Robust Guaranteed Cost Control for Uncertain Stochastic Systems with Multiple Decision Makers ⋆

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Abstract

The guaranteed cost control (GCC) problem for uncertain stochastic systems with N decision makers is investigated. It is noteworthy that the necessary conditions, which are determined from Karush-Kuhn-Tucker (KKT) conditions, for the existence of a guaranteed cost controller have been derived on the basis of the solutions of cross-coupled stochastic algebraic Riccati equations (CSAREs). It is shown that if CSAREs have an optimal solution, then the closed-loop system is exponentially mean square stable (EMSS) and has a cost bound. In order to simplify computations and attain a global optimum, the linear matrix inequality (LMI) technique is also considered. Finally, a numerical example for a practical megawatt-frequency control problem shows that the proposed methods can help in attaining an adequate cost bound. Furthermore, the features of these methods are characterized.

Key words: Guaranteed cost control (GCC); multiple decision makers; Pareto strategy; cross-coupled stochastic algebraic Riccati equations (CSAREs); linear matrix inequality (LMI).

1 Introduction

Over the last decade, stochastic control problems governed by Itô's differential equation have attracted considerable research interest. Recently, stochastic LQ and H_∞ control problems with state- and control-dependent noise have been investigated [5, 6, 10]. They have received much attention and have been widely used in various fields. In particular, the stochastic H_2/H_∞ control with state-dependent noise has been addressed [8]. Although the results of [8] are very elegant in theory and despite the fact that it is easy to obtain a strategy pair by solving the cross-coupled stochastic algebraic Riccati equations (CSAREs), a control problem with multiple decision makers is an issue that remains to be considered. Dynamic game theory is the most important approach for solving this problem. In [15], stochastic Nash games have been tackled for their deterministic disturbance and stochastic uncertainty for the first time. However, deterministic uncertainties such as modelling errors have not been focused on in this paper. This is because it is hard to define equilibrium for deterministic uncertain systems.

It is well known that uncertainty occurs in many dynamic systems and is frequently a cause of instability and performance degradation. In recent years, the problem of designing robust controllers for linear systems with parameter uncertainty has received considerable attention in control system literature. One design approach to solve this problem is the so-called guaranteed cost control (GCC) [1–3, 7]. This approach has the advantage of placing an upper bound on a given performance index, and it is guaranteed that the degradation of system's performance due to uncertainty is smaller than the cost bound. Although these design methodologies have proved to be efficient in solving various engineering problems, stochastic modelling with multiple decision makers has not yet been addressed.

In this paper, the GCC problem for a stochastic system with N decision makers and one that involves deterministic uncertainty is discussed. In particular, we introduce a new interpretation of the GCC problem in view of the optimization of the cost bound between the ordinary method [1–3] and the LMI technique. First, the stochastic algebraic Riccati inequality (SARI) is established such that the closed-loop system is exponentially mean square stable (EMSS) and has a cost bound. Sec-

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ond, the GCC problem is formulated on the basis of the Pareto strategy. In order to obtain the strategy set, new CSAREs are established as the necessary conditions using Karush-Kuhn-Tucker (KKT) conditions for the existence of a guaranteed cost controller. Furthermore, Newton's method is used for solving the CSAREs. Therefore, the exact cost bound can be computed directly. The linear matrix inequality (LMI) technique is also considered for the same problem. Although this approach results in the supremum, the global optimum is guaranteed because the considered design problem is based on convex optimization. Finally, a numerical example is solved for a practical megawatt-frequency control problem to validate the proposed technique.

Notation: The notations used in this paper are fairly standard. **block diag** denotes a block diagonal matrix. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $E[\cdot]$ denotes the expection operator. $\mathbf{Tr}$ denotes the trace of a matrix. $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and maximum eigenvalues for a matrix, respectively. $\mathbf{vec}M$ denotes the column vector of the matrix M [16]. $\otimes$ denotes the Kronecker product. U_{pq} denotes a permutation matrix in the Kronecker matrix sense [16] such that $U_{pq}\mathbf{vec}M = \mathbf{vec}M^T,\ M \in \mathbf{R}^{p\times q}$.

2 Definition and Preliminary

We first introduce the EMSS and the related facts (see e.g., [5,6,8,9] and the references therein for more details).

Definition 1 [5] *A stochastic system described by Itô's equation $dx(t) = Ax(t)dt + A_1x(t)dw(t)$ is said to be EMSS if it satisfies the following condition:*

$$\exists\rho,\ \psi > 0,\ E\|x(t)\|^2 \le \rho e^{-\psi(t-t_0)}E\|x(t_0)\|^2. \tag{1}$$

Definition 2 [8, 9] *$[\tilde{A},\ \tilde{A}_1,\ \ldots,\tilde{A}_M \mid \tilde{C}]$ is exactly observable, if for any $T > 0$, $y(t) \equiv 0$, $\forall t \in [0,\ T] \Rightarrow x_0 = 0$ holds with $y(t) = \tilde{C}x(t)$, where $x(t)$ is the solution of*

$$dx(t) = \tilde{A}x(t)dt + \sum_{p=1}^{M}\tilde{A}_px(t)dw_p(t),\ x(0) = x_0. \tag{2}$$

Moreover, $(\tilde{A},\ \tilde{A}_1,\ \ldots,\tilde{A}_M)$ is called stable, if the above system (2) is asymptotically mean square stable.

