Soft-Constrained Stochastic Nash Games for Weakly Coupled Large-Scale Systems ⋆

Hiroaki Mukaidani [a]

[a] *Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima, 739-8524 JAPAN.*

Abstract

In this paper, we discuss infinite-horizon soft-constrained stochastic Nash games involving state-dependent noise in weakly coupled large-scale systems. First, we formulate linear quadratic differential games in which robustness is attained against model uncertainty. It is noteworthy that this is the first time conditions for the existence of robust equilibria have been derived based on the solutions of sets of cross-coupled stochastic algebraic Riccati equations (CSAREs). After establishing an asymptotic structure along with positive definiteness for CSAREs solutions, we derive the formula for Newton's method for solving CSAREs. As another important feature, we propose a high-order approximate Nash strategy based on iterative solutions. Finally, we provide a numerical example to verify the efficiency of the proposed algorithms.

Key words: weakly coupled large-scale systems; soft-constrained stochastic Nash games; computational algorithm.

1 Introduction

Over the last decade, stochastic control problems governed by Itô's differential equation have attracted considerable research interest. Recently, the stochastic H_∞ control problem with state- and control-dependent noise was investigated (Ugrinovskii, 1998; Hinrichsen and Pritchard, 1998). It has attracted considerable attention and has been widely applied in various fields. In particular, the stochastic H_2/H_∞ control with state-dependent noise has been addressed (Chen and Zhang, 2004).

Linear quadratic Nash games and their applications have been widely investigated in many literatures (e.g. Petrovic and Gajić, 1988; Mukaidani, 2006,2007a,2007b for weakly coupled large-scale systems). Nash game as a concept has its roots in decision making and has been applied to various control fields (Basar and Olsder, 1999; Engwerda, 2005). However, robust Nash equilibrium in deterministic uncertain systems has not been investigated thus far. In contrast, robust equilibria in indefinite linear quadratic differential games under a deterministic disturbance input affecting systems have been discussed (Broek et al., 2003; Engwerda, 2005,2006). Although the results in (Broek et al., 2003; Engwerda, 2005,2006) are very elegant in theory and despite it being easy to obtain a strategy pair by solving the cross-coupled algebraic Riccati equations, stochastic uncertainty is an issue that remains to be considered.

On the other hand, stochastic Nash games have been widely studied (Buckdahn et al., 2004; Huang et al., 2006). Although many results are available on stochastic Nash games, they are limited to stochastic uncertainty, such as the Wiener process without deterministic disturbance. For practical applications, both deterministic and stochastic uncertainties should be considered. This can be convincing motivation to investigate soft-constrained stochastic Nash games with both state-dependent stochastic noise and unknown deterministic disturbance.

In this paper, we discuss theoretical and numerical aspects by extending the results of (Broek et al., 2003; Engwerda, 2005,2006) in the deterministic case to the soft-constrained stochastic Nash games governed by Itô's differential equations with state-dependent noise. It is noteworthy that earlier studies on weakly coupled stochastic Nash games (Srikant and Basar, 1992) did not take the state-dependent noise into consideration. Further, even the deterministic disturbance input was not considered. On the other hand, although the stochastic H_2/H_∞ control was considered, stochastic noise and unknown deterministic disturbance (Chen and Zhang,

⋆ This work was supported in part by the Research Foundation for the Electrotechnology of Chubu (REFEC) and a Grant-in-Aid for Young Scientists Research (B)-18700013 from the Ministry of Education, Culture, Sports, Science and Technology of Japan. This paper was not presented at any IFAC meeting. Corresponding author H. Mukaidani Tel. +81-82-424-7155. Fax +81-82-424-7155.

Email address: mukaida@hiroshima-u.ac.jp (Hiroaki Mukaidani).

2004) involving multiple players were not addressed. The main contributions of this paper are as follows. First, linear quadratic differential games are investigated with respect to an infinite horizon. After formulating the soft-constrained problem for the one-player case, a set of sufficient conditions is given as the saddle-point solution. Moreover, in order to guarantee the existence of strategy sets, sets of cross-coupled stochastic algebraic Riccati equations (CSAREs) are introduced for the first time. Second, for solving CSAREs, Newton's method is directly applied to find their solution. Another important feature is that a new high-order approximation strategy based on the numerical solution of CSAREs is established. Finally, in order to demonstrate the efficiency of the proposed algorithm, a numerical example is provided for practical megawatt-frequency control problems.

Notations: The notations used in this paper are fairly standard. δ_{ij} denotes the Kronecker delta. I_n denotes an $n \times n$ identity matrix. **block diag** denotes a block diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a matrix. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix such that $\|M\|_F^2 := \mathbf{Tr}[M^T M]$. E denotes the expectation. $\otimes$ denotes the Kronecker product. vec$M$ denotes the column vector of matrix M. The space of the $\mathbf{R}^k$-valued functions that are quadratically integrable on $(0, \infty)$ are denoted by $L_2^k(0, \infty)$. Finally, throughout this paper we have used the notation $\|x(t)\|_R^2$ instead of $x^T(t)Rx(t)$ for a real positive semidefinite symmetric matrix R and vector $x(t)$.

2 Soft-constrained Stochastic Nash Games

Consider stochastic linear time-invariant weakly coupled large-scale systems [1].

$$dx(t) = \left[A_\varepsilon x(t) + \sum_{j=1}^{N} B_{j\varepsilon} u_j(t) + E_\varepsilon v(t)\right] dt + \sum_{p=1}^{M} A_{p\varepsilon} x(t) dw_p(t),\ x(0) = x^0, \tag{1}$$

where

$$x(t) := \left[x_1^T(t) \cdots x_N^T(t)\right]^T,\ v(t) := \left[v_1^T(t) \cdots v_N^T(t)\right]^T,$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$A_{p\varepsilon} := \begin{bmatrix} A_{p11} & \varepsilon A_{p12} & \cdots & \varepsilon A_{p1N} \\ \varepsilon A_{p21} & A_{p22} & \cdots & \varepsilon A_{p2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{pN1} & \varepsilon A_{pN2} & \cdots & A_{pNN} \end{bmatrix},$$

$$E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix},\ B_{j\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1j}} B_{1j} \\ \varepsilon^{1-\delta_{2j}} B_{2j} \\ \vdots \\ \varepsilon^{1-\delta_{Nj}} B_{Nj} \end{bmatrix}.$$

$x_i(t) \in \mathbf{R}^{n_i},\ i = 1, \ldots, N$ represent the i-th state vectors. $u_j(t) \in \mathbf{R}^{m_j},\ j = 1, \ldots, N$ represent the j-th control inputs. $v_i(t) \in \mathbf{R}^{l_i},\ i = 1, \ldots, N$ represent the i-th disturbance. $w_p(t) \in \mathbf{R},\ p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space (Ugrinovskii, 1998; Hinrichsen and Pritchard, 1998; Chen and Zhang, 2004; Rami and Zhou, 2000). Moreover, $v_i(t) \in L_2^{l_i}(0, \infty)$ is considered to be an unknown finite-energy deterministic disturbance (Ugrinovskii, 1998; Chen and Zhang, 2004). Without loss of generality, it is assumed that $w_r(t)$ and $w_s(t)$ are mutually independent for all $r,\ s = 1, \ldots, M$ and $E[w(t)w^T(t)] = I_M$, where $w(t) := \left[w_1(t) \cdots w_M(t)\right]^T$. Here, ε denotes a relatively small positive coupling parameter that relates the linear system with the other subsystems [2]. It should be noted that the considered linear large-scale stochastic systems (1) cannot be treated by using the existing technique used in (Broek et al., 2003; Mukaidani 2006; Mukaidani 2007; Sagara et al, 2008) because the stochastic and deterministic uncertainties are both included.

The cost function for each strategy subset is defined by

$$J_i(u_1, \ldots, u_N, v, x(0)) = E \int_0^\infty \Big[\|x(t)\|_{Q_{i\varepsilon}}^2 + \|u_i(t)\|_{R_{ii}}^2 + \mu \sum_{j=1,\ j\neq i}^{N} \|u_j(t)\|_{R_{ij}}^2 - \|v(t)\|_{V_{i\mu}}^2\Big] dt, \tag{2}$$

where $i = 1, \ldots, N$,

$$Q_{i\varepsilon} = Q_{i\varepsilon}^T = \begin{bmatrix} \varepsilon^{1-\delta_{i1}} Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}} Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} Q_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n} \times \bar{n}},$$

$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i \times m_i},\ R_{ij} = R_{ij}^T \geq 0 \in \mathbf{R}^{m_j \times m_j},$
$V_{i\mu} = \mathbf{block\ diag}\left(\mu^{-(1-\delta_{i1})} V_{i1} \quad \mu^{-(1-\delta_{i2})} V_{i2}\right.$

[1] It should be noted that the generalized derivative of the Wiener process is called Gaussian white noise. Hence, we can also consider the problem by introducing the same equation that has been studied in (Gajić and Losada, 1999).

