LETTER

Unified Approach to Image Distortion: D-U and U-D Models

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SUMMARY We propose a unified view to deal with two formulations of image distortion and a method for estimating the distortion parameters for both of the formulations; So far the formulations have been developed separately. The proposed method is based on image registration and consists of nonlinear optimization to estimate parameters including view change and radial distortion. Experimental results demonstrate that our approach can deal with the two formulations simultaneously.

key words: *radial distortion, distortion model, undistortion, image registration*

1. Introduction

Calibrating the intrinsic camera parameters and correcting image distortion are important processes for computer vision. Much research on computer vision formulate the problems without considering distortion because of simplicity, but distortion is inevitable when we use an ordinary lens installed on an inexpensive camera; sometimes a point may be displaced more than ten pixels around the corner of the image. Pre-calibration of the intrinsic camera parameters and correction of distorted image are thus required for such research to produce better images.

For considering distortion, there is a practical problem. Based on a single distortion model proposed in an early study in photogrammetry [9], many studies on correcting distortion have been done by image registration [3], [4] or correspondences between corners [5], circles [6], curves [7] or feature points [8]. In such studies, two different formulations of the distortion model have been used by different papers. Computational cost depends if a model of distortion includes higher order terms, but the cost differs for different applications even when the order of a model of distortion is fixed.

In this paper, we show the relation between the two formulations, and propose a method for estimating the distortion parameters for both the formulations. Those formulations have been reported [6], [12], [17], but the parameters for them have never estimated at the same time. The proposed method is based on image registration, and estimates the parameters of the transformations of view change and radial distortion by a nonlinear optimization that minimizes residuals between the distorted image and a calibration pattern.

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In Sect. 2, we explain the distortion model and its formulations. Then how to estimate the distortion parameters and correct an image with both of the formulations are discussed in Sect. 3. Finally, we present experimental results in Sect. 4.

2. One Distortion Model and Two Formulations

Usually the distortion of image is observed by the following two steps. At first, a point in a three-dimensional space is projected onto the image plane through a camera lens (see Fig. 1). Let $\boldsymbol{p}^u = (x^u, y^u)^T$ be the projected, undistorted coordinates on an image. Then $\boldsymbol{p}^u$ is moved by the distortion to the distorted point $\boldsymbol{p}^d = (x^d, y^d)^T$ (see Fig. 2).

The relationship between $\boldsymbol{p}^u$ and $\boldsymbol{p}^d$ in an image is often modeled by five intrinsic camera parameters [10], [11] $\boldsymbol{\theta}^d = (\kappa_1, \kappa_2, c_x, c_y, s_x)^T$: the radial distortion parameters κ_1 and κ_2, the image center $(c_x, c_y)^T$, and the horizontal scale factor s_x. Although we consider only the radial distortion, the following discussion can be applied to other models involving higher-order term or decentering distortion [9], [12].

The radial distortion at a point $\boldsymbol{p} = (x, y)$ is represented by the following function with respect to the image center:

$$\boldsymbol{f}(\boldsymbol{p}, \boldsymbol{\theta}^d) = \begin{pmatrix} \dfrac{x - c_x}{s_x}(1 + \kappa_1 R(\boldsymbol{p})^2 + \kappa_2 R(\boldsymbol{p})^4) + c_x \\ (y - c_y)(1 + \kappa_1 R(\boldsymbol{p})^2 + \kappa_2 R(\boldsymbol{p})^4) + c_y \end{pmatrix}, \quad (1)$$

$$R(\boldsymbol{p}) = \sqrt{\left(\frac{x - c_x}{s_x}\right)^2 + (y - c_y)^2}. \quad (2)$$

Note that the inverse of $\boldsymbol{f}$ is not expressed in a closed-form.

