# Nondominated equilibrium solutions of multiobjective two-person nonzero-sum games in normal and extensive forms

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***Abstract*—In this paper, we review the development of studies on multiobjective noncooperative games, and particularly we focus on nondominated equilibrium solutions in multiobjective two-person nonzero-sum games in normal and extensive forms. After outlining studies related to multiobjective noncooperative games, we treat multiobjective two-person nonzero-sum games in normal form, and a mathematical programming problem yielding nondominated equilibrium solutions is shown. As for extensive form games, we first provide a game representation of the sequence form, and then formulate a mathematical programming problem for obtaining nondominated equilibrium solutions.**

## I. Introduction

An equilibrium solution based on the principle of rational responses is an important solution concept in a conventional noncooperative games. As an extension of the equilibrium solution, Pareto equilibrium solutions in multiobjective noncooperative games are defined on the basis of the concept of Pareto optimality from multiobjective optimization. The concept of Pareto optimal solutions is extended to nondominated solutions by using dominance cones [29], [22]. This review paper outlines the development of multiobjective noncooperative games and focuses on nondominated equilibrium solutions to multiobjective two-person nonzero-sum games in normal and extensive forms. Employing the concept of nondominated solutions, Nishizaki and Notsu define nondominated equilibrium solutions in multiobjective two-person nonzero-sum games in normal and extensive forms [15], [16], and give the necessary and sufficient conditions for a pair of mixed strategies to be a nondominated equilibrium solution. Moreover, they formulate mathematical programming problems yielding nondominated equilibrium solutions by using the necessary and sufficient conditions.

## II. Development of studies on multiobjective noncooperative games

Blackwell [1] investigates the properties of the set in which the payoffs of players converge through successive long-run plays in a multiobjective two-person zero-sum game. For a multiobjective two-person game, either zero-sum games or nonzero-sum, Shapley [21] defines a Pareto equilibrium solution by introducing the concept of Pareto optimality from multiobjective optimization. He proves the existence of the Pareto equilibrium solution from the scalarization via a weighting coefficient vector.

In multiobjective two-person zero-sum games, assuming that one player is the nature, Contini et al. [5] consider a multiobjective expected payoff maximization problem for a given probability distribution of strategies of the nature. Moreover, specifying a goal for each of the objectives, they formulate a joint probability maximization problem with respect to goal achievement. Zeleny [30] scalarizes a multiobjective two-person zero-sum game by using a weighting coefficient vector and obtains a minimax solution to the corresponding scalarized single-objective two-person zero-sum game. Especially, he shows that the formulated problem can be reduced to a linear programming problem when one player of the pair is the nature. Moreover, he points out that, because the set of Pareto equilibrium solutions is generally large, it is difficult to select a certain solution among the set and proposes a compromise strategy such that the distance from the ideal point, which is a vector of the maxima of the objectives, is minimized. Introducing a goal for each of the objectives in a multiobjective two-person zero-sum game, Cook [6] formulates the problem minimizing a weighted sum of the differences between the expected payoff vector and the corresponding goals; he shows that the formulated problem can be reduced to a linear programming problem.

Corley [7] provides the necessary and sufficient condition that a pair of mixed strategies is a Pareto equilibrium solution in a multiobjective two-person nonzero-sum game by using the Kuhn-Tucker condition [13] for optimality of the multiobjective mathematical programming problems. Moreover, he shows that a Pareto equilibrium solution is a solution of a parametric linear complementarity problem with parameters being the elements of the weighting coefficient vector.

Ghose and Prasad [9] propose a solution concept of Pareto optimal security strategies which is an extension of a minimax solution of a single-objective two-person zero-sum game. They give a necessary condition and a sufficient condition for a Pareto optimal security strategy from the relationship between a multiobjective game and the corresponding scalarized single-objective game. In a conventional single-objective two-person zero-sum game, a minimax solution is a saddle point, i.e.,

an equilibrium solution; but in a multiobjective two-person zero-sum game, there does not always exist a solution which is not only a Pareto optimal security strategy but also a Pareto equilibrium solution. Ghose [10] proves that all the Pareto optimal security strategies can be obtained through a finite number of scalarizations of a multiobjective games by showing that an extension set of vectors of security levels is polyhedral. Fernandez and Puerto [8] show that the necessary and sufficient condition that a pair of mixed strategies is a Pareto optimal security strategy in multiobjective two-person zero-sum games is that it is a Pareto optimal solution to a certain multiobjective linear programming problem; from this fact, they demonstrate that all the Pareto optimal security strategies can be obtained by finding all the Pareto optimal extreme solutions. Voorneveld [24] newly define a Pareto optimal security strategy from a different viewpoint. Without assuming that the opponent chooses a mixed strategy for each of the objectives separately, he considers a multiobjective two-person zero-sum game where the opponent is allowed to choose only one mixed strategy. By doing so, he constructs a standard matrix game arising from the multiobjective two-person zero-sum games.

Wierzbicki [28] investigates the relationship between the Pareto equilibrium solutions of a multiobjective $n$-person noncooperative game and the equilibrium solutions of the corresponding single-objective game scalerized by the generalized scalarizing functions including the scalarization by a weighting coefficient vector. For multiobjective $n$-person noncooperative games with cross-constrained continuum strategy sets, Charnes et al. [4] define a nondominated equilibrium solution and its extension by using the concept of nondominated solutions based on dominance cones in multiobjective mathematical programming problems; they give necessary conditions and sufficient conditions for an $n$-tuple of strategies to be a nondominated equilibrium solution. However, they do not deal with a multiobjective $n$-person noncooperative game with a discrete set of pure strategies and its probability mixture. Zhao [31] define a hybrid solution and a quasi-hybrid solution on the basis of a Pareto equilibrium solution of a multiobjective $n$-person noncooperative game and the core of an $n$-person cooperative game; he shows the existence of the solutions. Wang [27] investigate the existence of Pareto equilibrium solutions in a multiobjective $n$-person noncooperative game; he presents sufficient conditions to guarantee the existence of a Pareto equilibrium solution. Voorneveld et al. [26] study axiomatic properties of the Pareto equilibrium solutions by extending the axiomatization of the equilibrium solution of a single-objective $n$-person noncooperative game [19]. Voorneveld et al. [25] define ideal equilibrium solutions which maximize all the objectives for all players and examine some properties of the solutions.

Sakawa and Nishizaki [20] incorporate a fuzzy goal with respect to each of the objectives in a multiobjective two-person zero-sum game with fuzzy payoffs and examine a minimax strategy for degrees of attainment of the fuzzy goals. Nishizaki and Sakawa [17], [18] extend the results by Sakawa and Nishizaki to a multiobjective two-person nonzero-sum game without and with fuzzy payoffs; they formulate a mathematical programming problem yielding the equilibrium solutions.

Borm et al. [3] study the structure of a set of Pareto equilibrium solutions of a multiobjective two-person nonzero-sum game; they show that a set of Pareto equilibrium solutions is not always a union of polytopes if at least one player has two or more objectives and both players have three or more pure strategies. Moreover, defining a set of mixed strategies of the opponent in which a subset of pure strategies are optimal responses, they investigate the characteristics of Pareto equilibrium solutions.

Using the concept of nondominated solutions which is an extension of that of Pareto optimal solutions to multiobjective mathematical programming problems [29], [22], Nishizaki and Notsu [15] consider nondominated equilibrium solutions in multiobjective two-person nonzero-sum game in normal form.

Extension of games in extensive form under a multiobjective environment is made by Krieger [12], and existence of Pareto equilibrium solutions is considered. For multiobjective two-person nonzero-sum game in extensive form, Nishizaki and Notsu [15] define a nondominated equilibrium solution based on dominance cones by employing the sequence form [23], [11] which is a representation with compact mathematical formulation for games in extensive form.