[2] In general, ε is an arbitrary sign for weakly coupled systems. In this paper, since the sign of the coefficient matrix can be changed without loss of generality, it is assumed that ε has a positive sign.

$$\cdots\mu^{-(1-\delta_{iN})}V_{iN}\big)>0,\ \bar{n}:=\sum_{i=1}^{N}n_i,\ \bar{m}:=\sum_{i=1}^{N}m_i.$$

The weighting matrix $V_{i\mu}$ is always symmetric and positive definite for all $i=1,\ \dots,N$. On the other hand, the state weight matrices $Q_{i\varepsilon}$ are symmetric and assumed to be sign-indefinite (Broek et al., 2003; Engwerda, 2005,2006). In many applications of differential games, state changes that are beneficial to some players may be harmful to other players; thus we allow for the state weighting matrices $Q_{i\varepsilon}$ to be indefinite. Although this allowance causes considerable technical complications, this generality would be natural in the multiplayer context (Engwerda, 2005).

In this paper, $R_{ij}\neq 0$ is allowed, provided the parameter μ in the cost function is scaled. However, in order to simplify the algebra, it is assumed that μ denotes a small positive parameter, which is of the same order as the small parameter ε.

The following stabilizability, which is an essential assumption, has been introduced in (Chen and Zhang, 2004).

Definition 1 (Chen and Zhang, 2004) *A stochastically controlled system governed by Itô's equation $dx=(Fx+Gu)dt+G_1xdw_1$, $x(0)=x_0$ is considered stabilizable in the mean-square sense if there exists a feedback law $u=Kx$ such that for any x_0, the closed-loop system $dx=(F+GK)xdt+G_1xdw_1$, $x(0)=x_0$ is asymptotically mean-square stable, i.e., $\lim_{t\to\infty}Ex^T(t)x(t)=0$, where K is a constant matrix.*

It is noteworthy that in this study, the strategies u_i^* are restricted as linear feedback strategies (Basar, 1974) such as $u_i:=F_{i\varepsilon}x$. We consider the formulation of the objective functions of the players in order to express a desire for robustness.

$$\begin{aligned}&\bar{J}_i(u_1,\ \dots,u_N,\ x(0))\\&\quad:=\sup_{v\in L_2^{\bar{l}}(0,\ \infty)}J_i(F_{1\varepsilon}x,\ \dots,F_{N\varepsilon}x,\ v,\ x(0)),\end{aligned}\tag{3}$$

where

$$\begin{aligned}&J_i(F_{1\varepsilon}x,\ \dots,F_{N\varepsilon}x,\ v,\ x(0))\\&\quad=E\int_0^\infty(\|x(t)\|^2_{\boldsymbol{T}_1}-\|v(t)\|^2_{V_{i\mu}})dt,\end{aligned}$$

$$\boldsymbol{T}_1:=Q_{i\varepsilon}+F_{i\varepsilon}^TR_{ii}F_{i\varepsilon}+\mu\sum_{j=1,\ j\neq i}^{N}F_{j\varepsilon}^TR_{ij}F_{j\varepsilon},\ \bar{l}:=\sum_{i=1}^{N}l_i.$$

Since $V_{i\mu}$ has a minus sign in J_i, this matrix constrains the disturbance vector v in an indirect way; therefore, it can be used to describe the aversion to the model risk of player i (Broek et al., 2003).

Let $\mathcal{F}_N$ denote the set of all $(F_{1\varepsilon},\ \dots,F_{N\varepsilon})$ such that the following closed-loop stochastic system

$$dx(t)=\left[A_\varepsilon+\sum_{j=1}^{N}B_{j\varepsilon}F_{j\varepsilon}\right]x(t)dt+\sum_{p=1}^{M}A_{p\varepsilon}x(t)dw_p(t)\tag{4}$$

is asymptotically mean-square stable.

According to the feedback information structure, a set of equilibrium strategies should be independent of the initial state. Furthermore, the strategies should satisfy the usual equilibrium inequalities. A formal definition is given below.

Definition 2 (Broek et al., 2003; Engwerda, 2005,2006) *The strategy set $(u_1^*,\ \dots,u_N^*)$, $u_i^*(t):=F_{i\varepsilon}^*x(t)$ is a soft-constrained stochastic Nash equilibrium strategy set if for each $i=1,\ \dots,N$, the following inequality holds:*

$$\begin{aligned}&\bar{J}_i(u_1^*,\ \dots,u_N^*,\ x(0))\\&\quad\le\bar{J}_i(u_1^*,\ \dots,u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,u_N^*,\ x(0)),\end{aligned}\tag{5}$$

for all $x(0)$ and for all $(F_{1\varepsilon},\ \dots,F_{N\varepsilon})$ that satisfy $(F_{1\varepsilon},\ \dots,F_{N\varepsilon})\in\mathcal{F}_N$.

In the next section, we discuss the one-player case.

2.1 One-player Case

First, a one-player case is discussed. The result obtained for that particular case is used as the basis for the derivation of results for the general N-player case.

Consider a linear time-invariant stochastic stabilizable system

$$\begin{aligned}dx(t)=&[A_\varepsilon x(t)+B_{1\varepsilon}u_1(t)+E_\varepsilon v(t)]dt\\&+\sum_{p=1}^{M}A_{p\varepsilon}x(t)dw_p(t),\ x(0)=x^0,\end{aligned}\tag{6}$$

where $u_1(t,\ x):=F_{1\varepsilon}x(t)$, $F_{1\varepsilon}\in\mathcal{F}_1$.

It should be noted that strategy spaces involve a linear feedback strategy with a memory-less perfect-state information structure (Basar, 1974). The cost function is given below.

$$\begin{aligned}&J(u_1,\ v,\ x(0))\\&\quad=E\int_0^\infty(\|x(t)\|^2_{Q_{1\varepsilon}}+\|u(t)\|^2_{R_{11}}-\|v(t)\|^2_{V_{1\mu}})dt.\end{aligned}\tag{7}$$

Let us define the strategy spaces $\Gamma_u:=\{u_1(t,\ x):=F_{1\varepsilon}x(t)\mid F_{1\varepsilon}\in\mathcal{F}_1\}$ and $\Gamma_v:=\{v(t)\mid v(t)\in L^l(0,\ \infty)\}$.

Definition 3 *A strategy pair $(u_1^*,\ v^*)\in\Gamma_u\times\Gamma_v$ is in saddle-point equilibrium if*

$$J(u_1^*,\ v,\ x(0))\le J(u_1^*,\ v^*,\ x(0))\le J(u_1,\ v^*,\ x(0))\tag{8}$$

for all $(u_1^*,\ v)\in\Gamma_u\times\Gamma_v$ and $(u_1,\ v^*)\in\Gamma_u\times\Gamma_v$.

The following theorem generalizes the existing results of (Broek et al., 2003; Engwerda, 2005,2006); these results are very important in deterministic soft-constrained Nash games for a stochastic case.

Theorem 4 *Assume that for all $(u_1,\ v) \in \Gamma_u \times \Gamma_v$, the closed-loop system is asymptotically mean-square stable. Suppose the following stochastic algebraic Riccati equation (SARE) (9) has a solution $P_\varepsilon \geq 0$.*

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + \sum_{p=1}^{M} A_{p\varepsilon}^T P_\varepsilon A_{p\varepsilon} \\ -P_\varepsilon (S_{1\varepsilon} - M_{1\varepsilon}) P_\varepsilon + Q_{1\varepsilon} = 0, \quad (9)$$

where $S_{1\varepsilon} := B_{1\varepsilon} R_{11}^{-1} B_{1\varepsilon}^T$, $M_{1\varepsilon} := E_\varepsilon V_{1\mu}^{-1} E_\varepsilon^T$.

Furthermore, suppose there exists a real symmetric matrix W_ε that satisfies matrix inequality (10).

$$W_\varepsilon A_\varepsilon + A_\varepsilon^T W_\varepsilon + \sum_{p=1}^{M} A_{p\varepsilon}^T W_\varepsilon A_{p\varepsilon} - W_\varepsilon S_{1\varepsilon} W_\varepsilon + Q_{1\varepsilon} \geq 0. \quad (10)$$

The strategy pair

$$u_1^*(t,\ x) = F_{1\varepsilon}^* x(t) = -R_{11}^{-1} B_{1\varepsilon}^T P_\varepsilon x(t), \quad (11a)$$
$$v^*(t) = V_{1\mu}^{-1} E_\varepsilon^T P_\varepsilon \tilde{x}(t), \quad (11b)$$
$$d\tilde{x}(t) = [A_\varepsilon - (S_{1\varepsilon} - M_{1\varepsilon}) P_\varepsilon] \tilde{x}(t) dt \\ + \sum_{p=1}^{M} A_{p\varepsilon} \tilde{x}(t) dw_p(t),\ \tilde{x}(0) = x^0 \quad (11c)$$

is in saddle-point equilibrium. That is, if these conditions hold then inequality (8) related to cost function $J(u_1,\ v,\ x(0))$ is satisfied. Moreover, $J(u_1^*,\ v^*,\ x(0)) = x^T(0) P_\varepsilon x(0)$.

