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Portfolio Selection Problems with Normal Mixture Distributions Including Fuzziness

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Abstract— **In this paper, several portfolio selection problems with normal mixture distributions including fuzziness are proposed. Until now, many researchers have proposed portfolio models based on the stochastic approach, and there are some models considering both random and ambiguous conditions, particularly using fuzzy random or random fuzzy variables. However, the model including normal mixture distributions with fuzzy numbers has not been proposed yet. Our proposed problems are not well-defined problems due to randomness and fuzziness. Therefore, setting some criterions and introducing chance constrains, main problems are transformed into deterministic programming problems. Finally, we construct a solution method to obtain a global optimal solution of the problem.**

I. Introduction

In recent rapid expansions of investment and financial instability, the role of investment theory becomes more and more important, and so it is time to review the investment theory. Of course, it is easy to decide the most suitable financial assets allocation if investors can receive reliable information with respect to future returns a priori. However, there exist many cases that uncertainty from social conditions has a great influence on the future returns. In the real market, there are random factors derived from statistical analysis of historical data and ambiguous factors such as the psychological aspect of investors and lack of received efficient information. Under such uncertainty situations, they need to consider how to reduce a risk, and it becomes important whether they receive the greatest future profit.

Such a finance assets selection problem is generally called a portfolio selection problem, and various studies have been done till now. Markowitz [19] first proposed mean-variance model in the sense of the mathematical programming. Then, it has been central to research activity in the real financial field and numerous researchers have contributed to the development of the modern portfolio theory (for instance, Luenberger [18], Steinbach [22]). On the other hand, many researchers have proposed models of portfolio selection problems which extended Markowitz model; mean-absolute-deviation model (Konno [16], Konno, et al. [17]), semi-variance model (Bawa and Lindenberg [1]), safety-first model (Elton and Gruber [5]), Value at Risk and conditional Value at Risk model (Rockafellar and Uryasev [21]), etc..

In such previous researches, expected future return and variance of each asset are assumed to be known, and in this case, the mean-variance model is equivalent to a quadratic convex programming problem. Therefore, its optimal portfolio is analytically obtained. Particularly, in previous many studies in the sense of mathematical programming for the investment, future returns are assumed to be continuous random variables according to normal distributions. However, from recent experimental studies of investment markets, it is often shown that future returns do not occur according to normal distribution, but fat or heavy-tail distributions. Therefore, we need to consider portfolio selection problems with more general random distributions in the sense of mathematical programming. In this paper, we deal with a normal mixture distribution which is one of heavy-tail random distributions, and propose portfolio selection problems extending the previous models with normal distributions.

Furthermore, considering efficient or inefficient received information, the institution of expert decision maker and the existence of marginal random distribution, we need to consider that statistical distribution under these conditions includes some ambiguity and some flexibility. In this paper, we propose more extensional portfolio selection models including fuzzy factors. Until now, there are some basic researches under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol and Perez-Gladish [2], Carlsson et al. [3], Guo and Tanaka [6], Huang [10, 11], Inuiguchi et al. [12, 13], Katagiri et al. [14, 15], Tanaka et al. [23, 24], Watada [25]). We also proposed some portfolio models with both randomness and fuzziness [7, 8, 9]. However, there are few models considering both normal mixture distribution and ambiguity, simultaneously. Furthermore, there are no researches which are analytically extended and solved these types of portfolio selection problems.

Our proposal models are not well-defined problems due to including randomness and fuzziness in the sense of deterministic mathematical programming. Therefore, in this paper, we introduce chance constraints and transform main problems into the deterministic equivalent problems. Consequently, we construct the analytical solution method of proposed portfolio selection problems to apply those of previous portfolio models.

This paper is organized as follows. In Section 2, we introduce notations of parameters in this paper. Then, in Section 3, we formulate the basic mean-variance portfolio selection problems minimizing the total variance and maximizing the total future return with normal mixture distributions, respectively, and introduce the uncertainty sets. With respect to several portfolio selection problems including randomness and fuzziness, we construct the solution method. Finally, in Section 4, we conclude this paper and discuss future research problems.

