CALABI-YAU THREEFOLDS ARISING FROM FIBER PRODUCTS OF RATIONAL QUASI-ELLIPTIC SURFACES, I

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Abstract. In this paper, we construct some unirational Calabi-Yau threefolds in characteristic 3. We adopt the method by Schoen, but we use quasi-elliptic surfaces instead of elliptic surfaces. We found new examples which do not admit a lifting to characteristic zero.

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$. A Calabi-Yau threefold X is a nonsingular projective threefold over k which satisfies $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$. To a Calabi-Yau threefold X associated is a one-dimensional commutative formal group $\Phi^3(X, \mathbb{G}_m)$ called Artin-Mazur formal group [1], and we call X supersingular provided $p > 0$ and $\Phi^3(X, \mathbb{G}_m) \cong \hat{\mathbb{G}}_a$.

A Calabi-Yau threefold X is said to be unirational if there exists a dominant rational map, which is necessarily inseparable, from the three dimensional projective space $\mathbb{P}^3_k$. Unirational Calabi-Yau's are known to be supersingular. Typical examples can be found in the Fermat quintic $X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0$ in characteristic p with $p \equiv 2, 3, 4 \mod 5$ (cf. [27], [30], [31]). Among supersingular Calabi-Yau threefolds there are a class with the third l-adic Betti number $(l \neq p)$ vanished. Such Calabi-Yau's are non-liftable, that is, they do not admit any projective liftings to characteristic zero. At the moment examples are known only for $p = 2$ and $p = 3$, that is, one example in $p = 3$ with $e(X) = 84$ in [10], and examples in $p = 2, 3$ with $e(X) = 48$ in [25].

We continue to study some concrete examples of Calabi-Yau threefolds and their peculiar properties. We adopt Schoen's construction of Calabi-Yau threefolds with quasi-elliptic surfaces instead of elliptic surfaces. Since we encounter some difficulties in treating characteristic 2 and 3 cases uniformly, we deal with $p = 3$ case here and defer $p = 2$ case to the forthcoming paper.

We obtain examples of supersingular Calabi-Yau threefolds with topological Euler-Poincaré characteristic $e(X) = 36, 48, 60, 72, 84$ in $p = 3$. Two examples with $b_3(X) = 0$ are found, one of which coincides with the one in [10] in $e(X)$ as well as the Betti numbers, but is not isomorphic to it.

Theorem 1.1. *In characteristic* 3*, we have Calabi-Yau threefolds with the following properties:*

(1) *X is unirational, therefore supersingular.*
(2) $\rho(X) = b_2(X)$.
(3) $\pi_1^{alg}(X) = \{1\}$.

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(4) $(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 0, 20, 6, 20, 0, 1), (1, 0, 25, 4, 25, 0, 1), (1, 0, 30, 2, 30, 0, 1), (1, 0, 35, 0, 35, 0, 1), (1, 0, 41, 0, 41, 0, 1)$.
(5) *X admits at least two types of fibrations $X \to \mathbb{P}^1$ whose general fiber is 1) a non-normal rational surface, 2) a supersingular $K3$ surface with a rational double point of type A_2.*
(6) *X has a fibration whose general fiber is a rational curve with an ordinary cusp (quasi-elliptic fibration).*

One of the remaining problems (cf. [6]) is to see if there are any peculiarities of our examples in cohomologies associated with the Hodge spectral sequence

$$E_1^{ij} := H^j(\Omega_X^i) \Rightarrow H^{i+j}_{\mathrm{DR}}(X/k),$$

and the slope spectral sequence in the Hodge-Witt cohomologies ([13])

$$E_1^{ij} := H^j(W\Omega_X^i) \Rightarrow H^{i+j}_{\mathrm{crys}}(X/W).$$

Another fundamental question would be whether the number of topological types of supersingular Calabi-Yau threefolds is finite.

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2. Preliminaries

The l-adic Betti number $b_i(X)$ of a variety X complete over an algebraically closed field k of characteristic $p \geq 0$, with $i \geq 0$ and l prime not equal to p, is defined by $b_i(X) := \dim_{\mathbb{Q}_l} H^i_{\text{ét}}(X, \mathbb{Q}_l)$, which is known to be independent of l. The topological Euler-Poincaré characteristic $e(X)$ of X is

$$e(X) := \sum_{i=0}^{2\dim X} (-1)^i b_i(X).$$

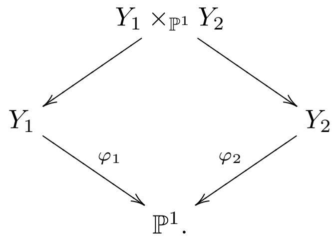
For the first Betti number $b_1(X)$ we have the equality $b_1(X) = 2q(X)$, where $q(X)$ is the dimension of the Albanese variety $\mathrm{Alb}\,(X)$.

A *quasi-elliptic surface* $\varphi : Y \to C$ is a nonsingular projective surface Y with a morphism to a nonsingular curve C, satisfying $\mathcal{O}_C = \varphi_* \mathcal{O}_Y$ and a general fiber is a rational curve with an ordinary cusp. Quasi-elliptic surfaces exist only in characteristic 2 and 3, enjoying properties analogous to elliptic surfaces (cf. [4]). A fiber Y_t of a quasi-elliptic surface will be called special if it is either a multiple fiber or not of type II in Kodaira's classification. A fiber Y_t is called nonspecial if it is not special. Let Σ be the closure of the nonsmooth locus of Y_η/η inside Y. We call it the moving cusp of $\varphi : Y \to C$.

A variety X of dimension n is said to be *unirational* if there exists a dominant rational map from the n-dimensional projective space $\mathbb{P}^n$ to X. X is said to be *separably* (resp. *purely inseparably*) *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ whose extension of function fields is separable (resp. purely inseparable).

3. Construction

Let $\varphi_1 : Y_1 \to \mathbb{P}^1$ and $\varphi_2 : Y_2 \to \mathbb{P}^1$ be relatively minimal rational quasi-elliptic surfaces with section. We fix the base curve $\mathbb{P}^1$ and take a fiber product:



This $Y_1 \times_{\mathbb{P}^1} Y_2$ is a local complete intersection and irreducible. It follows from the canonical bundle formula for quasi-elliptic surfaces that $Y_1 \times_{\mathbb{P}^1} Y_2 \in |-K_{Y_1 \times_k Y_2}|$, hence $K_{Y_1 \times_{\mathbb{P}^1} Y_2} = 0$ ([23]). We try to find a crepant resolution of singularities $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$, using the complete classification of rational quasi-elliptic surfaces with section up to isomorphism in $p = 2, 3$ ([14], [15]).

