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Masayuki Hirokado[1] · Hiroyuki Ito[2] · Natsuo Saito[1]

Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces, II

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Abstract. We construct examples of supersingular Calabi-Yau threefolds in characteristic 2 making use of the method by Schoen. Unirational Calabi-Yau threefolds of five different topological types are obtained. There are two examples with the third Betti number zero among them, and they are counted as other examples of non-liftable Calabi-Yau threefolds in characteristic 2 after the one given by Schröer.

Key words. Calabi-Yau threefolds – quasi-elliptic surfaces – crepant resolutions – supersingular $K3$ surfaces

1. Introduction

Let k be an algebraically closed field k of characteristic $p \geq 0$. A Calabi-Yau threefold X is a nonsingular projective threefold which satisfies $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$. When $p > 0$, we call a Calabi-Yau threefold X supersingular if the condition on its Artin-Mazur formal group $\Phi^3(X, \mathbf{G}_m) \cong \hat{\mathbf{G}}_a$ is satisfied (cf. [2]).

We are interested in understanding properties of Calabi-Yau threefolds in positive characteristic in comparison with those of complex Calabi-Yau manifolds or Calabi-Yau varieties in characteristic zero, especially the Hodge decomposition and the Hodge symmetry in $k = \mathbf{C}$,

$$H^d(X, \mathbf{C}) \cong \bigoplus_{i+j=d} H^j(\Omega^i_X), \quad \overline{H^j(\Omega^i_X)} \cong H^i(\Omega^j_X).$$

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M. Hirokado and N. Saito: Graduate School of Information Sciences, Hiroshima City University, 3-4-1 Ozuka-higashi, Asaminami-ku, Hiroshima 731-3194, Japan, e-mail: hirokado@math.its.hiroshima-cu.ac.jp, natsuo@math.its.hiroshima-cu.ac.jp

H. Ito: Department of Applied Mathematics, Graduate School of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan, e-mail: hiroito@amath.hiroshima-u.ac.jp

It is not clear what the analogous statements of these facts are in positive characteristic. Thus it would be interesting to calculate the possible combinations of the Betti numbers and the Hodge numbers of Calabi-Yau threefolds in positive characteristic.

Schoen constructed Calabi-Yau threefolds as fiber products and their small resolutions of two rational elliptic surfaces over the base $\mathbf{P}^1_{\mathbf{C}}$ [15]. We make a slight modification and consider fiber products of two rational quasi-elliptic surfaces, which exist only in $p = 2, 3$, over the base $\mathbf{P}^1_k$. In this situation, we can no longer hope neither that these fiber products be nonsingular, nor there exist any small resolutions.

This is a sequel to our article in characteristic 3 [8], and we will work in characteristic 2 throughout this paper.

Explicit constructions of supersingular Calabi-Yau threefolds with the following Betti numbers $(b_0, b_1, b_2, b_3, b_4, b_5, b_6)$ are given up to now:

- $(1, 0, 1, 204, 1, 0, 1)$ in $p \equiv 2, 3, 4 \bmod 5$ (Fermat quintic),
- $(1, 0, 41, 0, 41, 0, 1)$ in $p = 3$ [5],
- $(1, 0, 23, 0, 23, 0, 1)$ in $p = 2, 3$ [16],
- $(1, 0, 20, 6, 20, 0, 1)$, $(1, 0, 25, 4, 25, 0, 1)$, $(1, 0, 30, 2, 30, 0, 1)$, $(1, 0, 35, 0, 35, 0, 1)$, $(1, 0, 41, 0, 41, 0, 1)$ in $p = 3$ and $(1, 0, 27, 8, 27, 0, 1)$ in $p = 2$ in [8], [6].

We add the five more types in $p = 2$ as in Theorem 1.

Our result is as follows:

Theorem 1 *In characteristic* 2*, we construct Calabi-Yau threefolds with the following properties:*

1. *X is unirational, therefore supersingular.*
2. $\rho(X) = b_2(X)$.
3. $\pi_1^{\mathrm{alg}}(X) = \{1\}$.
4. $(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 0, 25, 4, 25, 0, 1), (1, 0, 36, 2, 36, 0, 1)$. $(1, 0, 47, 0, 47, 0, 1), (1, 0, 52, 2, 52, 0, 1), (1, 0, 63, 0, 63, 0, 1)$.
5. *X admits a fibration $X \to \mathbf{P}^1$ whose general fiber is a non-normal rational surface.*
6. *X admits a fibration whose general fiber is a supersingular $K3$ surface. In some cases, X has also a fibration whose general fiber is a supersingular $K3$ surface with three rational double points of type A_1.*
7. *Futhermore, X admits a fibration over a rational surface whose general fiber is a rational curve with an ordinary cusp (i.e., a quasi-elliptic fibration).*

Note that the examples with $b_3(X) = 0$ do not lift to characteristic zero. Since these examples have $e(X) = 96$ and 128, they are different from one given by Schröer in [16]. The examples with $e(X) = 72, 104$ have $b_3(X) = 2$, but we could not prove that these are rigid Calabi-Yau's, i.e. $H^1(T_X) = 0$. It is still an open question whether our Calabi-Yau's with positive b_3 are liftable to characteristic zero or not. It is still not known if there exist any Calabi-Yau threefolds with $b_3(X) = 0$ in $p \geq 5$.

2. Preliminaries

Let X be a nonsingular projective variety of dimension three over an algebraically closed field k of characteristic $p \geq 0$. The topological Euler-Poincaré characteristic $e(X)$ of X is defined as $e(X) := \sum_{i=0}^{6}(-1)^i b_i(X)$ with the i-th Betti number $b_i(X) := \dim_{\mathbf{Q}_l} H^i_{\text{et}}(X, \mathbf{Q}_l)$ $(l \neq p)$ and satisfies $e(X) = -2\chi(\Omega^1_X)$.

A *quasi-elliptic surface* $\varphi : Y \to C$ is a nonsingular projective surface Y with a morphism to a nonsingular curve C, satisfying $\mathcal{O}_C = \varphi_* \mathcal{O}_Y$ and the general fiber is a rational curve with an ordinary cusp. Quasi-elliptic surfaces exist only in characteristic 2 and 3, enjoying properties analogous to elliptic surfaces (cf. [4]). Let Σ be the closure of the nonsmooth locus of Y_η/η inside Y. We call it the moving cusp of $\varphi : Y \to C$.

A variety X is said to be *unirational* if there exists a dominant rational map from $\mathbf{P}^3$ to X. X is said to be *separably* (resp. *purely inseparably*) *unirational* if there exists a dominant rational map $\mathbf{P}^3 \dashrightarrow X$ whose extension of function fields is separable (resp. purely inseparable).

3. Construction

Let $\varphi_1 : Y_1 \to \mathbf{P}^1$ and $\varphi_2 : Y_2 \to \mathbf{P}^1$ be relatively minimal quasi-elliptic rational surfaces with section. Take a fiber product (cf. [8])

$$\begin{array}{ccccc} & & Y_1 \times_{\mathbf{P}^1} Y_2 & & \\ & \swarrow & & \searrow & \\ Y_1 & & & & Y_2 \\ & {\scriptstyle \varphi_1}\searrow & & \swarrow{\scriptstyle \varphi_2} & \\ & & \mathbf{P}^1. & & \end{array}$$

This fiber product $Y_1 \times_{\mathbf{P}^1} Y_2$ is irreducible and locally of complete intersection. It has the trivial canonical bundle $K_{Y_1 \times_{\mathbf{P}^1} Y_2} = 0$ and satisfies $H^1(\mathcal{O}_{Y_1 \times_{\mathbf{P}^1} Y_2}) = 0$, but necessarily has singularities. These singularities are hypersurface singularities, but not isolated. To get a Calabi-Yau threefold, we try to find a crepant resolution $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$. We use the complete classification of relatively minimal rational quasi-elliptic surfaces with section up to isomorphism.