II. NOTATIONS OF PARAMETERS

Notations of parameters in this paper are as follows:

$\boldsymbol{r}$: random column vector

$\boldsymbol{m}$: mean value column vector of random variable $\boldsymbol{r}$

$\mathbf{V}$: variance-covariance matrix of random variable $\boldsymbol{r}$

r_G : target value of the total future return

σ_G : target value of the total variance

a_j : cost coefficient

b : Upper limited value

b_j : Upper limited value of purchasing volume

$\boldsymbol{x}$: Purchasing volume (Decision variable)

III. MAIN RESULT

A. Formulation

In this paper, we deal with the following standard Markowitz model for portfolio selection problem involving normal mixture distributions with respect to the future returns:

$$\begin{aligned} &\text{Minimize} \quad \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{m}^t \boldsymbol{x} \geq r_G \\ &\qquad \boldsymbol{x} \in X \triangleq \left\{ \boldsymbol{x} \left| \begin{array}{l} \sum_{j=1}^{n} a_j x_j \leq b, \\ 0 \leq x_j \leq b_j, \ (j = 1, 2, ..., n) \end{array} \right. \right\} \end{aligned} \tag{1}$$

In the case that we obtain the strict value of parameters $\overline{\boldsymbol{r}}$ and $\mathbf{V}$, problem (3) is equivalent to a quadratic programming problem and we find an optimal portfolio using standard convex programming approaches. Furthermore, while problem (3) considers minimizing the total variance, the case maximizing the total future return is formulated as the following form:

$$\begin{aligned} &\text{Maximize} \quad \boldsymbol{m}^t \boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} \leq \sigma_G, \ \boldsymbol{x} \in X \end{aligned} \tag{2}$$

This problem is also a quadratic programming problem and so we obtain an optimal portfolio using the basic convex programming approach.

However, since it is rare that future returns occur according to the normal distribution in practical investment fields, we need to consider that future returns occur according to heavier tailed distributions than normal. Therefore, in this paper, we consider that each random variable r_j occurs according to the following normal mixture distribution which is heavier tailed than general normal distributions based on the study [20]:

$$\boldsymbol{r} = \boldsymbol{m} + \sqrt{W}\mathbf{AZ} \tag{3}$$

where each notation of parameters is as follows:

$\mathbf{Z}$: Random vector to occur according to the multi-dimensional normal distribution $N_n(0, I_n)$

$\mathbf{A}$: Matrix satisfying $\Sigma = \mathbf{A}^t\mathbf{A} \in \mathbb{R}^{n \times n}$

W : Non-negative scalar random variable

Subsequently, we assume matrix Σ to be a positive definite matrix. If parameter W is fixed, random variables r_j occur according to the following normal distribution:

$$\boldsymbol{r} | W = w \sim N_n(\boldsymbol{m}, w\Sigma) \tag{4}$$

where we assume that random variable W is independent on matrix Σ. In case (4), since random variables r_j are basic normal distributions, we analytically obtain an optimal portfolio of problems (1) and (2). However, W is a random variable, and so random variables r_j are not normal random distributions. Then, the expected value and covariance of random variable column vector $\boldsymbol{r}$ are as follows:

$$\begin{aligned} E(\boldsymbol{r}) &= E(E(\boldsymbol{r}|W)) = \boldsymbol{m} \\ \text{cov}(\boldsymbol{r}) &= E(\text{cov}(\boldsymbol{r}|W)) + \text{cov}(E(\boldsymbol{r}|W)) \\ &= E(W)\Sigma \end{aligned} \tag{5}$$

Therefore, using this expression (5), the expected return and covariance of random variable $\boldsymbol{r}^t\boldsymbol{x}$ are as follows:

$$\begin{aligned} E(\boldsymbol{r}^t\boldsymbol{x}) &= E(E(\boldsymbol{r}^t\boldsymbol{x}|W)) = \boldsymbol{m}^t\boldsymbol{x} \\ \text{cov}(\boldsymbol{r}^t\boldsymbol{x}) &= E(\text{cov}(\boldsymbol{r}^t\boldsymbol{x}|W)) + \text{cov}(E(\boldsymbol{r}^t\boldsymbol{x}|W)) \\ &= \boldsymbol{x}^t E(W)\Sigma\boldsymbol{x} \end{aligned} \tag{6}$$

Then, we equivalently transform main problems (1) and (2) into the following problems:

$$\begin{aligned} &\text{Minimize} \quad \boldsymbol{x}^t E(W)\Sigma\boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{m}^t\boldsymbol{x} \geq r_G, \ \boldsymbol{x} \in X \end{aligned} \tag{7}$$

$$\begin{aligned} &\text{Maximize} \quad \boldsymbol{m}^t\boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{x}^t E(W)\Sigma\boldsymbol{x} \leq \sigma_G, \ \boldsymbol{x} \in X \end{aligned} \tag{8}$$