## III. Nondominated equilibrium solutions of multiobjective two-person nonzero-sum games in normal form

### A. Multiobjective two-person nonzero-sum game

A multiobjective two-person nonzero-sum game can be represented by the following multiple $m \times n$ matrices:

$$A^{k_1} = \begin{bmatrix} a_{11}^{k_1} & \cdots & a_{1n}^{k_1} \\ \vdots & \ddots & \vdots \\ a_{m1}^{k_1} & \cdots & a_{mn}^{k_1} \end{bmatrix}, \ k_1 = 1, \ldots, r_1, \tag{1a}$$

$$B^{k_2} = \begin{bmatrix} b_{11}^{k_2} & \cdots & b_{1n}^{k_2} \\ \vdots & \ddots & \vdots \\ b_{m1}^{k_2} & \cdots & b_{mn}^{k_2} \end{bmatrix}, \ k_2 = 1, \ldots, r_2. \tag{1b}$$

In the game $(A,B)$, $A \triangleq (A^1, \ldots, A^{r_1})^T$, $B \triangleq (B^1, \ldots, B^{r_2})^T$, player 1 has $m$ pure strategies and $r_1$ objectives, and player 2 has $n$ pure strategies and $r_2$ objectives, where a superscription $T$ means the transposition of a vector or a matrix. Then, when player 1 chooses a pure strategy $i \in \{1, \ldots, m\}$ and player 2 chooses $j \in \{1, \ldots, n\}$, player 1 obtains a payoff vector $(a_{ij}^1, \ldots, a_{ij}^{r_1})$ and player 2 obtains a payoff vector $(b_{ij}^1, \ldots, b_{ij}^{r_2})$.

We define the following sets $X$ and $Y$ of mixed strategies

of players 1 and 2, respectively:

$$X \triangleq \left\{ x = (x_1, \ldots, x_m)^T \mid \sum_{i=1}^{m} x_i = 1,\ x_i \geqq 0,\ i = 1, \ldots, m \right\}, \tag{2a}$$

$$Y \triangleq \left\{ y = (y_1, \ldots, y_n)^T \mid \sum_{j=1}^{n} y_j = 1,\ y_j \geqq 0,\ j = 1, \ldots, n \right\}. \tag{2b}$$

When player 1 chooses a mixed strategy $x \in X$ and player 2 chooses $y \in Y$, expected payoff vectors of both players are expressed as follows:

$$x^T A y \triangleq (x^T A^1 y, \ldots, x^T A^{r_1} y)^T, \tag{3a}$$

$$x^T B y \triangleq (x^T B^1 y, \ldots, x^T B^{r_2} y)^T. \tag{3b}$$

*B. Nondominated solutions to a multiobjective mathematical programming problem*

Before examining nondominated equilibrium solutions in multiobjective two-person nonzero-sum games, we first review solutions concepts and related matters in multiobjective mathematical programming. For convenience, let us introduce the following notation: for any two vectors $z, z' \in \mathbb{R}^N$, $z = z' \Leftrightarrow z_i = z_i'$, $i = 1, \ldots, N$; $z \leqq z' \Leftrightarrow z_i \leqq z_i'$, $i = 1, \ldots, N$; $z < z' \Leftrightarrow z_i < z_i'$, $i = 1, \ldots, N$; $z \leq z' \Leftrightarrow z \leqq z'$ and $z \neq z'$.

Let $z$ be an $N$-dimension real decision variable. Consider a multiobjective mathematical programming problem minimizing $K$ objective functions $f(z) = (f_1(z), \ldots, f_K(z))^T$ subject to $M_1$ inequality constraints $g(z) = (g_1(z), \ldots, g_{M_1}(z))^T \leqq 0$ and $M_2$ equality constraints $h(z) = (h_1(z), \ldots, h_{M_2}(z))^T = 0$, where 0 is an appropriate dimensional zero vector $(0, \ldots, 0)^T$ corresponding to a dimension of the left hand side. Then, a multiobjective mathematical programming problem can be written as:

$$\min \quad f(z) \tag{4a}$$

$$\text{s. t.} \quad z \in Z \triangleq \{ z \in \mathbb{R}^N \mid g(z) \leqq 0,\ h(z) = 0 \}. \tag{4b}$$

Let $O = \{ f(z) \in \mathbb{R}^K \mid z \in Z \}$ be a feasible area of the multiple objective values in an objective space.

There does not generally exist a solution minimizing all the objectives simultaneously. Then, Pareto optimal solutions such that any improvement of one objective can be achieved only at the expense of another are introduced, and they are defined as follows.

*Definition 3.1:* $z^* \in Z$ is said to be a Pareto optimal solution if there does not exist another $z \in Z$ such that $f(z) \leq f(z^*)$.

As a slightly weaker solution concept than Pareto optimality, weak Pareto optimal solutions are defined by replacing $\leq$ with $<$ in the above definition.

Next, we present a definition of a nondominated solution proposed by Yu [29] which is a solution concept generalized from a Pareto optimal solution. To begin with, we give definitions of a cone and related concepts. A set $\Lambda \subset \mathbb{R}^K$ is said to be a cone if, for any vector $u \in \Lambda$ and nonnegative scalar $\eta \geqq 0$, $\eta u \in \Lambda$ holds. $\Lambda$ is a convex cone if, for any two vector $u^1, u^2 \in \Lambda$ and two nonnegative scalars $\eta^1, \eta^2 \geqq 0$, $\eta^1 u^1 + \eta^2 u^2 \in \Lambda$ holds. A polar cone of $\Lambda$ is given as

$$\Lambda^* = \{ \gamma \in \mathbb{R}^K \mid \gamma^T u \leqq 0,\ \forall u \in \Lambda \}. \tag{5}$$

We define a domination cone prescribing a preference relation. For $o, o' \in O \subset \mathbb{R}^K$, when $o$ is preferred to $o'$, it is denoted by $o \succ o'$. Then, a domination cone is defined as follows.

*Definition 3.2:* Given $o \in O \subset \mathbb{R}^K$, a nonzero vector $d \in \mathbb{R}^K$ is a domination factor for $o$ if $o \succ o + \rho d$ for all $\rho > 0$. Then, a domination cone $D(o)$ of $o$ is a set of all domination factors for $o$.

Throughout this paper, we use only a constant domination cone $\Lambda \triangleq D(o)$ for all $o \in O$, and simply call $\Lambda$ a domination cone. Furthermore, we restrict a domination cone to a polyhedral cone with nonempty interior which can be represented in the following by using its generator $\hat{V} = \{ \hat{v}^t \mid t = 1, \ldots, p \}$:

$$\Lambda = \left\{ \pi \in \mathbb{R}^K \;\middle|\; \pi = \sum_{t=1}^{p} \tau_t \hat{v}^t, \tau_t \geqq 0,\ t = 1, \ldots, p \right\}. \tag{6}$$

Then, a multiobjective mathematical programming problem can be defined by the three tuple $(Z, f(z), \Lambda)$, where $Z = \{ z \in \mathbb{R}^N \mid g(z) \leqq 0, h(z) = 0 \}$ is a feasible region, $f(z)$ is a vector of the multiple objectives, and $\Lambda \subset \mathbb{R}^K$ is a domination cone. A nondominated solution to a multiobjective mathematical programming problem $(Z, f(z), \Lambda)$ is defined as follows.

*Definition 3.3:* Given a multiobjective mathematical programming problem $(Z, f(z),\ \Lambda)$, $z^* \in Z$ is said to be a nondominated solution if there does not exist another $z \in Z$ such that

$$f(z^*) \in f(z) + \Lambda \quad \text{and} \quad f(z) \neq f(z^*). \tag{7}$$

If a domination cone $\Lambda$ is the negative quadrant, any nondominated solution is also a Pareto optimal solution.