Theorem 2 (Ito [10]) *A rational quasi-elliptic surface with section in $p = 2$ is given by one of the following:*

Type of degenerate fibers	Weierstrass form	MWG
(a) II^*	$y^2 = x^3 + t^5$	1
(b) I_4^*	$y^2 = x^3 + t^2x + t^5$	$\mathbf{Z}/2$
(c) $\mathrm{III}, \mathrm{III}^*$	$y^2 = x^3 + t^3x$	$\mathbf{Z}/2$
(d) *Two* I_0^**'s*	$y^2 = x^3 + at^2x + t^3,\ a \in k$	$\mathbf{Z}/2^{\oplus 2}$
(e) I_2^**, two* III*'s*	$y^2 = x^3 + (t^3 + t)x$	$\mathbf{Z}/2^{\oplus 2}$
(f) I_0^**, four* III*'s*	$y^2 = x^3 + (t^3 + at^2 + t)x,\ a \in k^*$	$\mathbf{Z}/2^{\oplus 3}$
(g) *Eight* III*'s*	$y^2 = x^3 + (t^3 + at^2 + bt)x + t^3,\ a \in k,\ b \in k^*$	$\mathbf{Z}/2^{\oplus 4}$

Here I_0^*, I_2^*, I_4^*, II^*, III *and* III^* *stand for the types of singular fibers in the sense of Kodaira.*

When either a fiber $\varphi_1^{-1}(t)$ or $\varphi_2^{-1}(t)$ does have all the components with multiplicity one for any $t \in \mathbf{P}^1$, the fiber product $Y_1 \times_{\mathbf{P}^1} Y_2$ is normal by Serre's criterion. We call the case (st) for the fiber product $Y_1 \times_{\mathbf{P}^1} Y_2$ when $\varphi_i : Y_i \to \mathbf{P}^1$ $(i = 1, 2)$ are of types (s) and (t) as in Theorem 2 respectively, with the condition above.

Our first result is that there do exist some candidates for Calabi-Yau.

Proposition 3 *For the following six cases, the fiber product* $Y_1 \times_{\mathbf{P}^1} Y_2$ *has a resolution of singularities* $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$ *such that* $X \setminus \pi^{-1}(\mathrm{Sing}\ (Y_1 \times_{\mathbf{P}^1} Y_2)) \cong Y_1 \times_{\mathbf{P}^1} Y_2 \setminus \mathrm{Sing}\ (Y_1 \times_{\mathbf{P}^1} Y_2)$ *and* $K_X = \pi^* K_{Y_1 \times_{\mathbf{P}^1} Y_2}$ *are satisfied.*

(bb): (b) *and* (b)*, the singular fiber of type* I_4^* *does not meet the singular fiber of type* I_4^**,*

(bc-1): (b) *and* (c)*, the singular fiber of type* I_4^* *meets the singular fiber of type* III*,*

(bd): (b) *and* (d)*, the singular fiber of type* I_4^* *does not meet the singular fibers of type* I_0^**,*

(be-1): (b) *and* (e)*, the singular fiber of type* I_4^* *meets one of the singular fibers of type* III*,*

(dd): (d) *and* (d)*, the singular fibers of type* I_0^* *do not meet the singular fibers of type* I_0^**,*

(de-1): (d) *and* (e)*, both the singular fibers of type* I_0^* *meet the singular fibers of type* III.

Remark 1. We do not claim that Proposition 3 is the exhaustive list. There may exist Calabi-Yau threefolds birational to the fiber product $Y_1 \times_{\mathbf{P}^1} Y_2$ in other combinations. In some cases, what would be called Calabi-Yau threefolds with simple singularities are observed.

To obtain equations of singularities, we need to observe the quasi-elliptic fibrations along the moving cusps and special fibers.

Let $\varphi : Y \to C$ be a quasi-elliptic surface in characteristic 2 which is given by $y^2 = x^3 + \phi(t)x + \psi(t)$. Bombieri and Mumford defined an invariant $\omega \in \Omega^1_{k(C)/k}$ by $\omega := (\mathrm{d}\phi/\mathrm{d}t)^3/((\mathrm{d}\psi/\mathrm{d}t)^2 + \phi(\mathrm{d}\phi/\mathrm{d}t)^2)\mathrm{d}t$ in [3].

Proposition 4 (Bombieri-Mumford [3]) *We take a point P in the moving cusp Σ and any local coordinate t on C at $\varphi(P)$. If P is chosen generally and the invariant ω is null, i.e., $\omega \equiv 0$, then in suitable formal coordinates x, y on Y at P, one has*

$$\varphi^* t = y^2 + x^3.$$

When $\omega \not\equiv 0$, the corresponding normal forms are given in [3], but the local parameter of the base curve cannot be chosen arbitrarily there. For our concrete equations, we have the following.

Proposition 5 *Consider the Weierstrass equations*

$$\begin{aligned} y^2 &= x^3 + tx, \\ y^2 &= x^3 + (t^2 + t)x, \\ y^2 &= x^3 + (at^4 + (a+1)t^2 + t)x, \end{aligned}$$

which correspond to types (c), (e), (f) in Theorem 2 respectively. These equations can be put into normal forms in some formal power series in $\tilde{x}, \tilde{y}$ along the moving cusp

$$\begin{aligned} t' &= \sqrt[4]{\lambda}\,\tilde{x}^3 + \tilde{y}^2 + \tilde{x}^2\tilde{y}, \\ t'^2 + t' &= \sqrt[4]{\lambda^2 + \lambda}\,\tilde{x}^3 + \tilde{y}^2 + \tilde{x}^2\tilde{y}, \\ at'^4 + (a+1)t'^2 + t' &= \sqrt[4]{a\lambda^4 + (a+1)\lambda^2 + \lambda}\,\tilde{x}^3 + \tilde{y}^2 + \tilde{x}^2\tilde{y}, \end{aligned}$$

where $t' := t + \lambda$, $\lambda \in k$. Here $\lambda = 0$ in (c), $\lambda^2 + \lambda = 0$ in (e) and $a\lambda^4 + (a+1)\lambda^2 + \lambda = 0$ in (f) correspond to singular fibers of type III.

Proof. Consider the case (c). Since the ordinary cusp on the fiber over $t = \lambda$ for any $\lambda \in k$ is given by $x + \sqrt{\lambda} = y = 0$, we change the coordinates $x' = x + \sqrt{\lambda}$, $y' = y$, $t' := t + \lambda$ to get the local equation

$$t'(x' + \sqrt{\lambda}) + {y'}^2 + {x'}^3 + \sqrt{\lambda}{x'}^2 = 0$$

from the Weierstrass form. Then,

$$t' = \frac{{y'}^2 + {x'}^3 + \sqrt{\lambda}{x'}^2}{x' + \sqrt{\lambda}} = \sqrt[4]{\lambda}\tilde{x}^3 + \tilde{y}^2 + \tilde{x}^2\tilde{y}$$

where $\tilde{x} := \dfrac{y'}{x' + \sqrt{\lambda}}$, $\tilde{y} := x' + \dfrac{\sqrt[4]{\lambda}y'}{x' + \sqrt{\lambda}}$ in $k[[x', y', t']]$. The other cases can be treated similarly. □

We need the following Weierstrass forms from Theorem 2 whose fiber over $t = 0$ corresponds to each special fiber:

III* : (c) $\quad y^2 + x^3 + t^3x = 0$,
I_0^* : (d) $\quad y^2 + x^3 + at^2x + t^3 = 0, \ a \in k$,
I_2^* : (e) $\quad y^2 + x^3 + (t+1)t^2x = 0$,
I_4^* : (b) $\quad y^2 + x^3 + t^2x + t^5 = 0$.