In this paper, we assume that the random variable W is the following discrete distribution introducing probabilities p_j :

$$W = \begin{cases} w_1 & \Pr\{W = w_1\} = p_1 \\ w_2 & \Pr\{W = w_2\} = p_2 \\ \vdots & \vdots \\ w_S & \Pr\{W = w_m\} = p_S \end{cases}, \ \sum_{i=1}^{S} p_i = 1 \tag{9}$$

where S is the total number of discrete values. Using this discrete distribution, we obtain the expected value as $E(W) = \sum_{i=1}^{S} p_i w_i$. Therefore, problems (8) and (9) are equivalently transformed into the following problems:

$$\begin{aligned} &\text{Minimize} \quad \boldsymbol{x}^t \left(\sum_{i=1}^{S} p_i w_i \right) \Sigma \boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{m}^t \boldsymbol{x} \geq r_G, \ \boldsymbol{x} \in X \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{Maximize} \quad \boldsymbol{m}^t \boldsymbol{x} \\ &\text{subject to} \quad \boldsymbol{x}^t \left(\sum_{i=1}^{S} p_i w_i \right) \Sigma \boldsymbol{x} \leq \sigma_G, \ \boldsymbol{x} \in X \end{aligned} \tag{11}$$

With respect to problems (10) and (11), if random distribution of parameter W is obtained and each parameter is constant, we may solve these problems analytically using similar methods to problems (1) and (2). However, considering ambiguity conditions such as subjectivity of the decision maker and the lack of received reliable information, it is natural that random variable W and each parameter include fuzziness. Therefore, we consider the following cases where problems (10) and (11) include fuzziness.

B. The portfolio model including fuzzy expected value

First, we consider the case where each expected value $\boldsymbol{m}$ includes fuzziness and is assumed to be a fuzzy number. This case is considered that the decision maker is a veteran investor and performs the more aggressive prediction than the statistical analysis derived from historical data.

In this subsection, since random distribution of parameter W is obtained and discrete value w_i and its probability p_i are constant, expected value $E(W)$ is also a constant. Then, the membership function of fuzzy numbers $\boldsymbol{m}$ is assumed to be a triangle fuzzy number and introduced by the following function:

$$\mu_{\tilde{m}_j}(\omega) = \begin{cases} \dfrac{\omega - (m_j - \gamma_j)}{\gamma_j} & (m_j - \gamma_j \leq \omega \leq m_j) \\ \dfrac{(m_j + \delta_j) - \omega}{\delta_j} & (m_j < \omega \leq m_j + \delta_j) \\ 0 & (\omega < m_j - \gamma_j \ , m_j + \delta_j < \omega) \end{cases} \tag{12}$$

where γ_j and δ_j are spreads of left and right side, respectively. In this paper, we assume that provided all fuzzy numbers are initially determined by the decision maker. Using these fuzzy numbers, the total future expected return $\tilde{R} = \tilde{\boldsymbol{m}}^t \boldsymbol{x}$ is also a fuzzy numbers characterized by the following membership function:

$$\mu_{\tilde{R}}(\omega) = \begin{cases} \dfrac{\omega - R_L}{\sum_{j=1}^{n} \gamma_j x_j} & \left(R_L \leq \omega \leq \sum_{j=1}^{n} m_j x_j \right) \\ \dfrac{R_U - \omega}{\sum_{j=1}^{n} \delta_j x_j} & \left(\sum_{j=1}^{n} m_j x_j < \omega \leq R_U \right) \\ 0 & (\omega < R_L, \ R_U < \omega) \end{cases} \tag{13}$$

$$R_L = \sum_{j=1}^{n} m_j x_j - \sum_{j=1}^{n} \gamma_j x_j, \ R_U = \sum_{j=1}^{n} m_j x_j + \sum_{j=1}^{n} \delta_j x_j$$

