A_{c\varepsilon} := A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon} - B_{2\varepsilon} F_2.$$

Since A_ε and $B_{i\varepsilon}$ have the term of ε^{-1} , the solution $M_{i\varepsilon}$ of the CALRE (7a) and (7b), if it exists, must contain terms of ε . Hence, in order to investigate the asymptotic structure of the CALRE (7), the following partitioned matrices are introduced.

$$M_{i\varepsilon} := \begin{bmatrix} M_{i1} & \varepsilon M_{i2} \\ \varepsilon M_{i2}^T & \varepsilon M_{i3} \end{bmatrix}, \quad N_i := \begin{bmatrix} N_{i1} & N_{i2} \\ N_{i2}^T & N_{i3} \end{bmatrix},$$

$$F_i := \begin{bmatrix} F_{i1} & F_{i2} \end{bmatrix}, \quad i = 1, 2,$$

where

$$M_{i1} := M_{i1}(\varepsilon), \quad M_{i2} := M_{i2}(\varepsilon), \quad M_{i3} := M_{i3}(\varepsilon),$$

$$N_{i1} := N_{i1}(\varepsilon), \quad N_{i2} := N_{i2}(\varepsilon), \quad N_{i3} := N_{i3}(\varepsilon),$$

$$F_{i1} := F_{i1}(\varepsilon), \quad F_{i2} := F_{i2}(\varepsilon).$$

In order to avoid the ill-conditioning caused by the large parameter ε^{-1} which is contained in the CALRE (7), the following useful lemma is introduced.

Lemma 1 *The CALRE (7) is equivalent to the following GCALRE (8)*

$$\mathcal{F}_1 := A_c^T M_1 + M_1^T A_c + M_1^T S_1 M_1 + F_2^T R_{12} F_2 + Q_1 = 0, \quad (8a)$$

$$\mathcal{F}_2 := A_c^T M_2 + M_2^T A_c + M_1^T S_2 M_1 + F_2^T R_{22} F_2 + Q_2 = 0, \quad (8b)$$

$$\mathcal{F}_3 := \Phi_\varepsilon N_1 A_c^T + A_c N_1 \Phi_\varepsilon - S_1 M_2 N_2 \Phi_\varepsilon - \Phi_\varepsilon N_2 M_2^T S_1 + S_2 M_1 N_2 \Phi_\varepsilon + \Phi_\varepsilon N_2 M_1^T S_2 = 0, \quad (8c)$$

$$\mathcal{F}_4 := \Phi_\varepsilon N_2 A_c^T + A_c N_2 \Phi_\varepsilon + \Phi_\varepsilon^2 = 0, \quad (8d)$$

$$\mathcal{F}_5 := R_{12} F_2 N_1 + R_{22} F_2 N_2 - B_2^T (M_1 N_1 + M_2 N_2) = 0, \quad (8e)$$

where

$$M_i := \begin{bmatrix} M_{i1} & \varepsilon M_{i2} \\ M_{i2}^T & M_{i3} \end{bmatrix}, \quad F_1 := R_{11}^{-1} B_1^T M_1, \quad A_c := A - S_1 M_1 - B_2 F_2, \\ \mathcal{F}_k := \mathcal{F}_k(M_1, M_2, F_2, N_1, N_2).$$

Proof: Firstly, by direct calculation we verify that $M_i = \Phi_\varepsilon M_{i\varepsilon}$. Hence,

$$A_c^T M_i = A_{c\varepsilon}^T \Phi_\varepsilon \Phi_\varepsilon^{-1} M_{i\varepsilon} = A_{c\varepsilon}^T M_{i\varepsilon}, \quad \Phi_\varepsilon N_1 A_c^T = \Phi_\varepsilon N_1 A_{c\varepsilon}^T \Phi_\varepsilon, \\ M_i^T S_i M_i = M_{i\varepsilon} \Phi_\varepsilon^{-1} \Phi_\varepsilon S_i \Phi_\varepsilon \Phi_\varepsilon^{-1} M_{i\varepsilon} = M_{i\varepsilon} S_{i\varepsilon} M_{i\varepsilon}.$$

By using the similar calculation, the CALRE (7) can be immediately rewritten as (8). \blacksquare

Setting $\varepsilon = 0$ for the previous equations (8), the following equations hold.

$$\bar{A}_c^T \bar{M}_1 + \bar{M}_1^T \bar{A}_c + \bar{M}_1^T S_1 \bar{M}_1 + \bar{F}_2^T R_{12} \bar{F}_2 + Q_1 = 0, \quad (9a)$$

$$\bar{A}_c^T \bar{M}_2 + \bar{M}_2^T \bar{A}_c + \bar{M}_1^T S_2 \bar{M}_1 + \bar{F}_2^T R_{22} \bar{F}_2 + Q_2 = 0, \quad (9b)$$

$$\Phi_0 \bar{N}_1 \bar{A}_c^T + \bar{A}_c \bar{N}_1 \Phi_0 - S_1 \bar{M}_2 \bar{N}_2 \Phi_0 - \Phi_0 \bar{N}_2 \bar{M}_2^T S_1 \\ + S_2 \bar{M}_1 \bar{N}_2 \Phi_0 + \Phi_0 \bar{N}_2 \bar{M}_1^T S_2 = 0, \quad (9c)$$

$$\Phi_0 \bar{N}_2 \bar{A}_c^T + \bar{A}_c \bar{N}_2 \Phi_0 + \Phi_0^2 = 0, \quad (9d)$$

$$R_{12} \bar{F}_2 \bar{N}_1 + R_{22} \bar{F}_2 \bar{N}_2 - B_2^T (\bar{M}_1 \bar{N}_1 + \bar{M}_2 \bar{N}_2) = 0, \quad (9e)$$