Lemma 3 [4] *For any matrices $D \in \Re^{n\times r}$, $E \in \Re^{s\times n}$, and $F(t) \in \Re^{r\times s}$, with $F^T(t)F(t) \le I_s$, and scalar $\beta > 0$, the following inequality holds: $DF(t)E + E^TF^T(t)D^T \le \beta DD^T + \beta^{-1}E^TE$.*

Lemma 4 [5, 6] *The trivial solution of a stochastic differential equation is as follows:*

$$dx(t) = f(t,\ x)dt + g(t,\ x)dw(t), \tag{3}$$

where $f(t,\ x)$ and $g(t,\ x)$ which are sufficiently differentiable maps, are EMSS if there exists a function $V(x(t))$, which satisfies the following inequalities:

$$a_1\|x(t)\|^2 \le V(x(t)) \le a_2\|x(t)\|^2,\ a_1,\ a_2 > 0, \tag{4a}$$

$$\begin{aligned}\mathcal{D}V(x(t)) &:= \frac{\partial V(x(t))}{\partial x}f(t,\ x) \\ &\quad + \frac{1}{2}\mathbf{Tr}\left[g^T(t,\ x)\frac{\partial^2 V(x(t))}{\partial x^2}g(t,\ x)\right] \\ &\le -c\|x(t)\|^2,\ c > 0\end{aligned} \tag{4b}$$

for $x(t) \ne 0$.

Lemma 5 *Consider an autonomous uncertain stochastic system*

$$\begin{aligned}&dx(t) = [\tilde{A} + \tilde{D}\tilde{F}(t)\tilde{E}]x(t)dt + \sum_{p=1}^{M}\tilde{A}_px(t)dw_p(t), \\ &x(0) = x^0,\end{aligned} \tag{5}$$

where $x(t) \in \Re^n$ is the state vector, and $\tilde{F}(t) \in \Re^{\tilde{r}\times\tilde{s}}$ is a Lebesgue measurable matrix of the uncertain parameters satisfying $\tilde{F}^T(t)\tilde{F}(t) \le I_{\tilde{s}}$. Associated with this stochastic system is the corresponding cost function

$$J = E\int_0^\infty x^T(t)\tilde{Q}x(t)dt,\ \tilde{Q} = \tilde{Q}^T > 0. \tag{6}$$

If for any given positive definite symmetric matrix $\tilde{Q}$, there exists a positive definite symmetric matrix $\tilde{X}$ and a positive scalar μ that satisfies the stochastic algebraic Riccati inequality (SARI)

$$\begin{aligned}&\tilde{X}\tilde{A} + \tilde{A}^T\tilde{X} + \sum_{p=1}^{M}\tilde{A}_p^T\tilde{X}\tilde{A}_p + \mu\tilde{X}\tilde{D}\tilde{D}^T\tilde{X} \\ &\quad + \mu^{-1}\tilde{E}^T\tilde{E} + \tilde{Q} \le 0,\end{aligned} \tag{7}$$

then the stochastic system (5) *is EMSS. Moreover,*

$$J = E\int_0^\infty x^T(t)\tilde{Q}x(t)dt \le E[x^T(0)\tilde{X}x(0)]. \tag{8}$$

Proof: Suppose there exists a symmetric positive definite matrix $\tilde{X} > 0$ such that the SARI (7) holds. In order to prove that the stochastic system (5) is EMSS, let us define the following Lyapunov function candidate:

$$V(x(t)) = V(x) := x^T(t)\tilde{X}x(t) > 0. \tag{9}$$

First, using the fact that

$$\lambda_{\min}(\tilde{X})\|x(t)\|^2 \le V(x) \le \lambda_{\max}(\tilde{X})\|x(t)\|^2, \tag{10}$$

the condition (4a) holds. Second, in order to prove formula (4b), the stochastic differential is given by

$$
\begin{aligned}
\mathcal{D}V(x) &= x^T(t)\Bigg[\tilde{X}[\tilde{A}+\tilde{D}\tilde{F}(t)\tilde{E}]+[\tilde{A}+\tilde{D}\tilde{F}(t)\tilde{E}]^T\tilde{X} \\
&\quad +\sum_{p=1}^{M}\tilde{A}_p^T\tilde{X}\tilde{A}_p\Bigg]x(t) \\
&\le x^T(t)\Bigg[\tilde{X}\tilde{A}+\tilde{A}^T\tilde{X}+\sum_{p=1}^{M}\tilde{A}_p^T\tilde{X}\tilde{A}_p \\
&\quad +\mu\tilde{X}\tilde{D}\tilde{D}^T\tilde{X}+\mu^{-1}\tilde{E}^T\tilde{E}\Bigg]x(t) \\
&\le -\lambda_{\min}(\tilde{Q})\|x(t)\|^2. \qquad (11)
\end{aligned}
$$