The proof of Theorem 4 is given in Appendix I.

Note that if $Q_{1\varepsilon} \geq 0$, condition (10) is trivially satisfied by choosing $W_\varepsilon = 0$. Otherwise, it is sufficient to have a symmetric solution W_ε of inequality (10). As a special case, $W_\varepsilon = -P_\varepsilon$ may be considered from a natural point of view.

The concept in inequality (10) is very important to generalize the result for a non-definite $Q_{1\varepsilon}$ matrix. Thus, we first introduced an arbitrary matrix W_ε. It should be noted that this condition is considerably less stringent than the condition $Q_{1\varepsilon} \geq 0$.

2.2 Soft-constrained Stochastic Nash Equilibrium

The soft-constrained stochastic Nash games are given below.

Theorem 5 *Assume that for all $u_i(t) \in \Gamma_u$, $i = 1,\ \ldots, N$ and $v(t) \in \Gamma_v$, the closed-loop system is asymptotically mean-square stable. Suppose that N real symmetric matrices $P_{i\varepsilon} \geq 0$ and N real symmetric matrices $W_{i\varepsilon}$ exist such that*

$$\boldsymbol{F}_i(\varepsilon,\ \mu,\ P_{1\varepsilon},\ \ldots, P_{N\varepsilon}) \\ = P_{i\varepsilon} \boldsymbol{A} + \boldsymbol{A}^T P_{i\varepsilon} + \sum_{p=1}^{M} A_{p\varepsilon}^T P_{i\varepsilon} A_{p\varepsilon} - P_{i\varepsilon} S_{i\varepsilon} P_{i\varepsilon} \\ + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon} + P_{i\varepsilon} M_{i\varepsilon} P_{i\varepsilon} + Q_{i\varepsilon} = 0, \quad (12a)$$

$$W_{i\varepsilon} \boldsymbol{A} + \boldsymbol{A}^T W_{i\varepsilon} + \sum_{p=1}^{M} A_{p\varepsilon}^T W_{i\varepsilon} A_{p\varepsilon} - W_{i\varepsilon} S_{i\varepsilon} W_{i\varepsilon} \\ + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon} S_{ij\varepsilon} P_{j\varepsilon} + Q_{i\varepsilon} \geq 0, \quad (12b)$$

where $i = 1,\ \ldots, N$, $\boldsymbol{A} := A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon} P_{j\varepsilon}$, $S_{i\varepsilon} := B_{i\varepsilon} R_{ii}^{-1} B_{i\varepsilon}^T$, $S_{ij\varepsilon} := B_{j\varepsilon} R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{j\varepsilon}^T$, $i \neq j$, $M_{i\varepsilon} := E_\varepsilon V_{i\mu}^{-1} E_\varepsilon^T$.

Define the set $(F_{1\varepsilon}^*,\ \ldots, F_{N\varepsilon}^*)$ by

$$u_i^*(t) := F_{i\varepsilon}^* x(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} x(t),\ i = 1,\ ..,N. \quad (13)$$

Then, $(F_{1\varepsilon}^*,\ \ldots, F_{N\varepsilon}^*) \in \mathcal{F}_N$, and this strategy set denotes the soft-constrained stochastic Nash equilibrium. Furthermore, $\bar{J}_i(F_{1\varepsilon}^* x,\ \ldots, F_{N\varepsilon}^* x,\ x(0)) = x^T(0) P_{i\varepsilon} x(0)$.

Proof: Now, let us consider the following problem in which the cost function (14) is minimal at $F_{i\varepsilon} = F_{i\varepsilon}^*$.

$$\phi(F_\varepsilon) := \sup_{v \in L_2^{\bar{l}}(0,\ \infty)} E \int_0^\infty (\|x(t)\|_{\boldsymbol{T}_2}^2 - \|v(t)\|_{V_{i\mu}}^2) dt, \quad (14)$$

where $\boldsymbol{T}_2 := Q_{i\varepsilon} + F_{i\varepsilon}^T R_{ii} F_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^T S_{ij} P_{j\varepsilon}$ and $x(t)$ follows from

$$dx(t) = \left[\left(A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon} P_{j\varepsilon} + B_{i\varepsilon} F_{i\varepsilon} \right) x(t) \right. \\ \left. + E_\varepsilon v(t) \right] dt + \sum_{p=1}^{M} A_{p\varepsilon} x(t) dw_p(t),\ x(0) = x^0. \quad (15)$$

Note that the function ϕ coincides with function J in Theorem 4. Applying Theorem 4 to this minimization problem as $P_{i\varepsilon} \Rightarrow P_\varepsilon$, $A_\varepsilon - \sum_{j=1,\ j\neq i}^{N} S_{j\varepsilon} P_{j\varepsilon} \Rightarrow A_\varepsilon$, $B_{i\varepsilon} \Rightarrow B_{1\varepsilon}$, $Q_{i\varepsilon} + \mu \sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^T S_{ij} P_{j\varepsilon} \Rightarrow Q_{1\varepsilon}$ and $R_{ii} \Rightarrow R_{11}$, $V_{i\mu} \Rightarrow V_{1\mu}$ yields the fact that the function ϕ is minimal at

$$F_{1\varepsilon}^* = -R_{11}^{-1} B_{1\varepsilon}^T P_\varepsilon \Rightarrow F_{i\varepsilon}^* = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon}. \quad (16)$$

Moreover, the minimal value is $x^T(0) P_{i\varepsilon} x(0)$. ■

It should be noted that if $Q_{i\varepsilon} \geq 0$ and $S_{ij\varepsilon} \geq 0$ for all $i = 1,\ \ldots, N$, matrix inequality (12b) is trivially satisfied by $W_{i\varepsilon} = 0$ (Broek et al., 2003; Engwerda, 2005,2006). In the case that one of these matrices is not positive semi-definite, one might consider $W_{i\varepsilon} = -P_{i\varepsilon}$ as a solution to (12b).

3 Asymptotic Structure of CSAREs

First, the asymptotic structure of CSAREs (12a) is established. Since A_ε, $A_{p\varepsilon}$, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $Q_{i\varepsilon}$ and $M_{i\varepsilon}$ include the parameter ε, the solution $P_{i\varepsilon}$ of CSAREs (12a), if it exists, should contain the parameter ε. By considering this fact, the solution $P_{i\varepsilon}$ of CSAREs (12a) is assumed to have the following structure.

$$P_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}} P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} P_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}}. \quad (17)$$

It should be noted that this assumption originated from the simulation results. In fact, when CSAREs (12a) are solved, the structure (17) can be predicted easily for various small ε.

The following reduced-order stochastic algebraic Riccati equation (SARE) is obtained by substituting the matrices A_ε, $A_{p\varepsilon}$, $S_{i\varepsilon}$, $S_{ij\varepsilon}$, $Q_{i\varepsilon}$, $M_{i\varepsilon}$ and $P_{i\varepsilon}$ into (12a); letting $\varepsilon = \mu = 0$; and partitioning CSAREs (12a). Here $\bar{P}_{ii},\ i = 1,\ \dots,N$ is the 0-order solution [3] of CSAREs (12a) as $\varepsilon = \mu = 0$.

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} + \sum_{p=1}^{M} A_{pii}^T\bar{P}_{ii}A_{pii} - \bar{P}_{ii}S_{ii}\bar{P}_{ii} + \bar{P}_{ii}M_{ii}\bar{P}_{ii} + Q_{ii} = 0, \quad (18)$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$ and $M_{ii} := E_{ii}V_{ii}^{-1}E_{ii}^T$.

It may be noted that there is a unique positive semi-definite solution of (18) (Dragan and Morozan, 1997).

The following condition is assumed.

Assumption 6 *The following matrices are nonsingular.*

$$\boldsymbol{D}_{ii} := \bar{\boldsymbol{\Theta}}^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes \bar{\boldsymbol{\Theta}}^T + \sum_{p=1}^{M} \bar{A}_p^T \otimes \bar{A}_p^T, \quad (19)$$

where $i = 1,\ \dots,N$, $\bar{\boldsymbol{\Theta}} := \bar{A} - \sum_{j=1}^{N} \bar{S}_j\bar{P}_j + \bar{M}_i\bar{P}_i$,

$$\begin{aligned} \bar{A} &:= \textbf{block diag}\left(A_{11}\ \cdots\ A_{NN}\right), \\ \bar{S}_j &:= \textbf{block diag}\left(0\ \cdots\ 0\ S_{jj}\ 0\ \cdots\ 0\right), \\ \bar{M}_i &:= \textbf{block diag}\left(0\ \cdots\ 0\ M_{ii}\ 0\ \cdots\ 0\right), \\ \bar{P}_i &:= \textbf{block diag}\left(0\ \cdots\ 0\ \bar{P}_{ii}\ 0\ \cdots\ 0\right), \\ \bar{A}_p &:= \textbf{block diag}\left(A_{p11}\ \cdots\ A_{pNN}\right). \end{aligned}$$

[3] The first order approximations $P_{i\varepsilon}$ corresponding to ε are called 0-order solutions.