Due to these fuzzy numbers, problems (10) and (11) are not well-defined problem in the sense of deterministic mathematical programming. Therefore, in order to solve main problem analytically, we need to set some criterion for fuzzy variables. In this paper, we consider the case where the decision maker usually has a goal to earn the total profit more than the target value. Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan in many real decision cases, we give a fuzzy goal to the target future return as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of probability $\mu_{\tilde{G}}(\omega)$ which is represented by,

$$\mu_{\tilde{G}}(\omega) = \begin{cases} 0 & \omega \leq f_L \\ \dfrac{\omega - f_L}{f_U - f_L} & f_L \leq \omega \leq f_U \\ 1 & f_U \leq \omega \end{cases} \tag{14}$$

Furthermore, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\prod\nolimits_{\tilde{R}} (\tilde{G}) = \sup_f \min \left\{ \mu_{\tilde{R}}(f), \ \mu_{\tilde{G}}(f) \right\} \tag{15}$$

Introducing this degree of possibility, we consider the following portfolio selection problems based on problems (10) and (11).

$$\begin{aligned} &\text{Minimize} \quad \boldsymbol{x}^t E(W) \Sigma \boldsymbol{x} \\ &\text{subject to} \quad \prod\nolimits_{\tilde{R}} (\tilde{G}) \geq h, \ \boldsymbol{x} \in X \end{aligned} \tag{16}$$

$$\begin{aligned}&\text{Maximize} \quad \prod_{\tilde{R}}\left(\tilde{G}\right)\\&\text{subject to} \quad E\left(W\right)\boldsymbol{x}^t\Sigma\boldsymbol{x} \le \sigma_G,\ \boldsymbol{x}\in X\\&\qquad \text{Maximize} \quad h\\&\Leftrightarrow \quad \text{subject to} \quad \prod_{\tilde{R}}\left(\tilde{G}\right)\ge h,\\&\qquad\qquad E\left(W\right)\boldsymbol{x}^t\Sigma\boldsymbol{x} \le \sigma_G,\ \boldsymbol{x}\in X\end{aligned} \tag{17}$$

In this problem, each constraint $\Pi_{\tilde{P}_i}\left(G_i\right)\ge h$ is transformed into the following inequality:

$$\begin{aligned}&\prod_{\tilde{R}}\left(\tilde{G}\right)\ge h\\&\Leftrightarrow \quad \sup_f \min\left\{\mu_{\tilde{R}}\left(f\right),\ \mu_{\tilde{G}}\left(f\right)\right\}\ge h\\&\Leftrightarrow \quad \exists f:\ \mu_{\tilde{R}}\left(f\right)\ge h,\ \mu_{\tilde{G}}\left(f\right)\ge h\\&\Leftrightarrow \quad \sum_{j=1}^{n} m_j x_j + (1-h)\sum_{j=1}^{n}\delta_j x_j \ge (1-h)\, f_L + f_U\end{aligned} \tag{18}$$

Therefore, problems (16) and (17) are equivalently transformed into the following problems:

$$\begin{aligned}&\text{Minimize} \quad \boldsymbol{x}^t E\left(W\right)\Sigma\boldsymbol{x}\\&\text{subject to} \quad \sum_{j=1}^{n} m_j x_j + (1-h)\sum_{j=1}^{n}\delta_j x_j \ge (1-h)\, f_L + h f_U,\\&\qquad \boldsymbol{x}\in X\end{aligned} \tag{19}$$

$$\begin{aligned}&\text{Maximize} \quad \frac{\sum_{j=1}^{n} m_j x_j + \sum_{j=1}^{n}\delta_j x_j - f_L}{\sum_{j=1}^{n}\delta_j x_j + f_U - f_L}\\&\text{subject to} \quad \boldsymbol{x}^t E\left(W\right)\Sigma\boldsymbol{x}\le\sigma_G,\ \boldsymbol{x}\in X\end{aligned} \tag{20}$$

Problem (19) is equivalent to problem (10), and so we analytically solve problem (19) using the same solution method to problem (10). Then, problem (20) is a standard fractional programming problem and the numerator and denominator of objective function are linear functions. Therefore, by performing the equivalent transformation using the fractional programming approach, problem (20) is a similar problem to problem (11), and so we also solve problem (20) analytically using the same solution method to problem (11).

C. The portfolio model including the fuzzy random variable

In the previous subsection, we considered the case where each expected value is assumed to be a fuzzy number. However, in real-world decision cases, it is difficult to set not only the expected value but also the possible value of random variable w_i or the occurrence probability p_i strictly. Therefore, in this subsection, we consider the case where random variable W also includes flexibility and is assumed to be a fuzzy number.