Here, there are two ways to apply the function to image coordinates. One is Distorted-to-Undistorted formulation (**D-U** model) in which the undistorted coordinates is expressed as a function of the distorted coordinates:

$$\boldsymbol{p}^u = \boldsymbol{f}(\boldsymbol{p}^d, \boldsymbol{\theta}^d). \quad (3)$$

The other is Undistorted-to-Distorted formulation (**U-D** model), the distorted coordinates is expressed by the undistorted coordinates:

$$\boldsymbol{p}^d = \boldsymbol{f}(\boldsymbol{p}^u, \boldsymbol{\theta}^d). \quad (4)$$

Historically, the D-U model was proposed to correct plate coordinates of a photographed point on a film [9] and has been used for long time in photogrammetry and computer vision [7], [8], [10], [11], [13]–[15]. Since the inverse

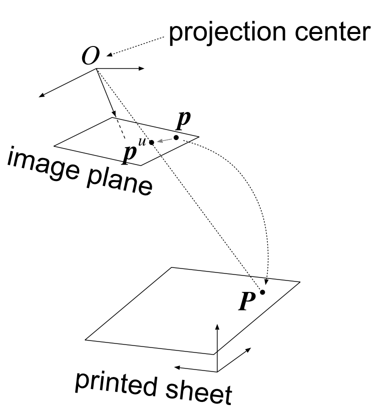


Fig. 1 View change.

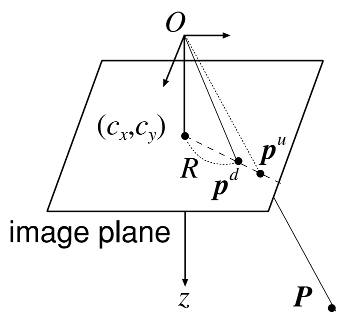


Fig. 2 Distortion.

of $\boldsymbol{f}$ is not an explicit function, it is inconvenient to make a combined transformation with projection and distortion. Therefore, sometimes the U-D model is used as an approximation of the D-U model [6].

However, some studies used the U-D model [4], [5], [16] as the exact formulation, and also used D-U model as an approximation of the U-D model [12], [17].

Usually computer vision applications require not the distortion parameters but just a corrected, distortion-free image. Therefore, any formulations or even the nonparametric approach [18] are employed if they can correct distortion under consideration.

Figure 3 illustrates an example where the barrel distortion is approximated by each of the formulations (here $|\boldsymbol{p}|$ denotes the distance between $\boldsymbol{p}$ and the image center because the distortion is usually represented with respect to the image center). In this case, the two formulations are close to the actual distortion for small $|\boldsymbol{p}^d|$. Since we assume that $|\boldsymbol{p}^d| \leq 300 \sim 400$ for ordinary digital images, the difference between the formulations is not significant and both of them can be used for correction.

Nevertheless, it is important to develop a method for the two formulations and to choose an appropriate one. As we will show in Sect. 3.5, different models are suitable for correcting distortion and distorting images in terms of computational cost, and a preferred model depends on a task. The proposed method described in the next section provides estimations for both the formulations in a single framework,

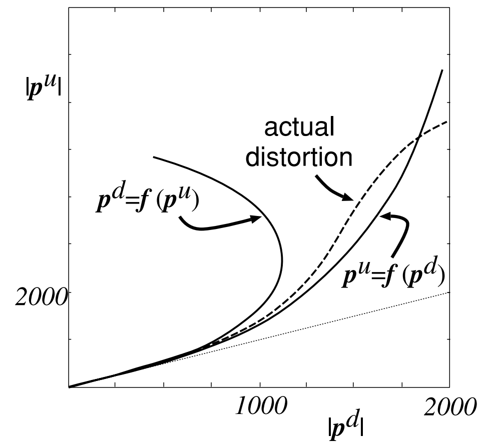


Fig. 3 Example of the distortion approximated by different formulations.

while existing studies [6], [12], [17] estimate parameters for one of the two formulations.

3. Correction Methods

In this section we describes details of the U-D and D-U models, and explain how to estimate distortion parameters and correct distortion for both the formulations.

The proposed method based on image registration establishes a correspondence between an ideal (distortion-free) calibration pattern I_1 and a distorted image I_2 of a printed pattern observed by a camera [1], [2]. The observation is modeled by successive two transformations; view change and distortion. I_2 is regarded as an image generated from I_1 by applying these transformations. The proposed method estimates parameters of the transformations by minimizing the difference between I_1 and I_2, that is, the sum of the squares of intensity residuals of the two images.

3.1 Modeling View Change

Given two images of a planar object from different viewpoints, the relationship between them is described by the planar perspective motion model with eight parameters [19], [20]. As shown in Fig. 1, I_1 and I_2 can be considered as different views of the same plane.