Starting from these equations, we obtain the following lemma:

Lemma 6 *For a relatively minimal rational quasi-elliptic surface $\varphi : Y \to \mathbf{P}^1$ of type (c), (d), (e) and (b), choose a point $P \in \varphi^{-1}(0) \subset Y$ outside the smooth part of φ. Then the pull-back by φ of the local parameter t of the base curve is expressed as in Figure 1 in suitable formal coordinates x, y, where u is a unit in $k[[x,y]] \cong \hat{\mathcal{O}}_{Y,P}$.*

Proof. A local equation of the singular fiber over $t = 0$ at the point where the component of multiplicity l intersects the component of multiplicity m can be written $\varphi^* t = x^l y^m$ provided either l or m is prime to characteristic p. All the other local equations can be obtained by direct calculations. We take the case (d) to illustrate them.

The surface of type (d) is locally defined in the Weierstrass form by $y^2 + x^3 + at^2x + t^3 = 0$ in $\operatorname{Spec} k[x,y][[t]]$. Recall that the singular fiber over $t = 0$ is of type I_0^*. After a blow-up at the singular point, we have a local equation

$$y_1{}^2 + x_1{}^3 t_1 + a t_1 x_1 + t_1 = 0,$$

where $x = x_1 t_1, y = y_1 t_1, t = t_1$. Since the exceptional divisor $E = \{y_1 = t_1 = 0\}$ is the component of multiplicity two, the roots of $x_1{}^3 + a x_1 + 1 = 0$ correspond to the points where components of multiplicity one pass through. The 1-form

$$dt = dt_1 = \frac{(x_1 + \sqrt{a})^2 t_1}{x_1{}^3 + a x_1 + 1} dx_1$$

shows that the moving cusp intersects the component of multiplicity two of the singular fiber at $x_1 = \sqrt{a}$, $y_1 = 0$.

For $\lambda \in k$, we set $\tilde{x_1} := x_1 + \lambda$ to get

$$y_1{}^2 + (\tilde{x_1}{}^3 + \lambda \tilde{x_1}{}^2 + 1) t_1 + (\lambda^2 + a)(\tilde{x_1} + \lambda) t_1 = 0.$$

Therefore, when $\lambda = \sqrt{a}$,

$$t = t_1 = \frac{y_1{}^2}{1 + \sqrt{a}\tilde{x_1}{}^2 + \tilde{x_1}{}^3} = \frac{y_1{}^2}{(1 + \sqrt{a}\tilde{x_1})^2}(1 + u\tilde{x_1}{}^3),$$

where we set $u := \displaystyle\sum_{l=0}^{\infty} \frac{\tilde{x_1}{}^l}{(1 + \sqrt{a}\tilde{x_1}{}^2)^{l-1}}$. Letting $x := \sqrt[3]{u}\tilde{x_1}$, $y := \dfrac{y_1}{1 + \sqrt{a}\tilde{x_1}}$, we get the local equation

$$\varphi^* t = y^2(1 + x^3).$$

When λ is not equal to either $\sqrt{a}$ or three roots of the equation $x_1{}^3 + a x_1 + 1 = 0$,

$$\begin{aligned} t = t_1 &= \frac{y_1{}^2}{\tilde{x_1}{}^3 + \lambda \tilde{x_1}{}^2 + 1 + (\lambda^2 + a)(\tilde{x_1} + \lambda)} \\ &= \left(\frac{y_1}{\sqrt{1 + a\lambda + \lambda^3}} \right)^2 \cdot \frac{1}{1 + \dfrac{a + \lambda^2 + \tilde{x_1}{}^2 + \lambda \tilde{x_1}}{1 + a\lambda + \lambda^3} \tilde{x_1}}. \end{aligned}$$

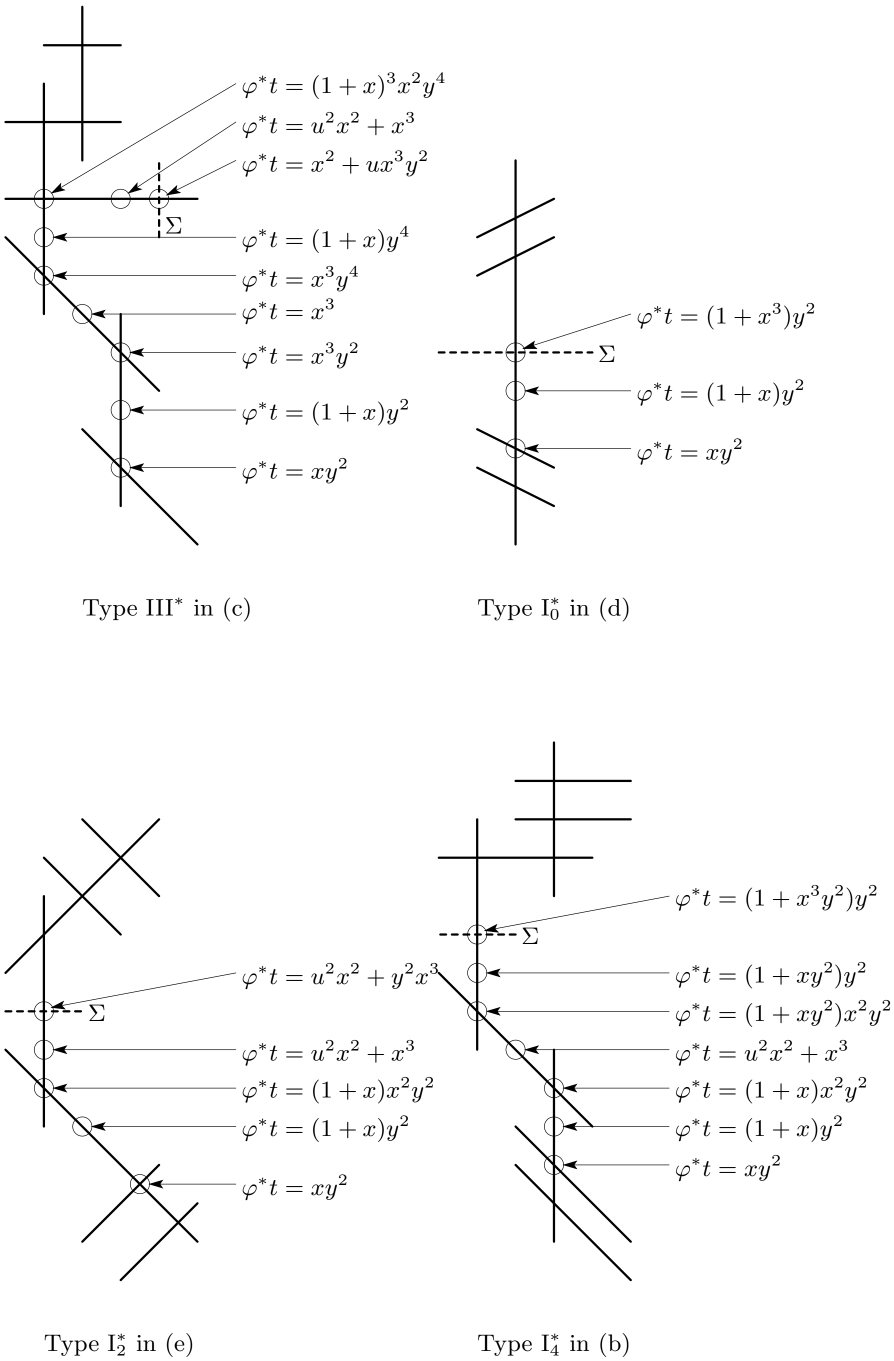
$\varphi^*t = (1+x)^3x^2y^4$
$\varphi^*t = u^2x^2 + x^3$
$\varphi^*t = x^2 + ux^3y^2$
Σ
$\varphi^*t = (1+x)y^4$
$\varphi^*t = x^3y^4$
$\varphi^*t = x^3$
$\varphi^*t = x^3y^2$
$\varphi^*t = (1+x)y^2$
$\varphi^*t = xy^2$
$\varphi^*t = (1+x^3)y^2$
Σ
$\varphi^*t = (1+x)y^2$
$\varphi^*t = xy^2$
Type III* in (c)
Type I$_0^*$ in (d)
$\varphi^*t = u^2x^2 + y^2x^3$
Σ
$\varphi^*t = u^2x^2 + x^3$
$\varphi^*t = (1+x)x^2y^2$
$\varphi^*t = (1+x)y^2$
$\varphi^*t = xy^2$
$\varphi^*t = (1+x^3y^2)y^2$
Σ
$\varphi^*t = (1+xy^2)y^2$
$\varphi^*t = (1+xy^2)x^2y^2$
$\varphi^*t = u^2x^2 + x^3$
$\varphi^*t = (1+x)x^2y^2$
$\varphi^*t = (1+x)y^2$
$\varphi^*t = xy^2$
Type I$_2^*$ in (e)
Type I$_4^*$ in (b)