The asymptotic expansion of CSAREs (12a) for $\varepsilon = \mu = 0$ is described by the following theorem.

Theorem 7 *Assume that the solutions $P_{i\varepsilon}$ of CSAREs* (12a) *have the structure* (17). *Suppose that SARE* (18) *has a positive definite solution. Under Assumption 6, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, CSAREs* (12a) *allows for a positive definite solution $P_{i\varepsilon}^*$, which can be written as*

$$P_{i\varepsilon} := P_{i\varepsilon}^* = \bar{P}_i + O(\varepsilon). \quad (20)$$

Proof: This can be proved by applying the implicit function theorem (Jittorntrum, 1978) on CSAREs (12a). In order to do this, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The derivative of the function $\boldsymbol{F}_i(\varepsilon,\ \mu,\ P_{1\varepsilon},\ \dots,P_{N\varepsilon})$ at matrix $P_{i\varepsilon}$ is given by

$$\begin{aligned} \boldsymbol{J}_{ii} &:= \frac{\partial}{\partial \mathrm{vec} P_{i\varepsilon}} \mathrm{vec}\boldsymbol{F}_i(\varepsilon,\ \mu,\ P_{1\varepsilon},\ \dots,P_{N\varepsilon})^T \\ &= \boldsymbol{\Theta}^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes \boldsymbol{\Theta} + \sum_{p=1}^{M} A_{p\varepsilon}^T \otimes A_{p\varepsilon}^T, \end{aligned} \quad (21a)$$

$$\begin{aligned} \boldsymbol{J}_{ij} &:= \frac{\partial}{\partial \mathrm{vec} P_{j\varepsilon}} \mathrm{vec}\boldsymbol{F}_i(\varepsilon,\ \mu,\ P_{1\varepsilon},\ \dots,P_{N\varepsilon})^T \\ &= -(S_{j\varepsilon}P_{i\varepsilon} - \mu S_{ij\varepsilon}P_{j\varepsilon})^T \otimes I_{\bar{n}} \\ &\quad - I_{\bar{n}} \otimes (S_{j\varepsilon}P_{i\varepsilon} - \mu S_{ij\varepsilon}P_{j\varepsilon})^T, \end{aligned} \quad (21b)$$

where $i \neq j,\ j = 1,\ \dots,N$ and $\boldsymbol{\Theta} := A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon} + M_{i\varepsilon}P_{i\varepsilon}$

Based on the fact that $S_{j\varepsilon}P_{i\varepsilon} = O(\varepsilon),\ i \neq j$, after performing some algebraic calculations, the Jacobian of CSAREs (12a) in the limit $\varepsilon \to +0$, $\mu \to +0$ can be expressed as

$$\begin{aligned} \boldsymbol{J} &= \begin{bmatrix} \boldsymbol{J}_{11}|_{\varepsilon=\mu=0} & \cdots & \boldsymbol{J}_{1N}|_{\varepsilon=\mu=0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{J}_{N1}|_{\varepsilon=\mu=0} & \cdots & \boldsymbol{J}_{NN}|_{\varepsilon=\mu=0} \end{bmatrix} \\ &= \textbf{block diag}\left(\boldsymbol{D}_{11}\ \cdots\ \boldsymbol{D}_{NN}\right). \end{aligned} \quad (22)$$

Obviously, $\boldsymbol{D}_{ii},\ i = 1,\ \dots,N$ are nonsingular under Assumption 6. Thus, $\det\boldsymbol{J} \neq 0$, i.e., $\boldsymbol{J}$ is nonsingular for $\varepsilon = \mu = 0$. As a consequence of the implicit function theorem, this implies that there exists a unique continuous mapping $P_{i\varepsilon} := \boldsymbol{H}_i(\varepsilon)$ that possesses the Taylor series expansion at $\varepsilon = 0$ (Kokotović et al., 1999); in other words $P_{i\varepsilon} := \boldsymbol{H}_i(0) + \sum_{l=1}^{\infty} \frac{\varepsilon^l}{l!}\boldsymbol{H}_i^{(l)}(0)$. Thus we have an equation with the form of (20). On the other hand, taking into account the fact that $\bar{P}_{ii}$ is a positive definite matrix, for sufficiently small parameters ε and μ, $P_{i\varepsilon}$ is also a positive definite solution. ■

It is noteworthy that the asymptotic structure of (20) can also be obtained by applying a result that will be given later as $k = 0$ in Theorem 8.

4 Newton's Method

In order to obtain the solution of CSAREs (12a), Newton's method can be applied.

$$
\begin{aligned}
&P_{i\varepsilon}^{(k+1)}A_{\varepsilon}^{(k)}+A_{\varepsilon}^{(k)T}P_{i\varepsilon}^{(k+1)}+\sum_{p=1}^{M}A_{p\varepsilon}P_{i\varepsilon}^{(k+1)}A_{p\varepsilon}\\
&\quad-\sum_{j=1,\ j\neq i}^{N}(P_{j\varepsilon}^{(k+1)}\Xi_{\varepsilon}^{(k)}+\Xi_{\varepsilon}^{(k)T}P_{j\varepsilon}^{(k+1)})\\
&\quad+\sum_{j=1,\ j\neq i}^{N}(P_{i\varepsilon}^{(k)}S_{j\varepsilon}P_{j\varepsilon}^{(k)}+P_{j\varepsilon}^{(k)}S_{j\varepsilon}P_{i\varepsilon}^{(k)})+P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)}\\
&\quad-\mu\sum_{j=1,\ j\neq i}^{N}P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)}-P_{i\varepsilon}^{(k)}M_{i\varepsilon}P_{i\varepsilon}^{(k)}\\
&\quad+Q_{i\varepsilon}=0,\ i=1,\ \dots,N,\ k=0,\ 1,\ \dots, \qquad (23)
\end{aligned}
$$

where $A_{\varepsilon}^{(k)}:=A_{\varepsilon}-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}^{(k)}+M_{i\varepsilon}P_{i\varepsilon}^{(k)}$ and $\Xi_{\varepsilon}^{(k)}:=S_{j\varepsilon}P_{i\varepsilon}^{(k)}-\mu S_{ij\varepsilon}P_{j\varepsilon}^{(k)}$ with the initial conditions

$$P_{i\varepsilon}^{(0)}=\bar{P}_i. \qquad (24)$$

It is easy to show that equation (23) is equivalent to the Newton's method. In fact, equation (23) can be revised as follows.

$$
\begin{aligned}
&\boldsymbol{P}^{(k+1)}=\boldsymbol{P}^{(k)}-[\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ P_{1\varepsilon}^{(k)},\ \dots,P_{N\varepsilon}^{(k)})]^{-1}[\mathrm{vec}\boldsymbol{F}^{(k)}],\\
&\boldsymbol{F}:=\left[\ \boldsymbol{F}_1\ \cdots\ \boldsymbol{F}_N\ \right]^T,\\
&\mathrm{vec}\boldsymbol{F}^{(k)}:=\left[\ [\mathrm{vec}\boldsymbol{F}_1^{(k)}]^T\ \cdots\ [\mathrm{vec}\boldsymbol{F}_N^{(k)}]^T\ \right]^T,\\
&\boldsymbol{F}_i^{(k)}:=\boldsymbol{F}_i^{(k)}(\varepsilon,\ \mu,\ P_{1\varepsilon}^{(k)},\ \dots,P_{N\varepsilon}^{(k)}),\\
&\boldsymbol{P}^{(k)}:=([\mathrm{vec}P_{1\varepsilon}^{(k)}]^T,\ \dots,[\mathrm{vec}P_{N\varepsilon}^{(k)}]^T).
\end{aligned}
$$

In general, it is well known that there exist several soft-constrained Nash equilibriums such as those in ordinary Nash games.

The following theorem give a feature of the uniqueness.