A condition that a point is a nondominated solution is given by Yu [29] and Tamura and Miura [22]. Because we restrict a domination cone to a polyhedral cone and the Tamura and Miura condition is a more natural extension of the Kuhn and Tucker condition [13] of optimality for a multiobjective mathematical programming problem, we employ the Tamura and Miura condition to develop a condition that a pair of mixed strategies is a nondominated equilibrium solution.

A polar cone $\Lambda^*$ for a domination cone can be represented in the following by using its generator $V = \{ v^t \mid t = 1, \ldots, q \}$:

$$\Lambda^* = \left\{ \omega \in \mathbb{R}^K \;\middle|\; \omega = \sum_{t=1}^{q} \zeta_t v^t,\ \zeta_t \geqq 0,\ t = 1, \ldots, q \right\}. \tag{8}$$

Let

$$F(z) = [\nabla f(z)^T v^1, \ldots, \nabla f(z)^T v^q], \tag{9a}$$

$$\nabla f(z)^T v^t = \begin{bmatrix} \frac{\partial f_1(z)}{\partial z_1} & \cdots & \frac{\partial f_K(z)}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1(z)}{\partial z_N} & \cdots & \frac{\partial f_K(z)}{\partial z_N} \end{bmatrix} \begin{bmatrix} v_1^t \\ \vdots \\ v_K^t \end{bmatrix}, \quad t \in \{1, \ldots, q\}. \tag{9b}$$

For a multiobjective mathematical programming problem $(Z, f(z), \Lambda)$, assume that $g(z)$ and $h(z)$ satisfy the Slater constraint qualification, ${v^t}^T f(z)$, $t = 1, \ldots, q$ are concave, and $Z$ is a convex set. Then, the following necessary and sufficient condition is given by Tamura and Miura [22]. $z \in Z$ is a nondominated solution if and only if there exist vectors $\mu \geq 0$, $\lambda \geqq 0$ and $\psi$ such that

$$F(z)\mu - \nabla g(z)^T \lambda - \nabla h(z)^T \psi = 0 \tag{10a}$$
$$g(z)^T \lambda = 0 \tag{10b}$$
$$g(z) \leqq 0 \tag{10c}$$
$$h(z) = 0. \tag{10d}$$

If the generator of the polar cone of the domination cone is specified by $V^0 = \{v^1 = (1,0,\ldots,0)^T, \ldots, v^K = (0,\ldots,0,1)^T\}$, the Tamura and Miura condition corresponds to the Kuhn and Tucker condition [13] for Pareto optimality to a multiobjective mathematical programming problem.

### *C. Nondominated equilibrium solutions of a multiobjective game*

First, we show a definition of Pareto equilibrium solutions given by Shapley [21], which can be considered as a special case of nondominated equilibrium solutions.

*Definition 3.4:* In a multiobjective two-person nonzero-sum game $(A, B)$, a pair of strategies $(x^*, y^*) \in X \times Y$ is said to be a Pareto equilibrium solution if there does not exist another $(x, y) \in X \times Y$ such that

$${x^*}^T A y^* \leq x^T A y^*, \quad {x^*}^T B y^* \leq {x^*}^T B y. \tag{11}$$

A multiobjective two-person nonzero-sum game $(A, B)$ can be reduced to a single-objective two-person nonzero-sum game by using a weighting coefficient vector $(w, v) \in \mathbb{R}^{r_1}_{++} \times \mathbb{R}^{r_2}_{++}$, where $\mathbb{R}^t_{++} = \{z \in \mathbb{R}^t \mid z > 0\}$. Because there exists at least one equilibrium solution in a single-objective two-person nonzero-sum game, it is known that there also exists at least one Pareto equilibrium solution [21], [7].

Let $f^1(x; y) \triangleq x^T A y$ and $f^2(y; x) \triangleq x^T B y$, and we define nondominated equilibrium solutions in the following.

*Definition 3.5:* Let $\Lambda^1$ and $\Lambda^2$ denote domination cones of players 1 and 2, respectively. Then, in a multiobjective two-person nonzero-sum game $(A, B)$, a pair of strategies $(x^*, y^*) \in X \times Y$ is said to be a nondominated equilibrium solution if there does not exist another $(x, y) \in X \times Y$ such that

$$f^1(x^*; y^*) \in f^1(x; y^*) + \Lambda^1, \quad f^2(y^*; x^*) \in f^2(y; x^*) + \Lambda^2. \tag{12}$$

In particular, by letting $\Lambda^1 = \mathbb{R}^{r_1}_-$ and $\Lambda^2 = \mathbb{R}^{r_1}_-$, any nondominated equilibrium solution is also a Pareto equilibrium solution, where $\mathbb{R}^t_- = \{z \in \mathbb{R}^t \mid z \leqq 0\}$.

The above definition means that $x^*$ is a nondominated response of player 1 for a strategy $y^*$ of player 2, and $y^*$ is a nondominated response of player 2 for a strategy $x^*$ of player 1. This can be explicitly expressed as follows. Sets of nondominated responses of players 1 and 2 are defined as

$$N^1(y, \Lambda^1) = \{x \in X \mid \text{there does not exist } x' \in X \text{ such that } f^1(x; y) \in f^1(x'; y) + \Lambda^1\}, \tag{13a}$$
$$N^2(x, \Lambda^2) = \{y \in Y \mid \text{there does not exist } y' \in Y \text{ such that } f^2(y; x) \in f^2(y'; x) + \Lambda^2\}. \tag{13b}$$

Then, by using the concept of nondominated responses, a set $N(\Lambda^1, \Lambda^2)$ of nondominated equilibrium solutions can be represented by

$$N(\Lambda^1, \Lambda^2) = \{(x^*, y^*) \mid x^* \in N^1(y^*, \Lambda^1),\ y^* \in N^2(x^*, \Lambda^2)\}. \tag{14}$$

A relation between the domination cones and the sets of nondominated equilibrium solutions is shown in the following proposition.

*Proposition 3.1:* Let $\Lambda^1$ and $\Lambda^{1'}$ denote domination cones of player 1, and $\Lambda^2$ and $\Lambda^{2'}$ denote domination cones of player 2 in a multiobjective two-person nonzero-sum game $(A, B)$. Then, if $\Lambda^1 \subset \Lambda^{1'}$ and $\Lambda^2 \subset \Lambda^{2'}$, $N(\Lambda^{1'}, \Lambda^{2'}) \subset N(\Lambda^1, \Lambda^2)$.

From the fact that there exists at least one Pareto equilibrium solution [21], we obtain the following theorem showing the existence of nondominated equilibrium solutions.

*Theorem 3.1:* In a multiobjective two-person nonzero-sum game $(A, B)$ in normal form, for any domination cones of players 1 and 2, there exists at least one nondominated equilibrium solution.

### *D. Necessary and sufficient condition for a nondominated equilibrium solution*

In a multiobjective two-person nonzero-sum game $(A, B)$ in normal form, given domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, the fact that a strategy $x^*$ of player 1 is a nondominated response for a strategy $y^*$ of player 2 corresponds to the fact that $x^*$ is a nondominated solution to a multiobjective mathematical programming problem $(X, f^1(x; y^*), \Lambda^1)$, and similarly the fact that a strategy $y^*$ of player 2 is a nondominated response for a strategy $x^*$ of player 1 corresponds to the fact that $y^*$ is a nondominated solution to a multiobjective mathematical programming problem $(Y, f^2(y; x^*), \Lambda^2)$. Then, the following theorem can be obtained by using the Tamura and Miura condition (10) to the two multiobjective mathematical programming problems $(X, f^1(x; y^*), \Lambda^1)$ and $(Y, f^2(y; x^*), \Lambda^2)$.