Fig. 1.

Since $\dfrac{a+\lambda^2+\tilde{x_1}^2+\lambda\tilde{x_1}}{1+a\lambda+\lambda^3}$ is a unit in $k[[\tilde{x_1}]]$, we have the desired result.

The other cases can be treated similarly. □

4. Crepant Resolutions

In this section we study certain hypersurface singularities of dimension three, that have crepant resolutions. All these singularities can be resolved by successive blow-ups along nonsingular curves. For the classification of rational double points in characteristic p, we refer the reader to [1].

Proposition 7 *The following hypersurface singularities $k[[x,y,z,w]]/(f)$ in $p=2$ have crepant resolutions.*

(1) $f=x^3+y^2+z^3$,
(2) $f=x^3+y^2+z^2w$,
(3) $f=x^2+xy^2+z^2+z^2w$,
(4) $f=x^3+y^2+z^4w$,
(5) $f=x^2+xy^2+z^2+z^4w$,
(6) $f=x^2+xy^2+z^2+z^3+z^3w$.

Remark 2. Let $\pi: X\to V$ be a resolution of three dimensional Gorenstein singularities which satisfies $\pi^*K_V=K_X$. A problem is if one can classify the complete local rings $\hat{\mathcal{O}}_{V,v}$ up to isomorphism, where v is a general point of a one dimensional singular locus of V. In characteristic zero, it is known (cf. [13, Corollary 1.14]) that the local ring in question is of the form

$$\hat{\mathcal{O}}_{V,v}\cong k[[x,y,z]]/(g)\otimes_k k[[w]],$$

for a rational double point $k[[x,y,z]]/(g)$. However, each resolution process indicates that the singularities (2), (3), (4), (5), (6) in Proposition 7 (and (7) in Remark 3 below) cannot be written in this form. This is a phenomenon specific to positive characteristic (cf. [8, Remark 4.2]).

Proposition 8 *The following hypersurface singularities $k[[x,y,z,w]]/(f)$ in $p=2$ have crepant resolutions.*

① $f=x^3+y^2+z^2w^3$,
② $f=x^3+y^2+z^3w^4$,
③ $f=x^2+xy^2+y^2z+z^2w^3$,
④ $f=x^2+xy^2+y^2z+z^4w^3$,
⑤ $f=x^2+xy^2+z^2w^2+z^2w^3$,
⑥ $f=x^2+xy^2+z^2w^2+z^4w^3$,
⑦ $f=x^2+xy^2+z^2w$.

Proof of Proposition 7. All the singularities have the singular locus consisting of a curve $C:=\{x=y=z=0\}$. We first blow up with the reduced center C.

For the singularity (1), this is a trivial deformation of the rational double point of type D_4^0, so after blowing up C there appear three distinct trivial deformations of the rational double point of type A_1, and additional three blow-ups give a crepant resolution. The singularity (2) is resolved by a single blow-up along C. For (3), after blowing up C, there appears a singularity which is locally a trivial deformation of a rational double point of type A_1. For the singularity (4), after a blow-up along C, one has the singularity $x^3z+y^2+z^2w=0$. Then blow up its reduced singular locus $\{x=y=z=0\}$, and we have another singularity $x^2z+y^2+z^2w=0$. Then blow up its reduced singular locus $\{x=y=z=0\}$, and there remains a trivial deformation of the rational double point of type A_1. For (5), after the blow-up along C, there appear two disjoint one-dimensional singularities, one is a trivial deformation of the rational double point of type A_1, another is given by $x^2+(x+1)y^2z+z^2w=0$, which can be resolved by two additional blow-ups. For (6), after the blow-up along C, there appear two disjoint singularities which are trivial deformations of A_1. □

Remark 3. We call the singularity $x^3z+y^2+z^2w=0$ which appeared in the proof above, of type (7):

(7) $x^3z+y^2+z^2w=0$.

Note that it is not isomorphic to any singularities in Proposition 7. Its general hyperplane section is a rational double point of type D_6^1, but this is not a trivial deformation of any rational double point. On the other hand, it can be checked that the singularities given by $x^2z+y^2+z^2w=0$ and $x^2+(x+1)y^2z+z^2w=0$, which appeared in the proof above, are isomorphic to the singularity of type (3) in Proposition 7.

Proof of Proposition 8. The singularity of type ⑦ has the singular locus $C_1 := \{x=y=z=0\}$. All the other singularities have the singular locus $C_1 := \{x=y=z=0\}$ and $C_2 := \{x=y=w=0\}$ intersecting at the origin O. We first blow up with the reduced center C_1.

For the singularity ①, after the blow-up along C_1, we have $y^2+w^3+x^3z=0$. Then blow up its singular locus $C_2=\{x=y=w=0\}$, and we have $xz+y^2+xw^3=0$. This is a trivial deformation of the rational double point of type A_1. Note that the morphism from this singular locus to C_2 is a triple cover which ramifies at the origin. For ②, after the blow-up along C_1, there appears an irreducible curve in the singular locus of the threefold. This curve maps to C_1 with the mapping degree three and a ramification point at the origin. After six additional blow-ups along the reduced nonsingular curves in the singular locus, we have a crepant resolution. For ③, after the blow-up along C_1, we have two disjoint singularities: one is a trivial deformation of A_1 which is resolved by a blow-up, another is given by $x^2+xy^2z+y^2z+w^3=0$, which is of type (2) in Proposition 7 and can be resolved by an additional blow-up. For ④, after the blow-up along C_1, we have two disjoint singularities: one is a trivial deformation of A_1 which is resolved by a blow-up, another is given by $x^2+xy^2z+y^2z+z^2w^3=0$, which is resolved by three additional blow-ups. For ⑤, after the blow-up along C_1, there remains the singularity whose locus is connected and consists of three irreducible curves. After five additional blow-ups along irreducible curves, we get a crepant resolution. For ⑥, after the blow-up along C_1, there remains the singularity whose locus is connected and consists of four irreducible curves. Then after eight additional blow-ups along the reduced nonsingular curves in the singular locus, we get a crepant resolution. For the singularity ⑦, after the blow-up along C_1, there appears the singularity $x^2 + xy + z^2 w = 0$ whose singular locus is $\{x = y = z = 0\}$. The additional blow-up along this reduced curve gives a crepant resolution. □

