リーマン多様体の等長埋め込みと剛性

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研究成果報告書

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平成16年度から平成18年度の間，科学研究費の補助を受けリーマン多様体の等長埋め込みと剛性に関する研究を行った．この報告書ではこの研究課題に関わる成果をまとめ，またこの課題に関連する研究代表者の論文を収録した．

任意のリーマン多様体は十分高い次元のユークリッド空間へ等長に埋め込める．この事実は，まず局所的な立場から Janet-Cartan により (実解析的の仮定のもとで) 初めて示され，更にその 30 年後，Nash により大域的な等長埋め込みの存在定理が示された．

このような状況のもとで自然に考えられる問題として,「与えられたリーマン多様体が等長に埋め込める“最小次元”のユークリッド空間を決定せよ」というものが考えられる．また，最小次元が確定すれば，その次に考えるべきこととして，最小次元の等長埋め込みに“剛性”があるか否かを判定する問題が考えられる．

本研究では，リーマン多様体として特に対称空間を取り上げ，その中のいくつかの空間について上記２つの問題の最良結果を得ることに成功した．これらの結果は既に論文の形で発表されており，また多くの研究集会等でその研究成果を発表した．

本研究成果報告書では，研究期間中に得られた成果を項目別にまとめ，更に主要成果となる論文・報告書類を収録した．特に本研究期間中， 新潟県湯沢における部分多様体の研究集会においてリーマン多様体の等長埋め込み全般に関する講演を行い，その講演記録にはガウスから現在に至るまでの等長埋め込み問題の研究史を簡潔にまとめたが，これは等長埋め込み問題に関する重要な文献を網羅・要約したものであり，この問題の現状を理解するための一つの参考資料として本報告書の最後に収録した．この資料が今後この方面の研究に資することがあれば幸いである．また，対称リーマン空間の局所等長埋め込みに関し，最新のデータを一覧表の形にして末尾に添付した．同様の表は雑誌「数学」(56 巻, 2004 年) の論説の中にも掲げたが，本報告書に掲載する表は論説の執筆以降得られた研究成果も盛り込んだ最新のものである．

本研究期間中，特に剛性問題に関しては研究分担者である兼田英二氏の多大な協力もあって予想していた以上の成果をあげることができた．論文も共同で数多く執筆することができ，実り豊かな３年間の研究期間であった．しかし，対称空間の局所等長埋め込み可能な次元の評価の改良等，まだ多くの未解決問題を残している．これらの未解決問題をこれからの課題とし，今後更に研究を深めてゆきたい．

この研究を遂行するにあたり, 研究分担者の方々には大変お世話になった．文末ではあるが，ここに記して深い感謝の意を表する．

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計	２,４００千円	０千円	２,４００千円

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研究成果

本研究期間中に得られた主要な結果を以下簡単に報告する．まず問題の基本的な位置づけ・状況について簡単に説明した後，本研究で得られた主結果について個別に述べる．その後に，本研究期間中に発表した論文 (発表予定も含む)・資料の中で特に重要なもの 12 編を収録した．

M を n 次元リーマン多様体とする．よく知られているように，M は十分に高い次元のユークリッド空間に等長に埋め込める．(局所的な場合は Janet-Cartan, 大域的な場合は Nash の結果．) 一方，リーマン多様体の中で特に重要な位置を占める “対称リーマン空間” と呼ばれるクラスがある．対称リーマン空間は，局所的には曲率の共変微分が 0 になる空間として特徴付けられ，ある意味で平坦な空間の次に単純な構造をもつリーマン多様体であり，定曲率空間もこのクラスに含まれている．これらの空間の局所等長埋め込みについて考察することが本研究の主題である．

対称リーマン空間の中の多くのものについて標準的な埋め込みが S. Kobayashi により構成されている．対称リーマン空間は平坦な空間に近いということもあり，これらの標準埋め込みは Janet-Cartan, Nash の与えた次元に比べ，一般にかなり低い次元のユークリッド空間への等長埋め込みを与えている．

ここで自然な問題として，これらの標準埋め込みは最小次元の等長埋め込みを与えているか否か，また最小次元の等長埋め込みである場合その埋め込みに本質的な一意性 (剛性) があるか否か，という問いが考えられる．例えば正の定曲率空間である n 次元球面 S^n については $\mathbf{R}^{n+1}$ への標準的な埋め込みが最小次元の等長埋め込みであることが知られており，また $n \geq 3$ のときはこの等長埋め込みは局所的な剛性をもつ．($n = 2$ の場合大域的な剛性はあるものの，局所的な剛性はない．) また，負の定曲率空間 H^n について，局所的に等長に埋め込める最小次元のユークリッド空間は $\mathbf{R}^{2n-1}$ であることが既に知られている．(大域的に H^n が $\mathbf{R}^{2n-1}$ に等長に埋め込めるか否かは，未解決の大問題である．) しかし球面の場合と異なり，H^n の $\mathbf{R}^{2n-1}$ への局所等長埋め込みに剛性はない．

以上のような状況のもとで，阿賀岡は研究分担者の兼田英二氏と長年にわたり対称リーマン空間の等長埋め込みの研究を続け，新たに多くの事実を示すことに成功した．本研究期間中においても，共同研究として幾多の成果をあげることができた．その中でも特に剛性に関してはいくつかの対称リーマン空間に対し最良の結果を得ることに成功した．

本研究成果報告書においては，新たに得られた結果を項目別にまとめ，更に既に論文として発表されているもの及び論文に準ずる資料を，研究成果の詳細にあたるものとして本文の続きに収録した．

1. $P^2(\mathbf{Cay})$, $P^2(\mathbf{H})$, $Sp(n)$, $Sp(n)/U(n)$ の標準埋め込みの剛性.

上記4つの対称リーマン空間について，S. Kobayashi の構成した標準埋め込みが局所的な剛性をもつことを示した.

例えば，この中で $Sp(n)/U(n)$ は n^2+n 次元のエルミート対称空間であり，これは正則等長変換群 $Sp(n)$ のリー環 $\mathfrak{sp}(n)$ へ大域的に等長に埋め込めることが知られている (Lichnérowicz, S. Kobayashi). またこの埋め込みは局所的な立場からも $Sp(n)/U(n)$ の最小次元の等長埋め込みを与えていることを兼田英二氏との共同研究で既に示している.

本研究期間中，この標準埋め込みは $n \geq 2$ のとき局所的な剛性をもつこと，つまり実質的な局所等長埋め込みの一意性の成り立つことを示した. ($n=1$ のときは $Sp(1)/U(1) \simeq S^2$ となるので，局所的な剛性はない.) 正確な主張は次の形に述べられる.

定理 (阿賀岡-兼田). U を $Sp(n)/U(n)$ ($n \geq 2$) の空でない連結な開リーマン部分多様体とする. f を U から $2n^2+n$ 次元ユークリッド空間への等長埋め込みとすると，ユークリッド空間 $\mathbf{R}^{2n^2+n}$ のある等長変換 a が存在して $f = a \circ f_0$ となる. ここに f_0 は $Sp(n)/U(n)$ の標準埋め込み.

他の空間 $P^2(\mathbf{Cay})$, $P^2(\mathbf{H})$, $Sp(n)$ についても同様の結果が成り立つ. 対称リーマン空間の中で，これに類した局所剛性が成り立つ例としては S^n ($n \geq 3$) しか知られておらず，これら4空間については球面と同様に局所等長埋め込みに関し決定的な結果が得られたことになる.

上記の定理は，標準埋め込みと同じ余次元におけるガウス方程式の実質的な解の一意性を示すことにより証明される. 特に $Sp(n)/U(n)$ はエルミート対称空間であるため，標準埋め込みの第二基本形式はエルミート性をもっているが，ガウス方程式の解はすべてエルミート性を有することを示すことにより解の実質的な一意性が証明される.

これに関連することとして，複素射影空間 $P^n(\mathbf{C})$ 以外のすべての既約エルミート対称空間の標準埋め込みは，$Sp(n)/U(n)$ と同様に局所的な剛性をもつものと予想される. この予想を示すことは今後の大きな課題である. (複素射影空間 $P^n(\mathbf{C})$ については，ガウス方程式が解をもつ最小余次元においてエルミート的でないガウス方程式の解が出現するため，$Sp(n)/U(n)$ と同列に扱うことはできない.)

この項はすべて兼田英二氏との共同研究によるものである. 兼田氏の労に対し，深く感謝する.

2. 多項式値 2-形式の分解可能性.

与えられた余次元においてガウス方程式の可解性を判定することは一般に代数的に難しい問題である. この困難な問題に対する一つの展望を与えるため，曲率を代数的に見てよ

り単純な構造をもつ多項式値 2-形式に書き換え，この定式化のもとで余次元が 1 のガウス方程式の可解性についての研究を行った．

余次元が 1 の場合は曲率に対応する 2-形式が分解可能であることが可解であるための必要十分条件であるから，Plücker の関係式が可解であるための一つの条件を与える．2-形式がスカラー値の場合にはこれは十分条件でもあるが，多項式値の場合は Plücker の関係式だけでは十分条件になり得ず，更にもう一つ新たな 3 次の条件を考察しなければならない．これら 2 条件を課することにより，ガウス方程式が可解となるための必要かつ十分な条件が得られることを示した．また証明の過程において，群作用で不変な代数多様体が様々な形で自然に出現するが，これらの包含関係・定義方程式についても調べ，ほぼ満足すべき成果が得られた．

しかし，幾何学的にはこの結果は余次元が 1 の場合の話にしかすぎず，今後の課題としてこの結果を高次元に拡張する問題が残されている．

3．曲率の新しい条件．

リーマン多様体 M が低次元のユークリッド空間に等長に埋め込めるためには M の曲率がある種の条件を満たさなくてはいけない．この条件を具体的に求めることは等長埋め込み問題における重要な課題の一つであるが，現在のところ余次元がおおよそ M の次元に等しいところまでしかそのような条件は知られていない．

より高い余次元における曲率の関係式を求めるための第一段階として，曲率と第二基本形式の満たすべきある種混合された高次の関係式を求めることに成功した．この関係式を基にすれば，余次元が高い場合のリーマン部分多様体の曲率の関係式が得られるはずであり，今後の課題としてその条件を具体的な対称リーマン空間に適用し，クラス数 (= 局所等長に埋め込み可能となるユークリッド空間の最小余次元) の評価の改良に役立てたい．

この項目に関しては，近日中に論文として発表する予定でいる．

4．$P^n(\mathbf{C})$, $P^n(\mathbf{H})$ のクラス数の改良．

複素射影空間 $P^n(\mathbf{C})$ 及び四元数射影空間 $P^n(\mathbf{H})$ はそれぞれ n^2+2n, $2n^2+3n$ 次元のユークリッド空間に大域的に等長に埋め込める (S. Kobayashi)．これはそれぞれの空間のクラス数が n^2 以下, $2n^2-n$ 以下であることを示している．一方複素射影空間 $P^n(\mathbf{C})$ のクラス数は $[6n/5]$ 以上であることが阿賀岡により示されており，これがクラス数に関して今までに知られていた下からの最良の評価式であった．本研究において，研究分担者の兼田英二氏と共にこの下からの評価の改良に取り組み，次の結果を得た．

定理 (阿賀岡-兼田)．$P^n(\mathbf{C})$ のクラス数は $2n-2$ 以上，$P^n(\mathbf{H})$ のクラス数は $4n-3$ 以上．

この結果はS. Kobayashiによる上からの評価式には及ばないものの，今までに知られていた下からの評価を大きく改良するものである．この定理は，それぞれ余次元が$2n-3$, $4n-4$におけるガウス方程式が解をもたないことを示すことにより証明される．具体的には，擬平坦数を実現する擬可換部分空間とルート部分空間との関係を詳しく調べることによりこの結果が得られる．

しかし，クラス数に関してS. Kobayashiの評価とのgapはまだ大きく，この溝を埋めることが今後の大きな課題として残されている．

5．今後の問題.

以上のまとめにより，今後の課題は自ずと浮かび上がってくるであろう．(残された対称リーマン空間について剛性の有無を調べること，また標準埋め込みの次元の改良等々．)

更にこれら以外にも等長埋め込みに関し解決すべき問題は多く残されている．その一部を公募問題集「21世紀の数学　幾何学の未踏峰」(宮岡礼子・小谷元子編, 日本評論社)に提出した．また湯沢における研究集会の報告集にも関連する未解決問題を多くまとめておいた．その中でも，特に非コンパクト型の対称リーマン空間の局所等長埋め込みを構成することは特筆すべき大きな課題である．(そのような例としては現在のところ，負定曲率空間H^nの$\mathbf{R}^{2n-1}$への埋め込みしか知られていない．)

以上，本研究期間中に得られた成果について簡単にまとめた．問題の解決に向けて大きな前進は得られたが，未解決問題が数多く残されている．問題の最終的な解決を目指す決意を新たにし，本研究終了後もこの研究を更に押し進めるつもりでいる．

以下，本研究の成果として得られた論文・資料類を添付する．

収録順は次頁に記す通りである．内容的なつながりを重要視したので，必ずしも発表年順にはなっていない．文献番号は「研究発表 (1) 学会誌等」にあるものに合わせた．この中で$*$印の文献は北海道大学で開催された研究集会の発表 (予定) 内容をまとめたものである．研究集会での講演自体は阿賀岡の私事のため直前にキャンセルとなったが，また内容的には今となってはやや古いものを含んではいるが，等長埋め込みの具体例等を多く含んでいることもあり，参考資料としてここに収録することとした．

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Decomposability of polynomial valued 2-forms

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Abstract: We give a characterization of decomposable polynomial valued 2-forms in terms of their components. Such 2-forms must satisfy some cubic condition in addition to Plücker's quadratic relation. Several $GL(n, K) \times GL(m, K)$-invariant varieties naturally appear during this characterization, and we state the mutual relation of these varieties and study their geometric properties in detail.

Key words: polynomial valued 2-form, decomposability, Plücker's relation, variety

Introduction

Let V be an n-dimensional vector space over the field K of real numbers or complex numbers ($n \geq 2$) and V^* be its dual space. As is well known, an element $C \in \wedge^2 V^*$ is *decomposable* i.e., it can be expressed as $C = \alpha \wedge \beta$ for some α, $\beta \in V^*$ if and only if C satisfies Plücker's relation

$$C(v_1, v_2)\, C(v_3, v_4) - C(v_1, v_3) C(v_2, v_4) + C(v_1, v_4) C(v_2, v_3) = 0$$

for any $v_i \in V$. (For example, see [9], [19]). The main purpose of this paper is to give a similar characterization of decomposable "polynomial valued" 2-forms. This problem is closely related to the existence of local isometric imbeddings of Riemannian manifolds into the Euclidean space with codimension 1 (cf. [2], [7]).