where

$$\Phi_0 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{M}_i := \begin{bmatrix} \bar{M}_{i1} & 0 \\ \bar{M}_{i2}^T & \bar{M}_{i3} \end{bmatrix}, \quad \bar{N}_i := \begin{bmatrix} \bar{N}_{i1} & \bar{N}_{i2} \\ \bar{N}_{i2}^T & \bar{N}_{i3} \end{bmatrix}, \\ \bar{F}_i := [\bar{F}_{i1} \quad \bar{F}_{i2}], \quad i = 1, 2, \quad \bar{A}_c := A - S_1 \bar{M}_1 - B_2 \bar{F}_2, \\ \bar{M}_{i1} := M_{i1}(0), \quad \bar{M}_{i2} := M_{i2}(0), \quad \bar{M}_{i3} := M_{i3}(0), \\ \bar{N}_{i1} := N_{i1}(0), \quad \bar{N}_{i2} := N_{i2}(0), \quad \bar{N}_{i3} := N_{i3}(0), \\ \bar{F}_{i1} := F_{i1}(0), \quad \bar{F}_{i2} := F_{i2}(0).$$

Then, taking the partial derivative of the function $\mathcal{F}_k := \mathcal{F}_k(M_1, M_2, F_2, N_1, N_2)$, $k = 1, \dots, 5$ with respect to M_i, N_i and F_2 results in (10).

$$\begin{aligned} & \mathcal{J}(\varepsilon, M_1, M_2, F_2, N_1, N_2) \\ & := \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial M_1} & \frac{\partial \mathcal{F}_1}{\partial M_2} & \frac{\partial \mathcal{F}_1}{\partial F_2} & \frac{\partial \mathcal{F}_1}{\partial N_1} & \frac{\partial \mathcal{F}_1}{\partial N_2} \\ \frac{\partial \mathcal{F}_2}{\partial M_1} & \frac{\partial \mathcal{F}_2}{\partial M_2} & \frac{\partial \mathcal{F}_2}{\partial F_2} & \frac{\partial \mathcal{F}_2}{\partial N_1} & \frac{\partial \mathcal{F}_2}{\partial N_2} \\ \frac{\partial \mathcal{F}_3}{\partial M_1} & \frac{\partial \mathcal{F}_3}{\partial M_2} & \frac{\partial \mathcal{F}_3}{\partial F_2} & \frac{\partial \mathcal{F}_3}{\partial N_1} & \frac{\partial \mathcal{F}_3}{\partial N_2} \\ \frac{\partial \mathcal{F}_4}{\partial M_1} & \frac{\partial \mathcal{F}_4}{\partial M_2} & \frac{\partial \mathcal{F}_4}{\partial F_2} & \frac{\partial \mathcal{F}_4}{\partial N_1} & \frac{\partial \mathcal{F}_4}{\partial N_2} \\ \frac{\partial \mathcal{F}_5}{\partial M_1} & \frac{\partial \mathcal{F}_5}{\partial M_2} & \frac{\partial \mathcal{F}_5}{\partial F_2} & \frac{\partial \mathcal{F}_5}{\partial N_1} & \frac{\partial \mathcal{F}_5}{\partial N_2} \end{bmatrix} \\ & = \begin{bmatrix} \Xi_{11} & & 0 \\ \Xi_{21} & & \Xi_{11} \\ \Xi_{31} & & \Xi_{32} \\ -S_1 \otimes N_2 - N_2 \otimes S_1 & & 0 \\ & -N_1 \otimes B_2^T & -N_2 \otimes B_2^T \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \Xi_{13} & 0 & 0 \\ \Xi_{23} & 0 & 0 \\ \Xi_{33} & \Xi_{34} & \Xi_{35} \\ \Xi_{43} & 0 & \Xi_{34} \\ N_2 \otimes R_{22} + N_1 \otimes R_{12} & \Xi_{54} & \Xi_{55} \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} \Xi_{11} &:= I_n \otimes (A - S_1 M_1 - B_2 F_2)^T + (A - S_1 M_1 - B_2 F_2)^T \otimes I_n, \\ \Xi_{13} &:= I_n \otimes (-B_2^T M_1 + R_{12} F_2)^T + [(-B_2^T M_1 + R_{12} F_2)^T \otimes I_n] U_{nm_i}, \\ \Xi_{21} &:= I_n \otimes (S_2 M_1 - S_1 M_2)^T + (S_2 M_1 - S_1 M_2)^T \otimes I_n, \\ \Xi_{23} &:= I_n \otimes (R_{22} F_2 - B_2^T M_2)^T + [(R_{22} F_2 - B_2^T M_2)^T \otimes I_n] U_{nm_i}, \\ \Xi_{31} &:= -S_1 \otimes (\Phi_\varepsilon N_1) - (\Phi_\varepsilon N_1) \otimes S_1 + S_2 \otimes (\Phi_\varepsilon N_2) + (\Phi_\varepsilon N_2) \otimes S_2, \\ \Xi_{32} &:= -S_1 \otimes (\Phi_\varepsilon N_2) - (\Phi_\varepsilon N_2) \otimes S_1, \\ \Xi_{33} &:= -B_2 \otimes (\Phi_\varepsilon N_1) - (\Phi_\varepsilon N_1) \otimes B_2, \\ \Xi_{34} &:= I_n \otimes (A - S_1 M_1 - B_2 F_2) + (A - S_1 M_1 - B_2 F_2) \otimes I_n, \\ \Xi_{35} &:= I_n \otimes (S_2 M_1 - S_1 M_2) + (S_2 M_1 - S_1 M_2) \otimes I_n, \\ \Xi_{43} &:= -B_2 \otimes (\Phi_\varepsilon N_2) - (\Phi_\varepsilon N_2) \otimes B_2, \\ \Xi_{54} &:= I_n \otimes (R_{12} F_2 - B_2^T M_1), \\ \Xi_{55} &:= I_n \otimes (R_{22} F_2 - B_2^T M_2). \end{aligned}$$

Using (10), the following asymptotic structure of the GICALRE (8) is established.