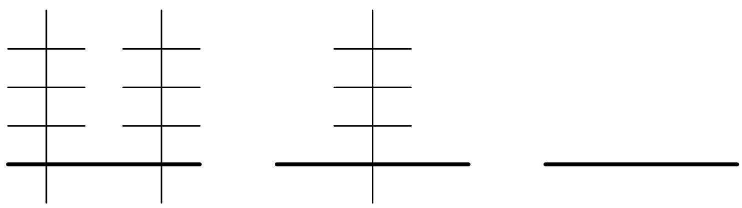
We restrict ourselves to $p = 3$ from now on.

Theorem 3.1 ([14]). *A rational quasi-elliptic surface with section in $p = 3$ is given by one of the following:*

Type of degenerate fibers		Weierstrass form
(a)	II^*	$y^2 = x^3 + t$
(b)	$\mathrm{IV}, \mathrm{IV}^*$	$y^2 = x^3 + t^2$
(c)	*Four* IV*'s*	$y^2 = x^3 + t^4 + t^2$

where II^*, IV *and* IV^* *stand for the types of singular fibers in the sense of Kodaira.*

In order to find a crepant resolution, we only consider the case where $Y_1 \times_{\mathbb{P}^1} Y_2$ is normal. If either $\varphi_1^{-1}(t)$ or $\varphi_2^{-1}(t)$ is reduced for any $t \in \mathbb{P}^1$, then it follows that $Y_1 \times_{\mathbb{P}^1} Y_2$ is normal from Serre's criterion for normality. Thus we treat the following ones:
(bb): (b) and (b), the singular fibers of type IV^* do not meet type IV or type IV^*.
(bc): (b) and (c), the singular fibers of type IV^* do not meet type IV.
(cc): (c) and (c).
Since the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$ arise from the non-smooth parts of φ_1 and φ_2, $\mathrm{Sing}(Y_1 \times_{\mathbb{P}^1} Y_2)$ consists of irreducible curves isomorphic to $\mathbb{P}^1$'s whose configuration is as in Figure 1. Note that the thick lines in Figure 1, which will be denoted by Γ, are derived from the moving cusps of quasi-elliptic surfaces.



Case (bb) Case (bc) Case (cc)

Figure 1

Proposition 3.2. *In the following eight sub-cases of (bb), (bc) and (cc), the fiber product admits a resolution of singularities $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$ with $K_X = \pi^* K_{Y_1 \times_{\mathbb{P}^1} Y_2}$, that is a crepant resolution.*

(bb-1) *the singular fiber of type* IV *meets the singular fiber of type* IV*,*
(bb-2) *the singular fiber of type* IV *does not meet the singular fiber of type* IV*,*
(bc-1) *the singular fiber of type* IV *meets a singular fiber of type* IV*,*
(bc-2) *the singular fiber of type* IV *does not meet any singular fiber of type* IV*,*
(cc-1) *four singular fibers of type* IV *meet singular fibers of type* IV*,*
(cc-2) *two singular fibers of type* IV *meet singular fiber of type* IV*,*
(cc-3) *one singular fiber of type* IV *meets singular fiber of type* IV*,*
(cc-4) *no singular fiber of type* IV *meets singular fibers of type* IV*.*

The proof will be given in the following section.

To obtain the defining equations of the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$, we use the local descriptions of the quasi-elliptic fibrations $\varphi : Y \to C$ at a point where φ is not smooth. The non-trivial part is:

Proposition 3.3. *Let $\varphi : Y \to C$ be a relatively minimal quasi-elliptic surface in characteristic 3. We take a point P on Y and any local coordinate t on C at $\varphi(P)$.*

(1) [3] *Suppose that P lies on the moving cusp Σ. If the fiber over $t = 0$ is nonspecial, then in suitable formal coordinates x, y on Y at P, we have $t = y^2 + x^3$.*
(2) *Suppose that P lies on the moving cusp Σ. If the fiber over $t = 0$ is of type* IV*, then in suitable formal coordinates x, y on Y at P, we have $t = xy^2 - x^3$.*
(3) *Suppose that the fiber over $t = 0$ is of type* IV^**. If P is an intersection point of the component of multiplicity three and a component of multiplicity two (resp. the moving cusp Σ), then there exist formal coordinates x, y such that $t = x^3y^2$ (resp. $t = x^3(1 + y^2)$). If P is on the component of multiplicity three but outside the four points described above, then $t = (1 + y)x^3$.*

Proof. (1) See [3].

(2) Any quasi-elliptic surface which has the degenerate fiber of type IV over $t = 0$ is locally defined in the Weierstrass form by

$$y^2 = x^3 + t^2u(t)$$

in $\operatorname{Spec} k[x, y][[t]]$, where $u(t) \in k[[t]]$ is a unit (cf. [16, p. 479]). We set $u(t) := 1 + \sum_{l\geqslant 1} a_l t^l$. After a blow-up at the singular point, we have a local equation

$$ {y_1}^2 - {t_1}^2 = x_1 \left(1 + \sum_{l\geqslant 1} a_l {x_1}^{l-1} {t_1}^{l+2} \right), $$

where $x = x_1, y = x_1y_1$, and $t = x_1t_1$. Thus

$$ t = t_1\tilde{y}_1^2 - {t_1}^3 \left(1 + \sum_{j=1}^{\infty} (-1)^j \left(\sum_{l\geqslant 1} a_l t^{l-1} {t_1}^3 \right)^j \right). $$

We substitute t recursively to get

$$ t = t_1\tilde{y}^2 - (\mathrm{unit})^3 {t_1}^3. $$

(3) Any quasi-elliptic surface which has the degenerate fiber of type IV^* over $t = 0$ is locally defined in the Weierstrass form by

$$y^2 = x^3 + t^4u(t)$$

in $\operatorname{Spec} k[x, y][[t]]$ (cf. [16, p. 479]). After a succession of blow-ups: $(x = x_1t_1, y = y_1t_1, t = t_1)$, $(x_1 = x_2, y_1 = x_2y_2, t_1 = x_2t_2)$, $(x_2 = x_3, y_2 = x_3y_3, t_2 = x_3t_3)$ and $(x_3 = x_4, y_3 = x_4y_4, t_3 = x_4t_4)$, we have

$$ t = {x_4}^3 t_4 \qquad \text{and} \qquad {y_4}^2 - t_4(1 + t_4u({x_4}^3t_4)) = 0. $$