To explain the results, we first fix the notations. We put $V = K^n$ ($K = \mathbf{R}$ or $\mathbf{C}$) and let A be a polynomial ring over K with m variables $x_1, \cdots, x_m$: $A = K[x_1, \cdots, x_m]$, and $A = \Sigma_{p \geq 0} A^p$ ($A^0 = K$) be the homogeneous decomposition of A. An element $\alpha \in V^* \otimes A^1$ may be considered as an A^1-valued 1-form on V. Then, for $\beta \in V^*$, the exterior product $\alpha \wedge \beta \in \wedge^2 V^* \otimes A^1$ is naturally defined as in the scalar valued case, and we say that $C \in \wedge^2 V^* \otimes A^1$ is *decomposable* if it is expressed as $\alpha \wedge \beta$ for some $\alpha \in V^* \otimes A^1$ and $\beta \in V^*$. In this polynomial valued case, decomposable 2-forms also satisfy Plücker's relation. But this relation is not sufficient to characterize decomposable 2-forms in contrast to the scalar valued case. In fact the algebraic set of $\wedge^2 V^* \otimes A^1$ defined by only Plücker's relation is not irreducible and it decomposes into two irreducible components, one of which just coincides with the set of decomposable 2-forms. To obtain a complete characterization of decomposable 2-forms, we must add one cubic condition on C. This additional condition is stated

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as follows: "For any $v_i \in V$, the polynomials $C(v_1, v_2)$, $C(v_1, v_3)$, $C(v_2, v_3)$ are linearly dependent in A^1". We here give one example: Consider the 2-form $C = x_1\omega_1 \wedge \omega_2 + x_2\omega_1 \wedge \omega_3 + x_3\omega_2 \wedge \omega_3$, where $\{\omega_i\}$ is a basis of V^*. Then, it is easy to see that C satisfies Plücker's relation, but does not satisfy the above cubic condition, and hence we know that this form C is not decomposable.

The other irreducible component of the algebraic set defined by Plücker's relation consists of A^1-valued 2-forms that can be reduced to some 3-dimensional subspace of V. As in the case of Plücker's relation, the algebraic set defined by the above cubic condition also decomposes into two irreducible components; one is the variety of decomposable forms, and the other is the variety consisting of 2-forms that take value in two variables x_1, x_2 after some variable transformation.

In order to understand the variety of decomposable 2-forms, it is natural to treat these three varieties simultaneously. All these varieties are characterized by two types of conditions on C, and they are related to each other by possessing one common defining equation for each pair (Theorem 1). In addition, the algebraic set defined by only one type of condition on C splits into two irreducible components (Theorem 2). In considering this mutual relation, another three varieties naturally appear as subsets of the above varieties. In this paper, we characterize these six varieties completely by giving their defining equations, inclusion relations, dimension, and clarify their geometric meaning by introducing a parametrization of each variety (Proposition 3 and Theorem 8).

The space $\wedge^2 V^* \otimes A^1$ may be considered as a sort of 3-tensor space, and the results of this paper possess some resemblance to the case of the 3-tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ studied in [3]. It is desirable and also interesting to extend our results to more general 3-tensor spaces such as $\wedge^3 V^*$, $\mathbf{C}^p \otimes \mathbf{C}^q \otimes \mathbf{C}^r$, etc (cf. [5], [6]).

As stated above, the decomposability of polynomial valued 2-forms C is naturally related to the problem of local isometric imbeddings of Riemannian manifolds through the notion of the partial Gauss equation that was introduced in [2]. By definition, the partial Gauss equation is expressed as

$$C = \alpha_1 \wedge \beta_1 + \cdots + \alpha_r \wedge \beta_r, \tag{A}$$

where $C \in \wedge^2 V^* \otimes A^1$ is a given 2-form and $\alpha_i \in V^* \otimes A^1$, $\beta_i \in V^*$. Roughly speaking, if an n-dimensional Riemannian manifold M^n ($n = \dim V$) is locally isometrically imbedded into $\mathbf{R}^{n+r}$, then certain 2-form C constructed from the curvature of M must be expressed in the above form (A). (For the precise statement, see [2].) In particular, the results stated in this paper is related to the case of hypersurfaces of $\mathbf{R}^{n+1}$ (the case $r=1$), and the conditions on the decomposability of C serve as obstructions to the existence of local isometric imbeddings of M into $\mathbf{R}^{n+1}$. For further applications in geometry, we must obtain a similar characterization of 2-forms C in (A) for larger r.

§1. Statement of the main results

In this section, after fixing some notations, we state the main results of this paper. The proof of Theorem 1 and Theorem 2 stated below will be given in the subsequent sections.

Let C be an element of $\wedge^2 V^* \otimes A^1$. We define two linear maps d_C and e_C as follows:

$$d_C : V \longrightarrow V^* \otimes A^1, \quad d_C(v) = v \lrcorner C,$$
$$e_C : \wedge^2 V \longrightarrow A^1, \quad e_C(v_1 \wedge v_2) = C(v_1, v_2),$$

where $v \lrcorner C$ implies the interior product. In terms of these maps, we define the following five subsets of $\wedge^2 V^* \otimes A^1$:

$$\Sigma_1 = \{C \in \wedge^2 V^* \otimes A^1 \mid C = \alpha \wedge \beta \text{ for some } \alpha \in V^* \otimes A^1,\ \beta \in V^*\},$$
$$\Sigma_2 = \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} d_C \leq 3\},$$
$$\Sigma_3 = \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} e_C \leq 2\},$$
$$\Sigma_4 = \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} d_C \leq 2\},$$
$$\Sigma_5 = \{C \in \wedge^2 V^* \otimes A^1 \mid \operatorname{rank} e_C \leq 1\}.$$

As we will see later, these five subsets are all irreducible varieties of $\wedge^2 V^* \otimes A^1$. We remark that if rank $d_C \leq k$, then C can be considered as an element of $\wedge^2 W^* \otimes A^1$ where W is a k-dimensional subspace of V. In fact, since dim Ker $d_C \geq n-k$, there exists a basis $\{e_1, \cdots, e_n\}$ of V satisfying $e_{k+1} \lrcorner C = \cdots = e_n \lrcorner C = 0$. Then, by using the dual basis $\{\omega_i\}$, the 2-form C is expressed as $\Sigma_{i,j=1}^{k} C_{ij}\, \omega_i \wedge \omega_j$, where $C_{ij} = C(e_i, e_j)$. Similarly, it is easy to see that if rank $e_C \leq l$, the number of variables m can be reduced to l after some variable transformation.

Next, we define several conditions on $C \in \wedge^2 V^* \otimes A^1$ in order to describe the defining equations of Σ_i. We say that C satisfies condition (C_P) if it satisfies classical Plücker's relation:

$$C(v_1, v_2)C(v_3, v_4) - C(v_1, v_3)C(v_2, v_4) + C(v_1, v_4)C(v_2, v_3) = 0 \in A^2$$

for any vectors $v_i \in V$. This condition is equivalent to $C \wedge C = 0 \in \wedge^4 V^* \otimes A^2$. Next, if the polynomials

$$C(v_1, v_2),\ C(v_1, v_3),\ C(v_2, v_3)$$

are linearly dependent in A^1 for any v_i, we say that C satisfies condition (C_Q). Using the components of C, this condition is expressed as cubic polynomial relations of C. Finally, for positive integer k, we say that C satisfies condition (C_k) if the polynomials

$$C(v_1, v_2),\ C(v_1, v_3),\ \cdots,\ C(v_1, v_{k+2})$$

are linearly dependent in A^1 for any v_i. It is easy to see that this condition is equivalent to

$$\operatorname{rank}\,(v \lrcorner C) \leq k \quad \text{for any } v \in V,$$

where "rank" means the usual rank of the (m, n)-matrix $v \lrcorner C \in V^* \otimes A^1$. In this paper, we use this condition only in the cases $k=1$ and 2. Note that condition (C_1) is quadratic and (C_2) is cubic, and clearly, condition (C_1) implies (C_2) and (C_Q). By using these four conditions (C_P), (C_Q), (C_1), (C_2), we can completely characterize the subset $\Sigma_i \subset \wedge^2 V^* \otimes A^1$ in the following way.

Theorem 1. (1) *$C \in \Sigma_1$ if and only if C satisfies (C_P) and (C_Q).*
(2) *$C \in \Sigma_2$ if and only if C satisfies (C_P) and (C_2).*
(3) *$C \in \Sigma_3$ if and only if C satisfies (C_Q) and (C_2).*
(4) *$C \in \Sigma_4$ if and only if C satisfies (C_P) and (C_1).*
(5) *$C \in \Sigma_5$ if and only if C satisfies (C_1).*
In addition, each subset Σ_i $(1 \leq i \leq 5)$ is an irreducible algebraic variety of $\wedge^2 V^* \otimes A^1$.

In particular, the decomposability of $C \in \wedge^2 V^* \otimes A^1$ is completely characterized by two types of conditions (C_P) and (C_Q). In the case $m \leq 2$, we remark that C is decomposable if and only if it satisfies condition (C_P) only, because condition (C_Q) is automatically satisfied in this case.

By definition, an element C belongs to Σ_2 if and only if rank $d_C \leq 3$, and hence, Σ_2 is defined by quartic polynomials. But, the above theorem asserts that this condition can be reduced to lower degree conditions (C_P) and (C_2).

By Theorem 1, we have clearly $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_3$, and C belongs to this algebraic set if and only if C satisfies three conditions (C_P), (C_Q), (C_2). In the following, we denote this algebraic set by Σ_6.

Next, we characterize the algebraic set of $\wedge^2 V^* \otimes A^1$ defined by one of (C_P), (C_Q), (C_2).

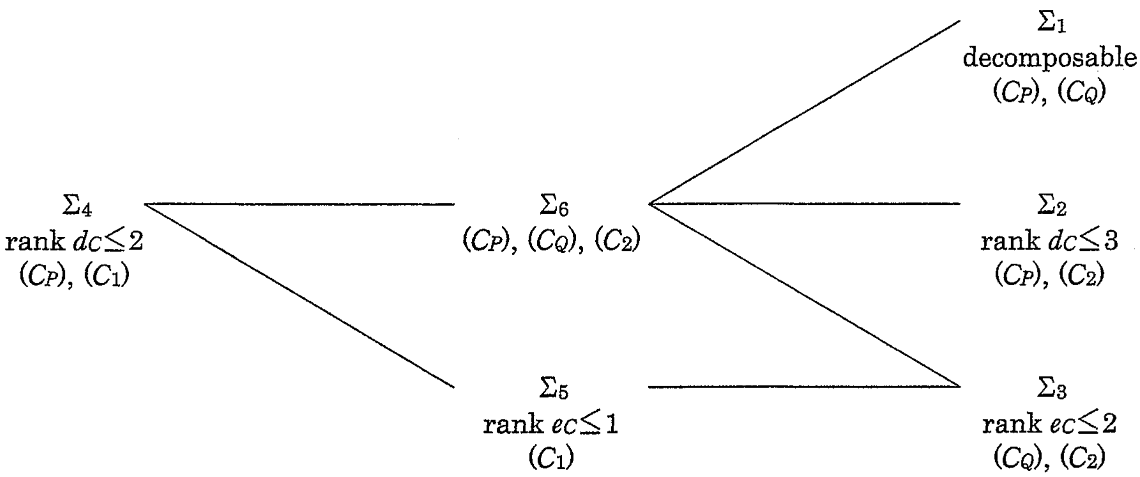
Theorem 2. (1) *C satisfies condition (C_P) if and only if $C \in \Sigma_1 \cup \Sigma_2$.*
(2) *C satisfies condition (C_Q) if and only if $C \in \Sigma_1 \cup \Sigma_3$.*
(3) *C satisfies condition (C_2) if and only if $C \in \Sigma_2 \cup \Sigma_3$.*

By definition, any element $C \in \Sigma_1$ can be parametrized by the pair $(\alpha, \beta) \in V^* \otimes A^1 \times V^*$ as $C = \alpha \wedge \beta$. Other varieties $\Sigma_2 \sim \Sigma_6$ also have similar parametrization, by which we can understand their geometric meaning.

Proposition 3. (1) *$C \in \Sigma_2$ if and only if $C = f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ for some $f_i \in A^1$ and $\beta_i \in V^*$.*
(2) *$C \in \Sigma_3$ if and only if $C = f_1\Omega_1 + f_2\Omega_2$ for some $f_i \in A^1$ and $\Omega_i \in \wedge^2 V^*$.*
(3) *$C \in \Sigma_4$ if and only if $C = f\beta_1 \wedge \beta_2$ for some $f \in A^1$ and $\beta_i \in V^*$.*
(4) *$C \in \Sigma_5$ if and only if $C = f\Omega$ for some $f \in A^1$ and $\Omega \in \wedge^2 V^*$.*
(5) *$C \in \Sigma_6$ if and only if $C = (f_1\beta_1 + f_2\beta_2) \wedge \beta_3$ for some $f_i \in A^1$ and $\beta_i \in V^*$.*

Proof. For the statements (1) and (3), "if" parts are easy to see. The converse parts are already proved after the definition of the varieties $\Sigma_1 \sim \Sigma_5$, where we show $C = \Sigma_{i,j=1}^k C_{ij}\,\omega_i \wedge \omega_j$ under the condition rank $d_C \leq k$. The statements (2) and (4) are almost trivial because the condition rank $e_C \leq l$ implies that the number of variables m can be reduced to l, as stated in the same place. For the statement (5), we assume for some time that the definition of Σ_6 is $\Sigma_2 \cap \Sigma_3$ (since we did not prove Theorem 1 yet). Then, if $C \in \Sigma_6$, C is expressed as $f_1\beta_1 \wedge \beta_2 + f_2\beta_1 \wedge \beta_3 + f_3\beta_2 \wedge \beta_3$ from the condition $C \in \Sigma_2$. Next, since the number of variables is reducible to two, we may put, by the symmetry, $f_3 = af_1 + bf_2$ $(a, b \in K)$. Then, after substituting this into the above expression, we have $C = (\beta_1 + b\beta_2 - a\beta_3) \wedge (f_1\beta_2 + f_2\beta_3)$. The converse part is trivial from (1) and (2). q.e.d.

This parametrization may be considered as a canonical form of each variety Σ_i. This proposition is useful in the proof of Theorems 1 and 2. We summarize the inclusion relations of Σ_i in the following figure:



(Note that condition (C_1) implies (C_2) and (C_Q), as stated before.)

Finally we state one remark. The group $GL(n, K) \times GL(m, K)$ acts naturally on the space $\wedge^2 V^* \otimes A^1$, and it is easy to see that the above varieties $\Sigma_1 \sim \Sigma_6$ are invariant under this group action. It is an interesting problem to classify all $GL(n, K) \times GL(m, K)$-invariant subvarieties of $\wedge^2 V^* \otimes A^1$ as in the case of the 3-tensor space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ (cf. [3]). Perhaps another new concept is required to solve this problem in addition to (C_P), (C_Q) and (C_k), and to know such fundamental concept is one important step to understand the 3-tensor space $\wedge^2 V^* \otimes A^1$.

§2. Preliminary lemmas

In this section we prepare several lemmas to prove the results in §1. Each lemma plays a crucial role in the proof of Theorems 1 and 2.

First, we prove the following lemma.

Lemma 4. *Assume* $C \in \wedge^2 V^* \otimes A^1$ *satisfies conditions* (C_P), (C_Q), *and there exists* $v \in V$ *such that* rank $(v \lrcorner C) \geq 2$. *Then, there exist a basis* $\{e_1, \cdots, e_n\}$ *and* $a_2 \sim a_n \in K$ *satisfying*

$$C_{ij} = a_j C_{1i} - a_i C_{1j},$$

for $1 \leq i, j \leq n$, *where* $C_{ij} = C(e_i, e_j)$ *and* $a_1 = -1$. *In addition, such* $\{a_i\}$ *uniquely exists if we fix a basis* $\{e_i\}$.