Theorem 1 *Assume that zeroth-order equations (9) have the solutions such that*

$$\begin{aligned} & \det \mathcal{J}(0, \bar{M}_1, \bar{M}_2, \bar{F}_2, \bar{N}_1, \bar{N}_2) \\ &= \begin{bmatrix} \bar{\Xi}_{11} & 0 \\ \bar{\Xi}_{21} & \bar{\Xi}_{11} \\ \bar{\Xi}_{31} & \bar{\Xi}_{32} \\ -S_1 \otimes N_2 - N_2 \otimes S_1 & 0 \\ -N_1 \otimes B_2^T & -N_2 \otimes B_2^T \\ \bar{\Xi}_{13} & 0 & 0 \\ \bar{\Xi}_{23} & 0 & 0 \\ \bar{\Xi}_{33} & \bar{\Xi}_{34} & \bar{\Xi}_{35} \\ \bar{\Xi}_{43} & 0 & \bar{\Xi}_{34} \\ N_2 \otimes R_{22} + N_1 \otimes R_{12} & \bar{\Xi}_{54} & \bar{\Xi}_{55} \end{bmatrix} \neq 0, \quad (11) \end{aligned}$$

where

$$\begin{aligned} \bar{\Xi}_{11} &:= I_n \otimes (A - S_1 \bar{M}_1 - B_2 \bar{F}_2)^T + (A - S_1 \bar{M}_1 - B_2 \bar{F}_2)^T \otimes I_n, \\ \bar{\Xi}_{13} &:= I_n \otimes (-B_2^T \bar{M}_1 + R_{12} \bar{F}_2)^T + [(-B_2^T \bar{M}_1 + R_{12} \bar{F}_2)^T \otimes I_n] U_{nm_i}, \\ \bar{\Xi}_{21} &:= I_n \otimes (S_2 \bar{M}_1 - S_1 \bar{M}_2)^T + (S_2 \bar{M}_1 - S_1 \bar{M}_2)^T \otimes I_n, \\ \bar{\Xi}_{23} &:= I_n \otimes (R_{22} \bar{F}_2 - B_2^T \bar{M}_2)^T + [(R_{22} \bar{F}_2 - B_2^T \bar{M}_2)^T \otimes I_n] U_{nm_i}, \\ \bar{\Xi}_{31} &:= -S_1 \otimes (\Phi_0 \bar{N}_1) - (\Phi_0 \bar{N}_1) \otimes S_1 + S_2 \otimes (\Phi_0 \bar{N}_2) + (\Phi_0 \bar{N}_2) \otimes S_2, \\ \bar{\Xi}_{32} &:= -S_1 \otimes (\Phi_0 \bar{N}_2) - (\Phi_0 \bar{N}_2) \otimes S_1, \end{aligned}$$

$$\begin{aligned}
\bar{\Xi}_{33} &:= -B_2 \otimes (\Phi_0 \bar{N}_1) - (\Phi_0 \bar{N}_1) \otimes B_2, \\
\bar{\Xi}_{34} &:= I_n \otimes (A - S_1 \bar{M}_1 - B_2 \bar{F}_2) + (A - S_1 \bar{M}_1 - B_2 \bar{F}_2) \otimes I_n, \\
\bar{\Xi}_{35} &:= I_n \otimes (S_2 \bar{M}_1 - S_1 \bar{M}_2) + (S_2 \bar{M}_1 - S_1 \bar{M}_2) \otimes I_n, \\
\bar{\Xi}_{43} &:= -B_2 \otimes (\Phi_0 \bar{N}_2) - (\Phi_0 \bar{N}_2) \otimes B_2, \\
\bar{\Xi}_{54} &:= I_n \otimes (R_{12} \bar{F}_2 - B_2^T \bar{M}_1), \\
\bar{\Xi}_{55} &:= I_n \otimes (R_{22} \bar{F}_2 - B_2^T \bar{M}_2).
\end{aligned}$$

Then there exists small $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, the GCALRE (8) admits the solutions $M_{i\varepsilon} = \Phi_\varepsilon M_i \geq 0$, F_i and N_i , which can be written as

$$M_i = \begin{bmatrix} \bar{M}_{i1} + O(\varepsilon) & \varepsilon \bar{M}_{i2}^T + O(\varepsilon^2) \\ \bar{M}_{i2} + O(\varepsilon) & \bar{M}_{i3} + O(\varepsilon) \end{bmatrix}, \quad (12a)$$

$$F_2 = \begin{bmatrix} \bar{F}_{i1} + O(\varepsilon) & \bar{F}_{i2} + O(\varepsilon) \end{bmatrix}, \quad (12b)$$

$$N_i = \begin{bmatrix} \bar{N}_{i1} + O(\varepsilon) & \bar{N}_{i2}^T + O(\varepsilon) \\ \bar{N}_{i2} + O(\varepsilon) & \bar{N}_{i3} + O(\varepsilon) \end{bmatrix}. \quad (12c)$$

Proof: It can be done by applying the implicit function theorem to the GCALRE (8). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. After some tedious algebra, the Jacobian (10) is derived. Setting $\varepsilon = 0$ for the Jacobian (10), the condition (11) is obtained. Finally, applying the implicit function theorem results in the result directly. \blacksquare

3 Newton's Method

In order to obtain the solutions of the GCALRE (8), the following new numerical computation that is based on the Newton's method is given.