Substituting y_4 with $y_4 - \lambda$ ($\lambda \in k$), we have

$$(y_4 - \lambda)^2 - t_4(1 + t_4 u(x_4{}^3 t_4)) = 0. \tag{3.1}$$

The component of multiplicity three is given by $x_4 = 0$. If $\lambda = 0$, then we have the expression $t = x_4{}^3 y_4{}^2/(1 + t_4 u(x_4{}^3 t_4))$ which we can put into $t = x_4{}^3 \tilde{y}_4^2$ by taking the square root of the unit. If $\lambda \neq 0$, then we know from (3.1)

$$t_4 = u_1 + y_4 u_2 \in k[[y_4, x_4{}^3]]^\times$$

with units $u_1 \in k[[x_4{}^3]]^\times, u_2 \in k[[y_4, x_4{}^3]]^\times$. By further coordinate changes this can be put into $t = \tilde{x}_4^3(1 + \tilde{y}_4)$. By a similar argument on another chart, one obtains the desired result.

□

Remark 3.4. (1) Note that in the assertion (2) in Proposition 3.3, we can choose the local parameter t of the base curve arbitrarily. Lang obtained similar results in [16, p. 479], but his assertion claims only the existence of a local parameter t which gives the normal form as above.

(2) By considering the automorphisms of $\mathbb{P}^1$, one knows that under (cc) the case where exactly three singular fibers of type IV meet singular fibers of type IV does not occur ([14]).

(3) The morphism $f : X \to \mathbb{P}^1$, which is the composition of π and the projection to the base curve $\varphi_1 \times_{\mathbb{P}^1} \varphi_2$, is a fibration and has a non-normal rational surface as a general fiber.

(4) The discriminant of a quasi-elliptic surface in $p = 3$ is given by $\Delta := (\phi')^2$ for the Weierstrass form $y^2 = x^3 + \phi(t)$.

4. Crepant Resolutions

We seek a crepant resolution of singularities $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$. Since $Y_1 \times_{\mathbb{P}^1} Y_2$ is a divisor of a nonsingular fourfold $Y_1 \times_k Y_2$, all the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$ are hypersurface singularities. In characteristic zero, if there exists a crepant resolution, any isolated singularity in codimension two is generically a trivial deformation of a rational double point [21, Corollary 1.14]. But in positive characteristic, this is not always the case.

Proposition 4.1. *The following hypersurface singularities in* $\mathbb{A}_k^4 = \operatorname{Spec} k[x, y, z, w]$ *with* $p = 3$ *have crepant resolutions.*

(1) $x^3 + y^2 + z^2 = 0$,
(2) $x^3 + y^2 + z^3 w = 0$,
(3) $x^3 + y^2 + z^3 w^2 = 0$,
(4) $x^3 + y^2 + z^2 w = 0$,
(5) $x^3 + xy^2 + y^2 z + zw^2 = 0$.

Remark 4.2. The singularity (1) is a trivial deformation of the rational double point of type A_2. The singularity (2) is an example which is not generically a trivial deformation of a rational double point, but has a crepant resolution.

Proof of Proposition 4.1. This is done by local calculation. Use the Jacobian criterion for regularity. In (1) and (4), blowing up with the center of the reduced singular locus $\{x = y = z = 0\}$ gives a resolution. In (2), blow up the reduced singular locus $\{x = y = z = 0\}$. There appears a one dimensional singular locus which is locally a trivial deformation of a rational double point of type A_1. Blowing up this singular locus gives a resolution. In (3), one can reduce to the case of type (2) after a blow-up along $\{x = y = w = 0\}$. In (5), blow up $\{x = y = w = 0\}$, there remain six ordinary double points. The reduced inverse image of the origin is $\mathbb{P}^2$

and blowing up this $\mathbb{P}^2$ gives a small resolution. One knows that all the resolutions are crepant using the following Lemma. $\square$

Lemma 4.3. *Let X be a hypersurface of an $(n+1)$-dimensional nonsingular variety Z $(n \geq 2)$. Suppose that $C \subset X$ is a nonsingular subvariety of dimension $n-2$ which is an irreducible component of* $\operatorname{Sing}(X)$. *Consider the blow-up $\pi : \tilde{Z} \to Z$ along C. If the total transform is expressed as $\pi^* X = \tilde{X} + 2E$, where $\tilde{X}$ is the strict transform and E is the exceptional divisor. Then the equality $K_{\tilde{X}} = \pi^* K_X$ holds.*

Proof. This follows from the canonical bundle formula, for example in [9, Chapter II, Exercise 8.5], and the adjunction formula. $\square$

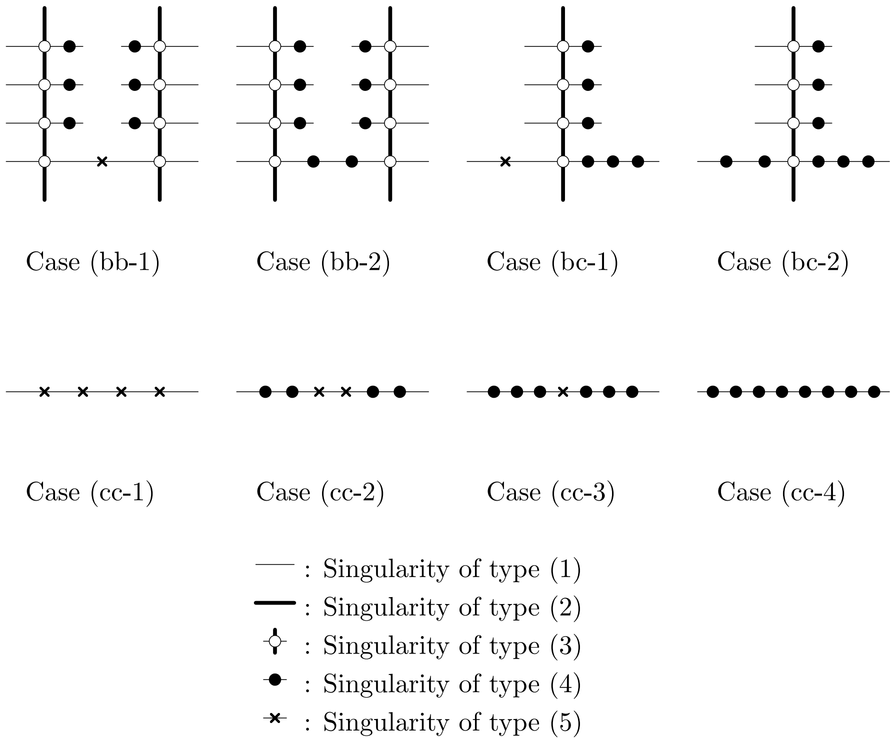


FIGURE 2

Proof of Proposition 3.2. Let $\Gamma \subset Y_1 \times_{\mathbb{P}^1} Y_2$ be the fiber product of the moving cusps of the quasi-elliptic surfaces $\varphi_i : Y_i \to \mathbb{P}^1$ $(i = 1, 2)$ (cf. Figure 1). We use Proposition 3.3 and obtain local equations of the singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$.