Proof. We choose a basis $\{e_1, \cdots, e_n\}$ such that $e_1 = v$, and let $\{\omega_1, \cdots, \omega_n\}$ be the dual basis of $\{e_i\}$. Then we have

$$v \rfloor C = C_{12}\omega_2 + \cdots + C_{1n}\omega_n.$$

By rearranging the indices if necessary, we may assume that $\{C_{12}, C_{13}\}$ is linearly independent because rank $(v \rfloor C) \geq 2$. Since $\{C_{12}, C_{13}, C_{23}\}$ is dependent from condition (C_Q), C_{23} is uniquely expressed as $C_{23} = a_3 C_{12} - a_2 C_{13}$. Next, for $4 \leq i \leq n$, we substitute this equality into Plücker's relation

$$C_{12}C_{3i} - C_{13}C_{2i} + C_{1i}C_{23} = 0.$$

Then we have immediately

$$C_{12}(C_{3i} + a_3 C_{1i}) = C_{13}(C_{2i} + a_2 C_{1i}).$$

Since $\{C_{12}, C_{13}\}$ is independent, the above expression is equal to $a_i C_{12} C_{13}$ for some $a_i \in K$. In particular, we have

$$C_{2i} = a_i C_{12} - a_2 C_{1i}.$$

(Note that this equality holds for $1 \leq i \leq n$.) Uniqueness of $a_4 \sim a_n$ is clear from this expression. We substitute this equality into

$$C_{12}C_{ij} - C_{1i}C_{2j} + C_{1j}C_{2i} = 0.$$

Then we have the desired equality $C_{ij} = a_j C_{1i} - a_i C_{1j}$ because $C_{12} \neq 0$. q.e.d.

Before proving the next lemma, we introduce a notation $|f_1 f_2 f_3|$ $(f_i \in A^1)$, which we often use in the following arguments. We express $f_i \in A^1$ as $\Sigma_{p=1}^{m} f_{ip} x_p$, and put

$$|f_1 f_2 f_3|_{pqr} = \begin{vmatrix} f_{1p} & f_{2p} & f_{3p} \\ f_{1q} & f_{2q} & f_{3q} \\ f_{1r} & f_{2r} & f_{3r} \end{vmatrix} \in A^3.$$

We define $|f_1 f_2 f_3|$ by

$$|f_1 f_2 f_3| = (|f_1 f_2 f_3|_{pqr})_{1 \leq p < q < r \leq m} \in \overbrace{A^3 \oplus \cdots \oplus A^3}^{\binom{m}{3}},$$

i.e., $|f_1 f_2 f_3|$ is the set of $\binom{m}{3}$ polynomials $|f_1 f_2 f_3|_{pqr}$ $(1 \leq p < q < r \leq m)$ arranged in some fixed order. Then, addition and scalar multiplication of $|f_1 f_2 f_3|$ is naturally defined. For example, we have the equalities

$$|f_1+f_2\, f_3\, f_4| = |f_1\, f_3\, f_4| + |f_2\, f_3\, f_4|,$$
$$|af_1\, f_2\, f_3| = a\,|f_1\, f_2\, f_3|.$$

Clearly, $|f_1 f_2 f_3|$ is skew symmetric with respect to $\{f_1, f_2, f_3\}$, and $|f_1 f_2 f_3|=0$ if and only if $\{f_1, f_2, f_3\}$ is linearly dependent in A^1.

Using vectors $v_i \in V$, we put $C_{ij}=C(v_i, v_j)$. Then, in terms of the above notation, condition (C_Q) is expressed as

$$|C_{ij}\ C_{ik}\ C_{jk}|=0.$$

By replacing the vector v_k by v_k+v_l, we have

$$|C_{ij}\ C_{ik}\ C_{jl}|+|C_{ij}\ C_{il}\ C_{jk}|=0.$$

In the same way, condition (C_2) is expressed in the form

$$|C_{ij}\ C_{ik}\ C_{il}|=0.$$

In this equality, we replace v_i by v_i+v_j. Then it follows that

$$|C_{ij}\ C_{ik}\ C_{jl}|+|C_{ij}\ C_{jk}\ C_{il}|=0.$$

In particular, if C satisfies both conditions (C_Q) and (C_2), we have from the above two equalities

$$|C_{ij}\ C_{ik}\ C_{jl}|=0$$

because $|C_{ij}\ C_{jk}\ C_{il}|=-|C_{ij}\ C_{il}\ C_{jk}|$. In addition, by replacing v_i by v_i+v_p in this equality, we have

$$|C_{ij}\ C_{pk}\ C_{jl}|+|C_{pj}\ C_{ik}\ C_{jl}|=0.$$

Now, using this notation, we prove the following lemma.

Lemma 5. *Assume $C \in \wedge^2 V^* \otimes A^1$ satisfies condition (C_P) or (C_2). In addition, there exist $v_1, v_2, v_3 \in V$ such that $\{C_{12}, C_{13}, C_{23}\}$ ($C_{ij}=C(v_i, v_j)$) is linearly independent in A^1. Then C is expressed in the form $C=f_1\beta_1\wedge\beta_2+f_2\beta_1\wedge\beta_3+f_3\beta_2\wedge\beta_3$ for some $f_i \in A^1$, $\beta_i \in V^*$.*

Proof. We fix a basis $\{e_1, \cdots, e_n\}$ of V such that $e_1=v_1$, $e_2=v_2$, $e_3=v_3$. Since $\{C_{12}, C_{13}, C_{23}\}$ is linearly independent, we may assume $C_{12}=x_1$, $C_{13}=x_2$, $C_{23}=x_3$ after some variable transformation. Now, we divide the proof into two cases.

(i) The case where C satisfies condition (C_P). In this case, we have for $4 \leq i \leq n$,

$$C_{12}C_{3i}-C_{13}C_{2i}+C_{1i}C_{23}=x_1C_{3i}-x_2C_{2i}+x_3C_{1i}=0.$$

From this equality, we have easily

$$\begin{aligned} C_{1i}&=a_i\, x_1-b_i\, x_2, \\ C_{2i}&=c_i\, x_1-b_i\, x_3, \\ C_{3i}&=c_i\, x_2-a_i\, x_3 \end{aligned}$$

for some a_i, b_i, c_i. Note that by putting $a_2=1$, $b_3=c_1=-1$, $a_1=a_3=b_1=b_2=c_2=c_3=0$, the above equalities hold for $1 \leq i \leq n$. Next, we substitute them into

$$C_{12}C_{ij}-C_{1i}C_{2j}+C_{1j}C_{2i}=0.$$

Then, we have

$$C_{ij}=(a_i\, c_j-a_j\, c_i)x_1+(b_j\, c_i-b_i\, c_j)x_2+(a_j\, b_i-a_i\, b_j)x_3.$$

Hence, by putting $\beta_1=\Sigma a_i\omega_i$, $\beta_2=\Sigma b_i\omega_i$, $\beta_3=\Sigma c_i\omega_i$ ($\{\omega_i\}$ is the dual basis of $\{e_i\}$), we have

$$C=x_1\beta_1\wedge\beta_3-x_2\beta_2\wedge\beta_3-x_3\beta_1\wedge\beta_2.$$

(ii) The case where C satisfies condition (C_2). In this case, as prepared above, we have $|C_{12}\ C_{13}\ C_{1i}|=|C_{21}\ C_{23}\ C_{2i}|=|C_{31}\ C_{32}\ C_{3i}|=0$, and hence, $C_{1i}\in\langle x_1, x_2\rangle$, $C_{2i}\in\langle x_1, x_3\rangle$, $C_{3i}\in\langle x_2, x_3\rangle$. In addition, from (C_2), we have

$$\begin{aligned} &|C_{12}\ C_{13}\ C_{2i}|+|C_{12}\ C_{23}\ C_{1i}|=0, \\ &|C_{23}\ C_{21}\ C_{3i}|+|C_{23}\ C_{31}\ C_{2i}|=0, \\ &|C_{31}\ C_{32}\ C_{1i}|+|C_{31}\ C_{12}\ C_{3i}|=0. \end{aligned}$$

Using these conditions, we obtain easily

$$\begin{aligned} C_{1i}&=a_i\, x_1-b_i\, x_2, \\ C_{2i}&=c_i\, x_1-b_i\, x_3, \\ C_{3i}&=c_i\, x_2-a_i\, x_3. \end{aligned}$$

Next, for $2 \leq i \leq n$ and $t \in K$, we have

$$\begin{aligned} (e_1+te_i)\lrcorner C&=C_{12}\omega_2+\cdots+C_{1n}\omega_n+t(C_{i1}\omega_1+\cdots+C_{in}\omega_n) \\ &=tC_{i1}\omega_1+(C_{12}+tC_{i2})\omega_2+\cdots+(C_{1n}+tC_{in})\omega_n. \end{aligned}$$

Since rank $((e_1+te_i)\rfloor C)\leq 2$ for any parameter t, we have in particular, dim $\langle C_{12}+tC_{i2}, C_{13}+tC_{i3}, C_{1j}+tC_{ij}\rangle\leq 2$ for $2\leq i, j\leq n$. If $|t|$ is sufficiently small, first two elements are linearly independent, and hence

$$C_{1j}+tC_{ij}\in\langle C_{12}+tC_{i2}, C_{13}+tC_{i3}\rangle\subset\langle x_1, x_2, x_3\rangle.$$

Since $C_{1j}\in\langle x_1, x_2, x_3\rangle$, we have $C_{ij}\in\langle x_1, x_2, x_3\rangle$, and we may put $C_{ij}=p_{ij}x_1+q_{ij}x_2+r_{ij}x_3$. Next, we take out the coefficient of x_1, x_2, x_3 in the above three elements $C_{12}+tC_{i2}, C_{13}+tC_{i3}, C_{1j}+tC_{ij}$. Then, since these vectors span the space of dimension≤ 2, it follows that

$$\begin{vmatrix} 1-tc_i & 0 & tb_i \\ 0 & 1-tc_i & ta_i \\ a_j+tp_{ij} & -b_j+tq_{ij} & tr_{ij} \end{vmatrix} = t(tc_i-1)\{(c_i r_{ij}+b_i p_{ij}+a_i q_{ij})t-r_{ij}+a_j b_i-a_i b_j\}=0$$

for any t. In particular, we have $r_{ij}=a_j b_i-a_i b_j$. Similarly, using the conditions rank $((e_2+te_i)\rfloor C)\leq 2$, rank $((e_3+te_i)\rfloor C)\leq 2$, we obtain $q_{ij}=b_j c_i-b_i c_j$, $p_{ij}=a_i c_j-a_j c_i$. Hence, it follows that

$$C_{ij}=(a_i c_j-a_j c_i)x_1+(b_j c_i-b_i c_j)x_2+(a_j b_i-a_i b_j)x_3.$$

Then, in the same way as in the case (i), we have the desired result. q.e.d.

We prepare one more lemma for later use.

Lemma 6. *Assume that* $C\in\wedge^2V^*\otimes A^1$ *satisfies condition* (C_Q) *and there exists a vector* $v\in V$ *such that* rank $(v\rfloor C)\geq 3$. *Then, there exist a basis* $\{e_1,\cdots, e_n\}$ *of* V *and* $a_2\sim a_n\in K$ *satisfying*

$$C_{ij}=a_jC_{1i}-a_iC_{1j}\quad(a_1=-1).$$

Proof. We fix a basis $\{e_1,\cdots, e_n\}$ such that $e_1=v$. Then, in the same way as in the proof of Lemma 4, we may assume that $\{C_{12},\cdots, C_{1p}\}$ is linearly independent and $C_{1,p+1}\sim C_{1n}\in\langle C_{12},\cdots, C_{1p}\rangle$. (Note that $p\geq 4$ because rank $(v\rfloor C)\geq 3$.) From condition (C_Q), the set $\{C_{1i}, C_{1j}, C_{ij}\}$ is linearly dependent for $2\leq i\neq j\leq p$, and hence, we may put

$$C_{ij}=a_{ij}C_{1i}-a_{ji}C_{1j}$$

for some $a_{ij}\in K$. (Note that $C_{ij}=-C_{ji}$.) In addition, we have from condition (C_Q)

$$|C_{1i}\ C_{1j}\ C_{ik}|+|C_{1i}\ C_{1k}\ C_{ij}|=0$$

for $2\leq i, j, k\leq p$ (i, j, k are all distinct). By substituting the above expression into this equality, we have immediately

$$(a_{ji}-a_{ki})\,|C_{1i}\ C_{1j}\ C_{1k}|=0.$$

Since $\{C_{1i}, C_{1j}, C_{1k}\}$ is linearly independent, we have $a_{ji}=a_{ki}$. Therefore, we may put $a_{ji}=a_i$ for $2\leq i\leq p$. Hence, by putting $a_1=-1$, we obtain $C_{ij}=a_jC_{1i}-a_iC_{1j}$ for $1\leq i, j\leq p$.

Next, we express $C_{1,p+1}\sim C_{1n}$ as

$$C_{1,p+1}=b_{p+1,2}C_{12}+\cdots+b_{p+1,p}C_{1p},$$
$$\cdots\cdots\cdots$$
$$C_{1n}=b_{n2}C_{12}+\cdots+b_{np}C_{1p}.$$

Then, for $2\leq i\neq j\leq p$, $p+1\leq\lambda\leq n$, we have from (C_Q)

$$\begin{aligned}0&=|C_{1i}\ C_{1j}\ C_{i\lambda}|+|C_{1i}\ C_{1\lambda}\ C_{ij}|\\&=|C_{1i}\ C_{1j}\ C_{i\lambda}|+|C_{1i}\ C_{1\lambda}\ a_jC_{1i}-a_iC_{1j}|\\&=|C_{1i}\ C_{1j}\ C_{i\lambda}+a_iC_{1\lambda}|.\end{aligned}$$

In particular, we have $C_{i\lambda}+a_iC_{1\lambda}\in\langle C_{1i}, C_{1j}\rangle$. Since $p\geq 4$, there exists an index k $(2\leq k\leq p)$, different from i, j. Hence, by replacing j by k, we have in the same way, $C_{i\lambda}+a_iC_{1\lambda}\in\langle C_{1i}, C_{1k}\rangle$, which implies $C_{i\lambda}+a_iC_{1\lambda}\in\langle C_{1i}\rangle$. Therefore, we may express

$$C_{i\lambda}=a_{i\lambda}C_{1i}-a_iC_{1\lambda}$$

for $1\leq i\leq p$, $p+1\leq\lambda\leq n$. (We may include the case $i=1$ because $a_1=-1$.) We will show that the value $a_{i\lambda}$ does not depend on i. For this purpose, we put $v_1=e_1$, $v_2=e_i+e_j$, $v_3=e_k+te_\lambda$ $(2\leq i, j, k\leq p$, i, j, k are all distinct, $p+1\leq\lambda\leq n$ and $t\in K$ is a parameter). Then, from (C_Q), we have

$$\begin{aligned}0&=|C(v_1, v_2)\ C(v_1, v_3)\ C(v_2, v_3)|\\&=|C_{1i}+C_{1j}\ \ C_{1k}+tC_{1\lambda}\ \ C_{ik}+tC_{i\lambda}+C_{jk}+tC_{j\lambda}|\\&=|C_{1i}+C_{1j}\ \ C_{1k}+tC_{1\lambda}\ \ a_kC_{1i}-a_iC_{1k}+a_kC_{1j}-a_jC_{1k}+t(a_{i\lambda}C_{1i}-a_iC_{1\lambda}+a_{j\lambda}C_{1j}-a_jC_{1\lambda})|\\&=t\,|C_{1i}+C_{1j}\ \ C_{1k}+tC_{1\lambda}\ \ a_{i\lambda}C_{1i}+a_{j\lambda}C_{1j}|\\&=t\,|C_{1i}+C_{1j}\ \ C_{1k}+t(b_{\lambda 2}C_{12}+\cdots+b_{\lambda p}C_{1p})\ \ (a_{j\lambda}-a_{i\lambda})C_{1j}|.\end{aligned}$$

Then, by taking out the coefficient of C_{1i}, C_{1j}, C_{1k}, we have

$$t\begin{vmatrix}1 & 1 & 0\\ tb_{\lambda i} & tb_{\lambda j} & 1+tb_{\lambda k}\\ 0 & a_{j\lambda}-a_{i\lambda} & 0\end{vmatrix}=t(a_{i\lambda}-a_{j\lambda})(1+tb_{\lambda k})=0$$

for any t, which implies $a_{i\lambda}=a_{j\lambda}$. In particular, we may put $a_{i\lambda}=a_\lambda$, and therefore,

$$C_{i\lambda}=a_\lambda C_{1i}-a_i C_{1\lambda}$$

for $1\leq i\leq p,\ p+1\leq\lambda\leq n$.