$$\begin{aligned}
&(A - S_1 M_1^{(n)} - B_2 F_2^{(n)})^T M_1^{(n+1)} + M_1^{(n+1)T} (A - S_1 M_1^{(n)} - B_2 F_2^{(n)}) \\
&\quad - (B_2^T M_1^{(n)} - R_{12} F_2^{(n)})^T F_2^{(n+1)} - F_2^{(n+1)T} (B_2^T M_1^{(n)} - R_{12} F_2^{(n)}) \\
&\quad + M_1^{(n)T} S_1 M_1^{(n)} + F_2^{(n)T} B_2^T M_1^{(n)} + M_1^{(n)T} B_2 F_2^{(n)} \\
&\quad - F_2^{(n)T} R_{12} F_2^{(n)} + Q_1 = 0, \quad (13a)
\end{aligned}$$

$$\begin{aligned}
&(S_2 M_1^{(n)} - S_1 M_2^{(n)})^T M_1^{(n+1)} + M_1^{(n+1)T} (S_2 M_1^{(n)} - S_1 M_2^{(n)}) \\
&\quad + (A - S_1 M_1^{(n)} - B_2 F_2^{(n)})^T M_2^{(n+1)} \\
&\quad + M_2^{(n+1)T} (A - S_1 M_1^{(n)} - B_2 F_2^{(n)}) \\
&\quad - (B_2^T M_2^{(n)} - R_{22} F_2^{(n)})^T F_2^{(n+1)} - F_2^{(n+1)T} (B_2^T M_2^{(n)} - R_{22} F_2^{(n)}) \\
&\quad + M_1^{(n)T} S_1 M_2^{(n)} + M_2^{(n)T} S_1 M_1^{(n)} - M_1^{(n)T} S_2 M_1^{(n)} \\
&\quad + F_2^{(n)T} B_2^T M_2^{(n)} + M_2^{(n)T} B_2 F_2^{(n)} - F_2^{(n)T} R_{22} F_2^{(n)} + Q_2 = 0, \quad (13b)
\end{aligned}$$

$$\begin{aligned}
&-\Phi_\varepsilon N_1^{(n)} M_1^{(n+1)T} S_1 - S_1 M_1^{(n+1)} N_1^{(n)} \Phi_\varepsilon \\
&\quad + S_2 M_1^{(n+1)} N_2^{(n)} \Phi_\varepsilon + \Phi_\varepsilon N_2^{(n)} M_1^{(n+1)T} S_2 \\
&\quad - S_1 M_2^{(n+1)} N_2^{(n)} \Phi_\varepsilon - \Phi_\varepsilon N_2^{(n)} M_2^{(n+1)T} S_1 \\
&\quad - \Phi_\varepsilon N_1^{(n)} F_2^{(n+1)T} B_2^T - B_2 F_2^{(n+1)} N_1^{(n)} \Phi_\varepsilon
\end{aligned}$$

$$\begin{aligned}
& +(A - S_1 M_1^{(n)} - B_2 F_2^{(n)}) N_1^{(n+1)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_1^{(n+1)} (A - S_1 M_1^{(n)} - B_2 F_2^{(n)})^T \\
& + \Phi_\varepsilon N_2^{(n+1)} (S_2 M_1^{(n)} - S_1 M_2^{(n)})^T + (S_2 M_1^{(n)} - S_1 M_2^{(n)}) N_2^{(n+1)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_1^{(n)} M_1^{(n)T} S_1 + S_1 M_1^{(n)} N_1^{(n)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_2^{(n)} M_2^{(n)T} S_1 + S_1 M_2^{(n)} N_2^{(n)} \Phi_\varepsilon \\
& - \Phi_\varepsilon N_2^{(n)} M_1^{(n)T} S_2 - S_2 M_1^{(n)} N_2^{(n)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_1^{(n)} F_2^{(n)T} B_2^T + B_2 F_2^{(n)} N_1^{(n)} \Phi_\varepsilon = 0, \tag{13c}
\end{aligned}$$

$$\begin{aligned}
& - \Phi_\varepsilon N_2^{(n)} M_1^{(n+1)T} S_1 - S_1 M_1^{(n+1)} N_2^{(n)} \Phi_\varepsilon \\
& - \Phi_\varepsilon N_2^{(n)} F_2^{(n+1)T} B_2^T - B_2 F_2^{(n+1)} N_2^{(n)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_2^{(n+1)} (A - S_1 M_1^{(n)} - B_2 F_2^{(n)})^T \\
& + (A - S_1 M_1^{(n)} - B_2 F_2^{(n)}) N_2^{(n+1)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_2^{(n)} M_1^{(n)T} S_1 + S_1 M_1^{(n)} N_2^{(n)} \Phi_\varepsilon \\
& + \Phi_\varepsilon N_2^{(n)} F_2^{(n)T} B_2^T + B_2 F_2^{(n)} N_2^{(n)} \Phi_\varepsilon + \Phi_\varepsilon^2 = 0, \tag{13d}
\end{aligned}$$

$$\begin{aligned}
& - B_2^T M_1^{(n+1)} N_1^{(n)} - B_2^T M_2^{(n+1)} N_2^{(n)} + R_{22} F_2^{(n+1)} N_2^{(n)} + R_{12} F_2^{(n+1)} N_1^{(n)} \\
& + (R_{12} F_2^{(n)} - B_2^T M_1^{(n)}) N_1^{(n+1)} + (R_{22} F_2^{(n)} - B_2^T M_2^{(n)}) N_2^{(n+1)} \\
& - R_{12} F_2^{(n)} N_1^{(n)} - R_{22} F_2^{(n)} N_2^{(n)} \\
& + B_2^T M_1^{(n)} N_1^{(n)} + B_2^T M_2^{(n)} N_2^{(n)} = 0, \tag{13e}
\end{aligned}$$

and the initial condition $F_2^{(0)}$ is chosen such that the closed-loop SPS is quadratically stable. Moreover, $M_i^{(0)}$ and $N_i^{(0)}$ satisfies the following generalized algebraic Lyapunov and Riccati equations, respectively.