First we consider the case (cc). The configuration consists of one irreducible curve Γ as in Figure 1. At a point in Γ which projects to both triple points of the singular fibers of type IV, the singularity is given by

$$xy^2 - x^3 + zw^2 - z^3 = 0,$$

which is isomorphic to the singularity of type (5) in Proposition 4.1. At a point in Γ which projects to the cusp of a nonspecial fiber and the triple point of the singular fiber of type IV, the singularity is given by

$$x^3 + y^2 + zw^2 - z^3 = 0,$$

which is isomorphic to the singularity of type (4). At a point in Γ which projects to both the cusps of nonspecial fibers of φ_1 and φ_2, we have the equation

$$x^3 + y^2 + z^3 + w^2 = 0,$$

which is isomorphic to the singularity of type (1) in Proposition 4.1. Consideration of $\mathrm{Aut}_k(\mathbb{P}^1_k)$ gives that the case (cc) is subdivided into (cc-1), (cc-2), (cc-3) and (cc-4) as in Figure 2.

For cases (bb) and (bc), Sing $(Y_1 \times_{\mathbb{P}^1} Y_2)$ consists of Γ and other $\mathbb{P}^1$'s which come from the cusp of a nonspecial fiber and components of the singular fiber of type IV* whose multiplicities are greater than one (cf. Figure 1). We already know the description of singularities of $Y_1 \times_{\mathbb{P}^1} Y_2$ along Γ. To describe the remaining singularities, let C (resp. D) be a component which comes from the cusp and the component of multiplicity three (resp. a component of multiplicity two) in the fiber of type IV*. Then it is known that Γ and C intersects at a point, and we know from Proposition 3.3 that the singularity at this point is given formally by

$$x^3 + y^2 + z^3(1 + w^2) = 0,$$

which is isomorphic to the singularity of type (3) in Proposition 4.1. C and D also intersects at a point, where the singularity is given by

$$x^3 + y^2 + z^3 w^2 = 0,$$

which is the singularity of type (3). At a point in C outside the four points described above, Proposition 3.3 gives

$$x^3 + y^2 + z^3(1 + w) = 0,$$

which is isomorphic to the singularity of type (2). At a point in D which projects to the cusp of a nonspecial fiber and a point where components of multiplicity one and two intersect in IV*, the equation is

$$x^3 + y^2 + z^2 w = 0,$$

which is the singularity of type (4). At a point in D outside the two points described above, we have a local equation

$$x^3 + y^2 + z^2 = 0,$$

which is the singularity of type (1) in Proposition 4.1. From the arguments above, we obtain the configurations as in Figure 2.

So we know by Proposition 4.1 that all the singularities have crepant resolutions locally. One then checks that there exists a sequence of blow-ups along the reduced centers $\mathbb{P}^1$'s in the singular loci followed by blow-ups along $\mathbb{P}^2$'s for ordinary double points which attain crepant resolutions $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$. □

5. Rationality of the singularities

For a crepant resolution $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$, whether the sheaf $R^1\pi_*\mathcal{O}_X$ vanishes or not is an important question. In characteristic zero, the vanishing follows from the Grauert-Riemenschneider vanishing theorem ([8]) and $K_X = 0$. The Leray spectral sequence $E_2^{i,j} := H^i(R^j\pi_*\mathcal{O}_X) \Rightarrow H^{i+j}(X, \mathcal{O}_X)$ gives an exact sequence

$$0 \to H^1(\pi_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(R^1\pi_*\mathcal{O}_X) \to H^2(\pi_*\mathcal{O}_X),$$

and $H^1(X, \mathcal{O}_X) = 0$ would follow from $H^1(Y_1 \times_{\mathbb{P}^1} Y_2, \mathcal{O}_{Y_1 \times_{\mathbb{P}^1} Y_2}) = 0$ under $R^1\pi_*\mathcal{O}_X = 0$.

Proposition 5.1. *For our examples of threefolds obtained in Proposition 3.2, we have $H^1(\mathcal{O}_X) = 0$.*

Proof. It is observed that X and $Y_1 \times_{\mathbb{P}^1} Y_2$ are anti-canonical members of nonsingular rational fourfolds, from which the vanishing follows (cf. [11]). □

We employ the definition of rational singularities as in [29], that is, a singular point x on a normal variety W is said to be a rational singularity if there exists a resolution of singularities $\pi : X \to W$ such that $(R^i\pi_*\mathcal{O}_X)_x = 0$ for all $i > 0$.

Proposition 5.2. *The sheaf $R^i\pi_*\mathcal{O}_X$ with $i = 1, 2$ is zero for a crepant resolution $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$ in Proposition 3.2. All the singularities given in Proposition 4.1 are rational singularities.*