Finally, we show the equality

$$C_{\lambda\mu}=a_\mu C_{1\lambda}-a_\lambda C_{1\mu}$$

for $p+1\leq\lambda,\ \mu\leq n$. In the same way as above, we put $v_1=e_1$, $v_2=e_i+se_\lambda$, $v_3=e_j+te_\mu$ $(2\leq i\neq j\leq p,\ p+1\leq\lambda\neq\mu\leq n$, and $s,t\in K$ are parameters), and apply condition (C_Q). Then, we have

$$\begin{aligned}
0 &= |\,C_{1i}+sC_{1\lambda}\ \ C_{1j}+tC_{1\mu}\ \ C_{ij}+tC_{i\mu}+sC_{\lambda j}+stC_{\lambda\mu}\,|\\
&= |\,C_{1i}+sC_{1\lambda}\ \ C_{1j}+tC_{1\mu}\ \ a_j C_{1i}-a_i C_{1j}+t(a_\mu C_{1i}-a_i C_{1\mu})-s(a_\lambda C_{1j}-a_j C_{1\lambda})+stC_{\lambda\mu}\,|\\
&= |\,C_{1i}+sC_{1\lambda}\ \ C_{1j}+tC_{1\mu}\ \ ta_\mu C_{1i}-sa_\lambda C_{1j}+stC_{\lambda\mu}\,|\\
&= |\,C_{1i}+sC_{1\lambda}\ \ C_{1j}+tC_{1\mu}\ \ st(C_{\lambda\mu}-a_\mu C_{1\lambda}+a_\lambda C_{1\mu})\,|\\
&= st\,|\,C_{1i}+sC_{1\lambda}\ \ C_{1j}+tC_{1\mu}\ \ C_{\lambda\mu}-a_\mu C_{1\lambda}+a_\lambda C_{1\mu}\,|\,.
\end{aligned}$$

Now, assume that $st\neq 0$ and $|s|$, $|t|$ are sufficiently small. Then, since $\{C_{1i}+sC_{1\lambda},\ C_{1j}+tC_{1\mu}\}$ is linearly independent, we have

$$C_{\lambda\mu}-a_\mu C_{1\lambda}+a_\lambda C_{1\mu}\in\langle C_{1i}+sC_{1\lambda},\ C_{1j}+tC_{1\mu}\rangle.$$

In particular, taking the limit $s, t\to 0$, it follows that

$$C_{\lambda\mu}-a_\mu C_{1\lambda}+a_\lambda C_{1\mu}\in\langle C_{1i},\ C_{1j}\rangle.$$

Using an index k $(2\leq k\leq p)$, which is different from i and j, we repeat the same procedure. Then, we have

$$C_{\lambda\mu}-a_\mu C_{1\lambda}+a_\lambda C_{1\mu}\in\langle C_{1i},\ C_{1j}\rangle\cap\langle C_{1i},\ C_{1k}\rangle\cap\langle C_{1j},\ C_{1k}\rangle=\{0\},$$

which implies $C_{\lambda\mu}=a_\mu C_{1\lambda}-a_\lambda C_{1\mu}$, and we complete the proof of the lemma. q.e.d.

§3. Proof of Theorems

Using the lemmas prepared in §2, we give a proof of Theorems 1 and 2 in this section.

Proof of Theorem 1. (5) If $C\in\Sigma_5$, then C is expressed as $f\,\Omega$ $(f\in A^1,\ \Omega\in\wedge^2V^*)$ by Proposition 3 (4), and hence, condition (C_1) clearly holds. Conversely, assume C satisfies condition (C_1). Then, for any vector $v\in V$, we have rank $(v\rfloor C)\leq 1$. If $C=0$, then the theorem holds trivially, and hence we may assume that there exists v such that rank $(v\rfloor C)=1$. We fix a basis $\{e_1,\cdots,e_n\}$ such that $e_1=v$, and by the symmetry, we may put $C_{12}=x_1$, $C_{1i}\in\langle x_1\rangle$. From the condition rank $(e_2\rfloor C)\leq 1$, we have $\dim\langle C_{21}, C_{23},\cdots, C_{2n}\rangle\leq 1$, in particular, $C_{2i}\in\langle x_1\rangle$. Next, for $2\leq i\leq n$, we have

$$(e_1+te_i)\rfloor C=tC_{i1}\omega_1+(C_{12}+tC_{i2})\omega_2+\cdots+(C_{1n}+tC_{in})\omega_n,$$

as in the proof of Lemma 5 (ii). Since rank $((e_1+te_i)\rfloor C)\leq 1$, we have dim $\langle C_{12}+tC_{i2},\ C_{1j}+tC_{ij}\rangle\leq 1$ for $2\leq i,j\leq n$. If $|t|$ is sufficiently small, $C_{12}+tC_{i2}$ is not zero, and hence

$$C_{1j}+tC_{ij}\in\langle C_{12}+tC_{i2}\rangle=\langle x_1\rangle.$$

In particular, we have $C_{ij}\in\langle x_1\rangle$ because $C_{1j}\in\langle x_1\rangle$. Therefore, the coefficients of C are all contained in the space $\langle x_1\rangle$, and hence, rank $ec\leq 1$, i.e., $C\in\Sigma_5$.

(4) If $C\in\Sigma_4$, C is expressed as $f\beta_1\wedge\beta_2$ by Proposition 3 (3). Then, we have clearly $C\wedge C=0$ and C satisfies condition (C_P). In addition, from Proposition 3 (4), we have clearly $C\in\Sigma_5$, which implies that C satisfies (C_1), just we showed above. Next, assume that C satisfies (C_P) and (C_1). From condition (C_1), we have $C\in\Sigma_5$, and we may express C as $f\Omega$ $(f\neq 0\in A^1,\ \Omega\in\wedge^2V^*)$. Then, from condition (C_P), we have $C\wedge C=f^2\Omega\wedge\Omega=0$, i.e., $\Omega\wedge\Omega=0$, which is equivalent to classical Plücker's relation. Hence Ω is decomposable, and C is expressed as $f\beta_1\wedge\beta_2$. Thus, by Proposition 3 (3), we have $C\in\Sigma_4$.

(1) Assume that $C\in\Sigma_1$. Then C is expressed as $\alpha\wedge\beta$, and hence it satisfies Plücker's relation $C\wedge C=0$. Next, for any vectors $v_i\in V$, we put $C_{ij}=C(v_i,v_j)$, $\beta_i=\beta(v_i)$. Then from the condition $\beta\wedge C=0$, we have $\beta_1C_{23}-\beta_2C_{13}+\beta_3C_{12}=0$, which implies that $\{C_{12},C_{13},C_{23}\}$ is linearly dependent in the case $(\beta_1,\ \beta_2,\ \beta_3)\neq 0$. If $\beta_1=\beta_2=\beta_3=0$, we have clearly $C_{12}=C_{13}=C_{23}=0$, and we obtain the same conclusion. Now, we prove the converse part. First, assume that there exists $v\in V$ such that rank $(v\rfloor C)\geq 2$. Then, by Lemma 4, C_{ij} is expressed as

$$C_{ij}=a_jC_{1i}-a_iC_{1j},$$

for some $a_i\in K$. Then, by putting $\alpha=\Sigma C_{1i}\omega_i$ and $\beta=\Sigma a_i\omega_i$, we have $C=\alpha\wedge\beta$, which implies that C is decomposable. Next, assume that rank $(v\rfloor C)\leq 1$ for any v. In this case, the 2-form C satisfies two conditions (C_P) and (C_1). Hence, by Theorem 1 (4), which we showed above, we have $C\in\Sigma_4$. In particular, from Proposition 3 (3), C is expressed as $f\beta_1\wedge\beta_2$, which implies that C is decomposable.

(3) By Proposition 3 (2), "only if" part is clear. We assume that C satisfies conditions (C_Q) and (C_2). From (C_2), we have rank $(v\rfloor C)\leq 2$ for any $v\in V$. If rank $(v\rfloor C)\leq 1$ for any v, then C satisfies condition (C_1), and in particular, $C\in\Sigma_5\subset\Sigma_3$ (cf. Proposition 3 (2), (4)). If there exists v such that rank $(v\rfloor C)=2$, then, as before, we can choose a basis $\{e_i\}$ such that $e_1=v$, $\{C_{12},C_{13}\}$ is linearly independent and $C_{14}\sim C_{1n}\in\langle C_{12},\ C_{13}\rangle$. Since two conditions (C_Q) and (C_2) hold, we have the following two equalities, which we showed in §2, after the proof of Lemma 4.

(B) $$|C_{ij}\ C_{ik}\ C_{jl}|=0,$$
(C) $$|C_{ij}\ C_{pk}\ C_{jl}|+|C_{pj}\ C_{ik}\ C_{jl}|=0.$$

From (B), we have $|C_{12}\ C_{13}\ C_{2l}|=0$, i.e., $C_{2l}\in\langle C_{12}, C_{13}\rangle$. From (C), we have $|C_{21}\ C_{pk}\ C_{13}|+|C_{p1}\ C_{2k}\ C_{13}|=0$. Since $C_{p1}, C_{2k}\in\langle C_{12}, C_{13}\rangle$, the second term is zero, and hence, we have $C_{pk}\in\langle C_{12}, C_{13}\rangle$, which shows that the 2-form C is $\langle C_{12}, C_{13}\rangle$-valued. In particular, the number of variables is reducible to two, and hence we have $C\in\Sigma_3$.

(2) If $C\in\Sigma_2$, it is expressed as $f_1\beta_1\wedge\beta_2+f_2\beta_1\wedge\beta_3+f_3\beta_2\wedge\beta_3$ by Proposition 3 (1). And in addition, without loss of generality, we may assume $\{\beta_1, \beta_2, \beta_3\}$ is linearly independent, by changing f_i if necessary. We extend $\{\beta_i\}$ to a basis of V^* and denote its dual by $\{e_i\}$. Then, for any vector $v=\Sigma a_i e_i\in V$, we have

$$\begin{aligned} v\rfloor C &= a_1 e_1\rfloor C+a_2 e_2\rfloor C+a_3 e_3\rfloor C \\ &= a_1(f_1\beta_2+f_2\beta_3)+a_2(-f_1\beta_1+f_3\beta_3)-a_3(f_2\beta_1+f_3\beta_2) \\ &= -(a_2 f_1+a_3 f_2)\beta_1+(a_1 f_1-a_3 f_3)\beta_2+(a_1 f_2+a_2 f_3)\beta_3. \end{aligned}$$

By using the equality

$$-a_1(a_2 f_1+a_3 f_2)+a_2(a_1 f_1-a_3 f_3)+a_3(a_1 f_2+a_2 f_3)=0,$$

we can easily check that rank $(v\rfloor C)\leq 2$, and hence, C satisfies condition (C_2). From the above expression of C, Plücker's relation $C\wedge C=0$ is clearly satisfied.

Next, we assume that C satisfies conditions (C_P) and (C_2). If there exist $v_i\in V$ such that $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is linearly independent, then by Lemma 5 and Proposition 3 (1), we have $C\in\Sigma_2$. If $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is dependent for any v_i, then C satisfies conditions (C_P), (C_Q), (C_2). Hence, by Theorem 1 (1), (3), it is decomposable and the number of variables can be reducible to two. Using these two facts, it is easy to see that C is in the form $(f_1\beta_1+f_2\beta_2)\wedge\beta_3$, and by Proposition 3 (1), we have $C\in\Sigma_2$.

Finally, we show that Σ_i is an irreducible variety. By definition and Theorem 1 (1), each Σ_i is an algebraic set of $\wedge^2V^*\otimes A^1$ because it is defined by the vanishing of some polynomials of C. In addition, by Proposition 3, it is just equal to the image of certain polynomial map from some affine space, and hence it is irreducible. q.e.d.

Proof of Theorem 2. For three statements, "if" parts are all clear from Theorem 1. We prove "only if" parts.

(1) Assuming that C satisfies (C_P) and $C\notin\Sigma_1$, we show $C\in\Sigma_2$. By Theorem 1 (1) and the condition $C\notin\Sigma_1$, C does not satisfy condition (C_Q), namely, there exist v_1, v_2, v_3 such that $\{C(v_1, v_2), C(v_1, v_3), C(v_2, v_3)\}$ is linearly independent. Then, by Lemma 5, C is expressed in the form $f_1\beta_1\wedge\beta_2+f_2\beta_1\wedge\beta_3+f_3\beta_2\wedge\beta_3$, and hence $C\in\Sigma_2$.

(2) We assume that C satisfies (C_Q) and $C\notin\Sigma_3$. Then, C does not satisfy (C_2), as above. Hence, there exists v such that rank $(v\rfloor C)\geq 3$, and by Lemma 6, we have $C_{ij}=a_jC_{1i}-a_iC_{1j}$ for some a_i. Using this expression, we have immediately $C\in\Sigma_1$ as we have done in the proof of Theorem 1 (1).

(3) Assume that the conditions (C_2) and $C\notin\Sigma_3$ hold. Then, since C does not satisfy (C_Q), we have $C\in\Sigma_2$ by Lemma 5, in the same way as (1). q.e.d.

§4. Dimension and the inverse formula

Scalar valued decomposable 2-forms $C\in\wedge^2V^*$ are expressed as $\beta_1\wedge\beta_2$. But two 1-forms β_1, β_2 $\in V^*$ are not uniquely determined from C. In contrast, for A^1-valued decomposable 2-forms $C=\alpha\wedge\beta$, two 1-forms $\alpha\in V^*\otimes A^1$ and $\beta\in V^*$ are essentially uniquely determined if C is sufficiently generic (precisely, if $C\in\Sigma_1\setminus\Sigma_4$). In this section, using this result, we express α and β explicitly in terms of the components of C. In addition, we determine the dimension of each variety Σ_i by using the results obtained in previous sections.

Proposition 7. *Assume* $C\in\Sigma_1\setminus\Sigma_4$ *and* $C=\alpha\wedge\beta=\alpha'\wedge\beta'$ $(\alpha,\ \alpha'\in V^*\otimes A^1,\ \beta,\ \beta'\in V^*)$. *Then there exist* $k\ (\neq 0)\in K$ *and* $f\in A^1$ *such that* $\alpha'=k\alpha+f\beta$, $\beta'=1/k\cdot\beta$.