Hence, $V(x(t))$ is a Lyapunov function for the stochastic system (5). By using (10), we also have

$$
\mathcal{D}V(x) \le -\lambda_{\min}(\tilde{Q})\|x(t)\|^2 \le -\frac{\lambda_{\min}(\tilde{Q})}{\lambda_{\max}(\tilde{X})}V(x). \qquad (12)
$$

Since the expectation of the stochastic integral vanishes, according to the the comparison lemma [20] the solution of equation (12) is represented as

$$
E[V(x)] \le E[V(x(0))]\exp\left[-\frac{\lambda_{\min}(\tilde{Q})}{\lambda_{\max}(\tilde{X})}t\right]. \qquad (13)
$$

Therefore, since the following inequality holds by using (10):

$$
E\|x(t)\|^2 \le \frac{\lambda_{\max}(\tilde{X})}{\lambda_{\min}(\tilde{X})}E\|x(0)\|^2\exp\left[-\frac{\lambda_{\min}(\tilde{Q})}{\lambda_{\max}(\tilde{X})}t\right], \qquad (14)
$$

the stochastic system (5) is EMSS. Moreover, applying Itô's formula gives

$$
\begin{aligned}
dV(x) &= \mathcal{D}V(x)dt + 2\sum_{p=1}^{M}x^T(t)\tilde{A}_p^T\tilde{X}dw_p \\
&\le -x^T(t)\tilde{Q}x(t)dt + 2\sum_{p=1}^{M}x^T(t)\tilde{A}_p^T\tilde{X}dw_p. \qquad (15)
\end{aligned}
$$

By integrating both sides of the inequality (15) from 0 to T, the following inequality holds:

$$
E[V(x(T))]-E[V(x(0))] \le -E\int_0^T x^T(t)\tilde{Q}x(t)dt. \qquad (16)
$$

Since the stochastic system (5) is EMSS, $E[V(x(T))] \to 0$ holds. Thus, inequality (8) holds. The proof of Lemma 5 is completed. ■

3 Problem Formulation

Consider the following uncertain stochastic systems with N-decision makers.

$$
\begin{aligned}
dx(t) &= \Bigg[[A+DF(t)E_a]x(t) \\
&\quad +\sum_{i=1}^{N}[B_i+DF(t)E_{bi}]u_i(t)\Bigg]dt \\
&\quad +\sum_{p=1}^{M}A_px(t)dw_p(t),\ x(0)=x^0, \qquad (17)
\end{aligned}
$$

where $x(t) \in \Re^n$ is the state vector, and $u_i(t) \in \Re^{m_i}$, $i = 1, \ldots, N$ are the control inputs. $w_p(t) \in \Re$, $p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space [6,8,10]. All the matrices are constant matrices with appropriate dimensions. $F(t) \in \Re^{r\times s}$ is a Lebesgue measurable matrix of uncertain parameters satisfying $F^T(t)F(t) \le I_s$. The initial state x_0 is assumed to be a random variable with a covariance matrix $E[x(0)x^T(0)] = M_0$.

The performance criterion is given by

$$
J_i(u_i) := E\int_0^\infty \left[x^T(t)Q_ix(t)+u_i^T(t)R_iu_i(t)\right]dt, \qquad (18)
$$

where $Q_i = Q_i^T > 0 \in \Re^{n\times n}$ and $R_i = R_i^T > 0 \in \Re^{m_i\times m_i}$.

Suppose that the Nash game approach is applied. Then, due to the presence of the deterministic uncertainty, the players cannot evaluate a Nash equilibrium solution. Hence, the purpose of this paper is to establish a class of robust strategies, which guarantee a given value of the cost function for each player.

First, we consider that the ith decision maker will design his control strategy based on the state information. The design specifications of the ith decision maker can be expressed in terms of a cost function J_i. We consider the situation in which decision makers decide their strategies through mutual cooperation. The solution to such a problem is found in the class of Pareto optimal strategies. It means that no deviation from the Pareto optimal strategy can decrease the costs of all decision makers [13, 14, 19].

A Pareto solution is a set $(u_1, \ldots, u_N)$, which minimizes

$$
J(u_1, \ldots, u_N) = \sum_{i=1}^{N}\gamma_iJ_i(u_i),\ 0<\gamma_i<1,\ \sum_{i=1}^{N}\gamma_i = 1, \qquad (19)
$$

for some γ_i, $i = 1, \ldots, N$.

Without loss of generality, the strategies are restricted to the following linear feedback strategies [12].

$$
u_i(t) = K_ix(t),\ i = 1, \ldots, N. \qquad (20)
$$

The definition of the GCC for uncertain stochastic systems with deterministic uncertainties is given below.

Definition 6 *A linear feedback strategy expressed as $u_i(t) = K_i x(t)$ is said to be the GCC in Pareto optimality with a guaranteed cost $\boldsymbol{J}$ for uncertain stochastic systems* (17) *and the cost function* (19) *if the closed-loop system is EMSS, and the closed-loop value of* (19) *satisfies the bound $J(u_1, \ldots, u_N) \leq \boldsymbol{J}$ for all admissible uncertainties.*

The objective of this paper is to introduce the GCC in Pareto optimal strategies $u_i(t) = K_i x_i(t)$, $i = 1, \cdots, N$ for uncertain stochastic systems (17) with deterministic uncertainties.