Proof. First recall $H^2(\mathcal{O}_{Y_1\times_{\mathbb{P}^1}Y_2}) = 0$, $H^3(\mathcal{O}_{Y_1\times_{\mathbb{P}^1}Y_2}) \cong k$ and $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ by Proposition 5.1. It follows $H^0(R^1\pi_*\mathcal{O}_X) = 0$ by the Leray spectral sequence. On the other hand, the support of the sheaf $R^1\pi_*\mathcal{O}_X$ is contained in the singular loci of $Y_1 \times_{\mathbb{P}^1} Y_2$. Straightforward arguments from the definition give that it is zero along a trivial deformation of a rational double point. We now prove that it is zero along the singularity of type (2). By replacing w by $z+w$, we have a flat morphism

$$W := \operatorname{Spec} k[x, y, z, w]/(x^3 + y^2 + z^4 + z^3w) \to \operatorname{Spec} k[w],$$

whose fibers are rational double points of type E_6^0. Then the first blow-up along the singular locus of W gives a family $W' \xrightarrow{\pi'} W \to \operatorname{Spec} k[w]$. Recall that W' has one-parameter trivial deformation of the rational double point of type A_1 as its singularity, so blowing up its locus as in the proof of Propositon 4.1 gives a resolution of singularities of W

$$X \xrightarrow{\pi''} W' \xrightarrow{\pi'} W \to \operatorname{Spec} k[w].$$

We know that $R^1\pi''_*\mathcal{O}_X = 0$. So we prove $R^1\pi'_*\mathcal{O}_{W'} = 0$. It can be checked that the fiber $W'_w \xrightarrow{\pi'_w} W_w$ for any $w \in \operatorname{Spec} k[w]$ is a reduced point blow-up of the rational double point of type E_6^0, and it satisfies $H^1(\mathcal{O}_{W'_w}) = 0$. This gives $R^1\pi'_*\mathcal{O}_{W'} = 0$. Then the exact sequence coming from the Leray spectral sequence $0 \to R^1\pi'_*(\pi''_*\mathcal{O}_X) \to R^1(\pi' \circ \pi'')_*\mathcal{O}_X \to \pi'_*R^1\pi''_*\mathcal{O}_X$ gives the vanishing along the singularity of type (2). So $R^1\pi_*\mathcal{O}_X$ is possibly supported on finite points, but this is ruled out by $H^0(R^1\pi_*\mathcal{O}_X) = 0$.

For $R^2\pi_*\mathcal{O}_X$, we know it is zero outside finite points by looking at the dimension of fibers of π. Then the spectral sequence says $H^0(R^2\pi_*\mathcal{O}_X) = 0$, so $R^2\pi_*\mathcal{O}_X$ vanishes. □

6. Unirationality and topological invariants

As is mentioned in the introduction, a unirational Calabi-Yau is supersingular. The converse is an open question, still unsolved for $K3$ surfaces. For our examples, we have:

Proposition 6.1. *Our examples of Calabi-Yau threefolds are purely inseparably unirational.*

Proof. It is observed that the base change of a quasi-elliptic surface $\varphi : Y \to \mathbb{P}^1$ by the Frobenius morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is a non-normal rational surface. X has the fibration $f : X \to \mathbb{P}^1$ induced from $\varphi_1 \times_{\mathbb{P}^1} \varphi_2$ as in Remark 3.4. Then the base change $X \times_{\mathbb{P}^1} \mathbb{P}^1$ of f by the Frobenius morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is a rational threefold. □

By standard arguments on étale coverings of X as in [20], we have the following proposition:

Proposition 6.2. *If a nonsingular projective threefold X is purely inseparably unirational, then its algebraic fundamental group $\pi_1^{alg}(X)$ is trivial.*

For a nonsingular Calabi-Yau threefold X, whether it satisfies an equality $\rho(X) = b_2(X)$ is a difficult question. In complex case, the exponential sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 1$ gives an affirmative answer, which is not available in our situation. Instead, Nygaard proves:

Theorem 6.3 ([20])**.** *If a nonsingular projective variety X is unirational, then the Picard number and the second Betti number of X coincide, i.e. $\rho(X) = b_2(X)$.*

Proposition 6.4. (1) *The Calabi-Yau threefolds obtained in the previous sections have the following invariants.*

	(bb-1)	(bb-2)	(bc-1)	(bc-2)	(cc-1)	(cc-2)	(cc-3)	(cc-4)
$e(X)$	72	60	60	48	84	60	48	36
$\rho(X)$	35	30	30	25	41	30	25	20

(2) *In cases* (bb-1) *and* (cc-1)*, X has $b_3(X) = 0$, hence does not lift to characteristic zero.*

We use the following lemma frequently in the proof of Proposition 6.4.

Lemma 6.5. *Let $f : X \to Y$ be a morphism of complete varieties, and $C \subset Y$ be a closed subvariety, and $E := C \times_Y X$ be a fiber product. Then we have a formula of topological Euler-Poincaré characteristics*

$$e(X) - e(Y) = e(E) - e(C).$$

In case Y is nonsingular and f is a blow-up along a smooth center $C \subset Y$, essentially the same statements can be found, for example, in [5, Proposition 4.4], [28, Lemma 2.1].

Proof. Consider the following two exact sequences and homomorphisms f^* induced from f (cf. [17, p. 94]):

$$\begin{array}{ccccccccc} \cdots \to H^i_{\mathrm{c}}(Y \setminus C, \mathbb{Q}_l) \to & H^i(Y, \mathbb{Q}_l) \to & H^i(C, \mathbb{Q}_l) & \to H^{i+1}_{\mathrm{c}}(Y \setminus C, \mathbb{Q}_l) \to & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \cdots \to H^i_{\mathrm{c}}(X \setminus E, \mathbb{Q}_l) \to & H^i(X, \mathbb{Q}_l) \to & H^i(E, \mathbb{Q}_l) & \to H^{i+1}_{\mathrm{c}}(X \setminus E, \mathbb{Q}_l) \to & \cdots \end{array}$$

Then the isomorphisms $H^i_{\mathrm{c}}(Y \setminus C, \mathbb{Q}_l) \cong H^i_{\mathrm{c}}(X \setminus E, \mathbb{Q}_l)$ give the desired result. □

Proof of Proposition 6.4. (1) The invariant $e(Y_1 \times_{\mathbb{P}^1} Y_2)$ is calculated from that of the normalization of $(Y_1 \times_{\mathbb{P}^1} Y_2) \times_{\mathbb{P}^1} \mathbb{P}^1$, which is the base change by the Frobenius morphism $\mathbb{P}^1 \to \mathbb{P}^1$. More precisely, let $\tilde{Y}_i$ be the normalization of the Frobenius base change of Y_i for $i = 1, 2$, and Z_i its resolution. Furthermore, let S_i be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ which is obtained from Z_i by blowing down (-1)-curves. In our cases, $\tilde{Y}_1 \times_{\mathbb{P}^1} \tilde{Y}_2$ is just the normalization of $(Y_1 \times_{\mathbb{P}^1} Y_2) \times_{\mathbb{P}^1} \mathbb{P}^1$, whose Euler-Poincaré characteristic is equal to that of $Y_1 \times_{\mathbb{P}^1} Y_2$. We can calculate $e(\tilde{Y}_1 \times_{\mathbb{P}^1} \tilde{Y}_2)$ from $e(S_1 \times_{\mathbb{P}^1} S_2)$, which is 8 since $S_1 \times_{\mathbb{P}^1} S_2$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $\mathbb{P}^1$, via $Z_1 \times_{\mathbb{P}^1} Z_2$:

$$S_1 \times_{\mathbb{P}^1} S_2 \leftarrow Z_1 \times_{\mathbb{P}^1} Z_2 \to \tilde{Y}_1 \times_{\mathbb{P}^1} \tilde{Y}_2.$$