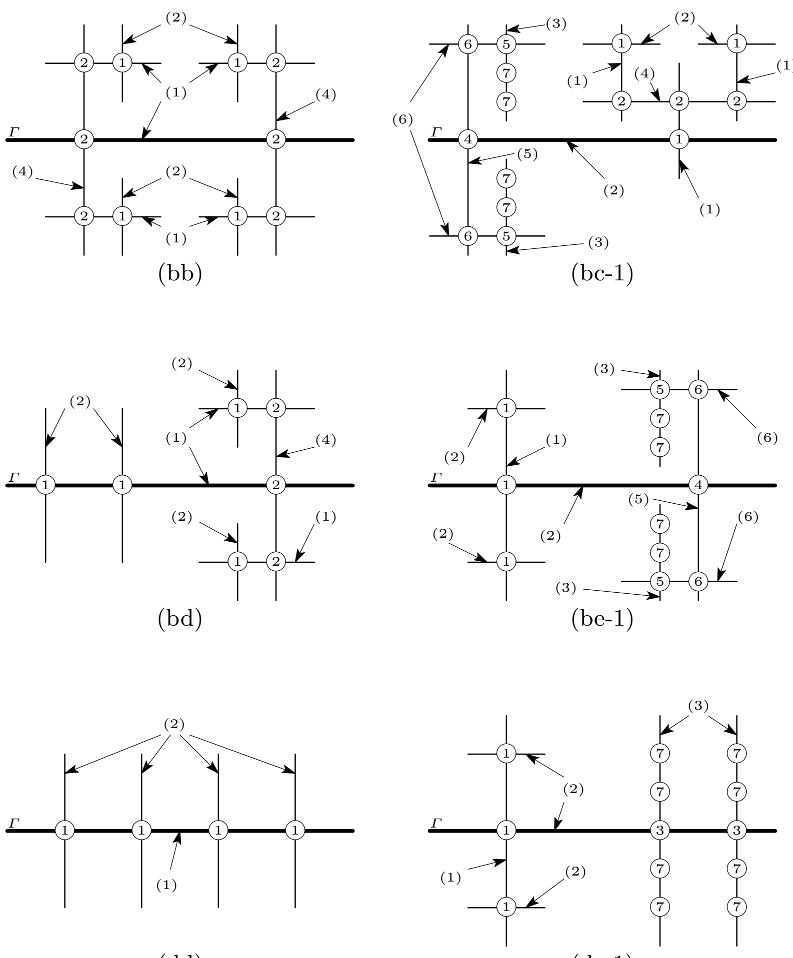

Fig. 2.

Proof of Proposition 3. Propositions 4, 5 and local calculation starting from the equations in Lemma 6 show that, in cases (bb), (bc-1), (bd), (be-1), (dd) and (de-1), all the singularities on the fiber product $Y_1 \times_{\mathbf{P}^1} Y_2$ are exhausted by those given in Propositions 7, 8 (cf. Figure 2). That is, for any $x \in \mathrm{Sing}(Y_1 \times_{\mathbf{P}^1} Y_2)$, there is an isomorphism

$$\hat{\mathcal{O}}_{Y_1 \times_{\mathbf{P}^1} Y_2, x} \cong k[[x, y, z, w]]/(f),$$

where f is one of the equations in Propositions 7, 8, and the configurations are as indicated in Figure 2. It can be checked that crepant resolutions can be obtained by successive blow-ups $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$ along reduced nonsingular curves in the singular loci.

Consider, for example, the case (dd). We know that the set of singular points $\mathrm{Sing}(Y_1 \times_{\mathbf{P}^1} Y_2)$ consists of five $\mathbf{P}^1$'s, that is, Γ which is the fiber product of the moving cusps, and $C_1, \ldots, C_4$ which correspond to the components of multiplicity two in fibers of type I_0^*. Then expressing the condition by the linear fractional functions, we see that the singularity along Γ is given by the equation $y^2 + x^3 + w^2 + z^3 = 0$ outside the four points where Γ intersects $C_1, \ldots, C_4$. This singularity is isomorphic to the singularity of type (1) in Proposition 7. At these four points, the singularities are given by $y^2 + x^3 + z^2(1 + w^3) = 0$, which is the singularity of type ① in Proposition 8. Outside these four points the singularity along C_i $(i = 1, \ldots, 4)$ is given either by $x^3 + y^2 + z^2 w = 0$ or $x^3 + y^2 + z^2(1 + w) = 0$. Both are isomorphic to the singularity of type (2) in Proposition 7.

We blow up $C_1, \ldots, C_4$, then blow up Γ. There still remains the singularity of this threefold, whose locus is a nonsingular curve of genus 2 and maps to Γ with its mapping degree three. By blowing up this reduced curve, we get a projective resolution of singularities $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$. □

5. Rationality of the singularities

We call a singular point x on a normal variety W a *rational singularity* if there exists a resolution of singularities $\pi : X \to W$ such that $(R^i \pi_* \mathcal{O}_X)_x = (R^i \pi_* \omega_X)_x = 0$ for all $i > 0$. In this section, we show that i) our examples of threefolds are indeed Calabi-Yau, ii) all the singularities treated in the previous section are rational singularities.

Remark 4. The definition of rational singularities is not well established in positive characteristic. Since we do not know whether the Grauert-Riemenschneider vanishing theorem holds or not, it may be natural to add an extra condition on $R^i \pi_* \omega_X$ to the definition of rational singularities in characteristic 0 (cf. [11, 5.9]).

Proposition 9 *For our examples of threefolds obtained in Proposition 3, we have* $H^1(\mathcal{O}_X) = 0$.

Proof. It is observed that X and $Y_1 \times_{\mathbf{P}^1} Y_2$ are anti-canonical members of nonsingular rational fourfolds, from which the vanishing follows (cf. [6]). □

Proposition 10 *The sheaf $R^i\pi_*\mathcal{O}_X$ with $i > 0$ is zero for the crepant resolution $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$ in Proposition 3. All the singularities treated in Propositions 7 and 8 are rational singularities.*

Lemma 11 *Let W be one of the singularities given in Proposition 7 or the singularity of type (7) in Remark 3. Let $C \subset W$ be the one-dimensional singular locus of W with the reduced scheme structure. Then there exists a flat morphism $W \to \operatorname{Spec} k[[t]]$ whose fiber has a rational double point along C and C is a section of this morphism.*

Proof. The assertion is obvious for the singularity of type (1) in Proposition 7. For singularities of other types, observation on local equations gives that a general hyperplane section has a rational double point of type D_4^0, D_4^0, E_8^0, D_8^0, D_4^0 and D_6^1 for type (2), (3), (4), (5), (6) and (7) respectively. The desired morphism can be constructed concretely as

$$W := \operatorname{Spec} k[[x, y, z, t]]/(f) \to \operatorname{Spec} k[[t]],$$

with

(2) $f = x^3 + y^2 + z^2(t + z)$,
(3) $f = x^2 + xy^2 + z^2 + z^2(t + z)$,
(4) $f = x^3 + y^2 + z^4(t + z)$,
(5) $f = x^2 + xy^2 + z^2 + z^4(t + z)$,
(6) $f = x^2 + xy^2 + z^2 + z^3 + z^3t$,
(7) $f = x^3z + y^2 + z^2(t + y)$. □

Proof of Proposition 10. Recall that our resolution of singularities $\pi : X \to Y_1 \times_{\mathbf{P}^1} Y_2$ is a succession of blow-ups along reduced nonsingular curves. Then by the previous lemma, we know that, for each blow-up $\pi' : W' \to W$, there exists a flat family $W' \to W \to \operatorname{Spec} k[[t]]$ such that the fiber $W'_t \to W_t$ is a partial resolution of a rational double point for any $t \in \operatorname{Spec} k[[t]]$. It follows that $H^1(\mathcal{O}_{W'_t}) = 0$. Then inductive arguments based on the Leray spectral sequence gives $R^1\pi_*\mathcal{O}_X = 0$ along the singularities treated in Proposition 7 (cf. [8]). So this sheaf is possibly supported on finite points. On the other hand, we know $H^0(R^1\pi_*\mathcal{O}_X) = 0$ from $H^1(\mathcal{O}_X) = 0$. So it follows that $R^1\pi_*\mathcal{O}_X = 0$. Since the inverse image of each point of $Y_1 \times_{\mathbf{P}^1} Y_2$ by π is of dimension smaller than two, we have $R^i\pi_*\mathcal{O}_X = 0$ for $i > 0$. The desired result follows from $\omega_X \cong \mathcal{O}_X$. □

6. Unirationality and topological invariants

In this section we calculate topological invariants of our examples of Calabi-Yau threefolds.