3.1 Necessary Conditions Obtained Using KKT Conditions

First, we establish the conditions for the existence of the GCC problem in Pareto optimality. Using Lemma 5 and the assumption that $E[x(0)x^T(0)] = M_0$, it is immediately deduced that the closed-loop stochastic system is EMSS and the integral portion of $J(u_1, \ldots, u_N)$ satisfies the relation

$$J(u_1, \ldots, u_N) \leq \boldsymbol{J} = \mathbf{Tr}[M_0 P], \tag{21}$$

if there exists a solution to the following SARI:

$$\begin{aligned} &\boldsymbol{F}(P, K_1, \ldots, K_N, \mu) \\ &:= PA_K + A_K^T P + \mu P D D^T P + \mu^{-1} E_K^T E_K \\ &\quad + \sum_{p=1}^{M} A_p^T P A_p + \sum_{i=1}^{N} \gamma_i K_i^T R_i K_i + Q \leq 0, \end{aligned} \tag{22}$$

where $A_K := A + \sum_{i=1}^{N} B_i K_i$, $E_K := E_a + \sum_{i=1}^{N} E_{bi} K_i$ and $Q := \sum_{i=1}^{N} \gamma_i Q_i$.

From the above problem, we obtain the following necessary optimality conditions:

Theorem 7 *Let us consider the uncertain stochastic systems* (17) *and cost function* (19)*. If the following two conditions hold*

i) $\left[A_K + \mu D D^T P,\ A_1, \ldots, A_M \,\middle|\, M_0 \right]$ *is exactly observable*

ii) $\left(A_K + \mu D D^T P,\ A_1, \ldots, A_M \right)$ *is stable*

and v^* is a local minimum that satisfies constraint qualification [1]*, then there exists a unique positive definite solution $G^* > 0$ such that*

$$\boldsymbol{M}(P^*, G^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \tag{23a}$$

$$\boldsymbol{H}_i(P^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \tag{23b}$$

$$\boldsymbol{I}(P^*, G^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \tag{23c}$$

$$\boldsymbol{F}(P^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \tag{23d}$$

where $v^* = ([\mathrm{vec} P^*]^T, [\mathrm{vec} K_1^*]^T, \ldots, [\mathrm{vec} K_N^*]^T, \mu^*)$,

$$\begin{aligned} &\boldsymbol{M}(P, G, K_1, \ldots, K_N, \mu) \\ &:= G(A_K + \mu D D^T P)^T + (A_K + \mu D D^T P) G \\ &\quad + \sum_{p=1}^{M} A_p G A_p^T + M_0, \end{aligned} \tag{24a}$$

$$\begin{aligned} &\boldsymbol{H}_i(P, K_1, \ldots, K_N, \mu) \\ &:= B_i^T P + \mu^{-1} E_{bi}^T E_a + \mu^{-1} \sum_{k=1}^{N} E_{bi}^T E_{bk} K_k \\ &\quad + \gamma_i R_i K_i,\ i = 1, \ldots, N, \end{aligned} \tag{24b}$$

$$\begin{aligned} &\boldsymbol{I}(P, G, K_1, \ldots, K_N, \mu) \\ &:= \mathbf{Tr}[G P D D^T P] - \mu^{-2} \mathbf{Tr}[G E_K^T E_K]. \end{aligned} \tag{24c}$$

In other words, let v^* be the solution set that gives a local minimum. Then, there exists a state feedback strategy $u_i(t) = K_i x(t)$, which is called the GCC in Pareto optimality such that these conditions (23) *that are described by the CSAREs hold.*

Proof: According to the KKT conditions [17], the problem of determining a strategy set that minimizes the cost bound (21) subject to the constraint of SARI (22) can be converted into the following optimization problem.

First, the cost of a closed-loop uncertain stochastic system with a state feedback controlle $u_i(t) = K_i x(t)$ can be obtained when $J \leq \mathbf{Tr}[M_0 P]$, where P is the solution of the SARI (22). Let us consider the Lagrangian $\boldsymbol{L}$

$$\begin{aligned} &\boldsymbol{L}(P, G, K_1, \ldots, K_N, \mu) \\ &= \mathbf{Tr}\,[M_0 P] + \mathbf{Tr}\,[G \boldsymbol{F}(P, K_1, \ldots, K_N, \mu)], \end{aligned} \tag{25}$$

where G is a symmetric positive definite matrix of Lagrange multipliers.

It is clear that $\mathbf{Tr}\,[M_0 P]$ and $\boldsymbol{F}(P, K_1, \ldots, K_N, \mu)$ are continuously differentiable at a point v^*. Using the KKT conditions, we have

$$\begin{aligned} &\frac{\partial \boldsymbol{L}}{\partial P} = \boldsymbol{M}(P^*, G^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \\ &\frac{\partial \boldsymbol{L}}{\partial K_i} = G^* \boldsymbol{H}_i(P^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \\ &\frac{\partial \boldsymbol{L}}{\partial \mu} = \boldsymbol{I}(P^*, G^*, K_1^*, \ldots, K_N^*, \mu^*) = 0, \\ &\boldsymbol{F}(P^*, K_1^*, \ldots, K_N^*, \mu^*) \leq 0,\ G^* \geq 0, \\ &G^* \boldsymbol{F}(P^*, K_1^*, \ldots, K_N^*, \mu^*) = 0. \end{aligned}$$

Hence, (23a) and (23c) can be derived. Applying the conditions i) and ii) to (23a), it follows immediately that (23a) has a unique positive definite solution $G^* > 0$. From the remaining equations, (23d) and (23b) hold. ■

[1] When the gradients of the active inequality and equality constraints are linearly independent at v^*, it is called constraint qualification.