Proof. Since C satisfies conditions (C_P) and $C\notin\Sigma_4$, it does not satisfy (C_1), and hence, there exists $v\in V$ such that rank $(v\rfloor C)\geq 2$. Then, by Lemma 4, C_{ij} is expressed as

$$C_{ij}=a_jC_{1i}-a_iC_{1j},$$

by using some $a_i\in K$, which is uniquely determined. In addition, as stated in the proof of Lemma 4, we may assume that $\{C_{12}, C_{13}\}$ is linearly independent by changing the indices if necessary. We put $\alpha_i=\alpha(e_i)$, $\beta_i=\beta(e_i)$, $\alpha'_i=\alpha'(e_i)$ and $\beta'_i=\beta'(e_i)$. If $\beta_1=0$, then we have $C_{12}=\alpha_1\beta_2$ and $C_{13}=\alpha_1\beta_3$, which implies that C_{12} and C_{13} are parallel. Hence, we have $\beta_1\neq 0$. In the same way, we have $\beta'_1\neq 0$. Then, from the condition $\beta\wedge C=\beta\wedge\alpha\wedge\beta=0$, we have

$$\beta_1C_{ij}-\beta_iC_{1j}+\beta_jC_{1i}=0,$$

namely,

$$C_{ij}=\frac{\beta_i}{\beta_1}C_{1j}-\frac{\beta_j}{\beta_1}C_{1i}.$$

Since the coefficient β_i/β_1 is uniquely determined from Lemma 4, we have $\beta_i/\beta_1=\beta'_i/\beta'_1$, which implies $\beta'=\beta'_1/\beta_1\cdot\beta$. Next, from the equality $C_{1i}=\alpha_1\beta_i-\alpha_i\beta_1$, we have

$$\alpha_i=\frac{1}{\beta_1}(\alpha_1\beta_i-C_{1i}).$$

Then, in terms of the dual basis $\{\omega_i\}$, we have

$$\begin{aligned}\alpha &=\Sigma\,\alpha_i\omega_i\\ &=\frac{1}{\beta_1}\Sigma(\alpha_1\beta_i-C_{1i})\,\omega_i\\ &=\frac{1}{\beta_1}(\alpha_1\beta-e_1\rfloor C).\end{aligned}$$

Using this equality, we obtain

$$\begin{aligned}\alpha' &=\frac{1}{\beta'_1}(\alpha'_1\beta'-e_1\rfloor C)\\ &=\frac{\alpha'_1}{\beta'_1}\cdot\frac{\beta'_1}{\beta_1}\beta-\frac{1}{\beta'_1}(\alpha_1\beta-\beta_1\alpha)\\ &=f\beta+k\alpha,\end{aligned}$$

where $f=\alpha'_1/\beta_1-\alpha_1/\beta'_1\in A^1$ and $k=\beta_1/\beta'_1$. q.e.d.

Remark. As is easy to see, we cannot drop the condition $C\notin\Sigma_4$ in this proposition. In particular, it is necessary $n\geq3$ and $m\geq2$ to hold the above condition, because rank $(v\rfloor C)\geq2$ for some $v\in V$. (Note that $(v\rfloor C)(v)=0$.)

Now, we give the explicit inverse formula for generic C. Using a basis $\{e_1,\cdots,e_n\}$, we put $C_{ij}=C(e_i,e_j)=\Sigma_{p=1}^{m}C_{ijp}\,x_p$. Then, since $\{C_{1i},C_{1j},C_{ij}\}$ is dependent, we have

$$\begin{vmatrix}C_{1ip}&C_{1jp}&C_{ijp}\\C_{1iq}&C_{1jq}&C_{ijq}\\C_{1ir}&C_{1jr}&C_{ijr}\end{vmatrix}=0,$$

and this equality implies

$$\begin{vmatrix}C_{1jp}&C_{ijp}\\C_{1jq}&C_{ijq}\end{vmatrix}C_{1i}-\begin{vmatrix}C_{1ip}&C_{ijp}\\C_{1iq}&C_{ijq}\end{vmatrix}C_{1j}+\begin{vmatrix}C_{1ip}&C_{1jp}\\C_{1iq}&C_{1jq}\end{vmatrix}C_{ij}=0.$$

Hence, if $\begin{vmatrix}C_{1ip}&C_{1jp}\\C_{1iq}&C_{1jq}\end{vmatrix}\neq0$, we have

$$C_{ij}=\frac{\begin{vmatrix} C_{1ip} & C_{ijp} \\ C_{1iq} & C_{ijq} \end{vmatrix} C_{1j} - \begin{vmatrix} C_{1jp} & C_{ijp} \\ C_{1jq} & C_{ijq} \end{vmatrix} C_{1i}}{\begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix}}.$$

Then, combining with the expression

$$C_{ij}=\frac{\beta_i}{\beta_1}C_{1j}-\frac{\beta_j}{\beta_1}C_{1i},$$

appeared in the proof of Proposition 7, we have

$$\frac{\beta_i}{\beta_1}=\frac{\begin{vmatrix} C_{1ip} & C_{ijp} \\ C_{1iq} & C_{ijq} \end{vmatrix}}{\begin{vmatrix} C_{1ip} & C_{1jp} \\ C_{1iq} & C_{1jq} \end{vmatrix}},$$

because the coefficient of C_{1j} is uniquely determined from Lemma 4. From this expression, the 1-form β is uniquely determined up to a non-zero constant, and this gives the inverse formula of β. Note that the right hand side of this expression does not depend on the choice of indices j, p, q unless the denominator is zero. The inverse formula for α is already given in the proof of Proposition 7:

$$\alpha=\frac{1}{\beta_1}(\alpha_1\beta - e_1\lrcorner C).$$

From this expression we know that the 1-form α is essentially equal to $e_1\lrcorner C$ up to a non-zero constant. We remark that $C\in\Sigma_1$ belongs to Σ_4 if and only if the determinant

$$\text{(D)} \qquad \begin{vmatrix} C_{ijp} & C_{ikp} \\ C_{ijq} & C_{ikq} \end{vmatrix}$$

is zero for all vectors v_i and indices p, q, where $C_{ij}=C(v_i, v_j)$. The denominator of the inverse formula of β is a special case of this determinant (D).

Finally, we determine the dimension of the varieties $\Sigma_1\sim\Sigma_6$.

Theorem 8. *The dimension of the variety* Σ_i *is given in the following table:*

	$n=2$	$m=1$	$n\geq 3$ and $m\geq 2$
Σ_1	m	$2n-3$	$(n-1)(m+1)$
Σ_2	m	$2n-3$	$3(n+m-3)$
Σ_3	m	$\binom{n}{2}$	$2\binom{n}{2}+2m-4$
Σ_4	m	$2n-3$	$2n+m-4$
Σ_5	m	$\binom{n}{2}$	$\binom{n}{2}+m-1$
Σ_6	m	$2n-3$	$3n+2m-7$

In the case $n=2$, *all varieties* Σ_i *are equal to the whole space* $\wedge^2V^*\otimes A^1\simeq A^1$, *and in the case* $m=1$, $\Sigma_3=\Sigma_5=\wedge^2V^*\otimes A^1\simeq\wedge^2V^*$ *and* $\Sigma_1=\Sigma_2=\Sigma_4=\Sigma_6$ *coincides with the set of scalar valued decomposable elements of* $\wedge^2V^*\otimes A^1\simeq\wedge^2V^*$.

Proof. If $n=2$, then any element $C\in\wedge^2V^*\otimes A^1$ can be expressed as $f\beta_1\wedge\beta_2$. Hence, by Proposition 3, we have $C\in\Sigma_i$ for $i=1\sim 6$, which implies $\Sigma_i=\wedge^2V^*\otimes A^1$. Next, in the case $m=1$, it is easy to see that any element of Σ_1, Σ_2, Σ_4, Σ_6 (resp. Σ_3, Σ_5) is expressed in the form $x_1\beta_1\wedge\beta_2$ (resp. $x_1\Omega$). (Note that $\beta_1\wedge\beta_2+\beta_1\wedge\beta_3+\beta_2\wedge\beta_3=(\beta_1+\beta_2)\wedge(\beta_2+\beta_3)$, and it is decomposable.) In particular, the variety $\Sigma_1=\Sigma_2=\Sigma_4=\Sigma_6$ coincides with the set of decomposable elements of $\wedge^2V^*$ and $\Sigma_3=\Sigma_5$ is equal to the whole space. The dimension of Σ_1 is easily determined by calculating the dimension of the isotropy subgroup of $\beta_1\wedge\beta_2(\neq 0)$ under the action of the general linear group $GL(n, K)$, because $GL(n, K)$ acts transitively on the set $\Sigma_1\backslash\{0\}$. We omit the explicit calculations.

Next, we consider the case $n\geq 3$ and $m\geq 2$. If $C=\alpha\wedge\beta\in\Sigma_1\backslash\Sigma_4$, then by Proposition 7, the parametrization of C by α and β has the freedom which is expressed uniquely by the pair $(k, f)\in K\times A^1$. Hence, we have $\dim\Sigma_1=\dim V^*\otimes A^1+\dim V^*-1-\dim A^1=(n-1)(m+1)$.

For the variety Σ_2, we first assume $m\geq 3$, and $C\in\Sigma_2\backslash\Sigma_6$, i.e., C satisfies (C_P), (C_2), but not (C_Q). Then, using a suitable basis $\{e_i\}$, the set $\{C_{12}, C_{13}, C_{23}\}$ is linearly independent, and as stated in the proof of Lemma 5, we have

$$C_{1i}=a_iC_{12}-b_iC_{13},$$
$$C_{2i}=c_iC_{12}-b_iC_{23},$$
$$C_{3i}=c_iC_{13}-a_iC_{23}$$

for $4\leq i\leq n$. In addition, other C_{ij} is also expressed in terms of $\{C_{12}, C_{13}, C_{23}\}$ and $\{a_i, b_i, c_i\}_{4\leq i\leq n}$. Since these parameters are uniquely determined by C, we have $\dim\Sigma_2=3m+3(n-3)=3(n+m-3)$. If $m=2$, any element $C\in\Sigma_2$ is contained in Σ_1 because it satisfies conditions (C_P) and (C_Q). (Note that condition (C_Q) is automatically satisfied in the case $m=2$.) Conversely, since any element $C\in\Sigma_1$ is expressed as $(f_1\beta_1+f_2\beta_2)\wedge\beta_3$ in the case $m=2$, we have $\Sigma_1\subset\Sigma_2$ by Proposition 3 (1). Hence, we have $\Sigma_1=\Sigma_2$, and in particular, $\dim\Sigma_2=\dim\Sigma_1=3(n-1)$, which is equal to $3(n+m-3)$.

For the variety Σ_3, we take an element $C\in\Sigma_3\backslash\Sigma_5$. Then, from the condition $C\notin\Sigma_5$, we may assume that $\{C_{12}, C_{13}\}$ is independent, and other C_{ij} is uniquely expressed as a linear combination of $\{C_{12}, C_{13}\}$ because the number of variables is reducible to two. Hence, we have $\dim\Sigma_3=2m+\{\binom{n}{2}-2\}\times 2=2\binom{n}{2}+2m-4$.

Next, any element of Σ_4 is expressed as $f\beta_1\wedge\beta_2$. As showed above, the dimension of the variety of decomposable elements of $\wedge^2 V^*$ is $2n-3$, and the degree of freedom of f is m. Since the scalar multiplication appears in common, we have dim $\Sigma_4=(2n-3)+m-1=2n+m-4$.

For the variety Σ_5, any element of Σ_5 is expressed in the form $f\Omega$, and by the same reason as above, we have dim $\Sigma_5=\binom{n}{2}+m-1$.

Finally, for the variety Σ_6, we take an element $C\in\Sigma_6\setminus\Sigma_4$. Then, since it does not satisfy (C_1), we can apply Lemma 4. As stated in the proof of this lemma, $\{C_{12}, C_{13}\}$ is linearly independent, and in addition, we have $C_{14}\sim C_{1n}\in\langle C_{12}, C_{13}\rangle$ from condition (C_2). Hence, we may put $C_{1i}=b_iC_{12}+c_iC_{13}$ for $4\leq i\leq n$. Since other C_{ij} is expressed as

$$C_{ij}=a_jC_{1i}-a_iC_{1j}$$

for some a_i $(a_1=-1)$, C is parametrized by $\{C_{12}, C_{13}, a_2,\cdots, a_n, b_4,\cdots,b_n, c_4,\cdots, c_n\}$. It is easy to check that these parameters are uniquely determined by C, and therefore, we have dim $\Sigma_6=2m+(n-1)+2(n-3)=3n+2m-7$. q.e.d.

We remark that the exceptional case $n=2$ or $m=1$ in this theorem corresponds to the case where the action of the product group $GL(n, K)\times GL(m, K)$ on $\wedge^2V^*\otimes A^1$ reduces to the single group $GL(m, K)$ or $GL(n, K)$, i.e., the case where the 3-tensor space $\wedge^2V^*\otimes A^1$ is reduced to a 1- or 2-tensor space. And so we must treat separately to determine the dimension of the variety, though two equalities dim $\Sigma_4=2n+m-4$ and dim $\Sigma_5=\binom{n}{2}+m-1$ always hold without the assumption $n\geq 3$ and $m\geq 2$.

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Local isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$

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Abstract. We investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces. We prove a non-existence theorem of local isometric imbeddings (see Theorem 2), by which we can conclude that the isometric imbeddings given in Kobayashi [8] are the least dimensional isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$.

Key words: Pseudo-nullity, isometric imbedding, projective plane.

1. Introduction

In this paper we investigate local isometric imbeddings of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ into the Euclidean spaces.

In [5], we determined the pseudo-nullity $p(G/K)$ for each compact rank one symmetric space G/K. (For the definition of the pseudo-nullity, see [5].) Utilizing $p(G/K)$, we have obtained the following result concerning the non-existence of isometric imbeddings of the complex projective spaces $P^n(\boldsymbol{C})$ $(n \geq 2)$, the quaternion projective spaces $P^n(\boldsymbol{H})$ $(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$ (see Theorem 5.6 of [5]).

Theorem 1 *Let G/K be one of the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 2)$, the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 2)$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$. Define an integer $q(G/K)$ by setting $q(G/K) = 2\dim G/K - p(G/K)$, i.e.,*

$$q(G/K) = \begin{cases} \min\{4n-2,\, 3n+1\}, & \text{if } G/K = P^n(\boldsymbol{C}) \ (n \geq 2), \\ \min\{8n-3,\, 7n+1\}, & \text{if } G/K = P^n(\boldsymbol{H}) \ (n \geq 2), \\ 25, & \text{if } G/K = P^2(\boldsymbol{Cay}). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^Q$ with $Q \leq q(G/K) - 1$.

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As is well known, $P^n(\boldsymbol{C})$ (resp. $P^n(\boldsymbol{H})$, resp. $P^2(\boldsymbol{Cay})$) can be globally isometrically imbedded into $\boldsymbol{R}^{n^2+2n}$ (resp. $\boldsymbol{R}^{2n^2+3n}$, resp. $\boldsymbol{R}^{26}$) (see Kobayashi [8]). By these facts, it follows that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then G/K can be isometrically imbedded into $\boldsymbol{R}^{q(G/K)+1}$. Then a natural question arises: Is there any isometric imbedding of $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ into the Euclidean space $\boldsymbol{R}^{q(G/K)}$?

