Transport Coefficients of a Gluon Plasma

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Transport coefficients of gluon plasma are calculated for an SU(3) pure gauge model by lattice QCD simulations on $16^3 \times 8$ and $24^3 \times 8$ lattices. Simulations are carried out at slightly above the deconfinement transition temperature T_c, where a new state of matter is currently being pursued in BNL RHIC experiments. Our results show that the ratio of the shear viscosity to the entropy is less than one and the bulk viscosity is consistent with zero in the region $1.4 \leq T/T_c \leq 1.8$.

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Introduction.—BNL Relativistic Heavy Ion Collider (RHIC) experiments have been providing us with many surprises. One of them is that the data are unexpectedly well described by the hydrodynamical model [1]. Experimental data and phenomenological analyses suggest that the quark-gluon plasma (QGP) or a new state of matter may be produced. See Ref. [2] for a review of RHIC experiments. Molnar and Gyulassy investigated the elliptic flow data using a Boltzmann-type equation for gluon scattering, and found that they needed a cross section about 50 times larger than expected in perturbative QCD [3]. This indicates that the QGP state above the phase transition temperature, T_c, is not a free gas of perturbed gluons. The QCD-TARO Collaboration measured the temporal meson propagators and found that their wave functions do not behave as free particles even at $T \sim 1.5T_c$; it was conjectured that the strong interactions between the thermal gluons and quarks may provide binding forces. Recently, more extended analyses of the temporal propagators were reported by three groups [4–6], and it was suggested that the charmonium state survives until around $2T_c$.

The new state of matter produced at high temperatures in RHIC experiments is most likely not a weakly interacting plasma, but a strongly interacting quark-gluon system. Investigating the results in Ref. [3], Teaney found that $\eta/s \sim 0.04$, where η and s are the shear viscosity and the entropy, respectively [7]. Shuryak and Zahed have proposed a "strongly coupled QGP" model for the new state of matter above T_c [8], and argued that the QGP studied in RHIC is the most perfect fluid ever measured. Policastro *et al.* calculated η for the finite-temperature $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the large N, strong-coupling regime, and obtained $\eta/s = 1/4\pi$ [9]. This value was found to be universal for theories with gravity duals, and it is conjectured that $\eta/s = 1/4\pi$ is a lower limit for all systems in nature [10].

It has now become highly desirable to study the nature of the quark-gluon system, particularly its hydrodynamical parameters such as the transport coefficients above T_c based on QCD in a nonperturbative manner. In this Letter, we calculate the transport coefficients of QGP at slightly above T_c by the lattice simulations. Simulations are carried out in the quenched approximation. For the calculation of the transport coefficients on a lattice, we apply the formulation based on the linear response theory [11–13], where the transport coefficients are calculated from Matsubara-Green's function of energy momentum tensors. Numerical simulations of transport coefficients with this formulation were first carried out by Karsch and Wyld [14]. In their pioneering work, they performed the simulation on an $8^3 \times 4$ lattice, but unfortunately, the size in the imaginary time direction was too small for the determination of the transport coefficients.

We report here our simulation on an $N_T = 8$ lattice with renormalization group (RG) improved action by Iwasaki. Our results are summarized as follows. (1) The ratio of the shear viscosity to the entropy η/s is small, i.e., less than one, but it is most probably larger than $1/4\pi$. (2) The bulk viscosity is less than the shear viscosity and is consistent with zero within the present statistics. For heat conductivity, we could obtain no meaningful result. This is because, in pure gauge theory, there is no conserved current which transports heat [15]. Preliminary results based on $16^3 \times 8$ and smaller lattices have been reported at lattice and quark matter conferences [16,17].

Transport coefficients in linear response theory.—The formulation for the transport coefficients of QGP in the framework of the linear response theory has been given in Refs. [11–13]. For the sake of consistency, we shall summarize the formulas which will be used in the following calculations.

Transport coefficients are calculated using the space-time integral of a retarded Green's function of energy momentum tensors,

$$\eta = -\int \left\langle T_{12}(\vec{x}, t) T_{12}(\vec{x}', t') \right\rangle_{\text{ret}}, \tag{1}$$

$$\frac{4}{3}\eta + \zeta = -\int \Big\langle T_{11}(\vec{x}, t) T_{11}(\vec{x}', t') \Big\rangle_{\rm ret}, \tag{2}$$