First, we assume that the possible value w_i includes the ambiguity and represents a fuzzy number. Then, the membership function of each value w_i is introduced by the following functions:

$$W=\begin{cases}\tilde{w}_1 & \Pr\left\{W=w_1\right\}=p_1\\ \tilde{w}_2 & \Pr\left\{W=w_2\right\}=p_2\\ \vdots & \vdots\\ \tilde{w}_m & \Pr\left\{W=w_m\right\}=p_m\end{cases}$$

$$\mu_{\tilde{w}_i}\left(\omega\right)=\begin{cases}\dfrac{\overline{w}_i-\omega}{\alpha_i} & \left(\overline{w}_i-\alpha_i\le\omega\le\overline{w}_i\right)\\ \dfrac{\omega-\overline{w}_i}{\beta_i} & \left(\overline{w}_i<\omega\le\overline{w}_i+\beta_i\right)\\ 0 & \left(\omega<\overline{w}_i-\alpha_i\ ,\overline{w}_i+\beta_i<\omega\right)\end{cases} \tag{21}$$

Using these membership functions and the extension principle of fuzzy theory, expected value of $E\left(W\right)$ is obtained as the following form:

$$\mu_{\tilde{E}(W)}\left(\omega\right)=\begin{cases}\dfrac{\omega-E_L}{\sum_{i=1}^{m}p_i\alpha_i} & \left(E_L\le\omega\le\sum_{i=1}^{m}p_i\overline{w}_i\right)\\ \dfrac{E_U-\omega}{\sum_{i=1}^{m}p_i\beta_i} & \left(\sum_{i=1}^{m}p_i\overline{w}_i<\omega\le E_U\right)\\ 0 & \left(\omega<E_L\ ,E_U<\omega\right)\end{cases} \tag{22}$$

$$E_L\left(W\right)=\sum_{i=1}^{m}p_i\overline{w}_i-\sum_{i=1}^{m}p_i\alpha_i,\ E_U\left(W\right)=\sum_{i=1}^{m}p_i\overline{w}_i+\sum_{i=1}^{m}p_i\beta_i$$

Therefore, we obtain the following membership function of the total variance:

$$\mu_{\tilde{V}}\left(\omega\right)=\begin{cases}\dfrac{\omega-V_L}{\boldsymbol{x}^t\left(\sum_{i=1}^{m}p_i\alpha_i\right)\Sigma\boldsymbol{x}}\\ \dfrac{V_U-\omega}{\boldsymbol{x}^t\left(\sum_{i=1}^{m}p_i\beta_i\right)\Sigma\boldsymbol{x}}\\ 0\end{cases} \tag{23}$$

$$V_L=\boldsymbol{x}^t\left(\left(\sum_{i=1}^{m}p_i\overline{w}_i\right)-\left(\sum_{i=1}^{m}p_i\alpha_i\right)\right)\Sigma\boldsymbol{x},\ V_U=\boldsymbol{x}^t\left(\left(\sum_{i=1}^{m}p_i\overline{w}_i\right)+\left(\left(\sum_{i=1}^{m}p_i\beta_i\right)\right)\right)\Sigma\boldsymbol{x}$$

Furthermore, in a way similar to subsection 2-B, using a concept of possibility measure to the total variance, we introduce the degree of possibility as follows:

$$\prod_{\tilde{V}}\left(\tilde{G}\right)=\sup_\sigma\min\left\{\mu_{\tilde{E}(W)}\left(\sigma\right),\ \mu_{\tilde{G}}\left(\sigma\right)\right\}$$

$$\mu_{\tilde{G}}\left(\omega\right)=\begin{cases}1 & \omega\le\sigma_L\\ \dfrac{\sigma_U-\omega}{\sigma_U-\sigma_L} & \sigma_L\le\omega\le\sigma_U\\ 0 & \sigma_U\le\omega\end{cases} \tag{24}$$