$$\begin{aligned}
& M_1^{(0)T} A_d + A_d^T M_1^{(0)} - M_1^{(0)T} S_1 M_1^{(0)} + F_2^{(0)T} R_{12} F_2^{(0)} + Q_1 = 0, \\
& M_2^{(0)T} (A_d - S_1 M_1^{(0)}) + (A_d - S_1 M_1^{(0)})^T M_2^{(0)} \\
& \quad + M_1^{(0)T} S_2 M_1^{(0)} + F_2^{(0)T} R_{22} F_2^{(0)} + Q_2 = 0, \\
& N_2^{(0)} (A_d - S_1 M_1^{(0)})^T + (A_d - S_1 M_1^{(0)}) N_2^{(0)} + I_n = 0, \\
& N_1^{(0)} (A_d - S_1 M_1^{(0)})^T + (A_d - S_1 M_1^{(0)}) N_1^{(0)} \\
& \quad - (S_1 M_2^{(0)} - S_2 M_1^{(0)}) N_2^{(0)} \Phi_\varepsilon - \Phi_\varepsilon N_2^{(0)} (S_1 M_2^{(0)} - S_2 M_1^{(0)})^T = 0,
\end{aligned}$$

where $A_d = A - B_2 F_2^{(0)}$.

In order to guarantee the existence of the gain $F_2^{(0)}$, the following assumption is needed.

Assumption 1 *The pairs (A, B_i) , $i = 1, 2$ are stabilizable.*

The algorithm (13) can be constructed by assuming $M_i^{(n+1)} = M_i^{(n)} + \Delta M_i^{(n)}$, $F_2^{(n+1)} = F_2^{(n)} + \Delta F_2^{(n)}$ and $N_i^{(n+1)} = N_i^{(n)} + \Delta N_i^{(n)}$ and neglecting $O(\Delta^2)$ term. The following theorem indicates that the algorithm (13) is the Newton's method.

Theorem 2 *Suppose that there exists a solution to the GCALRE (8). It can be obtained by performing the algorithm (13) that is equal to the Newton's method.*

Proof : First, the vec-operator transformation on both sides of (8) as

$$\text{vec}\mathcal{F}_k := \mathcal{F}_k(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) = 0$$

is applied. In addition, the vec-operator transformation on both sides of (13) is also carried out. Second, subtracting these equations, it is easy to verify that

$$\begin{aligned} & \begin{bmatrix} \text{vec}[M_1^{(n+1)} & M_2^{(n+1)}] \\ \text{vec}F_2^{(n+1)} \\ \text{vec}[N_1^{(n+1)} & N_2^{(n+1)}] \end{bmatrix} \\ &= \begin{bmatrix} \text{vec}[M_1^{(n)} & M_2^{(n)}] \\ \text{vec}F_2^{(n)} \\ \text{vec}[N_1^{(n)} & N_2^{(n)}] \end{bmatrix} \\ & - \mathcal{J}(\varepsilon, M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \\ & \times \begin{bmatrix} \text{vec}\mathcal{F}_1(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \\ \text{vec}\mathcal{F}_2(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \\ \text{vec}\mathcal{F}_3(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \\ \text{vec}\mathcal{F}_4(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \\ \text{vec}\mathcal{F}_5(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)}) \end{bmatrix}. \end{aligned} \quad (14)$$

This is the desired result. ■

Newton's method is well-known and is widely used to find a solution of algebraic nonlinear equations. Its local convergence properties are well understood [12]. Although the Newton's method guarantees the local convergence, it may not converge to the required solution if the initial condition is not suitably chosen. In order to guarantee the convergence to the required solution, the initial condition is chosen as follows.

$$M_i^{(0)} = \begin{bmatrix} \bar{M}_{i1} & \varepsilon \bar{M}_{i2}^T \\ \bar{M}_{i2} & \bar{M}_{i3} \end{bmatrix}, F_2^{(0)} = [\bar{F}_{i1} \quad \bar{F}_{i2}], N_i^{(0)} = \begin{bmatrix} \bar{N}_{i1} & \bar{N}_{i2}^T \\ \bar{N}_{i2} & \bar{N}_{i3} \end{bmatrix}. \quad (15)$$

The following theorem indicates that the algorithm attains the quadratic convergence under the above initial conditions.

Theorem 3 *Assume that the conditions of Theorem 1 hold. Then, there exists a small ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, Newton's method (13) converges to the exact solution of M_i^* , F_2^* and N_i^* with the rate of the quadratic convergence. Moreover, the convergence solution M_i^* , F_2^* and N_i^* is unique solution of the GCALRE (8) in the neighborhood of the initial condition (15). That is, the following relations are satisfied.*

$$\|M_i^{(n)} - M_i^*\| \leq O(\varepsilon^{2^n}), \quad n = 0, 1, \dots, \quad (16a)$$

$$\|F_2^{(n)} - F_2^*\| \leq O(\varepsilon^{2^n}), \quad n = 0, 1, \dots, \quad (16b)$$

$$\|N_i^{(n)} - N_i^*\| \leq O(\varepsilon^{2^n}), \quad n = 0, 1, \dots. \quad (16c)$$