The model warps a point $\boldsymbol{p} = (x, y)^T$ on I_1 to the corresponding point $\boldsymbol{p}^u = (x^u, y^u)^T$ on I_2 by

$$\boldsymbol{p}^u = \boldsymbol{u}(\boldsymbol{p}, \boldsymbol{\theta}^u) = \frac{1}{\theta_1^u x + \theta_2^u y + 1} \begin{pmatrix} \theta_3^u x + \theta_4^u y + \theta_5^u \\ \theta_6^u x + \theta_7^u y + \theta_8^u \end{pmatrix}, \qquad (5)$$

where $\boldsymbol{\theta}^u = (\theta_1^u, \ldots, \theta_8^u)^T$.

3.2 Distortion by U-D Formulation

At first, we consider the U-D formulation; the undistorted point $\boldsymbol{p}^u$ is further moved to $\boldsymbol{p}^d$ by Eq. (4). The Jacobian of $\boldsymbol{p}^d$ is derived straightforward by using the chain rule of vector differentiation [21]:

$$\frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{\theta}^d} & \frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{\theta}^u} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}^d} & \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}^u} \end{pmatrix}. \tag{6}$$

where $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{13})^T = (\boldsymbol{\theta}^u, \boldsymbol{\theta}^d)^T$. In this case, The Jacobian of the combined transformation with $\boldsymbol{u}$ and $\boldsymbol{f}$ is also derived straightforward.

3.3 Distortion by D-U Formulation

Next, we consider the D-U formulation. Eq. (3) is rewritten as follows:

$$\boldsymbol{p}^d = \boldsymbol{f}^{-1}(\boldsymbol{p}^u, \boldsymbol{\theta}^d) \equiv \boldsymbol{d}(\boldsymbol{p}^u, \boldsymbol{\theta}^d). \tag{7}$$

where $\boldsymbol{d}$ is the inverse function of $\boldsymbol{f}$ and is implemented by an iterative procedure [10] because $\boldsymbol{d}$ is not expressed in closed-form. Therefore, the Jacobian of $\boldsymbol{d}$ as well as that of the combined transformation with $\boldsymbol{u}$ and $\boldsymbol{d}$ seems to be difficult to calculate, and most researchers have tried to avoid the difficulty.

Here we introduce the *implicit function theorem* [22] for systems [23]. This theorem can represent the Jacobian of $\boldsymbol{d}$ as an explicit form through $\boldsymbol{f}$. Let $\boldsymbol{F}$ be a function of $\boldsymbol{q} = (\boldsymbol{p}^u, \boldsymbol{\theta}^d)$ and $\boldsymbol{p}^d$ represented by

$$\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{p}^d) = \boldsymbol{p}^u - \boldsymbol{f}(\boldsymbol{p}^d, \boldsymbol{\theta}^d). \tag{8}$$

If $\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{d}(\boldsymbol{q})) = 0$ is satisfied for $\forall \boldsymbol{q}$, $\boldsymbol{p}^d = \boldsymbol{d}(\boldsymbol{q})$ is called an implicit function determined by $\boldsymbol{F}(\boldsymbol{q}, \boldsymbol{p}^d) = 0$. In our case, the condition is theoretically always satisfied because we defined $\boldsymbol{d}$ as the inverse of $\boldsymbol{f}$, and numerically Eq. (8) is almost 0 (it can be less than 10^{-10}).

According to the theorem, the Jacobian is given by the following equations:

$$\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{q}} = -\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}} = -\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^d}^{-1}\begin{pmatrix} \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^u} & \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\theta}^d} \end{pmatrix}, \tag{9}$$

unless $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^d}$ is singular. On the other hand, the Jacobian is also decomposed into two parts as follows:

$$\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{q}} = \begin{pmatrix} \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{p}^u} & \frac{\partial \boldsymbol{d}}{\partial \boldsymbol{\theta}^d} \end{pmatrix}. \tag{10}$$

Therefore, the desired Jacobians of $\boldsymbol{d}$ are:

$$\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{\theta}^d} = -\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\theta}^d} = -\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}^d}, \tag{11}$$

$$\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{p}^u} = -\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}^u} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^d}^{-1}. \tag{12}$$