*Theorem 3.2:* In a multiobjective two-person nonzero-sum game $(A, B)$ in normal form, let $V^1 = \{v^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $W^2 = \{w^{t_2} \mid t_2 = 1, \ldots, q_2\}$ denote generators of polar cones $\Lambda^{1*}$ and $\Lambda^{2*}$ of the domination cones $\Lambda^1$ and $\Lambda^2$ of players 1

and 2, respectively, where $\Lambda^{1*}$ and $\Lambda^{2*}$ are represented as

$$\Lambda^{1*} = \left\{ \omega^1 \in \mathbb{R}^{r_1} \;\middle|\; \omega^1 = \sum_{t_1=1}^{q_1} \delta_{t_1} v^{t_1}, \delta_{t_1} \geqq 0, t_1 = 1, \ldots, q_1 \right\}, \tag{15a}$$

$$\Lambda^{2*} = \left\{ \omega^2 \in \mathbb{R}^{r_2} \;\middle|\; \omega^2 = \sum_{t_2=1}^{q_2} \varepsilon_{t_2} w^{t_2}, \varepsilon_{t_2} \geqq 0, t_2 = 1, \ldots, q_2 \right\}. \tag{15b}$$

Then, $(x^*, y^*)$ is a nondominated equilibrium solution if and only if there exist $\alpha^*$, $\beta^*$, $\delta^*$, and $\varepsilon^*$ satisfying the following condition, where $\alpha^*$ and $\beta^*$ are scalars and $\delta^*$ and $\varepsilon^*$ are $q_1$- and $q_2$-dimensional vectors, respectively.

$$\sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{t_1}^* v_{k_1}^{t_1} x_i^* a_{ij}^{k_1} y_j^* - \alpha^* = 0, \tag{16a}$$

$$\sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{t_2}^* w_{k_2}^{t_2} x_i^* b_{ij}^{k_2} y_j^* - \beta^* = 0, \tag{16b}$$

$$\sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{j=1}^{n} \delta_{t_1}^* v_{k_1}^{t_1} a_{ij}^{k_1} y_j^* - \alpha^* \leqq 0, \; i = 1, \ldots, m, \tag{16c}$$

$$\sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \varepsilon_{t_2}^* w_{k_2}^{t_2} x_i^* b_{ij}^{k_2} - \beta^* \leqq 0, \; j = 1, \ldots, n, \tag{16d}$$

$$\sum_{i=1}^{m} x_i^* - 1 = 0, \quad x^* \geqq 0, \tag{16e}$$

$$\sum_{j=1}^{n} y_j^* - 1 = 0, \quad y^* \geqq 0, \tag{16f}$$

$$\delta^* \geq 0, \quad \varepsilon^* \geq 0. \tag{16g}$$

If the domination cones of players 1 and 2 are the negative quadrant, any nondominated equilibrium solution is also a weak Pareto equilibrium solution and the generators of the polar cones of the domination cone is $V^1 = \{v^1 = (1,0,\ldots,0)^T, \ldots, v^{r_1} = (0,\ldots,0,1)^T\}$ and $W^2 = \{w^1 = (1,0,\ldots,0)^T, \ldots, w^{r_2} = (0,\ldots,0,1)^T\}$. Furthermore, if the multiplier vectors are strictly positive, i.e., $\delta > 0$, $\varepsilon > 0$, any nondominated equilibrium solution is also a Pareto equilibrium solution. From the above facts, we straightforwardly obtain the necessary and sufficient condition for a Pareto equilibrium solution.

### *E. Mathematical programming problem for obtaining nondominated equilibrium solutions*

Using the necessary and sufficient condition for a nondominated equilibrium solution, we formulate a mathematical programming problem whose optimal solutions are nondominated equilibrium solutions.

*Theorem 3.3:* In a multiobjective two-person nonzero-sum game $(A,B)$ in normal form, let $V^1 = \{v^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $W^2 = \{w^{t_2} \mid t_2 = 1, \ldots, q_2\}$ denote generators of polar cones $\Lambda^{1*}$ and $\Lambda^{2*}$ of the domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, where $\Lambda^{1*}$ and $\Lambda^{2*}$ are represented as (15). Then, $(x^*, y^*)$ is a nondominated equilibrium solution if and only if $(x^*, y^*, \alpha^*, \beta^*, \delta^*, \varepsilon^*)$ is an optimal solution to the following mathematical programming problem.

$$\begin{aligned} \max \quad & \sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{t_1} v_{k_1}^{t_1} x_i a_{ij}^{k_1} y_j \\ & + \sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{t_2} w_{k_2}^{t_2} x_i b_{ij}^{k_2} y_j - \alpha - \beta \end{aligned} \tag{17a}$$

$$\text{s. t.} \quad \sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{j=1}^{n} \delta_{t_1} v_{k_1}^{t_1} a_{ij}^{k_1} y_j - \alpha \leqq 0, \; i = 1, \ldots, m \tag{17b}$$

$$\sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \varepsilon_{t_2} w_{k_2}^{t_2} x_i b_{ij}^{k_2} - \beta \leqq 0, \; j = 1, \ldots, n \tag{17c}$$

$$\sum_{i=1}^{m} x_i - 1 = 0, \quad x \geqq 0 \tag{17d}$$

$$\sum_{j=1}^{n} y_j - 1 = 0, \quad y \geqq 0 \tag{17e}$$

$$\delta \geq 0, \quad \varepsilon \geq 0. \tag{17f}$$

By specifying the generators of the polar cones of the domination cone as $V^1 = \{v^1 = (1,0,\ldots,0)^T, \ldots, v^{r_1} = (0,\ldots,0,1)^T\}$ and $W^2 = \{w^1 = (1,0,\ldots,0)^T, \ldots, w^{r_2} = (0,\ldots,0,1)^T\}$, for the strictly positive multiplier vectors $\delta > 0$ and $\varepsilon > 0$, any nondominated equilibrium solution is also a Pareto equilibrium solution, and then we obtain the following corollary.

*Corollary 3.1:* For a multiobjective two-person nonzero-sum game $(A,B)$ in normal form, $(x^*, y^*)$ is a Pareto equilibrium solution if and only if $(x^*, y^*, \alpha^*, \beta^*, \delta^*, \varepsilon^*)$ is an optimal solution to the following mathematical programming problem.

$$\max \quad \sum_{k_1=1}^{r_1} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{k_1} x_i a_{ij}^{k_1} y_j + \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{k_2} x_i b_{ij}^{k_2} y_j - \alpha - \beta \tag{18a}$$

$$\text{s. t.} \quad \sum_{k_1=1}^{r_1} \sum_{j=1}^{n} \delta_{k_1} a_{ij}^{k_1} y_j - \alpha \leqq 0, \; i = 1, \ldots, m \tag{18b}$$

$$\sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \varepsilon_{k_2} x_i b_{ij}^{k_2} - \beta \leqq 0, \; j = 1, \ldots, n \tag{18c}$$

$$\sum_{i=1}^{m} x_i - 1 = 0, \quad x \geqq 0 \tag{18d}$$

$$\sum_{j=1}^{n} y_j - 1 = 0, \quad y \geqq 0 \tag{18e}$$

$$\delta > 0, \quad \varepsilon > 0. \tag{18f}$$

### *F. Scalarized Two-Person Nonzero-Sum Games*

By using weighting coefficient vectors $\lambda \in \mathbb{R}_+^{r_1}$ and $\theta \in \mathbb{R}_+^{r_2}$, where $\mathbb{R}_+^t = \{z \in \mathbb{R}^t \mid z \geqq 0\}$, a multiobjective two-person nonzero-sum game $(A,B) = ((A^1, \ldots, A^{r_1})^T, (B^1, \ldots, B^{r_2})^T)$ can be reduced to a single-objective two-person nonzero-sum game $(\lambda_1 A^1 + \cdots + \lambda_{r_1} A^{r_1}, \theta_1 B^1 + \cdots + \theta_{r_2} B^{r_2})$. In a multiobjective two-person nonzero-sum game $(A,B)$, let $V^1 = \{v^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $W^2 = \{w^{t_2} \mid t_2 = 1, \ldots, q_2\}$ denote