Proof : The proof of this theorem can be done by using Newton-Kantorovich theorem [12]. It is immediately obtained from the equation (10) that there exists the positive scalar constant \mathcal{L} such that for any $M_i^a, F_2^a, N_i^a, M_i^b, F_2^b$ and N_i^b ,

$$\begin{aligned} & \|\mathcal{J}(\varepsilon, M_1^a, M_2^a, F_2^a, N_1^a, N_2^a) - \mathcal{J}(\varepsilon, M_1^b, M_2^b, F_2^b, N_1^b, N_2^b)\| \\ & \leq \mathcal{L}\|(M_1^a, M_2^a, F_2^a, N_1^a, N_2^a) - (M_1^b, M_2^b, F_2^b, N_1^b, N_2^b)\|. \end{aligned} \quad (17)$$

Moreover, using (12), the following result holds

$$\begin{aligned} & \mathcal{J}(\varepsilon, M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)}) \\ & = \mathcal{J}(0, \bar{M}_1, \bar{M}_2, \bar{F}_2, \bar{N}_1, \bar{N}_2) + O(\varepsilon). \end{aligned} \quad (18)$$

Hence, it follows that $\mathcal{J}(\varepsilon, M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)})$ is nonsingular under the condition (11) for sufficiently small ε . Therefore, there exists β such that

$$\beta = \|\mathcal{J}(\varepsilon, M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)})^{-1}\|.$$

On the other hand, since

$$\mathcal{F}_k(M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)}) = O(\varepsilon),$$

there exists η such that

$$\begin{aligned} \eta & = \|\mathcal{J}(\varepsilon, M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)})^{-1}\| \\ & \quad \cdot \|\mathcal{F}_k(M_1^{(0)}, M_2^{(0)}, F_2^{(0)}, N_1^{(0)}, N_2^{(0)})\| = O(\varepsilon). \end{aligned}$$

Thus, there exists θ such that $\theta = \beta\eta\mathcal{L} < 2^{-1}$ because $\eta = O(\varepsilon)$. Finally, using the Newton-Kantorovich theorem, we can show that M_i^*, F_2^* and N_i^* are the unique solution in the subset. Moreover, using the Newton-Kantorovich theorem, the error estimate is given by (16). \blacksquare

4 High-order Approximate Strategy

The iterative solution of (13) will now be used to obtain a high-order approximate strategy for the Stackelberg game as compared to the exact strategy (6). The well-posedness property of the high-order approximate strategy $u_{i\text{app}}^{(n)} = -F^{(n)}z$ is established in the following theorem.

Theorem 4 *Assume that the conditions of Theorem 1 hold. If the reduced-order CALRE (9) possess a unique stabilizing solution, then*

$$\bar{J}_i(u_1^0(u_{2\text{app}}^{(n)}), u_{2\text{app}}^{(n)}) = \bar{J}_i(u_1^*, u_2^*) + O(\varepsilon^{2^n}), \quad (19a)$$

$$\bar{J}_i(u_{1\text{app}}^{(n)}, u_{2\text{app}}^{(n)}) = \bar{J}_i(u_1^0(u_{2\text{app}}^{(n)}), u_{2\text{app}}^{(n)}) + O(\varepsilon^{2^n}), \quad (19b)$$

$$\bar{J}_i(u_{1\text{app}}^{(n)}, u_{2\text{app}}^{(n)}) = \bar{J}_i(u_1^*, u_2^*) + O(\varepsilon^{2^n}), \quad (19c)$$

$$i = 1, 2, n = 0, 1, \dots$$

Proof: When the exact strategies $u_i^* = -F_i^* z$ are used, the resulting values of the cost function are $\bar{J}_i(u_1^*, u_2^*) = 1/2 \text{Trace } M_{i\varepsilon}$, where $M_{1\varepsilon}$ and $M_{2\varepsilon}$ are given by (7a) and (7b), respectively. Suppose now that the leader uses the high-order approximate strategy $u_{2\text{app}}^{(n)} = -F_2^{(n)} z$. Let the follower respond optimally by using $u_1 = u_1^0(u_{2\text{app}}^{(n)}) = -R_{11}^{-1} B_{1\varepsilon}^T V_{1\varepsilon} z$, where $V_{1\varepsilon}$ is the stabilizing solution of the following algebraic Riccati equation (ARE).

$$(A_\varepsilon - S_{1\varepsilon} V_{1\varepsilon} - B_{2\varepsilon} F_2^{(n)})^T V_{1\varepsilon} + V_{1\varepsilon} (A_\varepsilon - S_{1\varepsilon} V_{1\varepsilon} - B_{2\varepsilon} F_2^{(n)}) + V_{1\varepsilon} S_{1\varepsilon} V_{1\varepsilon} + F_2^{(n)T} R_{12} F_2^{(n)} + Q_1 = 0. \quad (20)$$

The resulting values of the cost function are $\bar{J}_i(u_1^0(u_{2\text{app}}^{(n)}), u_{2\text{app}}^{(n)}) = 1/2 \text{Trace } V_{i\varepsilon}$, where $V_{1\varepsilon}$ satisfies (20) and $V_{2\varepsilon}$ satisfies the following ARE.

$$(A_\varepsilon - S_{1\varepsilon} V_{1\varepsilon} - B_{2\varepsilon} F_2^{(n)})^T V_{2\varepsilon} + V_{2\varepsilon} (A_\varepsilon - S_{1\varepsilon} V_{1\varepsilon} - B_{2\varepsilon} F_2^{(n)}) + V_{1\varepsilon} S_{2\varepsilon} V_{1\varepsilon} + F_2^{(n)T} R_{22} F_2^{(n)} + Q_2 = 0. \quad (21)$$

Using the inequalities (16), the ARE (20) and (21) can be changed as follows, respectively.