Thus we have the following table:

	(bb-1)	(bb-2)	(bc-1)	(bc-2)	(cc-1)	(cc-2)	(cc-3)	(cc-4)
$e(Y_1 \times_{\mathbb{P}^1} Y_2)$	44	40	44	40	56	48	44	40

The crepant resolution π is a sequence of blow-ups as in the proof of Proposition 3.2, $X = W_r \to W_{r-1} \to \cdots \to W_0 := Y_1 \times_{\mathbb{P}^1} Y_2$. When the center of a blow-up $W_{i+1} \to W_i$ is isomorphic to $\mathbb{P}^1$, Lemma 6.5 says

$$e(W_{i+1}) = e(W_i) + (e(E_i) - 2), \tag{6.1}$$

where E_i is the exceptional set. If the exceptional set is isomorphic to $\mathbb{P}^1$, the topological Euler-Poincaré characteristic increases by one.

In (cc-1), the inverse image of the blow-up along the Γ, we have the exceptional set which consists of two rational ruled surfaces and four $\mathbb{P}^2$'s on which six ordinary double points sit each. Then we can calculate $e(X)$. For the exceptional set in (cc-2) (resp. (cc-3), (cc-4)), the inverse image of the blow-up along Γ consists of a non-normal surface and two $\mathbb{P}^2$'s (resp. a non-normal surface and one $\mathbb{P}^2$, a non-normal surface). The normalization of the non-normal surface is a ruled surface over a curve of genus 1 (resp. 2, 3). The topological Euler-Poincaré characteristic of the inverse image of Γ is 0 (resp. -2, -4). Taking into account the existence of the ordinary double points, we obtain $e(X)$.

Similarly, the inverse image of the blow-up along Γ consists of a non-normal surface and one $\mathbb{P}^2$ (resp. a non-normal surface, a non-normal surface and one $\mathbb{P}^2$, a non-normal surface) for the case (bb-1) (resp. (bb-2), (bc-1), (bc-2)). The normalization of the non-normal surface is a ruled surface over a curve of genus 0 (resp. $1, 1, 2$). Thus we can calculate the contribution to the topological Euler-Poincaré characteristic from them. Remaining contribution from the inverse image of the blow-ups along the components except Γ is 10 (resp. 20) for the cases (bc-1) and (bc-2) (resp. the cases (bb-1) and (bb-2)). Thus we get the results.

For the Picard number, we use the following formula:

$$\begin{aligned}\rho(X) = 3 \quad &+ \quad \sum_{t\in\mathbb{P}^1}(\#\{\text{irred. comp. of } f^{-1}(t)\} - 1) \\ &+ \quad \#\{\text{irred. excep. divisors w. r. t. } \pi \text{ which are horizontal to } f\}.\end{aligned}$$

This essentially comes from the exact sequence [23, (3.2), p. 182].

(2) The Betti numbers can be calculated by Theorem 6.3 and the Poincaré duality theorem because $b_1(X) = 0$ follows from $H^1(\mathcal{O}_X) = 0$. Thus $b_3(X) = 0$ in cases (bb-1) and (cc-1), which implies that X is not liftable to characteristic zero (cf. [10]). □

Proposition 6.6. *The Betti numbers of the example of Calabi-Yau threefold X with $e(X) = 84$ obtained in the previous proposition coincides with those of the example in* [10]*, but they are not isomorphic to each other.*

Proof. The Calabi-Yau threefold X we obtained here admits a fibration to $\mathbb{P}^1$. However, the example in [10] does not have a fibration to $\mathbb{P}^1$. Indeed, suppose that it has a fibration $X \to \mathbb{P}^1$. Then one can see from the construction that there exists a purely inseparable finite morphism of degree 3 from a nonsingular rational threefold S to X, which induces a fibration of S to $\mathbb{P}^1$. But since S is obtained by blowing up 40 distinct points on $\mathbb{P}^3$, it can be observed by arguments on intersection numbers that S does not have a non-trivial divisor D with D^2 trivial as an element of $N_1(S/k)$ (cf. [18, Chapter I]), and we have a contradiction. □

Remark 6.7.
(1) We cannot answer the question whether birationally equivalent Calabi-Yau threefolds in positive characteristic have the equal topological invariants such as Betti numbers (cf. [2]). We do not know whether the two examples in Proposition 6.6 are birationally equivalent to each other or not.
(2) For a fibration $f : X \to \mathbb{P}^1$, one has a formula (cf. [5])

$$e(X) = \sum_{t\in\mathbb{P}^1}(e(X_t) - e(X_{\overline{\eta}}) + d(X_t)) + e(X_{\overline{\eta}})e(\mathbb{P}^1).$$

From the above proposition, we know that $d(X_t)$, which comes from the Serre's measure of wild ramification, for the fibration $f := (\varphi_1 \times_{\mathbb{P}^1} \varphi_2) \circ \pi$ is zero (cf. [26]). However, if we could prove $d(X_t) = 0$ for all $t \in \mathbb{P}^1$ a priori, we could spare the tedious calculation in the proof of Proposition 6.4.

(3) As for other examples with $b_3(X) \neq 0$, we are not able to determine whether they are liftable to characteristic zero (cf. [24], [7]).

(4) For the unirational Calabi-Yau threefolds constructed from fiber products of elliptic and quasi-elliptic rational surfaces in [11], we calculate the topological Euler-Poincaré characteristic in the same method as above. Now we can compute the Betti numbers of these Calabi-Yau threefolds X as

$$\begin{aligned}(b_0, b_1, b_2, b_3, b_4, b_5, b_6) &= (1, 0, 27, 8, 27, 0, 1) \text{ in } p = 2,\\ &\quad\ (1, 0, 35, 0, 35, 0, 1) \text{ in } p = 3.\end{aligned}$$

The one in $p = 3$ has $b_3(X) = 0$ and is another example of non-liftable Calabi-Yau threefold.