Proposition 12 *The Calabi-Yau threefolds obtained in the previous sections are purely inseparably unirational.*

Proof. A general property of quasi-elliptic surfaces says that the base change of $\varphi : Y \to \mathbf{P}^1$ by the Frobenius morphism $\mathbf{P}^1 \to \mathbf{P}^1$ is a non-normal rational surface. X admits a fibration $f : X \to \mathbf{P}^1$ induced from the quasi-elliptic fibrations. Then the base change $X \times_{\mathbf{P}^1} \mathbf{P}^1$ by the Frobenius morphism $\mathbf{P}^1 \to \mathbf{P}^1$ is a rational threefold. □

For the algebraic fundamental group, the following is known (cf. [12]):

Proposition 13 *If a nonsingular projective threefold X is purely inseparably unirational, then its algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ is trivial.*

We calculate the topological Euler-Poincaré characteristic as well as the Betti numbers of our Calabi-Yau's. First we recall the following theorem.

Theorem 14 (Nygaard [12]) *If a nonsingular projective threefold X is unirational, then the Picard number and the second Betti number of X coincide, i.e. $\rho(X) = b_2(X)$.*

Proposition 15 *The Calabi-Yau threefolds obtained in the previous sections have the following invariants.*

	(bb)	(bc-1)	(bd)	(be-1)	(dd)	(de-1)
$e(X)$	96	128	72	104	48	72
$\rho(X)$	47	63	36	52	25	36

In case (bb) and (bc-1), it follows that the third Betti number $b_3(X) = 0$, hence X does not admit any projective lifting to characteristic 0.

Proof. Our crepant resolution is a sequence of blow-ups along nonsingular curves as we saw in Section 4. For a blow-up $\pi : X \to Y$ along a nonsingular curve $C \subset Y$, we have a formula

$$e(X) - e(Y) = e(E) - e(C),$$

where $E := \pi^{-1}(C)$ is the exceptional set (cf. [8, Lemma 6.5]). Now we calculate the Euler-Poincaré characteristic. We have the following table using the same method as in characteristic 3 case [8, Proposition 6.4]:

	(bb)	(bc-1)	(bd)	(be-1)	(dd)	(de-1)
$e(Y_1 \times_{\mathbf{P}^1} Y_2)$	40	48	40	48	40	48

In the case (bd), we have a total of 17 irreducible divisors as exceptional sets, whose normalizations are $\mathbf{P}^1$-bundles over $\mathbf{P}^1$, except the last one which is over an elliptic curve (cf. Figure 3). Thus $e(X) = 40 + (4-2) \times 16 + (4-4) \times 1 = 72$.

In the case (bc-1), we have a total of 36 irreducible divisors as exceptional sets. Among them, there are two divisors with Euler-Poincaré characteristic 6, and four divisors with Euler-Poincaré characteristic 5. All the remaining ones have Euler-Poincaré characteristic 4. Thus $e(X) = 48 + (6-2) \times 2 + (5-2) \times 4 + (4-2) \times 30 = 128$.

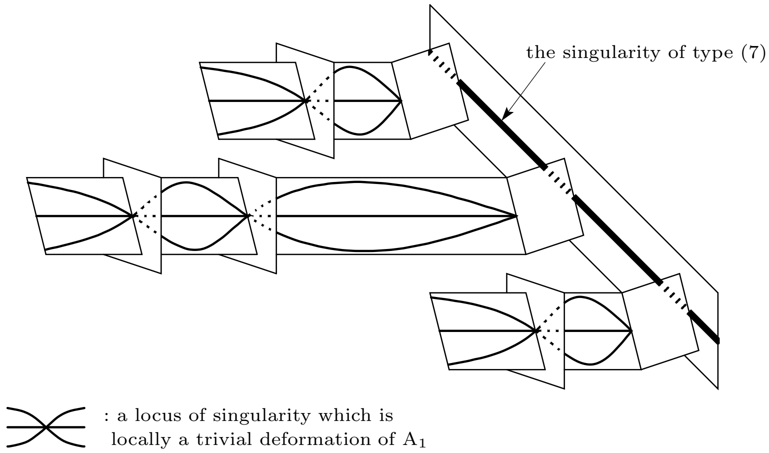

Fig. 3.

We can calculate similarly for the other cases analyzing the processes of the blow-ups carefully.

For the Picard number, we use the following formula for the fibration $f := (\varphi_1 \times_{\mathbf{P}^1} \varphi_2) \circ \pi : X \to \mathbf{P}^1$ as in [15, (3.2), p. 182]

$$\rho(X) = 3 + \sum_{t \in \mathbf{P}^1} (\#\{\text{irreducible components of } f^{-1}(t)\} - 1) + \#\{\text{irreducible exceptional divisors with respect to } \pi \text{ which are horizontal to } f\}.$$

Now the Betti numbers are calculated from Theorem 14 and the Poincaré duality theorem, thus we get the desired result. □

7. Fibrational structures

We call a morphism $f : X \to C$ from a Calabi-Yau threefold X to a curve C a fibration when the natural map $\mathcal{O}_C \to f_*\mathcal{O}_X$ is an isomorphism. Then it follows that $C \cong \mathbf{P}^1$ from the Leray spectral sequence and $H^1(\mathcal{O}_X) = 0$. By the adjunction formula, we see that a fiber has a trivial dualizing sheaf.

Quasi-elliptic fibrations on $K3$ surfaces in $p = 2$ played a central role in the proof of the equivalence between being unirational and being supersingular, i.e. $\rho = b_2$ for $K3$ surfaces [14]. On the other hand, it is observed that some Calabi-Yau threefolds admit fibrations which are not generically smooth.

Proposition 16 *The Calabi-Yau threefold X obtained as (bb), (bd), (dd), (bc-1), (be-1), (de-1) in the previous sections admits at least two types of fibrations $g : X \to \mathbf{P}^1$ whose general fibers are i) a non-normal rational* *surface, ii) a supersingular $K3$ surface. In (bc-1), (be-1), (de-1), there exists one more type of fibrations whose general fiber is a normal surface with three rational double points of type* A_1 *whose minimal resolution is a supersingular $K3$ surface.*

Proof. We denote by g the composition of $X \xrightarrow{\pi} Y_1 \times_{\mathbf{P}^1} Y_2 \xrightarrow{\mathrm{proj}_1} Y_1$ and a $\mathbf{P}^1$-fibration $\tau : Y_1 \to \mathbf{P}^1$. This g is indeed a fibration. Then a fiber $g^{-1}(t)$ with $t \in \mathbf{P}^1$ is factored by the following Cartesian products

$$\begin{array}{ccc}
\operatorname{Spec} k \times_{\mathbf{P}^1} X & \to & X \\
\downarrow & & \downarrow \pi \\
\operatorname{Spec} k \times_{\mathbf{P}^1} Y_1 \times_{\mathbf{P}^1} Y_2 & \to & Y_1 \times_{\mathbf{P}^1} Y_2 \\
\downarrow & & \downarrow \mathrm{proj}_1 \\
\operatorname{Spec} k \times_{\mathbf{P}^1} Y_1 & \to & Y_1 \\
\downarrow & & \downarrow \tau \\
\operatorname{Spec} k & \xrightarrow{[t]} & \mathbf{P}^1
\end{array}$$

$\operatorname{Spec} k \times_{\mathbf{P}^1} Y_1 \times_{\mathbf{P}^1} Y_2$ is the base change of $\varphi_2 : Y_2 \to \mathbf{P}^1$ by the double cover $\varphi_{1|\tau^{-1}(t)} : \operatorname{Spec} k \times_{\mathbf{P}^1} Y_1 \to \mathbf{P}^1$. To study a general fiber of g, we need to know the ramification points of this double cover $\mathbf{P}^1 \to \mathbf{P}^1$ for a general $t \in \mathbf{P}^1$, and the effects of blow-ups in π.