Theorem 8 *Under the conditions of Theorem 7, there exists a small $\bar{\sigma}$ such that for all $\varepsilon\in(0,\ \bar{\sigma})$, Newton's method* (23) *converges to the exact solution of $P_{i\varepsilon}$ at the same rate as that of quadratic convergence; here, $P_{i\varepsilon}^{(k)}$ is positive definite. Moreover, the convergence solutions attain a unique solution $P_{i\varepsilon}^*$ of CSAREs* (12a) *in the neighborhood of the initial condition $P_{i\varepsilon}^{(0)}=\bar{P}_i$. In other words, the following conditions are satisfied.*

$$\|P_{i\varepsilon}^{(k)}-P_{i\varepsilon}^*\|=O(\varepsilon^{2^k}),\ i=1,\ \dots,N,\ k=0,\ 1,\ \dots. \qquad (25)$$

Proof: The proof is given directly by applying the Newton-Kantorovich theorem (Ortega, 1990; Yamamoto, 1986) for CSAREs (12a). A positive scalar γ can be immediately obtained from CSAREs (12a) such that for any $P_{i\varepsilon}^a$ and $P_{i\varepsilon}^b$

$$
\begin{aligned}
&\|\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ P_{1\varepsilon}^a,\ \dots,P_{N\varepsilon}^a)-\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ P_{1\varepsilon}^b,\ \dots,P_{N\varepsilon}^b)\|\\
&\quad\le\gamma\|([\mathrm{vec}P_{1\varepsilon}^a]^T,\ \dots,[\mathrm{vec}P_{N\varepsilon}^a]^T)\\
&\qquad-([\mathrm{vec}P_{1\varepsilon}^b]^T,\ \dots,[\mathrm{vec}P_{N\varepsilon}^b]^T)\|,
\end{aligned}
$$

where

$$\gamma:=2(2N-1)\sum_{j=1}^{N}\|S_{j\varepsilon}\|+2\mu\sum_{i=1}^{N}\sum_{j=1,\ j\neq i}^{N}\|S_{ij\varepsilon}\|.$$

Moreover, it is easy to verify that $\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ \bar{P}_1,\ \dots,\bar{P}_N)=\boldsymbol{J}+O(\varepsilon)$ is nonsingular because for the small parameters ε and μ and using (20), $\boldsymbol{J}$ is also nonsingular. Therefore, there exists β such that $\beta=\|[\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ \bar{P}_1,\ \dots,\bar{P}_N)]^{-1}\|$. On the other hand, since $\|\boldsymbol{F}(\varepsilon,\ \mu,\ \bar{P}_1,\ \dots,\bar{P}_N)\|=O(\varepsilon)$, there exists η such that $\eta=\|[\nabla\boldsymbol{F}(\varepsilon,\ \mu,\ \bar{P}_1,\ \dots,\bar{P}_N)]^{-1}\|\cdot\|\boldsymbol{F}(\varepsilon,\ \mu,\ \bar{P}_1,\ \dots,\bar{P}_N)\|=O(\varepsilon)$. Thus, h exists such that $h=\beta\eta\gamma<2^{-1}$ because $\eta=O(\varepsilon)$. Finally, the Newton-Kantorovich theorem yields the desired results (25).

Second, the local uniqueness of the solution is discussed. Let us define $\boldsymbol{R}\equiv[1+\sqrt{1-2h}]/(\gamma\beta)$. Clearly, $S\equiv\{\ P_{i\varepsilon}\ :\ \|P_{i\varepsilon}-P_{i\varepsilon}^{(0)}\|<\boldsymbol{R}\ \}$ is in the certain convex set D. Subsequently, since the solution $P_{i\varepsilon}^*$ is unique in S, the local uniqueness of $P_{i\varepsilon}^*$ is guaranteed in the neighbourhood of $\varepsilon=0$ for a subset S by applying the Newton-Kantorovich theorem. ■

It should be noted that for solving the cross-coupled stochastic algebraic Lyapunov equations (CSALEs) (23) that appear in Newton's method, fixed-point algorithm can also be combined. See (Mukaidani, 2006; Sagara et al., 2007) for details.

5 High-order Approximate Soft-constrained Stochastic Nash Strategies

We focus our attention on the design of high-order approximate soft-constrained stochastic Nash strategies. Such strategies are obtained by using iterative solution (23).

$$
\begin{aligned}
&u_i^{(k)*}(t)=-R_{ii}^{-1}B_{i\varepsilon}^TP_{i\varepsilon}^{(k)}x(t)=F_{i\varepsilon}^{(k)*}x(t),\\
&i=1,\ \dots,N. \qquad (26)
\end{aligned}
$$

Theorem 9 *Suppose that SARE* (18) *has a positive definite solution such that $D_{ii}:=A_{ii}-S_{ii}\bar{P}_{ii}$ is stable. Then, there exists a small constant $\tilde{\sigma}$ and positive scalar parameters $\alpha>0$ and $\beta>0$ such that for all $\varepsilon\in(0,\ \tilde{\sigma})$, $\|\exp[(A_{\varepsilon}+\sum_{j=1}^{N}B_{j\varepsilon}F_{j\varepsilon}^{(k)*})t]\|\le\alpha e^{-\beta t}$. Moreover, if $\alpha^2/\beta\sum_{p=1}^{M}\|A_{p\varepsilon}\|^2\le\omega<2/(M+1)$, stochastic system* (1) *with $u_i^{(k)*}(t)$ given by* (26) *and $v(t)=0$ is exponentially mean-square stable.*

Proof: First, it is easy to verify that

$$A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*} = \bar{A} - \sum_{j=1}^{N} \bar{S}_j \bar{P}_j + O(\varepsilon)$$
$$= \textbf{block diag} \begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix} + O(\varepsilon) \quad (27)$$

Hence, using the stability assumption of D_{ii}, it can be shown that there exists a small constant $\tilde{\sigma}$ and positive scalar parameters $\alpha > 0$ and $\beta > 0$ such that for all $\varepsilon \in (0,\ \tilde{\sigma})$, $\|\exp[(A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*})t]\| \le \alpha e^{-\beta t}$. Let us consider the closed-loop stochastic system (28).

$$dx(t) = \left[A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*}\right] x(t)dt + \sum_{p=1}^{M} A_{p\varepsilon} x(t) dw_p(t),\ x(s) = x^s. \quad (28)$$

The representation of the solution of equation (28) is given as

$$x(t) = \exp\left[\left[A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*}\right](t-s)\right] x(s) + \sum_{p=1}^{M} \int_s^t \exp\left[\left[A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*}\right](t-\tau)\right] \times A_{p\varepsilon} x(\tau) dw_p(\tau). \quad (29)$$

Using inequality $\left\|\sum_{l=1}^{L} \boldsymbol{a}_l\right\|^2 \le L \sum_{l=1}^{L} \|\boldsymbol{a}_l\|^2$ and considering the independence of the Wiener processes $w_p(t)$ yields

$$E\|x(t)\|^2 \le (M+1) \left\|\exp\left[\left[A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*}\right](t-s)\right]\right\|^2 \times E\|x(s)\|^2 + (M+1) \times \sum_{p=1}^{M} \int_s^t \left\|\exp\left[\left[A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*}\right](t-\tau)\right]\right\|^2 \times \|A_{p\varepsilon}\|^2 E\|x(\tau)\|^2 d\tau. \quad (30)$$

Thus, the conditions $\|\exp[(A_\varepsilon + \sum_{j=1}^{N} B_{j\varepsilon} F_{j\varepsilon}^{(k)*})t]\| \le \alpha e^{-\beta t}$, $\exists\ \alpha,\ \beta > 0$ and $\alpha^2/\beta \sum_{p=1}^{M} \|A_{p\varepsilon}\|^2 \le \omega$ imply that

$$e^{2\beta(t-s)} E\|x(t)\|^2 \le (M+1)\alpha^2 E\|x(s)\|^2 + (M+1)\beta\omega \int_s^t e^{2\beta(\tau-s)} E\|x(\tau)\|^2 d\tau. \quad (31)$$

From the Bellman-Gronwall inequality (Wu et al., 1994) it follows that

$$E\|x(t)\|^2 \le (M+1)\alpha^2 E\|x(s)\|^2 e^{\beta[(M+1)\omega - 2](t-s)}. \quad (32)$$

Since ω has been selected such that $\omega < 2/(M+1)$, equation (28) is exponentially mean-square stable. ■

The degradation of the cost functional via new high-order approximate soft-constrained stochastic Nash strategies (26) is given as follows.

Theorem 10 *Under Assumption 6, the application of high-order approximate soft-constrained stochastic Nash strategies (26) to stochastic systems (1) results in $\bar{J}_i$ satisfying the relation*

$$\bar{J}_i(F_{1\varepsilon}^{*(k)} x,\ \dots,\ F_{N\varepsilon}^{(k)*} x,\ x(0)) = \bar{J}_i(F_{1\varepsilon}^{*} x,\ \dots,\ F_{N\varepsilon}^{*} x,\ x(0)) + O(\varepsilon^{2^k+1}),$$
$$k = 0,\ 1, \dots,\ i = 1,\ \dots,\ N. \quad (33)$$

The following lemma will play an important role in establishing performance degradation.