where $\int \equiv \int d^3x' \int_{-\infty}^{t} dt_1 e^{\epsilon(t_1 - t)} \int_{-\infty}^{t_1} dt'$ and η and ζ represent shear and bulk viscosities, respectively. $\Big\langle T_{\mu\nu} T_{\rho\sigma} \Big\rangle_{\rm ret}$ is the retarded Green's function of the energy momentum tensors at finite-temperature. For the pure gauge theory, $T_{\mu\nu}$'s are written using field strength tensors $F_{\mu\nu}$: $T_{\mu\nu} = 2{\rm Tr}[F_{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma}]$. $F_{\mu\nu}$'s are defined by plaquette variables on the lattice as $U_{\mu\nu}(x) = \exp[ia^2 g F_{\mu\nu}(x)]$, and are obtained either by taking the log of $U_{\mu\nu}$ directly, or by expanding $U_{\mu\nu}$ with respect to $a^2 g$. In the following, we use the latter method to calculate $F_{\mu\nu}$ [14,18].

It is difficult to calculate the retarded Green's functions in the lattice QCD in which Matsubara-Green's functions are measured. The retarded Green's functions are obtained from the analytic continuation. We obtain the numerical values of Matsubara-Green's functions at discrete variables $\omega_n = 2\pi n T$ in the momentum space, while the retarded Green's functions are functions of the continuous variable p_0. Therefore, we require a bridge for analytic continuation.

Matsubara-Green's functions G_β are expressed in a Fourier-transformed form with the spectral function ρ:

$$G_\beta(\vec{p}, t) = \sum_n e^{i\omega_n t} \int d\omega \frac{\rho(\vec{p}, \omega)}{i\omega_n - \omega}. \tag{3}$$

It is well known that the spectral function is common to both the retarded and Matsubara-Green's functions [19]. The expression for retarded Green's functions is obtained by setting $i\omega_n \to p_0 + i\epsilon$.

The determination of $\rho(\vec{p}, \omega)$ is not straightforward, because in a numerical simulation, Matsubara-Green's function has a finite number of points in the temperature direction, $N_T/2$. We must employ an ansatz for the spectral function with parameters, which are determined by fitting Matsubara-Green's function. The simplest nontrivial ansatz for the spectral function was proposed by Karsch and Wyld [14] as

$$\rho(\vec{p} = 0, \omega) = \frac{A}{\pi}\left(\frac{\gamma}{(m - \omega)^2 + \gamma^2} - \frac{\gamma}{(m + \omega)^2 + \gamma^2}\right), \tag{4}$$

where γ represents the effects of interactions and is related to the imaginary part of the self-energy. This ansatz is supported by perturbative calculations [11,13].

Once we use this ansatz for the spectral function, the space-time integral of the retarded Green's function can be calculated analytically. The result is

$$\alpha = 2A \frac{2\gamma m}{(\gamma^2 + m^2)^2}, \tag{5}$$

where α represents the shear viscosity η, bulk viscosity ζ, or heat conductivity $\chi \times T$. At least three independent data points for Matsubara-Green's functions are necessary to determine these parameters.

In Ref. [14], a simulation on a $8^3 \times 4$ lattice, where two independent data points in the temperature direction are available, was carried out. In this simulation, three parameters in the spectral function could not be determined. In order to determine A, γ, and m, we adopt $N_T = 8$.

Numerical simulations.—We calculate the transport coefficients in the SU(3) gauge theory for the regions slightly above the transition temperature, which are covered in RHIC experiments. We adopt Iwasaki's improved gauge action, which is closer to the renormalized trajectory than the plaquette action, and we obtain results close to the continuum limit on relatively coarse lattices [20]. We found that the fluctuation of Matsubara-Green's function is greatly suppressed compared with the standard plaquette action [21].

We should first determine the critical β of Iwasaki's improved action on the $N_T = 8$ lattice. For the $N_T = 4$ and 6 lattices, the critical β values for this action were determined by the Tsukuba group [22]. We have carried out a simulation for β_c on a $16^3 \times 8$ lattice [21]. However, the volume size was small, and we could obtain only a rough estimation of β_c, that is, $2.70 < \beta_c < 2.72$. Using the finite size scaling formula [22], β_c at $N_T = 8$ becomes $2.72 < \beta_c < 2.74$. The values of β_c determined by the simulation for $N_T = 4$, 6, and 8 do not yet satisfy the asymptotic two-loop scaling relation. We take $\beta = 3.05$, 3.2, and 3.3 as our simulation points.