For a $\mathbf{P}^1$-fibration τ on Y_1, consider the morphism $(\varphi_1, \tau) : Y_1 \to \mathbf{P}^1 \times_k \mathbf{P}^1$ of mapping degree two and its Stein factorization $Y_1 \to S \to \mathbf{P}^1 \times_k \mathbf{P}^1$. The case should be devided into three.

i) The Picard number $\rho(S) = 2$,
ii-1) $\rho(S) \neq 2$ and $\tau = \Phi_{|\Sigma_1|}$,
ii-2) $\rho(S) \neq 2$ and $\tau \neq \Phi_{|\Sigma_1|}$,

where $\Phi_{|\Sigma_1|}$ is the morphism on Y_1 induced from the complete linear system $|\Sigma_1|$ associated to the moving cusp Σ_1 on Y_1. In case i), the double cover $\varphi_{1|\tau^{-1}(t)} : \tau^{-1}(t) \to \mathbf{P}^1$ for a general $t \in \mathbf{P}^1$ is inseparable by Proposition 17 below. A general fiber of g comes from the base change of $Y_2 \to \mathbf{P}^1$ by the Frobenius morphism $\mathbf{P}^1 \to \mathbf{P}^1$ and is a non-normal rational surface.
In case ii-1), Proposition 17 tells that there exists a point on the base $s_0 \in \mathbf{P}^1$ such that a general double cover $\varphi_{1|\tau^{-1}(t)} : \tau^{-1}(t) \to \mathbf{P}^1$ is ramified over this $s_0 \in \mathbf{P}^1$. Recall that a special fiber of type III^* (resp. of type I_2^*) lies over $s_0 \in \mathbf{P}^1$ when Y_1 is of type (c) (resp. of type (e)) (cf. Proposition 3).
In case ii-2), a general fiber of g comes from the base change of $Y_2 \to \mathbf{P}^1$ by the double cover $\varphi|_{\tau^{-1}(t)} : \tau^{-1}(t) \to \mathbf{P}^1$ which is ramified at a general point of the base $\mathbf{P}^1$.

By local calculation we see that, in ii-1) and ii-2), the base change of $\varphi_2 : Y_2 \to \mathbf{P}^1$ by the double cover $\varphi_{1|\tau^{-1}(t)} : \tau^{-1}(t) \to \mathbf{P}^1$ with a general $t \in \mathbf{P}^1$ is a normal surface with one rational double point of type D_4^0 whose minimal resolution is a $K3$ surface with a quasi-elliptic fibration. So the next question is which component of $\operatorname{Sing}(Y_1 \times_{\mathbf{P}^1} Y_2)$ penetrates this rational double point (cf. Figure 2). In (bb), (bd), (dd), the component Γ goes through this rational double point of type D_4^0. The singularity of $Y_1 \times_{\mathbf{P}^1} Y_2$ along this component is of type (1) in Proposition 7. Since it is a trivial deformation of a rational double point of type D_4^0, the rational double point of type D_4^0 on a general fiber is resolved after successive blow-ups in π. In (bc-1), (be-1), (de-1), when $\tau = \Phi_{|\Sigma|}$, the singularity of $Y_1 \times_{\mathbf{P}^1} Y_2$ along the component which penetrates the D_4^0 is again of type (1), which is resolved after four blow-ups in π. In (bc-1), (be-1), (de-1), when $\tau \neq \Phi_{|\Sigma|}$, Γ penetrates this D_4^0-singularity. Since the singularities along Γ is of type (2) in Proposition 7, there will be a single blow-up along this component in π and there remain three rational double points of type A_1 on a general fiber. □

Remark 5. For a not generically smooth fibration in positive characteristic, some conditions on singularities which appear in general fiber are observed in [7] and [17]. In particular, the possible types of rational double points are determined.

For such fibrations of Calabi-Yau threefolds, the following examples are known.
i) $f : X \to \mathbf{P}^1$, whose general fiber is not normal.

1. A general fiber is a fiber product of a rational curve with an ordinary cusp and an elliptic curve in $p = 2, 3$ [6].
2. A general fiber is a non-normal rational surface in $p = 2$ in our examples in Proposition 3 and in $p = 3$ [8].

ii) $f : X \to \mathbf{P}^1$, whose general fiber is normal.

1. A general fiber is a normal surface with twelve rational double points of type A_1 (resp. a normal surface with one rational double point of type D_4^0) in $p = 2$; a general fiber is a normal surface with two rational double points of type A_2 in $p = 3$ in [6].
2. A general fiber is a normal surface with three rational double points of type A_1 in $p = 2$ in Proposition 16; a general fiber is a normal surface with one rational double point of type A_2 in $p = 3$ [8]. All these normal surfaces are birationally equivalent to supersingular $K3$ surfaces.

8. Geometry on rational surfaces

Let $\varphi : Y \to \mathbf{P}^1$ be a relatively minimal rational quasi-elliptic surface with section. Choose a $\mathbf{P}^1$-fibration $\tau : Y \to \mathbf{P}^1$ and take the Stein factorization of (φ, τ) as $Y \to S \to \mathbf{P}^1 \times_k \mathbf{P}^1$. The morphism $S \to \mathbf{P}^1 \times_k \mathbf{P}^1$ is a double cover and we are interested in its ramification locus. We put $\rho(S) := \dim_{\mathbf{Q}} \mathrm{NS}(S) \otimes \mathbf{Q}$. If $\rho(S) \neq \rho(\mathbf{P}^1 \times_k \mathbf{P}^1)$, it is easy to see that the above double cover is separable.

Proposition 17 *Let $\varphi : Y \to \mathbf{P}^1$ be a relatively minimal rational quasi-elliptic surface either of type (b), (c), (d), (e) in Theorem 2. Under the notation as above, we have the following:*

1. *The complete linear system $|\Sigma|$ associated with the moving cusp Σ gives a $\mathbf{P}^1$-fibration $\Phi_{|\Sigma|} : Y \to \mathbf{P}^1$. If we put $\tau = \Phi_{|\Sigma|}$, the Picard number $\rho(S)$ is two in (b) and (d), three in (c) and four in (e). In (c) (resp. (e)), the ramification divisor of the double cover corresponds to the irreducible component in the fiber of type* III^* *(resp.* I_2^**) which intersects the moving cusp.*
2. *There exist $\mathbf{P}^1$-fibrations τ which attain the following $\rho(S)$ under $\tau \neq \Phi_{|\Sigma|}$, that is, $\rho(S) = 2, 3$ in (b); $\rho(S) = 2$ in (c); $\rho(S) = 2, 4$ in (d); $\rho(S) = 2, 4$ in (e).*
3. *When $\rho(S) = 2$, the morphism $(\varphi, \tau) : Y \to \mathbf{P}^1 \times_k \mathbf{P}^1$ is purely inseparable.*
4. *When $\rho(S) > 2$ and $\tau \neq \Phi_{|\Sigma|}$, the moving cusp of φ corresponds to the ramification divisor of the double cover $S \to \mathbf{P}^1 \times_k \mathbf{P}^1$.*