3.4 Minimization

Image registration seeks to minimize the residuals r_i of intensities of I_1 and I_2. The function to be totally minimized is the sum of the squares of the residuals over the image I_1:

$$\min_{\boldsymbol{\theta}} \sum_i r_i^2, \quad \boldsymbol{p}_i \in I_1, \tag{13}$$

$$r_i = I_1(\boldsymbol{p}_i) - I_2(\boldsymbol{p}_i^d), \tag{14}$$

$$\boldsymbol{p}_i^d = \boldsymbol{f}(\boldsymbol{p}_i^u, \boldsymbol{\theta}^d) \quad \text{for U-D model}, \tag{15}$$

$$\boldsymbol{p}_i^d = \boldsymbol{d}(\boldsymbol{p}_i^u, \boldsymbol{\theta}^d) \quad \text{for D-U model}, \tag{16}$$

$$\boldsymbol{p}_i^u = \boldsymbol{u}(\boldsymbol{p}_i, \boldsymbol{\theta}^u). \tag{17}$$

Estimating the parameters $\boldsymbol{\theta}$, the objective function is minimized by Gauss-Newton method [21], an iterative optimization method. At each iteration of the optimization, the decent direction is calculated until the estimation converges.

To calculate the decent direction of the cost function, Jacobian of r with respect to $\boldsymbol{\theta}$:

$$\frac{\partial r}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \frac{\partial r}{\partial \boldsymbol{\theta}^u} & \frac{\partial r}{\partial \boldsymbol{\theta}^d} \end{pmatrix}, \tag{18}$$

is required. We show the derivations for both the formulations based on the discussions above.

For D-U model:

$$\frac{\partial r}{\partial \boldsymbol{\theta}^u} = \frac{\partial r}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{p}^u}{\partial \boldsymbol{\theta}^u} = -\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{d}}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}^u}$$
$$= -\nabla I_2(\boldsymbol{p}^d)\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}^u}, \tag{19}$$

$$\frac{\partial r}{\partial \boldsymbol{\theta}^d} = \frac{\partial r}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{\theta}^d} = \nabla I_2(\boldsymbol{p}^d)\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^d}^{-1}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}^d}. \tag{20}$$

For U-D model:

$$\frac{\partial r}{\partial \boldsymbol{\theta}^u} = \frac{\partial r}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{p}^u}{\partial \boldsymbol{\theta}^u} = -\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}^u}$$
$$= -\nabla I_2(\boldsymbol{p}^d)\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}^u}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\theta}^u}, \tag{21}$$

$$\frac{\partial r}{\partial \boldsymbol{\theta}^d} = \frac{\partial r}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{p}^d}\frac{\partial \boldsymbol{p}^d}{\partial \boldsymbol{\theta}^d} = -\nabla I_2(\boldsymbol{p}^d)\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}^d}. \tag{22}$$

3.5 Correcting Distortion

After the distortion parameters $\boldsymbol{\theta}^d$ are estimated, we can use them for correction. For every point $\boldsymbol{p}^u$ in the corrected image I_2', the intensity is decided by that of the corresponding point in the distorted image I_2 as follows:

$$I_2'(\boldsymbol{p}^u) = I_2(\boldsymbol{f}(\boldsymbol{p}^u, \boldsymbol{\theta}^d)) \quad \text{for U-D model}, \tag{23}$$

$$I_2'(\boldsymbol{p}^u) = I_2(\boldsymbol{d}(\boldsymbol{p}^u, \boldsymbol{\theta}^d)) \quad \text{for D-U model}. \tag{24}$$

Actually, U-D is faster than D-U because Eq. (24) involves the computation of the iterative procedure for $\boldsymbol{d}$ (see Sect. 3.3). Note that the Jacobian also involves $\boldsymbol{d}$, so computational cost for D-U is larger than that for U-D.