$$(A_{c\varepsilon} + O(\varepsilon^{2^n}))^T V_{1\varepsilon} + V_{1\varepsilon} (A_{c\varepsilon} + O(\varepsilon^{2^n})) + M_{1\varepsilon} S_{1\varepsilon} V_{1\varepsilon} + V_{1\varepsilon} S_{1\varepsilon} M_{1\varepsilon} - V_{1\varepsilon} S_{1\varepsilon} V_{1\varepsilon} + F_2^T R_{12} F_2 + Q_1 + O(\varepsilon^{2^n}) = 0, \quad (22a)$$

$$(A_{c\varepsilon} + O(\varepsilon^{2^n}))^T V_{2\varepsilon} + V_{2\varepsilon} (A_{c\varepsilon} + O(\varepsilon^{2^n})) + M_{1\varepsilon} S_{1\varepsilon} V_{2\varepsilon} + V_{2\varepsilon} S_{1\varepsilon} M_{1\varepsilon} - V_{1\varepsilon} S_{1\varepsilon} V_{2\varepsilon} - V_{2\varepsilon} S_{1\varepsilon} V_{1\varepsilon} + V_{1\varepsilon} S_{2\varepsilon} V_{1\varepsilon} + F_2^T R_{22} F_2 + Q_2 + O(\varepsilon^{2^n}) = 0. \quad (22b)$$

Subtracting (7b) and (7a) from (22a) and (22b) respectively, we find that $W_{i\varepsilon} = V_{i\varepsilon} - M_{i\varepsilon}$ satisfies the following AREs

$$A_{c\varepsilon}^T W_{1\varepsilon} + W_{1\varepsilon} A_{c\varepsilon} - W_{1\varepsilon} S_{1\varepsilon} W_{1\varepsilon} + O(\varepsilon^{2^n}) = 0, \quad (23a)$$

$$A_{c\varepsilon}^T W_{2\varepsilon} + W_{2\varepsilon} A_{c\varepsilon} - W_{1\varepsilon} S_{1\varepsilon} W_{2\varepsilon} - W_{2\varepsilon} S_{1\varepsilon} W_{1\varepsilon} + W_{1\varepsilon} S_{2\varepsilon} W_{1\varepsilon} + W_{1\varepsilon} S_{2\varepsilon} M_{1\varepsilon} + M_{1\varepsilon} S_{2\varepsilon} W_{1\varepsilon} + O(\varepsilon^{2^n}) = 0. \quad (23b)$$

It is easy to verify from (23a) that $W_{1\varepsilon} = V_{1\varepsilon} - M_{1\varepsilon} = O(\varepsilon^{2^n})$ because $A_{c\varepsilon}$ is stable. Moreover, substituting $W_{1\varepsilon} = O(\varepsilon^{2^n})$ into (23b) and taking the stability of $A_{c\varepsilon}$ into account, $W_{2\varepsilon} = O(\varepsilon^{2^n})$ holds. Thus, the equation (19a) holds.

If, instead of responding optimally, the follower uses the high-order approximate strategy $u_{1\text{app}} = -R_{11}^{-1} B_{1\varepsilon}^T M_{1\varepsilon}^{(n)} z$, the resulting values of the cost function will be $\bar{J}_i(u_{1\text{app}}, u_{2\text{app}}) = 1/2 \text{Trace } U_{i\varepsilon}$, where $U_{1\varepsilon}$ and $U_{2\varepsilon}$ satisfy the following AREs.

$$(A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon}^{(n)} - B_{2\varepsilon} F_2^{(n)})^T U_{1\varepsilon} + U_{1\varepsilon} (A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon}^{(n)} - B_{2\varepsilon} F_2^{(n)}) + M_{1\varepsilon}^{(n)} S_{1\varepsilon} M_{1\varepsilon}^{(n)} + F_2^{(n)T} R_{12} F_2^{(n)} + Q_1 = 0, \quad (24a)$$

$$(A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon}^{(n)} - B_{2\varepsilon} F_2^{(n)})^T U_{2\varepsilon} + U_{2\varepsilon} (A_\varepsilon - S_{1\varepsilon} M_{1\varepsilon}^{(n)} - B_{2\varepsilon} F_2^{(n)}) + M_{1\varepsilon}^{(n)} S_{2\varepsilon} M_{1\varepsilon}^{(n)} + F_2^{(n)T} R_{22} F_2^{(n)} + Q_2 = 0. \quad (24b)$$

Using the above similar manner, it can be shown that $U_{i\varepsilon} - V_{i\varepsilon} = O(\varepsilon^{2^n})$.

Finally, we have

$$\begin{aligned} & \bar{J}_i(u_{1\text{app}}^{(n)}, u_{2\text{app}}^{(n)}) - \bar{J}_i(u_1^*, u_2^*) \\ &= \bar{J}_i(u_{1\text{app}}^{(n)}, u_{2\text{app}}^{(n)}) - \bar{J}_i(u_1^0(u_{2\text{app}}^{(n)}), u_{2\text{app}}^{(n)}) \\ & \quad + \bar{J}_i(u_1^0(u_{2\text{app}}^{(n)}), u_{2\text{app}}^{(n)}) - \bar{J}_i(u_1^*, u_2^*). \end{aligned}$$

which proves (19c). ■

It is worth pointing out that the performance degradation (19) has been shown for the first time. Moreover, since we do not assume that A_{22} is non-singular compared with the existing result [7], our new results are applicable to both standard and nonstandard singularly perturbed systems.