It should be noted that the obtained conditions are necessary. Therefore, the result can only be expected to be a local minimum. However, we have found that the proposed technique works well in practice. In fact, the reliability of this method is shown with the help of a numerical example.

3.2 Newton's Method

In order to obtain the optimal solutions v^*, we have to solve CSAREs (23). We will discuss the numerical method on the basis of Newton's method to solve these sets of equations. In particular, the following special case of $N = 2$ and $M = 1$ is considered because it is easy to extend it to the general case.

Let us consider the following CSAREs (26).

$$\begin{aligned}
&\boldsymbol{f}_1(P,\ K_1,\ K_2,\ \mu) \\
&= P\bar{A}_K + \bar{A}_K^T P + \mu PDD^T P + \mu^{-1}\bar{E}_K^T \bar{E}_K + A_1^T P A_1 \\
&\quad + \sum_{i=1}^{2} \gamma_i [Q_i + K_i^T R_i K_i], \qquad (26a) \\
&\boldsymbol{f}_2(P,\ G,\ K_1,\ K_2,\ \mu) \\
&= G\left(\bar{A}_K + \mu DD^T P\right)^T + \left(\bar{A}_K + \mu DD^T P\right) G \\
&\quad + A_1 G A_1^T + M_0, \qquad (26b) \\
&\boldsymbol{f}_3(P,\ K_1,\ K_2,\ \mu) \\
&= B_1^T P + \mu^{-1} E_{b1}^T E_a + \mu^{-1} E_{b1}^T (E_{b1} K_1 + E_{b2} K_2) \\
&\quad + \gamma_1 R_1 K_1, \qquad (26c) \\
&\boldsymbol{f}_4(P,\ K_1,\ K_2,\ \mu) \\
&= B_2^T P + \mu^{-1} E_{b2}^T E_a + \mu^{-1} E_{b2}^T (E_{b1} K_1 + E_{b2} K_2) \\
&\quad + \gamma_2 R_2 K_2, \qquad (26d) \\
&\boldsymbol{f}_5(P,\ G,\ K_1,\ K_2,\ \mu) \\
&= \mathbf{Tr}[GPDD^T P] - \mu^{-2}\mathbf{Tr}[G\bar{E}_K^T \bar{E}_K], \qquad (26e)
\end{aligned}$$

where $\bar{A}_K := A + B_1K_1 + B_2K_2$ and $\bar{E}_K := E_a + E_{b1}K_1 + E_{b2}K_2$.

An iterative approach for solving CSAREs (26) is to use the following algorithm, which is based on Newton's method.

$$\boldsymbol{z}^{(n+1)} = \boldsymbol{z}^{(n)} - [\Delta(P^{(n)},\ G^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)})]^{-1} \times \begin{bmatrix} \mathbf{vec}\boldsymbol{f}_1(P^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)}) \\ \mathbf{vec}\boldsymbol{f}_2(P^{(n)},\ G^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)}) \\ \mathbf{vec}\boldsymbol{f}_3(P^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)}) \\ \mathbf{vec}\boldsymbol{f}_4(P^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)}) \\ \mathbf{vec}\boldsymbol{f}_5(P^{(n)},\ G^{(n)},\ K_1^{(n)},\ K_2^{(n)},\ \mu^{(n)}) \end{bmatrix} \qquad (27)$$

where

$$\boldsymbol{z}^{(n)} := \Big[\ [\mathbf{vec}P^{(n)}]^T\ [\mathbf{vec}G^{(n)}]^T\ [\mathbf{vec}K_1^{(n)}]^T\ [\mathbf{vec}K_2^{(n)}]^T\ \mu^{(n)}\ \Big]^T,$$

$$\Delta(P,\ G,\ K_1,\ K_2,\ \mu) := \begin{bmatrix} \Xi_{11} & 0 & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ \Xi_{21} & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} \\ \Xi_{31} & 0 & \Xi_{33} & \Xi_{34} & \Xi_{35} \\ \Xi_{41} & 0 & \Xi_{43} & \Xi_{44} & \Xi_{45} \\ \Xi_{51} & \Xi_{52} & \Xi_{53} & \Xi_{54} & \Xi_{55} \end{bmatrix},$$