generators of polar cones $\Lambda^{1*}$ and $\Lambda^{2*}$ of the domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, where $\Lambda^{1*}$ and $\Lambda^{2*}$ are represented as (15). Then, we consider a single-objective two-person nonzero-sum game scalarized by weighting coefficient vectors $\lambda \in \Lambda^{1*}$ and $\theta \in \Lambda^{2*}$. From the result by Mangasarian and Stone [14] and the parameter transformations $\lambda = \sum_{t_1=1}^{q_1} \delta_{t_1} v^{t_1}$, $\delta_{t_1} \geqq 0$, $t_1 = 1,\ldots,q_1$ and $\theta = \sum_{t_2=1}^{q_2} \varepsilon_{t_2} w^{t_2}$, $\varepsilon_{t_2} \geqq 0$, $t_2 = 1,\ldots,q_2$, it can be found that $(x^*, y^*)$ is an equilibrium solution of the scalarized game $(\lambda_1 A^1 + \cdots + \lambda_{r_1} A^{r_1}, \theta_1 B^1 + \cdots + \theta_{r_2} B^{r_2})$ if and only if $(x^*, y^*, \alpha^*, \beta^*)$ is an optimal solution to the following mathematical programming problem.

$$\max \quad \sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{t_1} v_{k_1}^{t_1} x_i a_{ij}^{k_1} y_j + \sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{t_2} w_{k_2}^{t_2} x_i b_{ij}^{k_2} y_j - \alpha - \beta \tag{19a}$$

$$\text{s. t.} \quad \sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{j=1}^{n} \delta_{t_1} v_{k_1}^{t_1} a_{ij}^{k_1} y_j - \alpha \leqq 0, \ i = 1,\ldots,m \tag{19b}$$

$$\sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{i=1}^{m} \varepsilon_{t_2} w_{k_2}^{t_2} x_i b_{ij}^{k_2} - \beta \leqq 0, \ j = 1,\ldots,n \tag{19c}$$

$$\sum_{i=1}^{m} x_i - 1 = 0, \quad x \geqq 0 \tag{19d}$$

$$\sum_{j=1}^{n} y_j - 1 = 0, \quad y \geqq 0. \tag{19e}$$

It should be noted that $\delta$ and $\varepsilon$ are not variables but given parameters.

By comparison with the problem (17), while all the optimal solutions of the problem (17) correspond to the set of nondominated equilibrium solutions, those of the problem (19) correspond to only a subset of nondominated equilibrium solutions with respect to the given parameters $\delta$ and $\varepsilon$ if $\delta$ and $\varepsilon$ are in the given polar cones of the domination cones.

Moreover, by the parameter transformations, assuming $\sum_{k_1=1}^{r_1} v_{k_1}^{t_1} a_{ij}^{k_1} > 0$, $t_1 = 1,\ldots,q_1$, $i = 1,\ldots,m$, $j = 1,\ldots,n$ and $\sum_{k_2=1}^{r_2} w_{k_2}^{t_2} b_{ij}^{k_2} > 0$, $t_2 = 1,\ldots,q_2$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, in a way similar to that of Corley [7], we can obtain the following a parametric linear complementarity problem.

$$\begin{bmatrix} 0 & \sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \varepsilon_{t_2} w_{k_2}^{t_2} B^{k_2 T} \\ \sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \delta_{t_1} v_{k_1}^{t_1} A^{t_1} & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} - \begin{bmatrix} \xi^2 \\ \xi^1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{20a}$$

$$\begin{bmatrix} y \\ x \end{bmatrix} \geqq 0, \ \begin{bmatrix} \xi^2 \\ \xi^1 \end{bmatrix} \geqq 0, \quad \begin{bmatrix} y \\ x \end{bmatrix}^T \begin{bmatrix} \xi^2 \\ \xi^1 \end{bmatrix} = 0. \tag{20b}$$

Of course, a set of solutions to the problem (20) also corresponds to only a subset of nondominated equilibrium solutions with respect to the given parameters $\delta$ and $\varepsilon$.

## IV. Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game in extensive form

### *A. A multiobjective two-person nonzero-sum game and sequences in the extensive form game*

A game in extensive form is characterized by a game tree, players, information sets, chance moves, and payoff functions. A game tree is represented by a graph with nodes including the root which is an initial node and directed edges. Particularly, a terminal node is called a leaf, and at each of leaves a vector of payoffs is assigned to each player in multiobjective games. An example of a multiobjective two-person nonzero-sum game in extensive form is given in Figure 1, where $n_i$, $i = 1,\ldots,31$ denote nodes; $m_i$, $l_i$, $i = 1,\ldots,6$ denote choices of player 1; $c_i$, $d_i$, $i = 1,2$ denote choices of player 2; and $p_i$, $i = 1,2$ denote probabilities of the chance move.

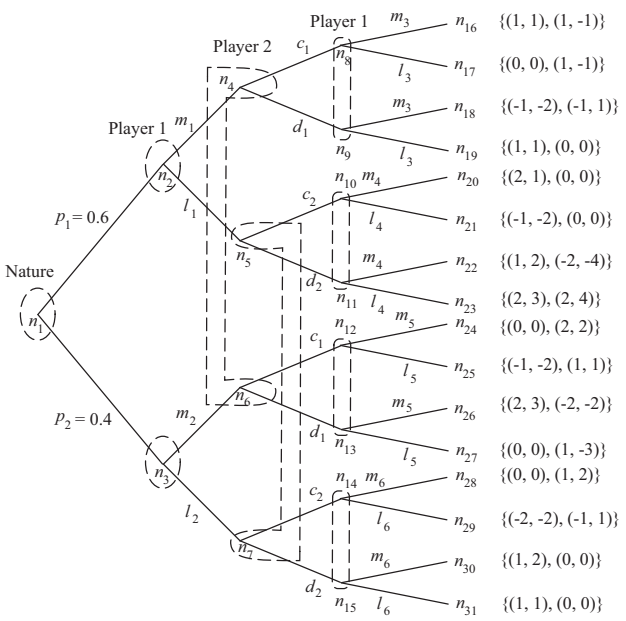


Fig. 1. A game tree of a multiobjective two-person nonzero-sum game

There are two representations of strategies in an extensive form game: behavior strategies and mixed strategies in the corresponding normal form game. An expected payoff as a function of behavior strategies becomes a high-degree nonlinear function when the number of levels of the game tree is large. When an extensive form game is transformed into a normal form game, the number of pure strategies increases exponentially with a size of game. On the assumption of perfect recall of players, von Stengel [23] and Koller et al. [11] propose a game representation of the sequence form which does not cause the mentioned above difficulties. Namely, the expected payoff as a function of realization plans is linear even if the game tree becomes multistage, and the number of sequences increases linearly with a size of game. Because the exponential increase of the number of pure strategies in the normal form game results from extreme increase of the number

of pure strategies such that players' choices are not consistent with behaviors of perfect recall, it can be interpreted that a set of pure strategies in sequence form corresponds to that of normal form excluding not perfect recall pure strategies.

A series of nodes and edges from the root to some node is called a path, and a sequence is defined by a set of labels of edges on the path to the node. For example, for node $n_{12}$ of the game tree depicted in Figure 1, a sequence of player 1 is $m_2$, that of player 2 is $c_1$, and that of chance player is $p_2$. For node $n_{25}$ which is a leaf, a sequence of player 1 is $m_2 l_5$, and those of player 2 and chance player are the same as the sequences for node $n_{12}$.