Matsubara-Green's function on $N_T = 8$ lattice.—For Matsubara-Green's functions G_{11} and G_{12}, from which the shear and bulk viscosities are calculated, we can obtain reliable signals from approximately 0.8×10^6 Monte Carlo simulation (MC) data on a $24^3 \times 8$ lattice. As an example, G_{12} is shown in Fig. 1 for $\beta = 3.3$. In the case of the $16^3 \times 8$ lattice, the errors are larger than the signal at

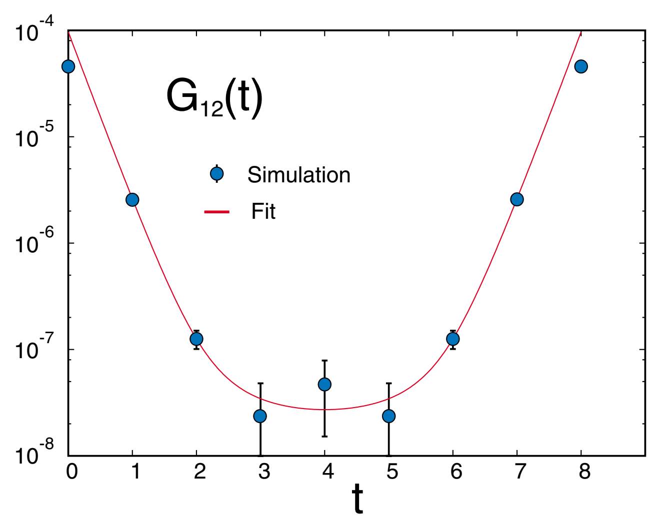

FIG. 1 (color online). Numerical data points and fitting results of Matsubara-Green's function $G_{12}(t)$ on a $24^3 \times 8$ lattice.

$\tau = 4$, even with more than 10^6 MC simulation data. The volume of 16^3 may be too small for $N_T = 8$.

Transport coefficients of gluon plasma.—The fitting of Matsubara-Green's function by Eq. (4) is carried out by applying a nonlinear least-square fitting program, SALS. Then the transport coefficients of the gluon plasma are calculated using Eq. (5). The errors are estimated by the jackknife method. The bin size in the jackknife analysis is changed from 5×10^4 to 1.2×10^5. The results are independent of the bin size. In the following, the bin size is 100 000.

In Table I, we show the fitting results of the spectral function, Eq. (4). The parameters A and m control overall size and slope of dumping rate of Green's function, while γ measures the deviation from simple pole behavior. If any of the three parameters is zero, the transport coefficient vanishes [Eq. (5)]. We see that large statistical errors originate from $A\gamma$.

The results for the shear and bulk viscosities are given in Table II. The bulk viscosity is equal to zero within the range of error bars, while the shear viscosity may be finite. We see little size dependence.

In the lattice calculations, the shear viscosity is calculated in the form $\eta \times a^3$. In order to express it in physical units, we need to know the lattice spacing a at each β value. For the estimation of a, we use the finite-temperature transition point β_c. We take $\beta_c = 2.73$ for $N_T = 8$. The transition temperature is $T_c = 276$ MeV [22], and asymptotic two-loop scaling for the region $\beta > 2.73$ is assumed. The lattice spacing and the shear and bulk viscosities in physical units are also listed in Table II. $\eta^{1/3}$ expressed in physical units is slightly less than the ordinary hadron masses around T_c.

Viscosity-entropy ratio.—As discussed in the introduction, the ratio of the shear viscosity and entropy density is estimated for QGP [7,8] and for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. To get the ratio in lattice QCD, we require lattice entropy data at the same coupling regions and N_T.

In a homogeneous system, the entropy density s can be obtained from the energy density ϵ and the pressure as

$$s = S/V = (\epsilon + p)/T. \qquad (6)$$

Using lattices with $N_T = 8$, CP-PACS collaboration calculated p and ϵ [23]. We reconstruct the results from their numerical raw data and calculate the entropy density in Eq. (6).

Concluding remarks.—In the high temperature limit, the transport coefficients have been calculated analytically by the perturbation method [11,24–28]. They are summarized as follows. (1) The bulk viscosity is smaller than the shear viscosity. This is consistent with our numerical results. (2) The shear viscosity in the next-to-leading log (NLL) is expressed by, $\eta_{\rm NLL} = (T^3/g^4)C_1/\log(\mu^*/m_D)$ [28], where $m_D = \sqrt{1 + N_f/6}gT$, and for the pure gluon system $C_1 = 27.126$ and $\mu^*/T = 2.765$.

There is a slight ambiguity in the relationship between coupling g and the temperature, and we use a simple form, $g^{-2} = 2b_0 \log(4T/\Lambda)$ with $b_0 = 11N_c/48\pi^2$. The scale parameter Λ on the lattice is set to be $\Lambda/T_c \simeq 1.5$. For the entropy density, we use the result by a hard-thermal loop calculation [29]. With these formulae, the perturbative η/s can be compared with the results of numerical calculations. The result is shown in Fig. 2.