In this paper, we will solve this problem. The main result of this paper is the following

Theorem 2 *Let G/K be the quaternion projective plane $P^2(\boldsymbol{H})$ or the Cayley projective plane $P^2(\boldsymbol{Cay})$. Then any open set of G/K cannot be isometrically imbedded into the Euclidean space $\boldsymbol{R}^{q(G/K)}$. Accordingly, $\boldsymbol{R}^{q(G/K)+1}$ is the least dimensional Euclidean space into which G/K can be locally isometrically imbedded.*

2. The Gauss equation

In the following G/K implies the quaternion projective plane $P^2(\boldsymbol{H}) = Sp(3)/Sp(2)\times Sp(1)$ or the Cayley projective plane $P^2(\boldsymbol{Cay}) = F_4/Spin(9)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair (G, K). We denote by $(\ , \ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. We assume that the G-invariant Riemannian metric g of G/K satisfies $g(X, Y) = (X, Y)$ $(X, Y \in \mathfrak{m})$. Then the curvature tensor R at o is given by

$$R(X, Y)Z = -\big[[X, Y], Z\big], \quad \forall X, Y, Z \in \mathfrak{m}. \tag{2.1}$$

Suppose that there is a local isometric imbedding of G/K into the Euclidean space $\boldsymbol{R}^Q$, i.e., there is an open set U of G/K and an isometric imbedding $\boldsymbol{f}$ of U into $\boldsymbol{R}^Q$. Because of homogeneity, we may assume that U contains the origin $o \in G/K$. Let $\boldsymbol{N}$ be the normal space of $\boldsymbol{f}(U)$ at $\boldsymbol{f}(o)$ and let $\langle \ , \ \rangle$ be the inner product of $\boldsymbol{N}$ induced from the canonical inner product of $\boldsymbol{R}^Q$. Then $\boldsymbol{N}$ is a vector space with $\dim \boldsymbol{N} = Q - \dim G/K$ and the second fundamental form $\boldsymbol{\Psi}$ of $\boldsymbol{f}$ at o, which is regarded as an $\boldsymbol{N}$-valued symmetric bilinear form on $\mathfrak{m}$, must satisfy the following Gauss equation:

$$\begin{aligned} -\big(R(X,Y)Z, W\big) = &\big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle \\ &-\big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle, \ \forall X,Y,Z,W \in \mathfrak{m}. \end{aligned} \tag{2.2}$$

On the contrary, we can prove

Theorem 3 *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. If $\dim \boldsymbol{N} \leq q(G/K) - \dim G/K$, then the Gauss equation (2.2) does not admit any solution, i.e., there is no $\boldsymbol{N}$-valued symmetric bilinear form Ψ on $\mathfrak{m}$ satisfying (2.2).*

Theorem 3 implies that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then there is no local isometric imbedding of G/K into $\boldsymbol{R}^{q(G/K)}$, proving Theorem 2.

We now make a preparatory discussion for the proof of Theorem 3. Take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then we have $\dim \mathfrak{a} = 1$, because $\operatorname{rank}(G/K) = 1$. We consider the root space decompositions of $\mathfrak{k}$ and $\mathfrak{m}$ with respect to $\mathfrak{a}$. Let $\lambda \in \mathfrak{a}$. We define subspaces $\mathfrak{k}(\lambda)$ $(\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)$ $(\subset \mathfrak{m})$ by setting

$$\mathfrak{k}(\lambda) = \{X \in \mathfrak{k} \mid [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a}\},$$
$$\mathfrak{m}(\lambda) = \{Y \in \mathfrak{m} \mid [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a}\}.$$

λ is called a *restricted root* when $\mathfrak{m}(\lambda) \neq 0$. We denote by Σ the set of non-zero restricted roots. In the case where $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, it is well known that there is a restricted root μ satisfying $\Sigma = \{\pm\mu, \pm 2\mu\}$ and

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.3}$$
$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)}, \tag{2.4}$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see § 5 of [5]). In the following discussions we fix this restricted root μ and the decompositions (2.3) and (2.4).

For convenience, for each integer i we set

$$\mathfrak{k}_i = \mathfrak{k}(|i|\mu), \ \mathfrak{m}_i = \mathfrak{m}(|i|\mu) \ (|i| \leq 2) \quad \text{and} \quad \mathfrak{k}_i = \mathfrak{m}_i = 0 \ (|i| > 2).$$

Then we have

Proposition 4 (1) *Let $i, j = 0, 1, 2$. Then:*

$$\begin{aligned} &[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \end{aligned} \tag{2.5}$$

(2) $\dim \mathfrak{k}_i = \dim \mathfrak{m}_i$ $(i = 1, 2)$.
(3) *The following table summarizes the basic data for $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$.*

G/K	$\dim G/K$	$\dim \mathfrak{m}_1$	$\dim \mathfrak{m}_2$	$q(G/K)$
$P^2(\boldsymbol{H})$	8	4	3	13
$P^2(\boldsymbol{Cay})$	16	8	7	25

Proof. (1) and (2) are well known (see Helgason [7], p. 335). (3) is obtained by Table 2 and Table 3 of [5]. □

3. Proof of Theorem 3

In this section we prove Theorem 3. Here we suppose that $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ and that there is a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.2).

Let $Y \in \mathfrak{m}$. We define a linear map $\boldsymbol{\Psi}_Y$ of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\boldsymbol{\Psi}_Y : \mathfrak{m} \ni Y' \longmapsto \boldsymbol{\Psi}(Y, Y') \in \boldsymbol{N}.$$

By $\mathbf{Ker}(\boldsymbol{\Psi}_Y)(\subset \mathfrak{m})$ we denote the kernel of the linear map $\boldsymbol{\Psi}_Y$. We now show a key proposition, which plays an important role in the following discussion.

Proposition 5 *Let $Y \in \mathfrak{m}$ $(Y \neq 0)$ and let $k \in K$ satisfy $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then*

$$\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2. \tag{3.1}$$

In particular, $\mathbf{Ker}(\boldsymbol{\Psi}_\mu) = \mathfrak{m}_2$.

Before proceeding to the proof of Proposition 5, we recall the notion of pseudo-abelian subspaces of $\mathfrak{m}$ defined in [5]. Let V be a subspace of $\mathfrak{m}$. Then, V is called *pseudo-abelian* if it satisfies $[V, V] \subset \mathfrak{k}_0$ (or equivalently, $\big[[V, V], \mathfrak{a}\big] = 0$). By (2.5) we can easily verify that $\mathfrak{m}_2$ is pseudo-abelian. On the contrary, we have

Lemma 6 *Let $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$. Then, any pseudo-abelian subspace V of $\mathfrak{m}$ with $\dim V > 2$ must be contained in $\mathfrak{m}_2$.*

Proof. Let V be a pseudo-abelian subspace of $\mathfrak{m}$ satisfying $V \not\subset \mathfrak{m}_2$. Then by Lemma 5.4 of [5], we obtain $\dim V \leq 1 + n(\mu)$, where $n(\mu)$ is the local pseudo-nullity associated with μ. (For the definition of the local pseudo-nullity, see § 3 in [5].) Moreover, we have $n(\mu) = 1$ if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$ (see Table 2 of [5]). Therefore, we get $\dim V \leq 2$, proving the lemma. □

We now start the proof of Proposition 5.

Proof of Proposition 5. We first note that $\dim \mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. In fact, since $\dim \boldsymbol{N} = q(G/K) - \dim G/K = \dim G/K - \dim \mathfrak{m}_2$, we have $\dim \mathbf{Ker}(\Psi_Y) \geq \dim G/K - \dim \boldsymbol{N} = \dim \mathfrak{m}_2 > 2$ (see Proposition 4 (3)).

In § 1 of [2], by considering the Gauss equation (2.2), we have proved

$$R(\mathbf{Ker}(\Psi_Y),\, \mathbf{Ker}(\Psi_Y))Y = 0. \tag{3.2}$$

Because of (2.1), the equality (3.2) means

$$\big[[\mathbf{Ker}(\Psi_Y),\, \mathbf{Ker}(\Psi_Y)],\, Y\big] = 0. \tag{3.3}$$

Applying $\mathrm{Ad}(k^{-1})$ to the both sides of (3.3), we get

$$\big[[\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y),\, \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)],\, \mu\big] = 0.$$

(Note that $\mathrm{Ad}(k^{-1})Y$ can be written as $\mathrm{Ad}(k^{-1})Y = c\mu$ for some $c \in \boldsymbol{R}$ ($c \neq 0$).) Since $\mathfrak{a} = \boldsymbol{R}\mu$, we know that $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$ with $\dim \mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) \geq \dim \mathfrak{m}_2 > 2$. Therefore, we have $\mathrm{Ad}(k^{-1})\mathbf{Ker}(\Psi_Y) = \mathfrak{m}_2$ (see Lemma 6). This proves (3.1). □

Utilizing Proposition 5, we will characterize solutions Ψ of the Gauss equation (2.2). For this purpose we need more informations about the action of the isotropy group $\mathrm{Ad}(K)$.

As is well known, any element of $\mathfrak{m}$ is conjugate to an element of $\boldsymbol{R}\mu(= \mathfrak{a})$ under the action of $\mathrm{Ad}(K)$. More strongly, under our assumption $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, we have

Proposition 7 (1) *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $Y_0 \neq 0$. Then there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Consequently, $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a} + \mathfrak{m}_2$, i.e.,*

$$\mathrm{Ad}(k_0)\mathfrak{m}_2 = \big\{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0',\, Y_0) = 0\big\}. \tag{3.4}$$

(2) *Let $Y_1 \in \mathfrak{m}_1$ satisfy $Y_1 \neq 0$. Then there is an element $k_1 \in K$ satisfying $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$. Consequently, $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$, i.e.,*

$$\mathrm{Ad}(k_1)\mathfrak{m}_2 = \big\{Y_1' \in \mathfrak{m}_1 \mid (Y_1',\, Y_1) = 0\big\}. \tag{3.5}$$

Under the same setting in Proposition 7 (2), we have

Proposition 8 *Let* $Y_1 \in \mathfrak{m}_1$ *satisfy* $Y_1 \neq 0$. *Then there is an element* $k_1' \in K$ *satisfying*

$$\mathrm{Ad}(k_1')\mu = \frac{1}{\sqrt{2}}\left\{\mu + \frac{|\mu|}{|Y_1|}Y_1\right\}, \tag{3.6}$$

$$\mathrm{Ad}(k_1')Y_2 = \frac{1}{\sqrt{2}}\left\{Y_2 + \frac{1}{|\mu|^3|Y_1|}\left[[\mu,\, Y_1],\, Y_2\right]\right\}, \quad \forall Y_2 \in \mathfrak{m}_2. \tag{3.7}$$

Here $|v|$ *denotes the norm of* $v \in \mathfrak{m}$, *i.e.*, $|v| = (v,\, v)^{1/2}$.

The proofs of Proposition 7 and Proposition 8 will be given in §4.

Utilizing Propositions 5, 7 and 8 we first show the following:

Proposition 9 *Assume that* $\dim \boldsymbol{N} = q(G/K) - \dim G/K$ *and that there is a solution* $\boldsymbol{\Psi}$ *of the Gauss equation* (2.2). *Then there exist two vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *satisfying*

$$\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \qquad \forall Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2, \tag{3.8}$$

$$\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \qquad \forall Y_1, Y_1' \in \mathfrak{m}_1, \tag{3.9}$$

$$\boldsymbol{\Psi}(Y_1, Y_2) = -\frac{1}{(\mu,\, \mu)^2}\boldsymbol{\Psi}\left(\mu,\, \left[[\mu,\, Y_1],\, Y_2\right]\right), \quad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2. \tag{3.10}$$

Proof. First we prove

$$\boldsymbol{\Psi}(Y_0,\, Y_0') = 0, \quad \forall Y_0,\, Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \text{ satisfying } (Y_0,\, Y_0') = 0. \tag{3.11}$$

We may assume that $Y_0,\, Y_0' \neq 0$. Then, by Proposition 7 (1), we know that there is an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. Since $(Y_0,\, Y_0') = 0$, we have $Y_0' \in \mathrm{Ad}(k_0)\mathfrak{m}_2$. Then, by Proposition 5, we know $Y_0' \in \mathbf{Ker}(\boldsymbol{\Psi}_{Y_0})$. Hence $\boldsymbol{\Psi}(Y_0,\, Y_0') = 0$, completing the proof of (3.11).

Now (3.8) can be proved by (3.11) as follows: Let Y_0 and Y_0' be two elements of $\mathfrak{a} + \mathfrak{m}_2$ of the same length. Since $(Y_0 + Y_0',\, Y_0 - Y_0') = 0$, we obtain $\boldsymbol{\Psi}(Y_0 + Y_0',\, Y_0 - Y_0') = 0$. Hence, we have $\boldsymbol{\Psi}(Y_0,\, Y_0) = \boldsymbol{\Psi}(Y_0',\, Y_0')$. This implies that $\boldsymbol{\Psi}(Y_0,\, Y_0)/(Y_0,\, Y_0)$ $(Y_0 \in \mathfrak{a} + \mathfrak{m}_2,\, Y_0 \neq 0)$ takes a constant value $\mathbf{A}$ $(\in \boldsymbol{N})$. Therefore, we have $\boldsymbol{\Psi}(Y_0,\, Y_0) = (Y_0,\, Y_0)\mathbf{A}$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Now (3.8) follows immediately from this equality.

In a similar manner, by applying Proposition 7 (2) we can prove (3.9).

Finally, we prove (3.10). Without loss of generality, we may assume that $Y_1 \neq 0$. Apply Proposition 8 to this Y_1 ($\in \mathfrak{m}_1$). Then there is an element $k_1' \in K$ satisfying (3.6) and (3.7). By (3.1) we have

$$0 = \boldsymbol{\Psi}\big(\mathrm{Ad}(k_1')\mu,\ \mathrm{Ad}(k_1')Y_2\big) = \frac{1}{2}\boldsymbol{\Psi}\Big(\mu + \frac{|\mu|}{|Y_1|}Y_1,\ Y_2 + \frac{1}{|\mu|^3|Y_1|}[[\mu,\ Y_1],\ Y_2]\Big).$$

Note that $\big[[\mu,\ Y_1],\ Y_2\big] \in \mathfrak{m}_1$ (see Proposition 4 (1)) and $\big[[\mu,\ Y_2],\ Y_1\big] = 2\big[[\mu,\ Y_1],\ Y_2\big]$ (see Lemma 5.3 of [5]). Then, we have

$$\big(Y_1,\ [[\mu,\ Y_1],\ Y_2]\big) = \frac{1}{2}\big(Y_1,\ [[\mu,\ Y_2],\ Y_1]\big) = -\frac{1}{2}\big([Y_1,\ Y_1],\ [\mu,\ Y_2]\big) = 0.$$

Hence by (3.9) we have $\boldsymbol{\Psi}\big(Y_1,\ [[\mu,\ Y_1],\ Y_2]\big) = 0$. This together with $\boldsymbol{\Psi}(\mu,\ Y_2) = 0$ proves (3.10). □

To calculate the left hand side of the Gauss equation (2.2), we prepare one more proposition, which will be proved in the last section of this paper.

Proposition 10 (1) *Let $Y_0,\ Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

$$\big[Y_0,\ [Y_0,\ Y_1]\big] = -(\mu,\ \mu)(Y_0,\ Y_0)Y_1, \tag{3.12}$$

$$\big[Y_0,\ [Y_0,\ Y_0']\big] = \begin{cases} -4(\mu,\ \mu)(Y_0,\ Y_0)Y_0', & \text{if } (Y_0,\ Y_0') = 0, \\ 0, & \text{if } Y_0' \in \boldsymbol{R}Y_0. \end{cases} \tag{3.13}$$

(2) *Let $Y_1,\ Y_1' \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then:*

$$\big[Y_1,\ [Y_1,\ Y_1']\big] = \begin{cases} -4(\mu,\ \mu)(Y_1,\ Y_1)Y_1', & \text{if } (Y_1,\ Y_1') = 0, \\ 0, & \text{if } Y_1' \in \boldsymbol{R}Y_1, \end{cases} \tag{3.14}$$

$$\big[Y_1,\ [Y_1,\ Y_0]\big] = -(\mu,\ \mu)(Y_1,\ Y_1)Y_0. \tag{3.15}$$

With these preparations, we start the proof of Theorem 3. We first show a series of lemmas by using the Gauss equation (2.2) and Proposition 9.