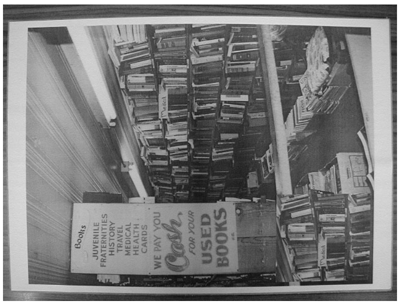
Note that the distortion parameters can be used to make an image distorted; e.g., CG is superimposed on a distorted real image. In this case, the above equations are used inversely as follows;

$$I(\boldsymbol{p}^d) = I'(\boldsymbol{d}(\boldsymbol{p}^d, \boldsymbol{\theta}^d)) \quad \text{for U-D model}, \tag{25}$$

$$I(\boldsymbol{p}^d) = I'(\boldsymbol{f}(\boldsymbol{p}^d, \boldsymbol{\theta}^d)) \quad \text{for D-U model}, \tag{26}$$

where I is a distorted image of an undistorted image I', and in turn the U-D model involves the iterative procedure.

(a)



(b)

Fig. 4 (a) Calibration pattern. (b) Captured image with distortion. (640 × 480)

Table 1 Estimated parameters.

	κ_1	κ_2	c_x	c_y	s_x
U-D	−4.96e − 07	7.49e − 13	298.7	241.2	0.762
D-U	5.07e − 07	−4.22e − 13	297.7	241.2	0.978

4. Experimental Results

We conducted experiments with the proposed method using a real distorted image. We used a photograph (Fig. 4 (a)) as the calibration pattern I_1, then printed it with a laser printer and captured a distorted image I_2 (Fig. 4 (b)) of the printed sheet by a digital camera.

Table 1 shows the estimated parameters of each of the formulations. The image centers are almost identical, however, the horizontal scales differ greatly. The reason is that s_x is theoretically absorbed into $\boldsymbol{\theta}^u$ for U-D formulation; the view change stretches the image horizontally while $s_x(< 1)$ makes the stretched image shrink. Therefore, it is difficult to estimate s_x accurately by U-D formulation.

Although the signs of the distortion parameters are inverted, in Fig. 5 we can see that κ_1 and κ_2 have the same effect on the distortion curves of the two formulations which are quite similar to each other for $|\boldsymbol{p}^d| < 400$ (the distance between a point in I_2 and the center is less than about 400) Note that we used $s_x = 1$ for U-D because of the reason above. The distorted images are corrected well (Fig. 6) by both the formulations ($s_x = 1$ for U-D). Therefore, both of them are comparable with each other except the estimation of s_x and the computational cost (see Sect. 3.5).

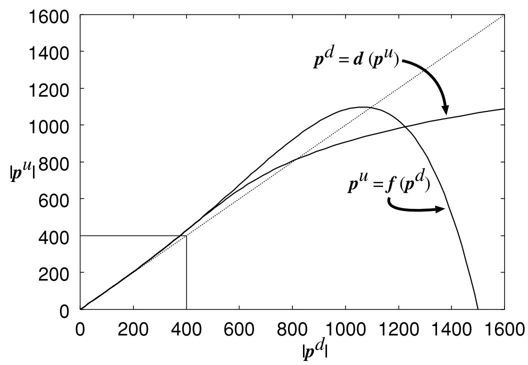
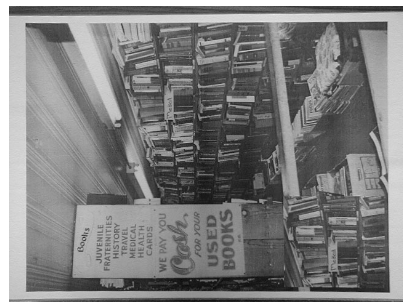
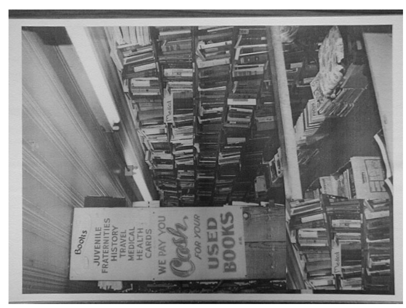


Fig. 5 Distortion curves of the two formulations.



(a)



(b)

Fig. 6 Corrected images by (a) U-D model ($s_x = 1$) and (b) D-U model.

5. Conclusions

We have proposed a unified approach to deal with two formulations of image distortion and a method for estimating the distortion parameters by using the two formulations. The proposed method is based on image registration to calculate Jacobians of U-D and D-U models by using implicit function theorem. Since the key of the proposed method is to use differentiation of a function that is not expressed in an explicit form, the discussion of the paper can be extended to other optimization based calibration methods other than image registration.

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