Let $L$ be a set of leaves. Payoff functions in extensive form are defined on the set $L$, and a vector of payoffs is assigned to each of the players at any leaf $l \in L$; let $H_1 : L \to \mathbb{R}^{r_1}$ be the payoff function of player 1, and let $H_2 : L \to \mathbb{R}^{r_2}$ be that of player 2, where $r_1$ and $r_2$ are the numbers of payoffs (objectives) of players 1 and 2, respectively. In contrast, payoff functions in sequence form are defined on a set of sequences. Let $S_0$, $S_1$, and $S_2$ be the sets of sequences of chance player, player 1, and player 2, respectively, and let $|S_0|$, $|S_1|$, and $|S_2|$ be the numbers of sequences of chance player, player 1, and player 2, respectively. Let $S = S_0 \times S_1 \times S_2$ be the space of sequences of all the players.

A payoff function of player 1 in sequence form is defined as $G_1 : S \to \mathbb{R}^{r_1}$, and if a sequence $s = (s_0, s_1, s_2) \in S$ is specified at a leaf $l \in L$, the payoff function is $G_1(s) = H_1(l)$ and otherwise it is $G_1(s) = 0$. A payoff function of player 2 $G_2 : S \to \mathbb{R}^{r_2}$ is also defined similarly. For example, for node $n_{12}$ of the game tree depicted in Figure 1, a sequence vector is $s^{12} = (p_2, m_2, c_1)$, and payoffs of players 1 and 2 are $G_1(s^{12}) = (0,0)$, $G_2(s^{12}) = (0,0)$, respectively. For node $n_{25}$ which is a leaf, a sequence vector is $s^{25} = (p_2, m_2 l_5, c_1)$, and payoffs of players 1 and 2 are $G_1(s^{25}) = (-1,-2)$, $G_2(s^{25}) = (1,1)$, respectively.

A set of all nodes in a game tree is divided into information sets. Let $U_1$ and $U_2$ be the sets of information sets of players 1 and 2, respectively, and let $|U_1|$ and $|U_2|$ be the numbers of the information sets of players 1 and 2, respectively. Each information set $u$ exactly belongs to one player $i$. All nodes in an information set $u$ have the same choices, and the set of choices at $u$ is denoted by $C_u$. Let $|C_u|$ be the number of choices at $u$.

Because it is assumed that perfect recall holds for all the players in a sequence form game, all nodes in an information set $u$ have the same sequence. Let the sequence be denoted by $\sigma_u$, and it leads the information set $u$. A choice $c \in C_u$ in $u$ extends the sequence $\sigma_u$, and the extended sequence is expressed by $\sigma_u c$, i.e.,

$$\sigma_u c = \sigma_u \cup \{c\},\ c \in C_u. \tag{21}$$

With this notation, a set of sequences of player $i$ can be represented by $S_i = \{\emptyset\} \cup \{\sigma_u c \mid u \in U_i, c \in C_u\}$.

In sequence form, a strategy is represented by giving a probability distribution to a set of sequences, and it is called a realization plan. A realization plan $\phi \in \mathbb{R}^{|S_1|}$ of player 1 is subject to the following constraints.

$$\phi(\emptyset) = 1 \tag{22a}$$
$$-\phi(\sigma_{u_1}) + \sum_{c_1 \in C_{u_1}} \phi(\sigma_{u_1} c_1) = 0,\ u_1 \in U_1 \tag{22b}$$
$$\phi(s_1) \geqq 0,\ s_1 \in S_1. \tag{22c}$$

Player 2's realization plan $\psi \in \mathbb{R}^{|S_2|}$ is also subject to the following constraints.

$$\psi(\emptyset) = 1 \tag{23a}$$
$$-\psi(\sigma_{u_2}) + \sum_{c_2 \in C_{u_2}} \psi(\sigma_{u_2} c_2) = 0,\ u_2 \in U_2 \tag{23b}$$
$$\psi(s_2) \geqq 0,\ s_2 \in S_2. \tag{23c}$$

By using the $(1+|U_1|) \times |S_1|$ constraint matrix $E^1$ and the $(1+|U_2|) \times |S_2|$ constraint matrix $E^2$, the above constraints (22) and (23) can be simply expressed by

$$E^1 \phi = e^1 \tag{24}$$
$$E^2 \psi = e^2, \tag{25}$$

respectively, where $e^1$ and $e^2$ are the $(1+|U_1|)$- and $(1+|U_2|)$-dimensional vectors such that the first element is 1 and the other elements are all 0, i.e., $(1,0,\ldots,0)^T$. Then, the sets $\Phi$ and $\Psi$ of realization plans of players 1 and 2 are defined by

$$\Phi = \left\{ \psi \in \mathbb{R}^{|S_1|} \mid E^1 \psi = e^1,\ \psi \geqq 0 \right\} \tag{26}$$
$$\Psi = \left\{ \psi \in \mathbb{R}^{|S_2|} \mid E^2 \psi = e^2,\ \psi \geqq 0 \right\}, \tag{27}$$

respectively.

Let $p = (p_1, \ldots, p_{|S_0|})$ be a realization plan of chance player. When players 1 and 2 choose sequences $s_1$ and $s_2$, respectively, the expected payoffs of them are

$$c_{s_1 s_2} = (c^1_{s_1 s_2}, \ldots, c^{r_1}_{s_1 s_2}) = \sum_{s_0 \in S_0} G_1(s_0, s_1, s_2) p(s_0) \in \mathbb{R}^{r_1} \tag{28}$$
$$d_{s_1 s_2} = (d^1_{s_1 s_2}, \ldots, d^{r_2}_{s_1 s_2}) = \sum_{s_0 \in S_0} G_2(s_0, s_1, s_2) p(s_0) \in \mathbb{R}^{r_2}. \tag{29}$$

Now, let $C$ and $D$ denote $|S_1| \times |S_2|$ matrices such that elements of the $s_1$th row and $s_2$th column are the above defined vectors $c_{s_1 s_2}$ and $d_{s_1 s_2}$, respectively. Then, for given realization plans $\phi \in \Phi$ and $\psi \in \Psi$ of players 1 and 2, the vectors of expected payoffs of them are represented by

$$\phi^T C \psi \triangleq \left( \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \phi_{s_1} c^1_{s_1 s_2} \psi_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \phi_{s_1} c^{r_1}_{s_1 s_2} \psi_{s_2} \right) \tag{30}$$
$$\phi^T D \psi \triangleq \left( \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \phi_{s_1} d^1_{s_1 s_2} \psi_{s_2}, \ldots, \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \phi_{s_1} d^{r_2}_{s_1 s_2} \psi_{s_2} \right), \tag{31}$$

respectively.

*B. Nondominated equilibrium solutions of a multiobjective two-person nonzero-sum game in extensive form*

First, in a multiobjective two-person nonzero-sum game in extensive form, we give a solution concept of Pareto equilibrium solutions, and then extend it to that of nondominated equilibrium solutions by using domination cones.

*Definition 4.1:* In a multiobjective two-person nonzero-sum game in extensive form, a pair of realization plans $(\phi^*, \psi^*) \in \Phi \times \Psi$ is said to be a Pareto equilibrium solution if there does not exist another $(\phi, \psi) \in \Phi \times \Psi$ such that

$$\phi^{*T} C \psi^* \leq \phi^T C \psi^* \tag{32a}$$

$$\phi^{*T} D \psi^* \leq \phi^{*T} D \psi. \tag{32b}$$

A multiobjective two-person nonzero-sum game in extensive form can be reduced to a single-objective two-person nonzero-sum game by using a weighting coefficient vector $(w, v) \in \mathbb{R}^{r_1}_{++} \times \mathbb{R}^{r_2}_{++}$, where $\mathbb{R}^{r_i}_{++} = \{z \in \mathbb{R}^{r_i} \mid z > 0\}$, $i = 1, 2$. Furthermore, because the single-objective game in extensive form can be transformed into a game in normal form and there exists at least one equilibrium solution in the game in normal form, in general there exists at least one Pareto equilibrium solution in a multiobjective two-person nonzero-sum game in extensive form [12].