Lemma 11 $\langle \mathbf{A},\ \mathbf{A}\rangle = \langle \mathbf{B},\ \mathbf{B}\rangle = 4(\mu,\ \mu)$.

Proof. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2,\ Y_2) = 1$. Put $X = Z = \mu$ and $Y = W = Y_2$ into the Gauss equation (2.2). Then, since $\boldsymbol{\Psi}(\mu,\ Y_2) = 0$, we have

$$\big(\big[[\mu,\ Y_2],\ \mu\big],\ Y_2\big) = \big\langle \boldsymbol{\Psi}(\mu,\ \mu),\ \boldsymbol{\Psi}(Y_2,\ Y_2)\big\rangle.$$

Since $\boldsymbol{\Psi}(\mu,\ \mu)/(\mu,\ \mu) = \boldsymbol{\Psi}(Y_2,\ Y_2) = \mathbf{A}$ and $\big(\big[[\mu,\ Y_2],\ \mu\big],\ Y_2\big) = 4(\mu,\ \mu)^2$, we have $\langle \mathbf{A}, \mathbf{A} \rangle = 4(\mu, \mu)$.

Next, we prove $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. Take elements Y_1, Y_1' of $\mathfrak{m}_1$ satisfying $(Y_1, Y_1) = (Y_1', Y_1') = 1$ and $(Y_1, Y_1') = 0$. Put $X = Z = Y_1$ and $Y = W = Y_1'$ into (2.2). Then, since $\mathbf{\Psi}(Y_1, Y_1') = 0$, we have

$$\big([[Y_1, Y_1'], Y_1], Y_1'\big) = \big\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{\Psi}(Y_1', Y_1')\big\rangle.$$

Since $\mathbf{\Psi}(Y_1, Y_1) = \mathbf{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1, Y_1'], Y_1\big] = 4(\mu, \mu)Y_1'$ (see (3.14)), we have $\langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$. □

Lemma 12 $\big\langle \mathbf{A}, \mathbf{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = \big\langle \mathbf{B}, \mathbf{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0.$

Proof. Let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_2 \in \mathfrak{m}_2$ satisfying $(Y_2, Y_2) = 1$. Put $X = Z = Y_2$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\mathbf{\Psi}(\mu, Y_2) = 0$, we have

$$\big([[Y_2, \mu], Y_2], Y_1\big) = \big\langle \mathbf{\Psi}(Y_2, Y_2), \mathbf{\Psi}(\mu, Y_1)\big\rangle.$$

Since $\mathbf{\Psi}(Y_2, Y_2) = \mathbf{A}$ and $\big[[Y_2, \mu], Y_2\big] = 4(\mu, \mu)\mu$ (see (3.13)), we have $\big\langle \mathbf{A}, \mathbf{\Psi}(\mu, Y_1)\big\rangle = 4(\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{A}, \mathbf{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0$.

Next, let Y_1 be an arbitrary element of $\mathfrak{m}_1$. Take an element $Y_1' \in \mathfrak{m}_1$ satisfying $(Y_1', Y_1) = 0$ and $(Y_1', Y_1') = 1$. Put $X = Z = Y_1'$, $Y = \mu$ and $W = Y_1$ into (2.2). Then, since $\mathbf{\Psi}(Y_1, Y_1') = 0$, we have

$$\big([[Y_1', \mu], Y_1'], Y_1\big) = \big\langle \mathbf{\Psi}(Y_1', Y_1'), \mathbf{\Psi}(\mu, Y_1)\big\rangle.$$

& Since $\mathbf{\Psi}(Y_1', Y_1') = \mathbf{B}$ and $\big[[Y_1', \mu], Y_1'\big] = (\mu, \mu)\mu$ (see (3.15)), we have $\big\langle \mathbf{B}, \mathbf{\Psi}(\mu, Y_1)\big\rangle = (\mu, \mu)(\mu, Y_1) = 0$. Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\big\langle \mathbf{B}, \mathbf{\Psi}_\mu(\mathfrak{m}_1)\big\rangle = 0$. □

Viewing Proposition 4 (3), we have $\dim \boldsymbol{N} = \dim \mathfrak{m}_1 + 1$. Since $\mathbf{Ker}(\mathbf{\Psi}_\mu) \cap \mathfrak{m}_1 = \mathfrak{m}_2 \cap \mathfrak{m}_1 = 0$, we have $\dim \mathbf{\Psi}_\mu(\mathfrak{m}_1) = \dim \mathfrak{m}_1 = \dim \boldsymbol{N} - 1$. Consequently, by Lemma 12 and Lemma 11, we easily have $\mathbf{B} = \pm\mathbf{A}$. More strongly, we can show

Lemma 13 $\mathbf{A} = \mathbf{B}$.

Proof. By the above discussion, it suffices to prove $\langle \mathbf{A}, \mathbf{B} \rangle > 0$. Let $Y_1 \in \mathfrak{m}_1$ satisfy $(Y_1, Y_1) = 1$. In (2.2), we put $X = Z = \mu$ and $Y = W = Y_1$. Then, we have

$$\big([[\mu, Y_1], \mu], Y_1\big) = \big\langle \mathbf{\Psi}(\mu, \mu), \mathbf{\Psi}(Y_1, Y_1)\big\rangle - \big\langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(Y_1, \mu)\big\rangle.$$

Since $\Psi(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\Psi(Y_1, Y_1) = \mathbf{B}$ and $\big[[\mu, Y_1], \mu\big] = (\mu, \mu)^2 Y_1$, we have

$$(\mu, \mu)\langle \mathbf{A}, \mathbf{B}\rangle = (\mu, \mu)^2(Y_1, Y_1) + \big\langle \Psi(\mu, Y_1), \Psi(\mu, Y_1)\big\rangle \geq (\mu, \mu)^2.$$

This proves $\langle \mathbf{A}, \mathbf{B}\rangle > 0$. □

Utilizing Lemma 13, we have

Lemma 14 *Let $Y_1, Y_1' \in \mathfrak{m}_1$. Then*

$$\big\langle \Psi(\mu, Y_1), \Psi(\mu, Y_1')\big\rangle = 3(\mu, \mu)^2(Y_1, Y_1'). \tag{3.16}$$

Proof. Put $X = Z = \mu$, $Y = Y_1$ and $W = Y_1'$ into (2.2). Then we have

$$\big(\big[[\mu, Y_1], \mu\big], Y_1'\big) = \big\langle \Psi(\mu, \mu), \Psi(Y_1, Y_1')\big\rangle - \big\langle \Psi(\mu, Y_1'), \Psi(Y_1, \mu)\big\rangle.$$

Since $\Psi(\mu, \mu) = (\mu, \mu)\mathbf{A}$, $\Psi(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}$ and $\mathbf{A} = \mathbf{B}$, the first term of the right hand side becomes $\big\langle \Psi(\mu, \mu), \Psi(Y_1, Y_1')\big\rangle = 4(\mu, \mu)^2(Y_1, Y_1')$ (see Lemma 11). Therefore, by $\big[[\mu, Y_1], \mu\big] = (\mu, \mu)^2 Y_1$, we have

$$\begin{aligned}\big\langle \Psi(\mu, Y_1), \Psi(\mu, Y_1')\big\rangle &= 4(\mu, \mu)^2(Y_1, Y_1') - (\mu, \mu)^2(Y_1, Y_1')\\ &= 3(\mu, \mu)^2(Y_1, Y_1').\end{aligned}$$

□

We are now in a position to complete the proof of Theorem 3. Let $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$ and $Y_2 \in \mathfrak{m}_2$ $(Y_2 \neq 0)$. Note that $[Y_1, Y_2] \in \mathfrak{k}_1$ (see Proposition 4 (1)). We also note that $[Y_1, Y_2] \neq 0$. In fact, if $[Y_1, Y_2] = 0$, then the 2-dimensional subspace generated by Y_1 and Y_2 forms an abelian subspace of $\mathfrak{m}$, which contradicts $\operatorname{rank}(G/K) = 1$. Now, set $Y_1' = \big[[Y_1, Y_2], \mu\big]$. Then it is clear that $Y_1' \in \mathfrak{m}_1$ (see Proposition 4 (1)). Moreover, we have $Y_1' \neq 0$, because $[\mu, Y_1'] = (\mu, \mu)^2[Y_1, Y_2] \neq 0$.

Now, put $X = Y_1$, $Y = Y_2$, $Z = \mu$ and $W = Y_1'$ into (2.2). Since $\Psi(Y_2, \mu) = 0$, we have

$$\big(\big[[Y_1, Y_2], \mu\big], Y_1'\big) = \big\langle \Psi(Y_1, \mu), \Psi(Y_2, Y_1')\big\rangle. \tag{3.17}$$

By (3.10) and (3.16), the right hand side of (3.17) becomes

$$\begin{aligned}\big\langle \Psi(Y_1, \mu), \Psi(Y_2, Y_1')\big\rangle &= -\big\langle \Psi(\mu, Y_1), \Psi(\mu, \big[[\mu, Y_1'], Y_2\big])\big\rangle/(\mu, \mu)^2\\ &= -3\big(Y_1, \big[[\mu, Y_1'], Y_2\big]\big)\\ &= 3\big([Y_1, Y_2], [\mu, Y_1']\big)\end{aligned}$$

$$= 3\big([[Y_1, Y_2], \mu], Y_1'\big).$$

Putting this equality into (3.17), we have $\big([[Y_1, Y_2], \mu], Y_1'\big) = 0$, which contradicts our assumption $\big([[Y_1, Y_2], \mu], Y_1'\big) = (Y_1', Y_1') \neq 0$.

As we have shown above, starting from the assumption that the Gauss equation (2.2) admits a solution $\boldsymbol{\Psi}$, we finally arrive at a contradiction. Accordingly, we can conclude that if $G/K = P^2(\boldsymbol{H})$ or $P^2(\boldsymbol{Cay})$, then the Gauss equation (2.2) does not admit any solution in case $\dim \boldsymbol{N} = q(G/K) - \dim G/K$. This completes the proof of Theorem 3. □

4. The action of the isotropy group $\mathrm{Ad}(K)$

In this section we prove Propositions 7, 8 and 10, which are needed in the proof of Theorem 3.

Lemma 15 *Let $X_i \in \mathfrak{k}_i$ $(i = 1, 2)$. Then*

$$\big[X_i, [X_i, \mu]\big] = -i^2(\mu, \mu)(X_i, X_i)\mu. \tag{4.1}$$

&

Proof. By (2.5) we have $\big[X_i, [X_i, \mu]\big] \in \mathfrak{a} + \mathfrak{m}_{2i}$. By the Jacobi identity we have

$$\Big[\mu, \big[X_i, [X_i, \mu]\big]\Big] = \big[[\mu, X_i], [X_i, \mu]\big] + \Big[X_i, \big[[\mu, X_i], \mu\big]\Big] = 0,$$

because $\big[[\mu, X_i], \mu\big] \in \boldsymbol{R}X_i$. Therefore, we have $\big[X_i, [X_i, \mu]\big] \in \mathfrak{a}$. Since $\mathfrak{a} = \boldsymbol{R}\mu$, there is a scalar $c \in \boldsymbol{R}$ satisfying $\big[X_i, [X_i, \mu]\big] = c\mu$. Then we have $c = -i^2(\mu, \mu)(X_i, X_i)$, because

$$\begin{aligned} c(\mu, \mu) &= \big(\big[X_i, [X_i, \mu]\big], \mu\big) = \big(X_i, \big[[X_i, \mu], \mu\big]\big) \\ &= -(i\mu, \mu)^2(X_i, X_i). \end{aligned}$$

□

By the above lemma, we obtain

Lemma 16 *Let $X_i \in \mathfrak{k}_i$ $(i = 1, 2)$ satisfy $X_i \neq 0$. Then*

$$\begin{aligned} \mathrm{Ad}\big(\exp(tX_i)\big)\mu = {} & \cos(i|\mu||X_i|t)\mu \\ & + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i, \mu], \quad \forall t \in \boldsymbol{R}. \end{aligned} \tag{4.2}$$

Proof. Let n be a non-negative integer. By induction of n, we can easily show

$$(\operatorname{ad} X_i)^{2n}\mu = (-1)^n(i|\mu||X_i|)^{2n}\mu,$$
$$(\operatorname{ad} X_i)^{2n+1}\mu = (-1)^n(i|\mu||X_i|)^{2n}[X_i, \mu].$$

Consequently, for all $t \in \boldsymbol{R}$ we have

$$\begin{aligned}
\operatorname{Ad}(\exp(tX_i))\mu &= \sum_{n=0}^{\infty}\left\{\frac{t^{2n}}{(2n)!}(\operatorname{ad} X_i)^{2n}\mu + \frac{t^{2n+1}}{(2n+1)!}(\operatorname{ad} X_i)^{2n+1}\mu\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(i|\mu||X_i|t)^{2n}\mu \\
&\qquad + \frac{1}{i|\mu||X_i|}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(i|\mu||X_i|t)^{2n+1}[X_i, \mu] \\
&= \cos(i|\mu||X_i|t)\mu + \frac{\sin(i|\mu||X_i|t)}{i|\mu||X_i|}[X_i, \mu].
\end{aligned}$$

□

With these preparations, we proceed to the proof of Proposition 7. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. If $Y_0 \in \mathfrak{a}$, then we have only to set $k_0 = e$, where e is the identity element of K.

Now we assume that $Y_0 \notin \mathfrak{a}$ and write $Y_0 = c\mu + Y_2$ ($c \in \boldsymbol{R}$, $Y_2 \in \mathfrak{m}_2$, $Y_2 \neq 0$). Set $X_2 = [Y_0, \mu]$. Then we easily have $X_2 = [Y_2, \mu] \in \mathfrak{k}_2$ and $[X_2, \mu] = -4(\mu, \mu)^2 Y_2$. Moreover, we have $|X_2| = 2|\mu|^2|Y_2|$, because

$$(X_2, X_2) = ([Y_2, \mu], [Y_2, \mu]) = -([[Y_2, \mu], \mu], Y_2) = 4(\mu, \mu)^2(Y_2, Y_2).$$

Putting this X_2 into Lemma 16, we have

$$\operatorname{Ad}(\exp(tX_2))\mu = \cos(4|\mu|^3|Y_2|t)\mu - \frac{|\mu|}{|Y_2|}\sin(4|\mu|^3|Y_2|t)Y_2, \quad \forall t \in \boldsymbol{R}.$$

Take $t_0 \in \boldsymbol{R}$ satisfying $\cos(4|\mu|^3|Y_2|t_0) = c(|\mu|/|Y_0|)$ and $\sin(4|\mu|^3|Y_2|t_0) = -|Y_2|/|Y_0|$. Let us set $k_0 = \exp(t_0X_2)$. Then we have $k_0 \in K$ and

$$\operatorname{Ad}(k_0)\mu = \operatorname{Ad}(\exp(t_0X_2))\mu = \frac{|\mu|}{|Y_0|}(c\mu + Y_2) = \frac{|\mu|}{|Y_0|}Y_0.$$

Thus we get $\operatorname{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$. By (2.5) we immediately have $[X_2, \mathfrak{a} + \mathfrak{m}_2] \subset$

$\mathfrak{a}+\mathfrak{m}_2$. Hence, we have $\mathrm{Ad}(k_0)(\mathfrak{a}+\mathfrak{m}_2)=\mathfrak{a}+\mathfrak{m}_2$. Since $\mathrm{Ad}(k_0)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_0)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_0$ in $\mathfrak{a}+\mathfrak{m}_2$. This finishes the proof of Proposition 7 (1).