Proof. 1) Recall that on a relatively minimal rational quasi-elliptic surface Y, the self-intersection number of a $\mathbf{P}^1$ is always greater than -3, and any $\mathbf{P}^1$ with self-intersection number -2 (resp. -1) is in a fiber (resp. a section) of the quasi-elliptic fibration. Since $(\Sigma^2) = 0$, it follows from the Riemann-Roch theorem that the complete linear system $|\Sigma|$ has a positive dimension and base point free. So it induces a $\mathbf{P}^1$-fibration. The Picard number $\rho(S)$ can be calculated by counting (-2)-curves on Y contracted by (φ, τ). For the second assertion for (c) (resp. (e)), consider the double cover $\Phi^{-1}_{|\Sigma|}(t) \to \mathbf{P}^1$ for a general $t \in \mathbf{P}^1$. Then it can be checked by arguments on intersection numbers that this cover is one-to-one at the unique intersection point of $\Phi^{-1}_{|\Sigma|}(t)$ and the component in question.

2) It can be seen that the dual graph of a singular fiber of a $\mathbf{P}^1$-fibration on a relatively minimal rational quasi-elliptic surface Y is one of the graphs in Figure 4. Finding such subgraphs in the dual graph consisting of all the (-1)-curves and (-2)-curves on each Y (cf. Figure 5), we can determine all the $\mathbf{P}^1$-fibrational structures on Y.

3) We shall show that $\mathrm{rank}(\Omega^1_{Y/\mathbf{P}^1 \times_k \mathbf{P}^1}) = 1$ in the following exact sequence under $\rho(S) = 2$.

$$(\varphi, \tau)^* \Omega^1_{\mathbf{P}^1 \times_k \mathbf{P}^1} \to \Omega^1_Y \to \Omega^1_{Y/\mathbf{P}^1 \times_k \mathbf{P}^1} \to 0.$$

Consider the exact sequences of torsion free sheaves

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \tau^*\Omega^1_{\mathbf{P}^1}(B) & & & & \\ & & & & \downarrow & \searrow^{\alpha} & & & \\ 0 & \longrightarrow & \varphi^*\Omega^1_{\mathbf{P}^1}(2\Sigma + A) & \longrightarrow & \Omega^1_Y & \longrightarrow & \Omega^1_{Y/\mathbf{P}^1}/\mathrm{tor} & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \Omega^1_{Y/\mathbf{P}^1}/\mathrm{torsion} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0, & & & & \end{array}$$

where A (resp. B) is an effective divisor whose support coincides with that of irreducible components of reducible fibers of φ (resp. of τ) with multiplicity ≥ 2 (cf. [3]). Suppose that α is a non-zero map. Then it follows that the linear system $|\varphi^{-1}(s) + 2\tau^{-1}(t) - 2\Sigma - A - B|$ is non-empty. Take the intersection number with $\varphi^{-1}(s)$, then we have

$$(\varphi^{-1}(s), \varphi^{-1}(s) + 2\tau^{-1}(t) - 2\Sigma - A - B) = -(\varphi^{-1}(s), B).$$

By using the classification of the dual graphs given above, we see that B contains at least one (-1)-curve and the above intersection number is negative. But this contradicts the fact that $\varphi^{-1}(s)$ is a nef divisor.

4) We have $(\varphi^{-1}(s), \tau^{-1}(t)) = 2$ and $(\Sigma, \tau^{-1}(t)) \geq 1$. We claim that $\Sigma \cap \tau^{-1}(t)$ consists of a single point for a general $t \in \mathbf{P}^1$ and it is the unique ramification point of the double cover $\varphi|_{\tau^{-1}(t)} : \tau^{-1}(t) \to \mathbf{P}^1$. Indeed, since $p = 2$, the double cover $\varphi|_{\tau^{-1}(t)} : \tau^{-1}(t) \cong \mathbf{P}^1 \to \mathbf{P}^1$ has a single ramification point. Pick up any point Q in $\Sigma \cap \tau^{-1}(t)$, then the fiber $\varphi^{-1}(\varphi(Q))$ has an ordinary cusp at Q. The local intersection number satisfies $(\varphi^{-1}(\varphi(Q)), \tau^{-1}(t))_Q \geq 2$. Since we have $(\varphi^{-1}(\varphi(Q)), \tau^{-1}(t)) \geq (\varphi^{-1}(\varphi(Q)), \tau^{-1}(t))_Q$, it follows that the above intersection numbers are two and Q is the unique ramification point. □

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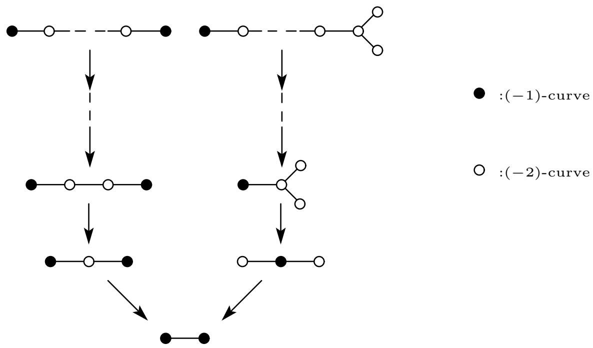

Fig. 4.

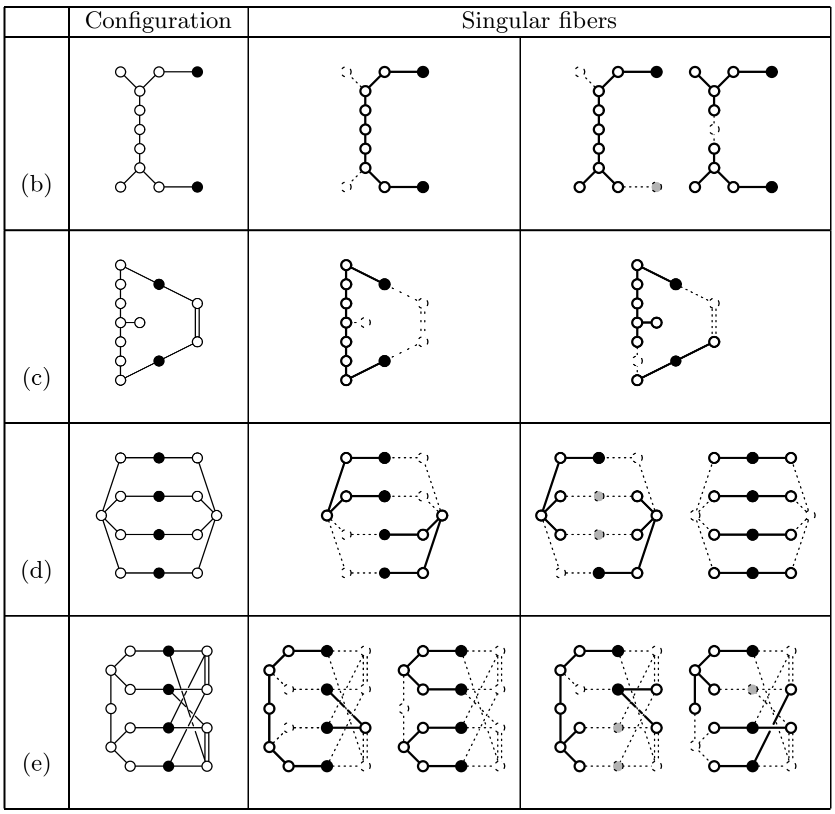

Fig. 5.

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