For simplicity, let $g^1(\phi; \psi) \triangleq \phi^T C \psi$ and $g^2(\psi; \phi) \triangleq \phi^T D \psi$, and we define nondominated equilibrium solutions in the following.

*Definition 4.2:* Let $\Lambda^1$ and $\Lambda^2$ denote domination cones of players 1 and 2, respectively. Then, in a multiobjective two-person nonzero-sum game in extensive form, a pair of realization plans $(\phi^*, \psi^*) \in \Phi \times \Psi$ is said to be a nondominated equilibrium solution if there does not exist another $(\phi, \psi) \in \Phi \times \Psi$ such that

$$g^1(\phi^*; \psi^*) \in g^1(\phi; \psi^*) + \Lambda^1, \tag{33a}$$

$$g^2(\psi^*; \phi^*) \in g^2(\psi; \phi^*) + \Lambda^2. \tag{33b}$$

In particular, by letting $\Lambda^1 = \mathbb{R}^{r_1}_-$ and $\Lambda^2 = \mathbb{R}^{r_2}_-$ where $\mathbb{R}^{r_i}_- = \{z \in \mathbb{R}^{r_i} \mid z \leqq 0\}$, $i = 1, 2$, any nondominated equilibrium solution with respect to the domination cones $\mathbb{R}^{r_1}_-$ and $\mathbb{R}^{r_2}_-$ is also a Pareto equilibrium solution.

The above definition means that $\phi^*$ is a nondominated response of player 1 for a strategy $\psi^*$ of player 2, and $\psi^*$ is a nondominated response of player 2 for a strategy $\phi^*$ of player 1. This can be explicitly expressed as follows. The sets of nondominated responses of players 1 and 2 are defined as

$$N^1(\psi, \Lambda^1) = \{\phi \in \Phi \mid \text{there does not exist } \phi' \in \Phi \text{ such that } g^1(\phi; \psi) \in g^1(\phi'; \psi) + \Lambda^1\}, \tag{34a}$$

$$N^2(\phi, \Lambda^2) = \{\psi \in \Psi \mid \text{there does not exist } \psi' \in \Psi \text{ such that } g^2(\psi; \phi) \in g^2(\psi'; \phi) + \Lambda^2\}. \tag{34b}$$

Then, by using the concept of nondominated responses, the set $N(\Lambda^1, \Lambda^2)$ of nondominated equilibrium solutions can be represented by

$$N(\Lambda^1, \Lambda^2) = \{(\phi^*, \psi^*) \mid \phi^* \in N^1(\psi^*, \Lambda^1),\ \psi^* \in N^2(\phi^*, \Lambda^2)\}. \tag{35}$$

A relation between the domination cones and the sets of nondominated equilibrium solutions is shown in the following proposition.

*Proposition 4.1:* Let $\Lambda^1$ and $\Lambda^{1'}$ denote domination cones of player 1, and $\Lambda^2$ and $\Lambda^{2'}$ denote domination cones of player 2 in a multiobjective two-person nonzero-sum game in extensive form. Then, if $\Lambda^1 \subset \Lambda^{1'}$ and $\Lambda^2 \subset \Lambda^{2'}$, $N(\Lambda^{1'}, \Lambda^{2'}) \subset N(\Lambda^1, \Lambda^2)$.

From the fact that there exists at least one Pareto equilibrium solution [12], we obtain the following theorem showing the existence of nondominated equilibrium solutions.

*Theorem 4.1:* In a multiobjective two-person nonzero-sum game in extensive form, for any domination cones of players 1 and 2, there exists at least one nondominated equilibrium solution.

*C. Necessary and sufficient condition for a nondominated equilibrium solution*

In a multiobjective two-person nonzero-sum game in extensive form, given domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, the fact that a realization plan $\phi^*$ of player 1 is a nondominated response for a realization plan $\psi^*$ of player 2 corresponds to the fact that $\phi^*$ is a nondominated solution to a multiobjective mathematical programming problem $(\Phi, g^1(\phi; \psi^*), \Lambda^1)$, and similarly the fact that a realization plan $\psi^*$ of player 2 is a nondominated response for a realization plan $\phi^*$ of player 1 corresponds to the fact that $\psi^*$ is a nondominated solution to a multiobjective mathematical programming problem $(\Psi, g^2(\psi; \phi^*), \Lambda^2)$. Assume that $\Phi$, $\Psi$, $g^1(\phi; \psi^*) = \phi^T C \psi^*$, and $g^2(\psi; \phi^*) = \phi^{*T} D \psi$ are represented by (26), (27), (30), and (31), respectively, and $\Lambda^1$ and $\Lambda^2$ are polyhedral domination cones. Then, the following theorem can be obtained by using the Tamura and Miura condition (10) to the two multiobjective mathematical programming problems $(\Phi, g^1(\phi; \psi^*), \Lambda^1)$ and $(\Psi, g^2(\psi; \phi^*), \Lambda^2)$.

*Theorem 4.2:* In a multiobjective two-person nonzero-sum game in extensive form, let $V^1 = \{v^{t_1} \mid t_1 = 1, \ldots, q_1\}$ and $W^2 = \{w^{t_2} \mid t_2 = 1, \ldots, q_2\}$ denote generators of polar cones $\Lambda^{1*}$ and $\Lambda^{2*}$ of the domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, where $\Lambda^{1*}$ and $\Lambda^{2*}$ are represented as (15). Then, $(\phi^*, \psi^*)$ is a nondominated equilibrium solution if and only if there exist $\alpha^*$, $\beta^*$, $\delta^*$, and $\varepsilon^*$ satisfying the following condition, which are $|U_1|$-, $|U_2|$-, $q_1$-, and $q_2$-dimensional vectors, respectively.

$$\sum_{t_1=1}^{q_1} \sum_{k_1=1}^{r_1} \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \delta^*_{t_1} v^{t_1}_{k_1} \phi^*_{s_1} c^{k_1}_{s_1 s_2} \psi^*_{s_2} - \sum_{s_1=1}^{|S_1|} \sum_{u_1=0}^{|U_1|} \alpha^*_{u_1} e^1_{u_1 s_1} \phi^*_{s_1} = 0 \tag{36a}$$

$$\sum_{t_2=1}^{q_2} \sum_{k_2=1}^{r_2} \sum_{s_1=1}^{|S_1|} \sum_{s_2=1}^{|S_2|} \varepsilon^*_{t_2} w^{t_2}_{k_2} \phi^*_{s_1} d^{k_2}_{s_1 s_2} \psi^*_{s_2} - \sum_{s_2=1}^{|S_2|} \sum_{u_2=0}^{|U_2|} \beta^*_{u_2} e^2_{u_2 s_2} \psi^*_{s_2} = 0 \tag{36b}$$

$$\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta_{t_1}^{*}v_{k_1}^{t_1}c_{s_1s_2}^{k_1}\psi_{s_2}^{*}-\sum_{u_1=0}^{|U_1|}\alpha_{u_1}^{*}e_{u_1s_1}^{1}\leqq 0,\ s_1=1,\ldots,|S_1| \tag{36c}$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\varepsilon_{t_2}^{*}w_{k_2}^{t_2}\phi_{s_1}^{*}d_{s_1s_2}^{k_2}-\sum_{u_2=0}^{|U_2|}\beta_{u_2}^{*}e_{u_2s_2}^{2}\leqq 0,\ s_2=1,\ldots,|S_2| \tag{36d}$$

$$\sum_{s_1=1}^{|S_1|}e_{u_1s_1}^{1}\phi_{s_1}^{*}-e_{u_1}^{1}=0,\ u_1=0,\ldots,|U_1| \tag{36e}$$