Therefore, from this degree of possibility, problems (10) and (11) are equivalently transformed into the following problems performing the method similar to (18):

$$\begin{aligned}
&\text{Maximize} \quad \prod_{\tilde{V}}\left(\tilde{G}\right) \\
&\text{subject to} \quad \sum_{j=1}^{n} m_j x_j + (1-h)\sum_{j=1}^{n}\delta_j x_j \geq (1-h)\, f_L + h f_U, \\
&\qquad \boldsymbol{x} \in X
\end{aligned}$$

$$\begin{aligned}
&\text{Minimize} \quad \frac{\boldsymbol{x}^t\left(\sum_{i=1}^{m} p_i \overline{w}_i - \left(\sum_{i=1}^{m} p_i \alpha_i\right)\right)\Sigma\boldsymbol{x} - \sigma_U}{\boldsymbol{x}^t\left(\sum_{i=1}^{m} p_i \alpha_i\right)\Sigma\boldsymbol{x} + (\sigma_U - \sigma_L)} \\
\Leftrightarrow\ &\text{subject to} \quad \sum_{j=1}^{n} m_j x_j + (1-h)\sum_{j=1}^{n}\delta_j x_j \geq (1-h)\, f_L + h f_U, \\
&\qquad \boldsymbol{x} \in X,\ \overline{W}(\mu) = \left(\left(\sum_{i=1}^{m} p_i \overline{w}_i\right) - \mu\left(\sum_{i=1}^{m} p_i \alpha_i\right)\right)
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} m_j x_j + \sum_{j=1}^{n}\delta_j x_j - f_L}{\sum_{j=1}^{n}\delta_j x_j + f_U - f_L} \\
&\text{subject to} \quad \prod_{\tilde{V}}\left(\tilde{G}\right) \geq \overline{h},\ \boldsymbol{x} \in X
\end{aligned}$$

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} m_j x_j + \sum_{j=1}^{n}\delta_j x_j - f_L}{\sum_{j=1}^{n}\delta_j x_j + f_U - f_L} \\
\Leftrightarrow\ &\text{subject to} \quad \boldsymbol{x}^t\left(\sum_{i=1}^{m} p_i \overline{w}_i - (1-\overline{h})\left(\sum_{i=1}^{m} p_i \alpha_i\right)\right)\Sigma\boldsymbol{x} \leq \overline{h}\sigma_L + (1-\overline{h})\sigma_U, \\
&\qquad \boldsymbol{x} \in X
\end{aligned} \tag{26}$$

Problem (26) is equivalent to problem (20), and so we analytically and efficiently solve it using the same solution method to problem (20). Then, with respect to problem (25), the numerator and denominator of objective function are convex functions. Therefore, we analytically solve it using the solution method proposed by Dinkelbach [4].

On the other hand, we also consider the case where each occurrence probability includes an ambiguity and is assumed to be the following membership function in a way similar to (21):

$$W = \begin{cases} w_1 & \Pr\{W = w_1\} = \tilde{p}_1 \\ w_2 & \Pr\{W = w_2\} = \tilde{p}_2 \\ \vdots & \qquad \vdots \\ w_m & \Pr\{W = w_m\} = \tilde{p}_m \end{cases}$$

$$\mu_{\tilde{p}_i}(\omega) = \begin{cases} \dfrac{\omega - (\overline{p}_i - \xi_i)}{\xi_i} & (\overline{p}_i - \xi_i \leq \omega \leq \overline{p}_i) \\ \dfrac{(\overline{p}_i + \zeta_i) - \omega}{\zeta_i} & (\overline{p}_i < \omega \leq \overline{p}_i + \zeta_i) \\ 0 & (\omega < \overline{p}_i - \xi_i\ ,\ \overline{p}_i + \zeta_i < \omega) \end{cases} \tag{27}$$

In this case, we assume that each possible occurrence probability p_i includes the h-cut set of the fuzzy number $\tilde{p}_i$. Then, we also apply similar transformations and degree of possibility to this case, and we equivalently transform problems (10) and (11) into the following problems:

$$\begin{aligned}
&\text{Minimize} \quad \boldsymbol{x}^t E(W)\Sigma\boldsymbol{x} \\
&\text{subject to} \quad \sum_{j=1}^{n} m_j x_j + (1-h)\sum_{j=1}^{n}\delta_j x_j \geq (1-h)\, f_L + h f_U,\ \boldsymbol{x} \in X, \\
&\qquad E(W) = \left\{ \min \sum_{i=1}^{m} p_i w_i \left| \begin{array}{l} \overline{p}_i - (1-\overline{h})\xi_i \leq p_i \leq \overline{p}_i + (1-\overline{h})\zeta_i \\ \sum_{i=1}^{m} p_i = 1 \end{array} \right. \right\}
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} m_j x_j + \sum_{j=1}^{n}\delta_j x_j - f_L}{\sum_{j=1}^{n}\delta_j x_j + f_U - f_L} \\
&\text{subject to} \quad \boldsymbol{x}^t E(W)\Sigma\boldsymbol{x} \leq \overline{h}\sigma_L + (1-\overline{h})\sigma_U,\ \boldsymbol{x} \in X, \\
&\qquad E(W) = \left\{ \min \sum_{i=1}^{m} p_i w_i \left| \begin{array}{l} \overline{p}_i - (1-\overline{h})\xi_i \leq p_i \leq \overline{p}_i + (1-\overline{h})\zeta_i \\ \sum_{i=1}^{m} p_i = 1 \end{array} \right. \right\}
\end{aligned} \tag{29}$$