Lemma 11 *Consider the stochastic system*

$$dx(t) = [A_\varepsilon x(t) + E_\varepsilon v(t)]dt + \sum_{p=1}^{M} A_{p\varepsilon} x(t) dw_p(t),\ x(0) = x^0 \quad (34)$$

and the corresponding cost function

$$\hat{J}(v,\ x(0)) = E \int_0^\infty (\|x(t)\|_{Q_\varepsilon}^2 - \|v(t)\|_{V_\mu}^2) dt. \quad (35)$$

Assume that stochastic system (34) is asymptotically mean-square stable. Suppose that the following SARE has a solution $L_\varepsilon \ge 0$.

$$L_\varepsilon A_\varepsilon + A_\varepsilon^T L_\varepsilon + \sum_{p=1}^{M} A_{p\varepsilon}^T L_\varepsilon A_{p\varepsilon} + L_\varepsilon M_\varepsilon L_\varepsilon + Q_\varepsilon = 0, \quad (36)$$

where $M_\varepsilon := E_\varepsilon V_\mu^{-1} E_\varepsilon^T$.

Then, the maximum of $\hat{J}(v,\ x(0))$ is uniquely attained by

$$\bar{v}(t) = V_\mu^{-1} E_\varepsilon^T L_\varepsilon \underline{x}(t), \quad (37a)$$
$$d\underline{x}(t) = [A_\varepsilon + M_\varepsilon L_\varepsilon]\underline{x}(t)dt + \sum_{p=1}^{M} A_{p\varepsilon} \underline{x}(t) dw_p(t),\ \underline{x}(0) = x^0 \quad (37b)$$

Moreover, $\hat{J}(\bar{v},\ x(0)) = x^T(0) L_\varepsilon x(0)$.

The proof of Lemma 11 is given in Appendix II.

Proof of Theorem 10: This can be proved directly by applying the Newton-Kantorovich theorem (Ortega, 1990; Yamamoto, 1986). When $u_i^{(k)*}(t)$ is used, the equilibrium values of the cost functional is given by (38) because of Lemma 11.

$$\bar{J}_i(F_{1\varepsilon}^{*(k)}x,\ \dots,F_{N\varepsilon}^{(k)*}x,\ x(0)) = x^T(0)Z_{i\varepsilon}x(0), \quad (38)$$

where $Z_{i\varepsilon}$ is a positive definite solution of the following SALEs

$$Z_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(k)}\right)^T Z_{i\varepsilon} + \sum_{p=1}^{M} A_{p\varepsilon}^T Z_{i\varepsilon}A_{p\varepsilon} + P_{i\varepsilon}^{(k)}S_{i\varepsilon}P_{i\varepsilon}^{(k)} + \mu\sum_{j=1,\ j\neq i}^{N} P_{j\varepsilon}^{(k)}S_{ij\varepsilon}P_{j\varepsilon}^{(k)} + Z_{i\varepsilon}M_{i\varepsilon}Z_{i\varepsilon} + Q_{i\varepsilon} = 0. \quad (39)$$

Subtracting (12a) from (39) and using the result of (25), $\boldsymbol{Z}_{i\varepsilon} = Z_{i\varepsilon} - P_{i\varepsilon}$ satisfies the following SALE

$$\boldsymbol{G}(\boldsymbol{Z}_{i\varepsilon}) := \boldsymbol{Z}_{i\varepsilon}\tilde{\boldsymbol{\Theta}}_\varepsilon + \tilde{\boldsymbol{\Theta}}_\varepsilon^T\boldsymbol{Z}_{i\varepsilon} + \sum_{p=1}^{M} A_{p\varepsilon}^T\boldsymbol{Z}_{i\varepsilon}A_{p\varepsilon} + \boldsymbol{Z}_{i\varepsilon}M_{i\varepsilon}\boldsymbol{Z}_{i\varepsilon} + O(\varepsilon^{2^k+1}) = 0, \quad (40)$$

where $\tilde{\boldsymbol{\Theta}}_\varepsilon := A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} + M_{i\varepsilon}P_{i\varepsilon}^{(k)} + O(\varepsilon^{2^k})$.

Since the function $\boldsymbol{G}(\boldsymbol{Z}_{i\varepsilon})$ is continuous at any $\boldsymbol{Z}_{i\varepsilon}$, taking the partial derivative of the function $\boldsymbol{G}(\boldsymbol{Z}_{i\varepsilon})$ with respect to $\boldsymbol{Z}_{i\varepsilon}$ yields

$$\nabla\boldsymbol{G}(\boldsymbol{Z}_{i\varepsilon}) := I_{\bar{n}} \otimes (\tilde{\boldsymbol{\Theta}}_\varepsilon + M_{i\varepsilon}\boldsymbol{Z}_{i\varepsilon})^T + (\tilde{\boldsymbol{\Theta}}_\varepsilon + M_{i\varepsilon}\boldsymbol{Z}_{i\varepsilon})^T \otimes I_{\bar{n}} + \sum_{p=1}^{M} A_{p\varepsilon}^T \otimes A_{p\varepsilon}^T. \quad (41)$$

Thus, by using Assumption 6, there exists a small constant $\hat{\sigma}$ such that for all $\varepsilon \in (0,\ \hat{\sigma})$, $\nabla\boldsymbol{G}(0) = I_{\bar{n}} \otimes \tilde{\boldsymbol{\Theta}}_\varepsilon^T + \tilde{\boldsymbol{\Theta}}_\varepsilon^T \otimes I_{\bar{n}} + \sum_{p=1}^M A_{p\varepsilon}^T \otimes A_{p\varepsilon}^T$ is nonsingular. Then, for any matrices $\boldsymbol{X}_\varepsilon$ and $\boldsymbol{Y}_\varepsilon$ that belong to $\boldsymbol{Z}_{i\varepsilon}$, it is immediately obtained from equation (41) that

$$\|\nabla\boldsymbol{G}(\boldsymbol{X}_\varepsilon) - \nabla\boldsymbol{G}(\boldsymbol{Y}_\varepsilon)\| \leq \bar{\gamma}\|\boldsymbol{X}_\varepsilon - \boldsymbol{Y}_\varepsilon\|, \quad (42)$$

where $\bar{\gamma} := 2\|M_{i\varepsilon}\|$.

Moreover, there exists $\bar{\eta}$ such that $\|[\nabla\boldsymbol{G}(0)]^{-1}\boldsymbol{G}(0)\| < O(\varepsilon^{2^k+1}) = \bar{\eta}$ because of $\boldsymbol{G}(0) = O(\varepsilon^{2^k+1})$. Using the Newton-Kantorovich theorem, the estimate is given by

$$\|\boldsymbol{Z}_{i\varepsilon} - 0\| = \|\boldsymbol{Z}_{i\varepsilon}\| \leq 2\bar{\eta} = O(\varepsilon^{2^k+1}). \quad (43)$$

Hence, since $\boldsymbol{Z}_{i\varepsilon} = O(\varepsilon^{2^k+1})$, $x(0)^T\boldsymbol{Z}_{i\varepsilon}x(0) = O(\varepsilon^{2^k+1})$ results in (33). ■

6 Numerical Example

In order to demonstrate the efficiency of the soft-constrained stochastic Nash games, we present results for the megawatt-frequency control problem of multiarea electric energy systems. The model is based on the multi-stage decomposition of two interconnected areas (Elgerd and Fosha, 1970). The detailed physical significances of the model were also developed in (Gajić et al., 1990). The system matrices are given as follows.

$$A_{11} = \begin{bmatrix} 0 & 0.315 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & 0 & -3.333 & 3.333 \\ 0 & 0 & -5.2083 & 0 & -12.5 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -3.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & -3.333 & 3.333 \\ 0 & -5.2083 & 0 & -12.5 \end{bmatrix},$$

$$A_{111} = \textbf{block diag}\begin{pmatrix} 0 & 0 & 0.00249 & 0 & 0 \end{pmatrix},$$
$$A_{122} = \textbf{block diag}\begin{pmatrix} 0 & 0.00249 & 0 & 0 \end{pmatrix},$$
$$B_{11}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 33.333 \end{bmatrix},\ B_{22}^T = \begin{bmatrix} 0 & 0 & 0 & 33.333 \end{bmatrix},$$
$$E_{11}^T = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 \end{bmatrix},\ E_{22}^T = \begin{bmatrix} 0 & -0.6 & 0 & 0 \end{bmatrix},$$
$$Q_1 = \textbf{block diag}\begin{pmatrix} I_5 & \varepsilon I_4 \end{pmatrix},$$
$$Q_2 = \textbf{block diag}\begin{pmatrix} \varepsilon I_5 & I_4 \end{pmatrix},$$
$$R_1 = R_2 = 0.1,\ V_{11} = V_{22} = 10,$$
$$A_{1ij} = 0,\ B_{ij} = 0,\ E_{ij} = 0,\ V_{ij} = 0,\ i \neq j.$$

Small parameters $\varepsilon = 0.1$ and $\mu = 0$ are selected. In order to guarantee the feasibility of the solution, we consider that the difference in the area power angle for the model presented in (Gajić et al., 1990) is 60 degrees.