$$\begin{aligned}
\Xi_{11} &:= (\bar{A}_K + \mu DD^T P)^T \otimes I_n \\
&\quad + I_n \otimes (\bar{A}_K + \mu DD^T P)^T + A_1^T \otimes A_1^T, \\
\Xi_{13} &:= I_n \otimes (PB_1) + [(PB_1) \otimes I_n]U_{nm_1} \\
&\quad + \mu^{-1}\Big[I_n \otimes (E_a^T E_{b1}) + [(E_a^T E_{b1}) \otimes I_n]U_{nm_1} \\
&\quad + I_n \otimes (K_1^T E_{b1}^T E_{b1}) + [(K_1^T E_{b1}^T E_{b1}) \otimes I_n]U_{nm_1} \\
&\quad + I_n \otimes (K_2^T E_{b2}^T E_{b1}) + [(K_2^T E_{b2}^T E_{b1}) \otimes I_n]U_{nm_1} \Big] \\
&\quad + I_n \otimes (\gamma_1 K_1^T R_1) + [(\gamma_1 K_1^T R_1) \otimes I_n]U_{nm_1}, \\
\Xi_{14} &:= I_n \otimes (PB_2) + [(PB_2) \otimes I_n]U_{nm_2} + \mu^{-1} \\
&\quad \Big[I_n \otimes (E_a^T E_{b2}) + [(E_a^T E_{b2}) \otimes I_n]U_{nm_2} \\
&\quad + I_n \otimes (K_1^T E_{b1}^T E_{b2}) + [(K_1^T E_{b1}^T E_{b2}) \otimes I_n]U_{nm_2} \\
&\quad + I_n \otimes (K_2^T E_{b2}^T E_{b2}) + [(K_2^T E_{b2}^T E_{b2}) \otimes I_n]U_{nm_2} \Big] \\
&\quad + I_n \otimes (\gamma_2 K_2^T R_2) + [(\gamma_2 K_2^T R_2) \otimes I_n]U_{nm_2}, \\
\Xi_{15} &:= \mathbf{vec}[PDD^T P - \mu^{-2}\bar{E}_K^T \bar{E}_K], \\
\Xi_{21} &:= (\mu DD^T) \otimes G + G \otimes (\mu DD^T P), \\
\Xi_{22} &:= (\bar{A}_K + \mu DD^T P) \otimes I_n \\
&\quad + I_n \otimes (\bar{A}_K + \mu DD^T P) + A_1 \otimes A_1, \\
\Xi_{23} &:= G \otimes B_1 + [B_1 \otimes G]U_{nm_1}, \\
\Xi_{24} &:= G \otimes B_2 + [B_2 \otimes G]U_{nm_2}, \\
\Xi_{25} &:= \mathbf{vec}[GPDD^T + DD^T PG], \\
\Xi_{31} &:= I_n \otimes B_1^T,\ \Xi_{33} := I \otimes (\mu^{-1} E_{b1}^T E_{b1} + \gamma_1 R_1), \\
\Xi_{34} &:= I_n \otimes (\mu^{-1} E_{b1}^T E_{b2}),\ \Xi_{35} := -\mu^{-2}\bar{E}_K^T E_{b1}, \\
\Xi_{41} &:= I_n \otimes B_2^T,\ \Xi_{43} := I_n \otimes (\mu^{-1} E_{b2}^T E_{b1}), \\
\Xi_{44} &:= I_n \otimes (\mu^{-1} E_{b2}^T E_{b2} + \gamma_2 R_2),\ \Xi_{45} := -\mu^{-2}\bar{E}_K^T E_{b2}, \\
\Xi_{51} &= \Xi_{25}^T,\ \Xi_{52} = \Xi_{15}^T,\ \Xi_{53} := -2\mu^{-2} E_{b1}^T \bar{E}_K G, \\
\Xi_{54} &:= -2\mu^{-2} E_{b2}^T \bar{E}_K G,\ \Xi_{55} := 2\mu^{-3}\mathbf{Tr}[G\bar{E}_K^T \bar{E}_K].
\end{aligned}$$

It is well known that Newton's method is quadratic if the iteration starts near the desired optimum solutions v^*. However, if the Δ of (27) is close to a non-singular matrix, the inverted Δ can be numerically unstable and the iterative solution may diverge. Moreover, large dimensions of the associated inverse of the Jacobian matrix cause computational complexity; for example, a large amount of computer memory is required for computation. In other words, the computation of the inverse matrix with the sparse and large dimensions leads to numerical complexity. On the other hand, the derivation of (27) appears to be tedious, even though it provides the exact bound of the guaranteed cost $\mathbf{Tr}[M_0 P]$. Since the necessary conditions of CSAREs are derived from KKT conditions, its solution could be obtained as a local minimum; this is a major drawback of this approach. Therefore, in order to overcome this shortcoming and reduce the number of required storage elements, LMI conditions will be used instead of KKT conditions, as discussed in the next section.

4 LMI Approach

It is difficult to obtain the global optimum solution analytically because the approach given in the previous section is not a convex optimization. However, if the exact bound of the guaranteed cost is not needed, the same problem can be formulated by LMI optimization as a suboptimization problem. It is well known that the LMI optimization problem can be solved by convex optimization, and thus, the global minimum can always be found. First, we have the following corollary on the basis of Lemma 5.