In this Letter, we reported the first lattice QCD result of the transport coefficients in the vicinity of the critical temperature. Although it still contains large errors, it may provide useful information for understanding QGP in these temperature regions. Especially we find that $0 \leq \eta/s \leq 1$, i.e., we give the upper bound for the viscosity-entropy ratio of QGP. This is one- or two-orders of magnitude smaller than ordinary liquid such as water and He [10]. The small η/s supports the success of the hydrodynamical description for QGP. Applicability conditions of the hydrodynamical model in the quantum field theory were first considered in Ref. [30]. Together with experimental and phenomenological studies, the field theoretical approach will enrich our understanding of the new state of matter. We have shown here that the lattice QCD numerical simulations can provide useful information.

TABLE I. Parameters of the spectral functions in Matsubara-Green's functions G_{11} and G_{12}.

	β	m	γ	$A\gamma$
G_{11}				
16^3	3.05	2.94(18)	0.0087(127)	0.0061(97)
	3.2	3.35(67)	0.048(61)	0.054(89)
	3.3	2.89(19)	0.014(27)	0.0099(213)
24^3	3.05	3.04(29)	0.036(29)	0.028(30)
	3.3	3.43(40)	0.058(39)	0.069(63)
G_{12}	β	m	γ	$A\gamma$
16^3	3.05	3.24(19)	0.018(28)	0.016(27)
	3.2	3.43(57)	0.058(45)	0.063(75)
	3.3	2.87(24)	0.012(26)	0.0079(188)
24^3	3.05	3.16(30)	0.036(37)	0.029(38)
	3.3	3.69(49)	0.067(24)	0.093(75)

TABLE II. Shear and bulk viscosities nondimensional and in physical units. The lattice scales, $a^{-1} = 3.09$, 3.62, and 4.03 GeV for $\beta = 3.05$, 3.20, and 3.30, respectively.

	β	ηa^3	ζa^3	ηGeV^3	ζGeV^3
	3.05	0.0018(28)	−0.0015(29)	0.054(82)	−0.044(85)
16^3	3.2	0.0059(46)	−0.0025(20)	0.281(223)	−0.122(90)
	3.3	0.0013(27)	−0.0001(42)	0.084(175)	−0.008(276)
24^3	3.05	0.0036(36)	−0.00095(288)	0.106(108)	−0.028(85)
	3.3	0.0072(30)	−0.0031(26)	0.471(194)	−0.201(167)

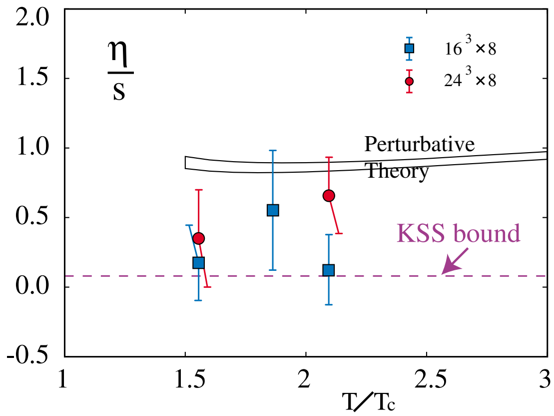

FIG. 2 (color online). The ratio of the shear viscosity to the entropy as a function of T/T_c Kovtun-Son-Starinets bound is $1/4\pi$ [10]. "Perturbative theory" is constructed from η in Ref. [28] and s in Ref. [29].

The next step is to obtain data with smaller systematic and statistical errors. If we can reduce the error bars in Fig. 2 by a factor of 2 or 3, we may realistically compare the data with the conjecture in Ref. [10]. We observed that Matsubara-Green's function suffers from large fluctuations, but by using the improved action, the fluctuations are significantly reduced. Another possibility for reducing the fluctuations may be to employ improved operators for $T_{\mu\nu}$ [31].

The results here depend on the ansatz of the spectral function of the Fourier transform of Matsubara-Green's function. In order to test the functional form of the spectral function, we need more data points for Matsubara-Green's function in the temperature direction. To this end, the most effective approach will be to apply an anisotropic lattice. If we can obtain a sufficient number of data points, the maximum entropy method is a promising way of determining a spectral function [32] that is free from the ansatz. Aarts and Martinez-Resco pointed out, however, that it is difficult to extract transport coefficients in weakly coupled theories from the Euclidean lattice, since Green's function is insensitive to details of the spectral function $\rho(\omega)$ at small ω [33]. New concepts will be necessary to overcome this difficulty.

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