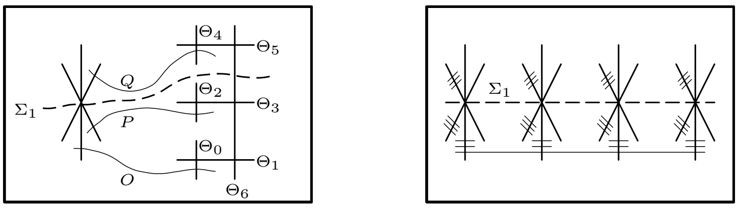
7. Fibrational structures

A fibration of a Calabi-Yau threefold X is a surjective morphism $g : X \to S$ with S normal and $\mathcal{O}_S \cong g_*\mathcal{O}_X$, hence $K_{g^{-1}(s)_{\mathrm{red}}} = 0$. In characteristic zero, it follows from generic smoothness of g and classification theories that a general fiber of g is an elliptic curve when $\dim S = 2$, and either a $K3$ surface or an abelian surface when $\dim S = 1$. In positive characteristic, we need to add quasi-elliptic fibrations, however it is not known whether there exists a fibration g whose general fiber is not reduced (cf. [22], [19]). When $\dim S = 1$, very little is known about what kind of surfaces appear as a general fiber of g.

Proposition 7.1. *In characteristic* 3*, the Calabi-Yau threefold X obtained in the previous sections admits a fibration $g : X \to \mathbb{P}^1$ whose general fiber is a supersingular $K3$ surface with one rational double point of type A_2. Moreover, if Y_1 or Y_2 is of type (b), then X also has another fibration $X \to \mathbb{P}^1$ whose general fiber is a smooth supersingular $K3$ surface.*

Proof. Let $\rho_1 : Y_1 \to \mathbb{P}^1$ be one of the $\mathbb{P}^1$-fibrations on the rational quasi-elliptic surface Y_1. We consider the composition $g_1 : X \xrightarrow{\pi} Y_1 \times_{\mathbb{P}^1} Y_2 \xrightarrow{\mathrm{proj}_1} Y_1 \xrightarrow{\rho_1} \mathbb{P}^1$, which indeed is a fibration.

Let F_{φ_1} and F_{ρ_1} be general fibers of $\varphi_1 : Y_1 \to \mathbb{P}^1$ and $\rho_1 : Y_1 \to \mathbb{P}^1$, respectively. By the canonical bundle formula for Y_1, we observe $F_{\varphi_1}.F_{\rho_1} = 2$. This means that a general fiber of the composition $Y_1 \times_{\mathbb{P}^1} Y_2 \xrightarrow{\mathrm{proj}_1} Y_1 \xrightarrow{\rho_1} \mathbb{P}^1$ is obtained as the base change of Y_2 by a double cover $\varphi_1|_{F_{\rho_1}} : \mathbb{P}^1 \to \mathbb{P}^1$, which is ramified at two points by the Hurwitz formula. We consider a double cover $\psi_1 := (\varphi_1, \rho_1) : Y_1 \to \mathbb{P}^1 \times \mathbb{P}^1$. We investigate the ramification divisor R_1 of ψ_1. Note that the configurations of special fibers and sections on rational quasi-elliptic surfaces of type (b) and (c) are given in [14, p. 11].



type (b) type (c)

Figure 3. Configurations of sections

Case 1. Y_1 is of type (b). Y_1 has three sections O, P and Q, whose self-intersection numbers are all -1. Let Θ_i $(i = 0, \ldots, 6)$ be components of the fiber of type IV* (see Figure 3). We can blow down eight curves O, P, Q and Θ_i $(i = 0, \ldots, 4)$ in this order to get a Hirzebruch surface of degree 1, and denote by $\rho_1 : Y_1 \to \mathbb{P}^1$ the $\mathbb{P}^1$-fibration induced by the $\mathbb{P}^1$-bundle structure on the above Hirzebruch surface. Then $2\Theta_5$ is the pull-back of a fiber of $\mathrm{proj}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ by the finite part of the Stein factorization of ψ_1. This means that Θ_5 is a component of the ramification divisor R_1. Moreover, if we denote the moving cusp of the quasi-elliptic surface Y_1 by Σ_1, then it follows from $\Sigma_1.F_{\rho_1} = 1$ and $F_{\varphi_1}.F_{\rho_1} = 2$ that ψ_1 is also ramified along Σ_1. Thus taking a crepant resolution $\pi : X \to Y_1 \times_{\mathbb{P}^1} Y_2$, we have a smooth $K3$ surface as a general fiber of $g_1 : X \to \mathbb{P}^1$, which is supersingular since it has a quasi-elliptic fibration.

We can choose other configurations of eight curves to obtain Hirzebruch surfaces of degree $0, 1$ and 2, such that no fiber of $\varphi_1 : Y_1 \to \mathbb{P}^1$ has a component of R_1. In these cases, a general fiber of $g_1 : X \to \mathbb{P}^1$ has one rational double point of type A_2.

Case 2. Y_1 is of type (c). As in the Case 1, we can observe that the moving cusp Σ_1 is a component of R_1 for any $\mathbb{P}^1$-fibration $\rho_1 : Y_1 \to \mathbb{P}^1$. On the other hand, since all components of fibers are reduced in this case, no fiber of $\varphi_1 : Y_1 \to \mathbb{P}^1$ contains a component of the ramification divisor, hence the other irreducible component of R_1 exists horizontally to φ_1. Thus a general fiber of $g_1 : X \to \mathbb{P}^1$ is a supersingular $K3$ surface with one rational double point of type A_2. □

Remark 7.2.
1. In Case 1 of the proof above, we can choose other configurations of eight curves to get Hirzebruch surfaces of degree 0 and 2, and a general fiber of $g_1 : X \to \mathbb{P}^1$ is smooth.
2. We note that the quasi-elliptic fibrational structure on a general fiber of $g_1 : X \to \mathbb{P}^1$ depend on Y_2. Figure 4 illustrates the case Y_1 and Y_2 are of type (b) and (c), respectively.
3. As for the classification of singularities on general fibers of fibrations on Fano and Calabi-Yau threefolds over $\mathbb{P}^1$, a partial answer in rational double points and simple elliptic singularities is obtained in [12].