Corollary 8 *Consider the uncertain stochastic system* (17) *with the state feedback strategy* $u_i(t) = K_i x(t)$ *and the corresponding cost function* (19). *If there exists a positive definite symmetric matrix* $P > 0$ *that satisfies the following stochastic algebraic Riccati strict inequality* (SARSI)

$$PA_K + A_K^T P + \mu PDD^T P + \mu^{-1} E_K^T E_K + \sum_{p=1}^{M} A_p^T P A_p + \sum_{i=1}^{N} \gamma_i K_i^T R_i K_i + Q < 0, \quad (28)$$

then the closed loop stochastic system is EMSS. Moreover, $J < E[x^T(0)Px(0)] = \mathbf{Tr}[M_0 P]$.

Taking into consideration the aforementioned property, we have the following result.

Theorem 9 *Consider the uncertain stochastic systems described by* (17) *and cost function* (19). *Suppose that the following LMI* (29) *has a solution set of a symmetric positive definite matrix* $X \in \Re^{n\times n}$, *matrix* $Y_i \in \Re^{m_i \times n}$ *and positive scalar* μ.

$$\Lambda := \begin{bmatrix} \Phi & XA_1^T & \cdots & XA_M^T & \Psi^T & X & Y_1^T & \cdots & Y_N^T \\ A_1 X & -X & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ A_M X & 0 & \cdots & -X & 0 & 0 & 0 & \cdots & 0 \\ \Psi & 0 & \cdots & 0 & -\mu I_s & 0 & 0 & \cdots & 0 \\ X & 0 & \cdots & 0 & 0 & -Q^{-1} & 0 & \cdots & 0 \\ Y_1 & 0 & \cdots & 0 & 0 & 0 & \Gamma_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \ddots & 0 \\ Y_N & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Gamma_N \end{bmatrix} < 0, \quad (29)$$

where $\Phi := XA^T + AX + \sum_{i=1}^{N}(Y_i^T B_i^T + B_i Y_i) + \mu DD^T$, $\Psi := E_a X + \sum_{i=1}^{N} E_{bi} Y_i$ *and* $\Gamma_i := -(\gamma_i R_i)^{-1}$.

If such conditions are satisfied, $K_i = Y_i X^{-1}$ *is the GCC gain matrix for closed-loop uncertain stochastic systems. Furthermore, the value of* (19) *satisfies the following inequality:*

$$J(u_1, \ldots, u_N) \le \mathbf{Tr}[M_0 P] = \mathbf{Tr}[M_0 X^{-1}]. \quad (30)$$

Proof: Pre- and post-multiplying inequality (29) by the positive definite matrix

$$\textbf{block diag}\left(P\ I_n\ \cdots\ I_n\ I_s\ I_n\ I_{m_1}\ \cdots\ I_{m_N}\right) > 0,$$

denoting $X = P^{-1}$ and $Y_i = K_i P^{-1}$ and applying the Schur complement [18], we obtain (28). On the other hand, since the results of the cost bound (30) can be proved by using an argument similar to the one used in the proof of Lemma 5, the proof is omitted. ∎

Since the LMI (29) consists of a solution set of $\boldsymbol{X} \in (\mu,\ X_i,\ Y_1,\ \ldots, Y_N)$, various efficient convex optimization algorithms can be applied. Moreover, its solutions represent a set of guaranteed cost controllers. This parameterized representation can be exploited to design guaranteed cost controllers, which minimize the value of the guaranteed cost for the closed-loop stochastic systems. Consequently, solving the following optimization problem allows us to determine the optimal bound.

$$J(u_1, \ldots, u_N) \le \mathbf{Tr}[M_0 X^{-1}] < \min_{\boldsymbol{X}} \mathbf{Tr}[M_0 Z] = \boldsymbol{J}^*, \quad \boldsymbol{X} \in (\mu,\ X,\ Y_1,\ \ldots, Y_N) \quad (31)$$

such that the LMI (29) and

$$\begin{bmatrix} -Z & I_n \\ I_n & -X \end{bmatrix} < 0. \quad (32)$$

That is, the problem addressed in this paper is as follows: "Find $K_i = Y_i X^{-1}$, $i = 1, \ldots, N$ such that the LMIs (29) and (32) are satisfied and the cost $\mathbf{Tr}[M_0 Z]$ becomes as small as possible."

Finally, the optimization problem that should be solved is given.

Theorem 10 *If the above optimization problem has the solutions* μ, X, Y_i, $i = 1, \ldots, N$ *and* Z, *then the strategy* $u_i(t) = K_i x(t) = Y_i X^{-1} x(t)$ *is the linear state feedback strategy, which ensures the minimization of the guaranteed cost* (31) *for the uncertain stochastic systems.*

Proof: Using Theorem 9, the control strategies $u_i(t) = K_i x(t) = Y_i X^{-1} x(t)$ that consist of the feasible solutions μ, X, Y_i, $i = 1, \ldots, N$ and Z are the guaranteed cost controllers of the uncertain stochastic systems (17). Using the Schur complement of the LMI (32) results in $X^{-1} < Z$. Thus, the minimization of $\mathbf{Tr}[M_0 Z]$ implies the supremum $\boldsymbol{J}$ of the guaranteed cost for the uncertain stochastic systems (17). The optimality of the solution of the optimization problem follows from the convexity of the objective function under the LMI constraints. This is the desired result. ■

This method on the basis of the LMI results provides a conceptually simple procedure and saves considerable computation time as compared to CSAREs.