It should be noted that both the deterministic disturbance distribution and the state-dependent noise related to the load frequency constant (Elgerd and Fosha, 1970) are considered. That is, the nature of stochastic uncertainty considered here is not addressed in (Gajić et al., 1990). We suppose that the error of the load frequency constant is within 5% of the nominal value. Therefore, the proposed design method is very useful because the resulting strategy can be implemented to more practical weakly coupled large-scale stochastic systems.

The proposed computational algorithm was developed in MATLAB 5.2 and Control System Toolbox 5.2 for simulation purposes. Moreover, the computer used in this simulation was a Pentium M 1200 MHz with 632MB RAM, running Windows XP.

It is easy to verify that for the two-players case, algorithm (23) converges to the exact solution of CSAREs (12a) with an accuracy of $\|\boldsymbol{F}^{(k)}\| < 1.0e-11$ after seven iterations, where $\|\boldsymbol{F}^{(k)}\| := \sum_{i=1}^2 \|\boldsymbol{F}_i(\varepsilon,\ \mu,\ P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)})\|$.

Table 1. Errors per iterations.

k	$\|\boldsymbol{F}^{(k)}\|$	k	$\|\boldsymbol{F}^{(k)}\|$
0	2.7022	4	$5.0196e-02$
1	$1.1057e+01$	5	$4.3264e-04$
2	2.8660	6	$2.6499e-08$
3	$6.0292e-01$	7	$6.2248e-14$

In order to verify the accuracy of the solution, the remainder per iteration is substituted by $P_{i\varepsilon}^{(k)}$ into CSAREs (12a). In Table 1, the results of the error $\|\boldsymbol{F}^{(k)}\|$ per iteration are given. It can be seen that algorithm (23) yields quadratic convergence.

Finally, high-order approximate soft-constrained stochastic Nash strategies are obtained as follows.

$$\begin{aligned} F_{1\varepsilon}^{(7)*} &= \big[\, -2.8614e \ -1.7457 \ -3.9435 \ -5.0595 \\ &\quad -2.9645 \ -1.4043 \ -7.1326e-02 \\ &\quad 4.9675e-02 \ 1.9434e-03 \,\big]\,, \qquad (44a) \\ F_{2\varepsilon}^{(7)*} &= \big[\, 3.3869e-01 \ -2.1367 \ -3.4923e-01 \\ &\quad -1.5753e-01 \ -4.5787e-03 \ -9.9669e-01 \\ &\quad -3.6633 \ -4.8556 \ -2.9584 \,\big]\,. \qquad (44b) \end{aligned}$$

As a result, although both state-dependent stochastic noise and unknown deterministic disturbance are included, the high-order approximate Nash strategy was obtained.

7 Conclusion

In this paper, infinite-horizon stochastic Nash games were discussed. First, conditions required for the existence of Nash equilibrium were established by utilizing CSAREs. Second, after establishing an asymptotic structure along with positive definiteness for solving CSAREs, Newton method was adopted. As a result, it was shown that quadratic convergence can be attained. Thus, the proposed algorithm is expected to be very useful and reliable for a sufficiently small value of ε. As another important feature, a high-order approximate strategy that yields better cost performance was attained. In fact, the cost degradation by using the proposed approximate strategy was demonstrated.

Acknowledgments

The author would like to thank Prof. J. Engwerda and Prof. V. Dragan for their helpful comments with regard to Theorem 4 and the uniqueness of SARE (18), respectively. He would also like to thank the anonymous reviewers for their constructive and insightful comments, which led to significant improvements in the quality of this paper.

Appendix I: Proof of Theorem 4

Let us consider the scalar function $V(t,\ x(t)) := x^T(t)P_\varepsilon x(t)$. By applying Itô's formula to $V(t,\ x(t))$ and considering (9), we have $dV(t,\ x(t)) = \Big[\|u_1(t) + R_{11}^{-1}B_{1\varepsilon}P_\varepsilon x(t)\|_{R_{11}}^2 - \|v(t) - V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon x(t)\|_{V_{1\mu}}^2 - [\|x(t)\|_{Q_{1\varepsilon}}^2 + \|u(t)\|_{R_{11}}^2 - \|v(t)\|_{V_{1\mu}}^2]\Big]dt + 2\sum_{p=1}^M x^T(t)P_\varepsilon A_{p\varepsilon}x(t)dw_p(t)$. Since we assume that for all $u_1(t)$ and $v(t)$, the closed-loop system is asymptotically mean-square stable, $\lim_{t\to\infty} Ex^T(t)x(t) = 0$. Thus, integrating both sides of the above equation and using $E\int_0^\infty x^T(t)P_\varepsilon A_{p\varepsilon}x(t)dw_p(t) = 0$ results in $J(u_1,\ v,\ x(0)) = x^T(0)P_\varepsilon x(0) + E\int_0^\infty \Big[\|u_1(t) - u_1^*(t)\|_{R_{11}}^2 - \|v(t) - V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon x(t)\|_{V_{1\mu}}^2\Big]dt$. From this, it follows that $J(u_1^*,\ v,\ x(0)) = x^T(0)P_\varepsilon x(0) - E\int_0^\infty \|v(t) - V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon \check{x}(t)\|_{V_{1\mu}}^2 dt \le x^T(0)P_\varepsilon x(0)$, where $\check{x}(t)$ is governed by $d\check{x}(t) = \Big[(A_\varepsilon - S_{1\varepsilon}P_\varepsilon)\check{x}(t) + E_\varepsilon v(t)\Big]dt + \sum_{p=1}^M A_{p\varepsilon}\check{x}(t)dw_p(t)$, $\check{x}(0) = x^0$. Furthermore, if $J(u_1^*,\ v,\ x(0)) = x^T(0)P_\varepsilon x(0)$, then $v(t) = v^*(t)$. Hence, $J(u_1^*,\ v,\ x(0)) < x^T(0)P_\varepsilon x(0)$, for all $v(t) \neq v^*(t)$ and $J(u_1^*,\ v^*,\ x(0)) = x^T(0)P_\varepsilon x(0)$. Then, $\check{x}(t)$ is governed by (11c) as $\tilde{x}(t) = \check{x}(t)$.