$$\sum_{s_2=1}^{|S_2|}e_{u_2s_2}^{2}\psi_{s_2}^{*}-e_{u_2}^{2}=0,\ u_2=0,\ldots,|U_2| \tag{36f}$$

$$\phi^{*}\geqq 0 \tag{36g}$$

$$\psi^{*}\geqq 0 \tag{36h}$$

$$\delta^{*}\geq 0 \tag{36i}$$

$$\varepsilon^{*}\geq 0 \tag{36j}$$

If the domination cones of players 1 and 2 are the negative quadrant, any nondominated equilibrium solution with respect to the domination cones is also a weak Pareto equilibrium solution and the generators of the polar cones of the domination cone are $V^1=\{v^1=(1,0,\ldots,0)^T,v^2=(0,1,0,\ldots,0)^T,\ldots,v^{r_1}=(0,\ldots,0,1)^T\}$ and $W^2=\{w^1=(1,0,\ \ldots,0)^T,w^2=(0,1,0,\ldots,0)^T,\ldots,w^{r_2}=(0,\ldots,0,1)^T\}$. Furthermore, if the multiplier vectors are strictly positive, i.e., $\delta>0$, $\varepsilon>0$, any nondominated equilibrium solution is also a Pareto equilibrium solution. From the above facts, we straightforwardly obtain the necessary and sufficient condition for a Pareto equilibrium solution.

### *D. Nondominated equilibrium solutions and corresponding mathematical programming problem*

Using the necessary and sufficient condition for a nondominated equilibrium solution, we formulate a mathematical programming problem whose optimal solutions are nondominated equilibrium solutions.

*Theorem 4.3:* In a multiobjective two-person nonzero-sum game in extensive form, let $V^1=\{v^{t_1}\mid t_1=1,\ldots,q_1\}$ and $W^2=\{w^{t_2}\mid t_2=1,\ldots,q_2\}$ denote generators of polar cones $\Lambda^{1*}$ and $\Lambda^{2*}$ of the domination cones $\Lambda^1$ and $\Lambda^2$ of players 1 and 2, respectively, where $\Lambda^{1*}$ and $\Lambda^{2*}$ are represented as (15). Then, $(\phi^{*},\psi^{*})$ is a nondominated equilibrium solution if and only if $(\phi^{*},\psi^{*},\alpha^{*},\beta^{*},\delta^{*},\varepsilon^{*})$ is an optimal solution to the following mathematical programming problem.

$$\begin{aligned}\max\ &\sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{t_1}v_{k_1}^{t_1}\phi_{s_1}c_{s_1s_2}^{k_1}\psi_{s_2}\\&+\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{t_2}w_{k_2}^{t_2}\phi_{s_1}d_{s_1s_2}^{k_2}\psi_{s_2}\\&-\sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1}e_{u_1s_1}^{1}\phi_{s_1}-\sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2}e_{u_2s_2}^{2}\psi_{s_2}\end{aligned} \tag{37a}$$

$$\text{s. t.}\quad \sum_{t_1=1}^{q_1}\sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta_{t_1}v_{k_1}^{t_1}c_{s_1s_2}^{k_1}\psi_{s_2}-\sum_{u_1=0}^{|U_1|}\alpha_{u_1}e_{u_1s_1}^{1}\leqq 0,\quad s_1=1,\ldots,|S_1| \tag{37b}$$

$$\sum_{t_2=1}^{q_2}\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\varepsilon_{t_2}w_{k_2}^{t_2}\phi_{s_1}d_{s_1s_2}^{k_2}-\sum_{u_2=0}^{|U_2|}\beta_{u_2}e_{u_2s_2}^{2}\leqq 0,\quad s_2=1,\ldots,|S_2| \tag{37c}$$

$$\sum_{s_1=1}^{|S_1|}e_{u_1s_1}^{1}\phi_{s_1}-e_{u_1}^{1}=0,\ u_1=0,\ldots,|U_1| \tag{37d}$$

$$\sum_{s_2=1}^{|S_2|}e_{u_2s_2}^{2}\psi_{s_2}-e_{u_2}^{2}=0,\ u_2=0,\ldots,|U_2| \tag{37e}$$

$$\phi\geqq 0 \tag{37f}$$

$$\psi\geqq 0 \tag{37g}$$

$$\delta\geq 0 \tag{37h}$$

$$\varepsilon\geq 0. \tag{37i}$$

By specifying the generators of the polar cones of the domination cone as $V^1=\{v^1=(1,0,\ldots,0)^T,v^2=(0,1,0,\ldots,0)^T,\ldots,v^{r_1}=(0,\ldots,0,1)^T\}$ and $W^2=\{w^1=(1,0,\ \ldots,0)^T,w^2=(0,1,0,\ldots,0)^T,\ldots,w^{r_2}=(0,\ldots,0,1)^T\}$, for the strictly positive multiplier vectors $\delta>0$ and $\varepsilon>0$, any nondominated equilibrium solution with respect to the domination cones is also a Pareto equilibrium solution, and then we obtain the following corollary.

*Corollary 4.1:* For a multiobjective two-person nonzero-sum game in extensive form, $(\phi^{*},\psi^{*})$ is a Pareto equilibrium solution if and only if $(\phi^{*},\psi^{*},\alpha^{*},\beta^{*},\delta^{*},\varepsilon^{*})$ is an optimal solution to the following mathematical programming problem.

$$\begin{aligned}\max\ &\sum_{k_1=1}^{r_1}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\delta_{k_1}\phi_{s_1}c_{s_1s_2}^{k_1}\psi_{s_2}+\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\sum_{s_2=1}^{|S_2|}\varepsilon_{k_2}\phi_{s_1}d_{s_1s_2}^{k_2}\psi_{s_2}\\&-\sum_{u_1=0}^{|U_1|}\sum_{s_1=1}^{|S_1|}\alpha_{u_1}e_{u_1s_1}^{1}\phi_{s_1}-\sum_{u_2=0}^{|U_2|}\sum_{s_2=1}^{|S_2|}\beta_{u_2}e_{u_2s_2}^{2}\psi_{s_2}\end{aligned} \tag{38a}$$

$$\text{s. t.}\quad \sum_{k_1=1}^{r_1}\sum_{s_2=1}^{|S_2|}\delta_{k_1}c_{s_1s_2}^{k_1}\psi_{s_2}-\sum_{u_1=0}^{|U_1|}\alpha_{u_1}e_{u_1s_1}^{1}\leqq 0,\ s_1=1,\ldots,|S_1| \tag{38b}$$

$$\sum_{k_2=1}^{r_2}\sum_{s_1=1}^{|S_1|}\varepsilon_{k_2}\phi_{s_1}d_{s_1s_2}^{k_2}-\sum_{u_2=0}^{|U_2|}\beta_{u_2}e_{u_2s_2}^{2}\leqq 0,\ s_2=1,\ldots,|S_2| \tag{38c}$$

$$\sum_{s_1=1}^{|S_1|}e_{u_1s_1}^{1}\phi_{s_1}-e_{u_1}^{1}=0,\ u_1=0,\ldots,|U_1| \tag{38d}$$

$$\sum_{s_2=1}^{|S_2|}e_{u_2s_2}^{2}\psi_{s_2}-e_{s_2}^{2}=0,\ u_2=0,\ldots,|U_2| \tag{38e}$$

$$\phi\geqq 0 \tag{38f}$$

$$\psi\geqq 0 \tag{38g}$$

$$\tag{38h}$$

$$\delta > 0 \tag{38i}$$

$$\varepsilon > 0. \tag{38j}$$

## V. Conclusions

In this paper, we outlined the development of multiobjective noncooperative game theory. In particular, we focused on the nondominated equilibrium solutions of multiobjective two-person nonzero-sum games in normal and extensive forms, and showed the mathematical programming problems for deriving the nondominated equilibrium solutions.

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