5 Computational Example

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is given. The system matrices are given as follows [9].

$$\begin{aligned} A_\varepsilon &= \begin{bmatrix} 0 & 1 \\ -\varepsilon^{-1} & -\varepsilon^{-1} \end{bmatrix}, \quad B_{1\varepsilon} = B_{2\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{-1} \end{bmatrix}, \\ Q_1 = Q_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_{11} = 1, \quad R_{12} = 2, \quad R_{21} = 1, \quad R_{22} = 1. \end{aligned}$$

The small parameter is chosen as $\varepsilon = 0.1$. It should be noted that the algorithm (13) converges to the exact solution with accuracy of $\|\mathcal{F}^{(k)}(\varepsilon)\| < 1.0e - 8$ after four iterations, where

$$\|\mathcal{F}^{(n)}(\varepsilon)\| := \sum_{k=1}^5 \|\mathcal{F}_k(M_1^{(n)}, M_2^{(n)}, F_2^{(n)}, N_1^{(n)}, N_2^{(n)})\|. \quad (25)$$

In order to verify the exactitude of the solutions, the remainder per iteration by substituting these solutions into the GCALRE (8) is computed. In Table 1, the results of the error $\|\mathcal{F}^{(n)}(\varepsilon)\|$ per iteration are given for several values ε . As a result, it can be seen that the algorithm (13) has the quadratic convergence. On the other hand, when the existing algorithm [1] is applied to this problem, the computation in the case of $\varepsilon = 0.5$ and $\varepsilon = 0.1$ converge through 7 and 14 iterations, respectively. Moreover, the computation in the case of $\varepsilon = 1.0e - 02$ and $\varepsilon = 1.0e - 03$ do not converge, while the proposed method converges after five iterations.

The required iterations of the proposed algorithm (13) versus the existing method [1] are presented in Table 2. It can be seen from Table 2 that the proposed algorithm (13) succeed in reducing the iterations for the different values of ε . Particularly, for large ε the required iterations are small.

In Table 3, the results of the CPU time are given when the existing method [1] are used. The CPU time represents the average based on the computations of ten runs. From Table 3, the existing method [1] takes a lot of CPU time compared with the iterative algorithm (13).

Table 1.

k	$\ \mathcal{F}^{(n)}(0.5)\ $	$\ \mathcal{F}^{(n)}(1.0e-01)\ $	$\ \mathcal{F}^{(n)}(1.0e-02)\ $	$\ \mathcal{F}^{(n)}(1.0e-03)\ $
1	$1.8425e-02$	$8.2280e-02$	2.4626	$3.9402e+01$
2	$5.6567e-05$	$1.2227e-03$	$1.6370e-01$	3.7784
3	$1.9560e-10$	$9.8064e-08$	$7.7667e-04$	$8.0144e-02$
4	-	$1.7447e-14$	$1.4652e-08$	$7.8100e-06$
5	-	-	$4.8112e-12$	$1.7135e-10$

Table 2. Number of iterations

ε	Newton's Method	Existing algorithm[1]
0.5	3	7
$1.0e-01$	4	14
$1.0e-02$	5	-
$1.0e-03$	5	-

In view of the above results, it can be said that the proposed algorithm is very reliable and useful compared with the existing one in the sense that the resulting algorithm can converge even for a small perturbation parameter with small CPU time.

Remark 1 *From this numerical example and the reference [9], it can be seen that the existing algorithm does not converge to an exact solution [1]. It has been shown that such combined algorithm can be solved for small perturbation parameters [9]. However, it is clear that the CPU time will be increased because for each iteration the recursive computation is needed and the step size related to corresponding to the cost performance has to be updated.*

Finally, for $\varepsilon = 1.0e-03$, the exact strategies F_i , $i = 1, 2$ and the solutions of the CALRE (7) are given below.

$$\begin{aligned}
F_1 &= [1.1674e-02 \quad 3.8745e-01], \\
F_2 &= [-1.0268e-01 \quad 1.8824e-01], \\
M_{1\varepsilon} &= \begin{bmatrix} 4.0472e-01 & 1.1674e-05 \\ 1.1674e-05 & 3.8745e-04 \end{bmatrix}, \\
M_{2\varepsilon} &= \begin{bmatrix} 3.6603e-01 & 5.8747e-06 \\ 5.8747e-06 & 3.7620e-04 \end{bmatrix}, \\
N_1 &= \begin{bmatrix} -6.5462e-04 & 0 \\ 0 & 2.2087e-04 \end{bmatrix}, \\
N_2 &= \begin{bmatrix} 8.6704e-01 & -5.0000e-01 \\ -5.0000e-01 & 2.8876e-01 \end{bmatrix}.
\end{aligned}$$

It is worth pointing out that the proposed strategies F_i can be computed for the small parameter ε .

6 Conclusion

The numerical algorithm for solving the linear closed-loop Stackelberg strategies of the SPS has been investigated. First, the uniqueness and boundedness of the

Table 3. CPU Time [sec]

ϵ	Newton's Method	Existing algorithm[1]
0.5	$4.0700e - 02$	$4.2100e - 02$
$1.0e - 01$	$4.0700e - 02$	$6.7100e - 02$
$1.0e - 02$	$1.7190e - 01$	-
$1.0e - 03$	$1.4380e - 01$	-

solution to the GCALE and its asymptotic structure have been studied. Second, the new numerical algorithm for solving the GCALE has been developed. As a result, the local quadratic convergence of the proposed algorithm that is different from the existing algorithm [1, 9] have been proven. Moreover, it is worth pointing out that since there is no need to determine the step size for each iteration, the iterative computation can be done directly. The simulation result has shown that the proposed algorithm succeeds in reducing the CPU time.

Although the algorithm is derived from the Newton's method, the uniqueness, the boundedness and the asymptotic structure of the solution for the GCALE have been proven for the first time by using the Newton-Kantorovich theorem [12]. It is worth pointing out that the proposed algorithm is the first algorithm to solve the GCALE in the sense that its convergence property has been proven rigorously.

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