Let $\hat{x}(t)$ and $\bar{x}(t)$ be governed by $d\hat{x}(t) = [A_\varepsilon\hat{x}(t) + B_{1\varepsilon}F_{1\varepsilon}\hat{x}(t) + E_\varepsilon v^*(t)]dt + \sum_{p=1}^M A_{p\varepsilon}\hat{x}(t)dw_p(t)$, $\hat{x}(0) = x^0$, $d\bar{x}(t) = [A_\varepsilon\bar{x}(t) + B_{1\varepsilon}F_{1\varepsilon}^*\bar{x}(t) + E_\varepsilon v^*(t)]dt + \sum_{p=1}^M A_{p\varepsilon}\bar{x}(t)dw_p(t)$, $\bar{x}(0) = x^0$, respectively. Furthermore, we define two new variables $\nu(t) := (F_{1\varepsilon}^* - F_{1\varepsilon})\hat{x}(t)$ and $\eta(t) := v^*(t) - V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon\hat{x}(t)$. Then, $J(u_1,\ v^*,\ x(0)) - J(u_1^*,\ v^*,\ x(0)) = E\int_0^\infty \Big[\|\nu(t)\|_{R_{11}}^2 - \|\eta(t)\|_{V_{1\mu}}^2\Big]dt$. Introducing $\xi(t) := \bar{x}(t) - \hat{x}(t)$ yields $d\xi(t) = [(A_\varepsilon - S_{1\varepsilon}P_\varepsilon)\xi(t) + B_{1\varepsilon}\nu(t)]dt + \sum_{p=1}^M A_{p\varepsilon}\xi(t)dw_p(t)$, $\xi(0) = 0$, $\eta(t) = V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon\xi(t)$. Hence, taking into account the fact that for any $\nu(t)$, the closed-loop system is asymptotically mean-square stable, $E\int_0^\infty d\|\xi(t)\|_{P_\varepsilon}^2 = 0$. Furthermore, taking $E\int_0^\infty \xi^T(t)P_\varepsilon A_{p\varepsilon}\xi(t)dw_p(t) = 0$ and the SARE (9) into account, we have $J(u_1,\ v^*,\ x(0)) - J(u_1^*,\ v^*,\ x(0)) = E\int_0^\infty \Big[[\|\nu(t)\|_{R_{11}}^2 - \|\eta(t)\|_{V_{1\mu}}^2]dt - d\xi^T(t)P_\varepsilon\xi(t)\Big] = E\int_0^\infty \Big[\|\nu(t)\|_{R_{11}}^2 - \xi^T(t)\Big[P_\varepsilon E_\varepsilon V_{1\mu}^{-1}E_\varepsilon^T P_\varepsilon + P_\varepsilon(A_\varepsilon - S_{1\varepsilon}P_\varepsilon) + (A_\varepsilon - S_{1\varepsilon}P_\varepsilon)^T P_\varepsilon + \sum_{p=1}^M A_{p\varepsilon}^T P_\varepsilon A_{p\varepsilon}\Big]\xi(t) - 2\xi^T(t)P_\varepsilon B_{1\varepsilon}\nu(t)\Big]dt = E\int_0^\infty [\|\nu(t) + F_{1\varepsilon}^*\xi(t)\|_{R_{11}}^2 + \|\xi(t)\|_{Q_{1\varepsilon}}^2]dt$. Next, assuming that $v(t) := \nu(t) + F_{1\varepsilon}^*\xi(t) = F_{1\varepsilon}^*\bar{x}(t) - F_{1\varepsilon}\hat{x}(t)$ yields $d\xi(t) = [A_\varepsilon\xi(t) + B_{1\varepsilon}v(t)]dt + \sum_{p=1}^M A_{p\varepsilon}\xi(t)dw_p(t)$. Since $\xi(0) = 0$ and $\xi(t)$ is asymptotically mean-square stable, $E\int_0^\infty d\|\xi(t)\|_{W_\varepsilon}^2 = 0$. Thus, $J(u_1,\ v^*,\ x(0)) - J(u_1^*,\ v^*,\ x(0)) = E\int_0^\infty \Big[[\|v(t)\|_{R_{11}}^2 + \|\xi(t)\|_{Q_{1\varepsilon}}^2]dt + d\|\xi(t)\|_{W_\varepsilon}^2\Big] = E\int_0^\infty [\|v(t) + R_{11}^{-1}B_{1\varepsilon}^T W_\varepsilon\xi(t)\|_{R_{11}}^2 + \|\xi(t)\|_{\boldsymbol{W}}^2]dt$, where $\boldsymbol{W} := W_\varepsilon A_\varepsilon + A_\varepsilon W_\varepsilon^T + \sum_{p=1}^M A_{p\varepsilon}^T W_\varepsilon A_{p\varepsilon} - W_\varepsilon S_{1\varepsilon}W_\varepsilon + Q_{1\varepsilon}$.

Since inequality (10) holds, the desired result is obtained. ■

Appendix II: Proof of Lemma 11

Since it is assumed that the stochastic system is asymptotically mean-square stable, $\lim_{t\to\infty} Ex^T(t)x(t) = 0$. Thus, applying Itô's formula to $x^T(t)L_\varepsilon x(t)$ and using a completion of squares yields $\hat{J}(v,\ x(0)) = x^T(0)L_\varepsilon x(0) - E\int_0^\infty \|v(t) - V_\mu^{-1}E_\varepsilon^T L_\varepsilon x(t)\|_{V_\mu}^2 dt$. Hence, $\hat{J}(v,\ x(0)) \le x^T(0)L_\varepsilon x(0)$, and equality holds iff (37). Furthermore, the maximum value $\hat{J}(\bar{v},\ x(0)) = x^T(0)L_\varepsilon x(0)$ is obtained uniquely by $\bar{v}(t)$. ■

References

Basar, T. (1974) A counterexample in linear–quadratic games: Existence of nonlinear Nash solutions, *J. Optimization Theory and Applications*, 14(4) 425-430.

Basar T. and Olsder G. J. (1999) *Dynamic noncooperative game theory*, second edition, Philadelphia: SIAM.

Broek, W. A. V. D., Engwerda, J. C., & Schumacher, J. M. (2003) Robust equilibria in indefinite linear-quadratic differential games, *J. Optimization Theory and Applications*, 119(3), 565-595.

Buckdahn, R., Cardaliaguet, P., & Rainer, C. (2004) Nash equilibrium payoffs for nonzero-sum stochastic differential games, *SIAM J. Control and Optimization*, 43(2), 466-476.

Chen, B. S., & Zhang, W. (2004) Stochastic H_2/H_∞ control with state-dependent noise, *IEEE Trans. Automatic Control*, 49(1) 45-57.

Dragan, V., and Morozan, T. (1997) Global solutions to a game theoretic Riccati equation of stochastic control, *J. Differential Equations*, 138(10), 328-350.

Elgerd, O. I. and Fosha, C. E. JR. (1970) Optimum megawatt-frequency control of multiarea electric energy systems, *IEEE Trans. Power Apparatus and Systems*, 89(4), 556-563.

Engwerda, J. C. (2005) *LQ dynamic optimization and differential games*, Chichester: John Wiley and Sons.

Engwerda, J. C. (2006) A numerical algorithm to find soft-constrained Nash equilibria in scalar LQ-games, *Int. J. Control*, 79(6) 592-603.

Gajić, Z., & Losada, R. (1999) Solution of the state-dependent noise optimal control problem in terms of Lyapunov iterations, *Automatica*, 35(5) 951-954.

Gajić, Z., Petkovski, D., & Shen, X. (1990) *Singularly perturbed and weakly coupled linear system - A recursive approach*. Lecture Notes in Control and Information Sciences, vol.140, Berlin: Springer-Verlag.

Hinrichsen, D., & Pritchard, A.J. (1998) Stochastic H_∞, *SIAM J. Control and Optimization*, 36(5), 1504-1538.

Huang, M., Malhamé, R. P., & Caines, P. E. (2006) Nash certainty equivalence in large population stochastic dynamic games: connections with the physics of interacting particle systems, In *Proceedings of IEEE Conf. Decision and Control*, San Diego (pp. 4921-4926).

Jittorntrum, K. (1978) An implicit function theorem, *J. Optimization Theory and Applications*, 25(4), 285-288.

Kokotović, P. V., Khalil, H. K. & O'Reilly, J. (1999) *Singular perturbation methods in control: analysis and design*, Philadelphia: SIAM.

Mukaidani, H. (2006) A numerical analysis of the Nash strategy for weakly coupled large-scale systems, *IEEE Trans. Automatic Control*, 51(8), 1371-1377.

Mukaidani, H. (2007a) Newton's method for solving cross-coupled sign-indefinite algebraic Riccati equations for weakly coupled large-scale systems, *Applied Mathematics and Computation*, 188(1) 103-115.

Mukaidani, H. (2007b) Numerical computation of sign indefinite linear quadratic differential games for weakly coupled large-scale systems, *Int. J. Control*, 80(1), 75-86.

Ortega, J. M. (1990) *Numerical analysis, A second course*, Philadelphia: SIAM.

Petrovic, B., & Gajić, Z. (1988) Recursive solution of linear-quadratic Nash games for weakly interconnected systems, *J. Optimization Theory and Applications*, 56(3), 463-477.

Rami, M. A., & Zhou, X. Y. (2000) Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls, *IEEE Trans. Automatic Control*, 45(6), 1131-1143.

Sagara, M., Mukaidani, H., & Yamamoto, T. (2008) Numerical solution of stochastic Nash games with state-dependent Noise for weakly coupled large-scale systems, *Applied Mathematics and Computation*, 197(2), 844-857.

Srikant, R., & Basar, T. (1992) Asymptotic solutions to weakly coupled stochastic teams with nonclassical information, *IEEE Trans. Automatic Control*, 37(2), 163-173.

Ugrinovskii, V. A. (1998) Robust H_∞ control in the presence of stochastic uncertainty, *Int. J. Control*, 71(2), 219-237.

Wu, H. S., Willgoss, R. A., & Mizukami, K. (1994) Robust stabilization for a class of uncertain dynamical systems with time delay, *J. Optimization Theory and Applications*, 82(2), 361-378.

Yamamoto, T. (1986) A method for finding sharp error bounds for Newton's method under the Kantorovich assumptions, *Numerische Mathematik*, 49(2-3), 203-220.

Hiroaki Mukaidani received his B.S. degree in integrated arts and sciences from Hiroshima University, Japan, in 1992 and his M.Eng. and Dr.Eng. degrees in information engineering from Hiroshima University, Japan, in 1994 and 1997, respectively. He worked with Hiroshima City University as a Research Associate from 1998 to 2002. Since 2002, he has been with the Graduate School of Education, Hiroshima University, Japan, as an Assistant Professor and currently as an Associate Professor. During 2007 to 2008, he spent ten months as a Japan Society for the Promotion of Science research fellow in the Department of Electrical and Computer Engineering, University of Waterloo, Canada. His current research interests include robust control, dynamic game and its application of singularly perturbed systems and large-scale systems. He is a member of the Institute of Electrical and Electronics Engineers (IEEE).