References

[1] M. Artin and B. Mazur, *Formal groups arising from algebraic varieties*, Ann. Sci. École Norm. Sup. **10** (1977), no. 4, 87–131.

[2] V.V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, New trends in algebraic geometry (Warwick, 1996), Cambridge University Press, Cambridge, 1999, pp. 1–11.

[3] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p, III*, Invent. Math. **35** (1976), 197–232.

[4] ———, *Enriques' classification of surfaces in char. p, II*, Complex Analysis and Algebraic Geometry (W. L. Baily Jr. and T. Shioda, eds.), Iwanami Shoten, Tokyo, and Cambridge University Press, Cambridge, 1977, pp. 23–42.

[5] I.V. Dolgachev, *The Euler characteristic of algebraic varieties*, Mat. Sb. **89** (1972), no. 2, 297–312 (Russian). English transl.: Math. USSR Sbornik **18** (1972), 303–319.

[6] T. Ekedahl, *On non-liftable Calabi-Yau threefolds*, `arXiv:math.AG/0306435 v2`, 2004.

[7] T. Ekedahl and N.I. Shepherd-Barron, *Tangent lifting of deformations in mixed characteristic*, J. Algebra **291** (2005), 108–128.

[8] H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.

[9] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York - Berlin - Heidelberg, 1977.

[10] M. Hirokado, *A non-liftable Calabi-Yau threefold in characteristic* 3, Tohoku Math. J. **51** (1999), no. 2, 479–487.

[11] ———, *Calabi-Yau threefolds obtained as fiber products of elliptic and quasi-elliptic rational surfaces*, J. Pure and Appl. Algebra **162** (2001), 251–271.

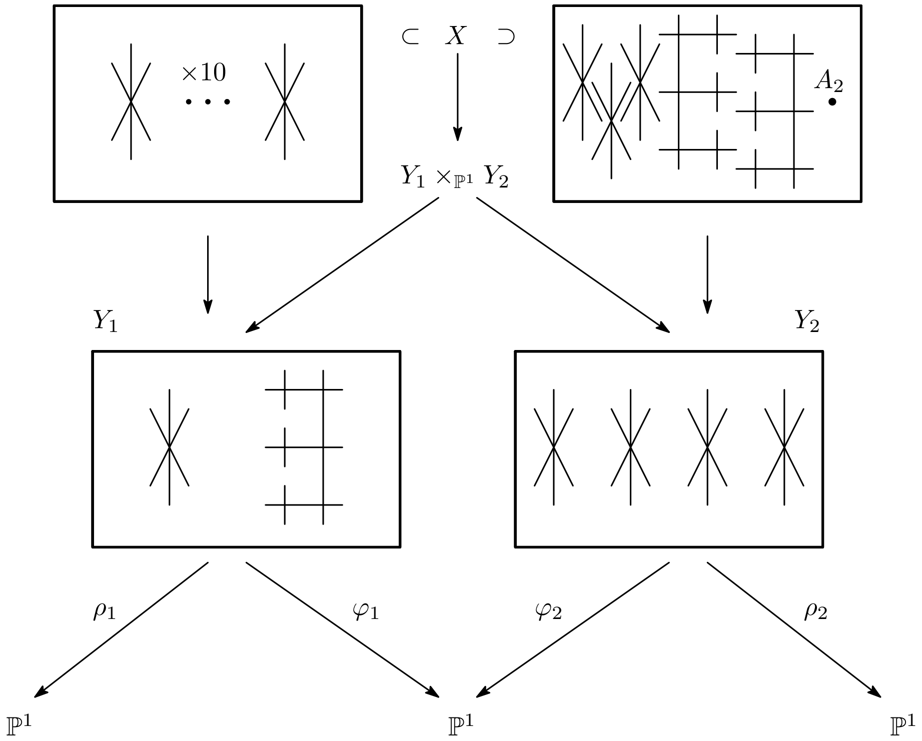


FIGURE 4. The case Y_1, Y_2 are of type (b) and (c)

[12] ———, *Deformations of rational double points and simple elliptic singularities in characteristic p*, Osaka J. Math. **41** (2004), 605–616.

[13] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. **4** (1979), 501–661.

[14] H. Ito, *The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic* 3, Math. Z. **211** (1992), 1–40.

[15] ———, *The Mordell-Weil groups of unirational quasi-elliptic surfaces in characteristic* 2, Tôhoku Math. J. **46** (1994), 221–251.

[16] W. E. Lang, *Quasi-elliptic surfaces in characteristic three*, Ann. scient. Éc. Norm. Sup. **12** (1979), no. 4, 473–500.

[17] J. S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[18] S. Mori and J. Kollár, *Birational Geometry of Algebraic Varieties*, Cambridge University Press, Cambridge, 1998.

[19] S. Mori and N. Saito, *Fano threefolds with wild conic bundle structures*, Proc. of Japan Acad. **79** (2003), 111–114.

[20] N. Nygaard, *On the fundamental group of a unirational* 3*-fold*, Invent. Math. **44** (1978), 75–86.

[21] M. Reid, *Canonical* 3*-folds*, Journées de géométrie algébrique d'Angers (A. Beauville, ed.), Sijthoff and Nordhoff, 1980, pp. 273–310.

[22] N. Saito, *Fano threefolds with Picard number* 2 *in positive characteristic*, Kodai Math. J. **26** (2003), 147–166.

[23] C. Schoen, *On fiber products of rational elliptic surfaces*, Math. Z. **197** (1988), 177–199.

[24] S. Schröer, *The* T^1*-lifting theorem in positive characteristic*, J. Algebraic Geom. **12** (2003), 699–714.

[25] ———, *Some Calabi-Yau threefolds with obstructed deformations over the Witt vectors*, Compositio Math. **140** (2004), 1579–1592.

[26] J.-P. Serre, *Représentations Linéaires des Groupes Finis*, 3rd ed., Hermann, Paris, 1978.

[27] I. Shimada, *Unirationality of certain complete intersections in positive characteristics*, Tôhoku Math. J. **44** (1992), 379–393.

[28] T. Shioda and T. Katsura, *On Fermat varieties*, Tôhoku Math. J. **31** (1979), 97–115.

[29] K.E. Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), 159–180.

[30] G. van der Geer and T. Katsura, *An invariant for varieties in positive characteristic*, Contemporary Mathematics, vol. 300, AMS, 2002.
[31] ———, *On the height of Calabi-Yau varieties in positive characteristic*, Doc. Math. **8** (2003), 97–113.

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