In these problems, the mathematical programming

$$E(W) = \left\{ \min \sum_{i=1}^{m} p_i w_i \left| \begin{array}{l} \overline{p}_i - (1-\overline{h})\xi_i \leq p_i \leq \overline{p}_i + (1-\overline{h})\zeta_i \\ \sum_{i=1}^{m} p_i = 1 \end{array} \right. \right\}$$

is a linear programming problem, and so it is easy to solve this problem. Consequently, problems (28) and (29) are equivalent to problems (25) and (26), respectively. Therefore, we obtain the optimal portfolio with respect to the various types of fuzzy portfolio selection problems with normal mixture random distributions.

IV. NUMERICAL EXAMPLE

In this section, we consider problem (29) and provide a brief numerical example to comparing our proposed model with the basic model. Table 1 shows that we assume four decision variables and three scenarios in the numerical example. Then, all fuzzy numbers are assumed to be symmetric triangle fuzzy numbers and Σ is a symmetric positive definite matrix satisfying $\sigma_{ij} = 0$.

TABLE 1. EXPECTED VALUES AND VARIANCES OF ALL SCENARIOS

	x_1	x_2	x_3	x_4
m_j	0.05	0.13	0.16	0.22
δ_j	0.02	0.08	0.03	0.06
Σ	0.121	0.225	0.283	0.438

$$W = \begin{cases} 0.6 & \Pr\{W = 0.6\} = \langle 0.3, 0.1 \rangle \\ 1.2 & \Pr\{W = 1.2\} = \langle 0.4, 0.2 \rangle \\ 1.6 & \Pr\{W = 1.6\} = \langle 0.3, 0.2 \rangle \end{cases}$$

Furthermore, fuzzy goals are provided as the following forms:

$$\mu_{\tilde{G}}(\omega)=\begin{cases}0\\ \dfrac{\omega-0.13}{0.02}\\ 1\end{cases},\ \mu_{\tilde{G}}(\omega)=\begin{cases}1\\ \dfrac{0.1-\omega}{0.02}\\ 0\end{cases},$$

Using these parameters and fuzzy goals and introducing the feasible area $X \triangleq \left\{ \boldsymbol{x} \left| \sum_{j=1}^{5} x_j = 1,\ 0 \le x_j \le 0.5,\ (j=1,...,5) \right. \right\}$, we solve the basic model (11) and our proposed model (29) in the case that $\sigma_G = 0.1,\ \bar{h} = 0.8$, and obtain the following optimal portfolio as Table 2.

TABLE 2. OPTIMAL PORTFOLIO TO EACH PROBLEM

	x_1	x_2	x_3	x_4
Problem (11)	0.099	0.306	0.319	0.275
Problem (29)	0.198	0.235	0.322	0.245

From Table2, we find that the rate to decision variable x_1 of optimal portfolio in our proposed model (29) is much larger than that of problem (11). This means that our proposed model tends to select assets with small variance and speared value to fuzzy numbers satisfying the future return more than the target value because decision variable x_1 has the smallest value of variance and spread among all decision variables.

V. CONCLUSIONS

In this paper, we have considered portfolio selection problems with normal mixture random distributions involving ambiguous factors, and proposed the new portfolio models extending mean-variance model. Since our proposed models are not well-defined problems due to randomness and fuzziness, we have set some criterion to stochastic and fuzzy aspects, and performing the equivalent transformations. Finally, we have constructed the efficient solution methods based on the standard mean-variance approaches. Therefore, we have developed more versatile portfolio models with randomness and fuzziness than previous standard portfolio models, and we may obtain more beneficial knowledge for the investment theory.

As the future studies, we are going to consider the case where normal mixture distributions are more general patterns than in this paper. Then, we also need to consider the case that the optimal solutions are restricted to be integers.

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