5 Numerical Example

In order to demonstrate the efficiency of the proposed design methodologies, we present results for the megawatt-frequency control problem of multiarea electric energy systems [11] with a trivial modification. The system matrices are given as follows, where the difference of the area power angle is 60° as compared with [11].

$$A = \begin{bmatrix} 0 & 0.315 & 0 & 0 & 0 & -3.15 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 6 & 0 & 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.333 & 3.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & -13.9 & 0 & -33.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 18.88 & 0 & 0 & 0 & -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.333 & 3.333 \\ 0 & 0 & 0 & 0 & 0 & 0 & -13.9 & 0 & -33.333 \end{bmatrix},$$

$$A_1 = \textbf{block diag}\left(0\ 0\ 0.00249\ 0\ 0\ 0\ 0.00249\ 0\ 0 \right),$$
$$B_1^T = \left[0\ 0\ 0\ 0\ 33.333\ 0\ 0\ 0\ 0 \right],$$
$$B_2^T = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 33.333 \right],$$
$$D^T = \begin{bmatrix} 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix},$$
$$E_a = \begin{bmatrix} 0 & 0 & 1.4029 & 0 & 3.367 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.4029 & 0 & 3.367 \end{bmatrix},$$
$$E_{b1} = \begin{bmatrix} 3.367 \\ 0 \end{bmatrix},\ E_{b2} = \begin{bmatrix} 0 \\ 3.367 \end{bmatrix},$$
$$Q_1 = \textbf{block diag}\left(I_5\ 0.1I_4 \right),$$
$$Q_2 = \textbf{block diag}\left(0.1I_5\ I_4 \right),$$
$$R_1 = R_2 = 0.1,\ \gamma_1 = \gamma_2 = 0.5,\ M_0 = I_9.$$

It should be noted that both the deterministic uncertainty that represents the time constant of the governor and the stochastic uncertainty as the state-dependent noise related to the load frequency constant [11] are considered. That is, suppose that there is uncertainty in the time constant of the governor and the variation is within 10%. This indicates deterministic uncertainty in closed-loop uncertain stochastic systems. On the other hand, we suppose that the error of the load frequency constant is within 5% of the nominal value. Hence, this uncertainty represents the state-dependent noise. Therefore, the proposed design method is very useful because the resulting strategy can be implemented in more practical stochastic uncertain systems.

By applying Theorem 7 and using the Newton's method (27), the linear optimal state feedback strategies are given as

$$K_{1\text{new}}^* = \big[-3.4109e-01\ -2.3887e+01\ -8.0996\ -7.7020\ -2.7589\ -3.3496e+01\ -5.7029\ -3.4447\ -1.0310e-01 \big],$$
$$K_{2\text{new}}^* = \big[3.4706\ -3.7227e+01\ -6.0771\ -3.4642\ -1.0310e-01\ -2.9379e+01\ -9.2869\ -8.4071\ -2.7799 \big],$$
$$\textbf{Tr}[P] = 2.923541327673542e+01,$$
$$\mu_{\text{new}}^* = 3.218409801834177e+03.$$

On the other hand, using the LMI technique of Theorem 9 gives the following results:

$$K_{1\text{lmi}}^* = \big[-3.4112e-01\ -2.3887e+01\ -8.0996\ -7.7020\ -2.7589\ -3.3496e+01\ -5.7028\ -3.4447\ -1.0307e-01 \big],$$
$$K_{2\text{lmi}}^* = \big[3.4706\ -3.7227e+01\ -6.0771\ -3.4642\ -1.0310e-01\ -2.9379e+01\ -9.2869\ -8.4071\ -2.7799 \big],$$
$$\boldsymbol{J}^* = 2.923541341535887e+01,$$
$$\mu_{\text{lmi}}^* = 3.218397065823378e+03.$$

Therefore, it can be easily observed that the exact cost bound by means of Newton's method is smaller than the cost bound obtained by the LMI technique. However, since these results are very close and the technique does not require complex calculations unlike the Newton's method, it is preferable to use the LMI technique for solving the GCC problem with multiple decision makers.

6 Conclusion

The guaranteed cost control (GCC) problem for uncertain stochastic systems with N decision makers has been considered by using two different optimization techniques. In particular, a new interpretation of the optimization method for the GCC problem has been introduced. First, the necessary conditions for the existence of a guaranteed cost controller have been established on the basis of the KKT conditions. As a result, it has been shown that the optimal solutions can be obtained by solving CSAREs.

It should be noted that the same methodology can be applied to the existing GCC problem [1]. Second, in order to simplify the computations, the LMI technique has also been considered. Since this method is based on convex optimization, the global minimum can be attained, as compared to the previous approach, whose solution could be a local minimum. Finally, for a practical megawatt-frequency control problem, we have confirmed that the cost bound obtained via the LMI technique can be reduced by Newton's method. In other words, we can obtain the exact cost bound by using Newton's method instead of the LMI method. However, the LMI technique would still be reliable and useful because the obtained supremum is very close to the exact cost bound and it is very easy to solve the problem by using software such as MATLAB's LMI control Toolbox.

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