To prove Proposition 7 (2), we first show

Lemma 17 *Let* $X_1\in\mathfrak{k}_1$. *Then*

$$[X_1,[X_1,Y_2]]=-(\mu,\mu)(X_1,X_1)Y_2,\quad \forall Y_2\in\mathfrak{m}_2.$$

Proof. By (4.1), we have

$$[X_1,[X_1,\mu]]=-(\mu,\mu)(X_1,X_1)\mu. \tag{4.3}$$

Let Y_2 be a non-zero element of $\mathfrak{m}_2$. Then, as in the proof of Proposition 7 (1), we know that there is a scalar $t_0\in\boldsymbol{R}$ such that the element $k_0=\exp(t_0X_2)\in K$ satisfies $\mathrm{Ad}(k_0)\mu\in\boldsymbol{R}Y_2$, where we set $X_2=[Y_2,\mu]\in\mathfrak{k}_2$. Then, we have $\mathrm{Ad}(k_0)\mathfrak{k}_1=\mathfrak{k}_1$, because $[X_2,\mathfrak{k}_1]\subset\mathfrak{k}_1$ (see Proposition 4 (1)).

Now, applying $\mathrm{Ad}(k_0)$ to the both sides of (4.3), we have

$$\begin{aligned}[\mathrm{Ad}(k_0)X_1,[\mathrm{Ad}(k_0)X_1,Y_2]]&=-(\mu,\mu)(X_1,X_1)Y_2\\&=-(\mu,\mu)(\mathrm{Ad}(k_0)X_1,\mathrm{Ad}(k_0)X_1)Y_2.\end{aligned}$$

Writing X_1 instead of $\mathrm{Ad}(k_0)X_1\in\mathfrak{k}_1$, we get the lemma. □

Now we return to the proof of Proposition 7 (2). Set $X_1=[Y_1,\mu]$. In the same way as in the proof of (1), we can easily prove $X_1\in\mathfrak{k}_1$, $[X_1,\mu]=-(\mu,\mu)^2Y_1$ and $|X_1|=|\mu|^2|Y_1|$. Applying Lemma 16 to this X_1, we have

$$\begin{aligned}\mathrm{Ad}(\exp(tX_1))\mu&=\cos(|\mu|^3|Y_1|t)\mu\\&\quad-\frac{|\mu|}{|Y_1|}\sin(|\mu|^3|Y_1|t)Y_1,\quad\forall t\in\boldsymbol{R}.\end{aligned}\tag{4.4}$$

Let $Y_2\in\mathfrak{m}_2$. By Lemma 17, we have

$$\begin{aligned}(\mathrm{ad}\,X_1)^{2n}Y_2&=(-1)^n(|\mu||X_1|)^{2n}Y_2,\\(\mathrm{ad}\,X_1)^{2n+1}Y_2&=(-1)^n(|\mu||X_1|)^{2n}[X_1,Y_2].\end{aligned}$$

From these equalities, it follows

$$\begin{aligned}\mathrm{Ad}(\exp(tX_1))Y_2&=\cos(|\mu|^3|Y_1|t)Y_2\\&\quad+\frac{\sin(|\mu|^3|Y_1|t)}{|\mu|^3|Y_1|}[[Y_1,\mu],Y_2],\quad\forall t\in\boldsymbol{R}.\end{aligned}\tag{4.5}$$

Let us take $t_1 \in \boldsymbol{R}$ satisfying $|\mu|^3|Y_1|t_1 = -\pi/2$ and set $k_1 = \exp(t_1 X_1)$. Then we can easily show that $k_1 \in K$, $\mathrm{Ad}(k_1)\mu = (|\mu|/|Y_1|)Y_1 \in \mathfrak{m}_1$ and

$$\mathrm{Ad}(k_1)Y_2 = -\frac{1}{|\mu|^3|Y_1|}\big[[Y_1, \mu], Y_2\big]. \tag{4.6}$$

Hence, we have $\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1$ and $\mathrm{Ad}(k_1)\mathfrak{m}_2 \subset \big[[Y_1, \mu], \mathfrak{m}_2\big]$. Since $\big[[Y_1, \mu], \mathfrak{m}_2\big] \subset \mathfrak{m}_1$ (see Proposition 4 (1)), we have $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) \subset \mathfrak{m}_1$. Therefore, we have $\mathrm{Ad}(k_1)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{m}_1$, because $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1$ (see Proposition 4 (3)). Since $\mathrm{Ad}(k_1)$ is an orthogonal transformation of $\mathfrak{m}$, we know that $\mathrm{Ad}(k_1)\mathfrak{m}_2$ coincides with the orthogonal complement of $\boldsymbol{R}Y_1$ in $\mathfrak{m}_1$. This completes the proof of Proposition 7 (2). □

Next we prove Proposition 8. Under the same situation as in the proof of Proposition 7 (2), let us set $k_1' = \exp(t_1 X_1/2)$. Then by the equalities (4.4) and (4.5) we easily obtain (3.6) and (3.7). □

Finally, we prove Proposition 10. First we show Proposition 10 (1). If $Y_0 \in \mathfrak{a}$, then there is nothing to prove. Hence we may assume that $Y_0 \notin \mathfrak{a}$. Applying Proposition 7 (1), we have an element $k_0 \in K$ satisfying $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{a} + \mathfrak{m}_2) = \mathfrak{a} + \mathfrak{m}_2$. Then, it is easily seen that $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$. If we write $\mathrm{Ad}(k_0)\mu = cY_0$ $(c \in \boldsymbol{R})$, then we have $c^2 = (\mu, \mu)/(Y_0, Y_0)$. Let Y_i be an element of $\mathfrak{m}_i$ $(i = 1, 2)$. Apply $\mathrm{Ad}(k_0)$ to the both sides of the equality $\big[\mu, [\mu, Y_i]\big] = -i^2(\mu, \mu)^2 Y_i$ $(i = 1, 2)$. Then, since $c^2 = (\mu, \mu)/(Y_0, Y_0)$, we have

$$\big[Y_0, [Y_0, \mathrm{Ad}(k_0)Y_i]\big] = -i^2(\mu, \mu)(Y_0, Y_0)\,\mathrm{Ad}(k_0)Y_i, \quad i = 1, 2.$$

Now, (3.12) and (3.13) follow immediately from the above equality. (Note the equality (3.4) and the fact $\mathrm{Ad}(k_0)\mathfrak{m}_1 = \mathfrak{m}_1$.)

By applying Proposition 7 (2), Proposition 10 (2) can be also shown in a similar manner. Details are left to the readers. □

Thus, we have completed the proofs of Propositions 7, 8 and 10.

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Rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(Cay)$

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Abstract. In [7], we have proved that $P^2(\boldsymbol{Cay})$ cannot be isometrically immersed into $\boldsymbol{R}^{25}$ even locally. In this paper, we investigate isometric immersions of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$ and prove that the canonical isometric imbedding f_0 of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$, which is defined in Kobayashi [17], is rigid in the following strongest sense: Any isometric immersion f_1 of a connected open set $U(\subset P^2(\boldsymbol{Cay}))$ into $\boldsymbol{R}^{26}$ coincides with f_0 up to a euclidean transformation of $\boldsymbol{R}^{26}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{26}$ satisfying $f_1 = af_0$ on U.

Key words: curvature invariant, isometric immersion, Cayley projective plane, rigidity.

1. Introduction

In the previous paper [7], we investigated the problem of (local) isometric immersions of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$. In particular, we proved the following non-existence theorem of (local) isometric immersions:

Theorem 1 *Any open set of the Cayley projective plane $P^2(\boldsymbol{Cay})$ cannot be isometrically immersed into $\boldsymbol{R}^{25}$.*

As is well-known, there is an isometric immersion f_0 of $P^2(\boldsymbol{Cay})$ into the euclidean space $\boldsymbol{R}^{26}$, which is called the canonical isometric imbedding of $P^2(\boldsymbol{Cay})$ (Kobayashi [17]). This fact, together with Theorem 1, implies that $\boldsymbol{R}^{26}$ is the least dimensional euclidean space into which $P^2(\boldsymbol{Cay})$ can be (locally) isometrically immersed.

In this paper, we consider (local) isometric immersions of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$ and discuss the rigidity of the canonical isometric imbedding f_0. Concerning the rigidity of f_0 Kaneda [15] has shown that the canonical isometric imbedding f_0 is of finite type, i.e., the space of local infinitesimal isometric deformations of f_0 is of finite dimension. However, it seems to the authors that any further result concerning the rigidity of f_0 has not been

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obtained.

In the present paper, we will show the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$ in the following strongest form:

Theorem 2 *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $P^2(\boldsymbol{Cay})$ into the euclidean space $\boldsymbol{R}^{26}$. Then, for any isometric immersion $\boldsymbol{f}_1$ defined on a connected open set U of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$, there exists a euclidean transformation a of $\boldsymbol{R}^{26}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

To prove Theorem 2, we first establish a rigidity theorem for an isometric immersion of a Riemannian manifold. Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the m-dimensional euclidean space $\boldsymbol{R}^m$. We will prove that if the Gauss equation in codimension r $(= m - n)$ admits essentially one solution everywhere on M, then $\boldsymbol{f}_0$ is rigid, i.e., for any isometric immersion $\boldsymbol{f}_1$ of M into $\boldsymbol{R}^m$ there exists a euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ (see Theorem 5). This theorem may be established by various methods; for example, by combining the results of Nomizu [19] and Szczarba [21], [22] (cf. Agaoka [1]) or by solving a differential system of Pfaff (cf. Bishop–Crittenden [10], Ch. X). In this paper, we will give a simple proof based on a congruence theorem of differentiable mappings, which is easy to understand and gives a clear view on the geometric meaning (see Theorem 6).

Next, we will show that for the Cayley projective plane $P^2(\boldsymbol{Cay})$ the Gauss equation in codimension 10 $(= 26 - \dim P^2(\boldsymbol{Cay}))$ admits essentially one solution (see Theorem 10). To show this, we utilize the results obtained in [6] and [7]. Among all, the result concerning pseudo-abelian subspaces (Proposition 8) plays an important role in our proof.

Then, Theorem 2 is a direct consequence of Theorem 5 and Theorem 10.

Throughout this paper we assume the differentiability of class C^∞. Notations for Lie algebras are the same as those used in [6] and [7].

2. The Gauss equation

Let M be a Riemannian manifold and $T(M)$ the tangent bundle of M. We denote by g the Riemannian metric of M and by R the Riemannian curvature tensor of type (1, 3) with respect to g.

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\boldsymbol{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $p \in M$ and let $S^2T_p^*(M) \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $T_p(M)$. We call the following equation on $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$ the *Gauss equation* at $p \in M$:

$$-g_p(R_p(x, y)z, w) = \langle \boldsymbol{\Psi}(x, z), \boldsymbol{\Psi}(y, w) \rangle - \langle \boldsymbol{\Psi}(x, w), \boldsymbol{\Psi}(y, z) \rangle, \quad (2.1)$$

where x, y, z, $w \in T_p(M)$. We denote by $\mathcal{G}_p(\boldsymbol{N})$ the set of all solutions of (2.1), which is called the *Gaussian variety* associated with $\boldsymbol{N}$ at $p \in M$. As is well-known, $\mathcal{G}_p(\boldsymbol{N}) = \emptyset$ happens in case the dimensionality r $(= \dim \boldsymbol{N})$ is so small, however, $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ if r is sufficiently large (see Cartan [11] or Kaneda–Tanaka [16]).

Let $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ be two euclidean vector spaces and let φ be a linear mapping of $\boldsymbol{N}_1$ to $\boldsymbol{N}_2$. Define a linear map $\widehat{\varphi}$ of $S^2T_p^*(M) \otimes \boldsymbol{N}_1$ to $S^2T_p^*(M) \otimes \boldsymbol{N}_2$ by

$$(\widehat{\varphi}\boldsymbol{\Psi})(x, y) = \varphi(\boldsymbol{\Psi}(x, y)), \ \ \boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}_1, \ x, y \in T_p(M). \quad (2.2)$$

Then, we can easily verify

Lemma 3 *Let φ be a linear mapping of a euclidean vector space $\boldsymbol{N}_1$ to a euclidean vector space $\boldsymbol{N}_2$. Assume that φ is isometric, i.e., $\langle \varphi(x), \varphi(y) \rangle_2 = \langle x, y \rangle_1$ $(x, y \in \boldsymbol{N}_1)$, where $\langle \ , \ \rangle_i$ $(i = 1, 2)$ denotes the inner product of $\boldsymbol{N}_i$. Then $\widehat{\varphi}\mathcal{G}_p(\boldsymbol{N}_1) \subset \mathcal{G}_p(\boldsymbol{N}_2)$. In particular, if* $\dim \boldsymbol{N}_1 = \dim \boldsymbol{N}_2$, *then $\widehat{\varphi}\mathcal{G}_p(\boldsymbol{N}_1) = \mathcal{G}_p(\boldsymbol{N}_2)$.*

In view of Lemma 3, the solvability of the Gauss equation (2.1) substantially depends on the dimensionality of $\boldsymbol{N}$. To emphasize $\dim \boldsymbol{N}$ we call (2.1) the Gauss equation in codimension r $(= \dim \boldsymbol{N})$.

Let $\boldsymbol{N}$ be a euclidean vector space and let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2T_p^*(M) \otimes \boldsymbol{N}$ by

$$(h\boldsymbol{\Psi})(x, y) = h(\boldsymbol{\Psi}(x, y)),$$

where $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$, x, $y \in T_p(M)$. We say that two elements $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are *equivalent* if there is an element $h \in O(\boldsymbol{N})$ such that $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$. It is easily seen that if $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are equivalent and $\boldsymbol{\Psi} \in \mathcal{G}_p(\boldsymbol{N})$, then $\boldsymbol{\Psi}' \in \mathcal{G}_p(\boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}_p(\boldsymbol{N})$ is *EOS* if $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ and if it consists of essentially one solution, i.e., any solutions of the Gauss equation (2.1) are equivalent to each other under the action of $O(\boldsymbol{N})$.

Proposition 4 *Let M be a Riemannian manifold and let $p \in M$. Let $\boldsymbol{N}$ be an r-dimensional euclidean vector space such that $\mathcal{G}_p(\boldsymbol{N})$ is EOS. Then:*
(1) *Let $\boldsymbol{\Psi}$ be an arbitrary element of $\mathcal{G}_p(\boldsymbol{N})$. Then, the vectors $\boldsymbol{\Psi}(x, y)$ $(x, y \in T_p(M))$ span the whole space $\boldsymbol{N}$.*
(2) *Let $\boldsymbol{N}_1$ be a euclidean vector space. Then:*
 (2a) $\mathcal{G}_p(\boldsymbol{N}_1) = \emptyset$ *if* $\dim \boldsymbol{N}_1 < r$;
 (2b) $\mathcal{G}_p(\boldsymbol{N}_1)$ *is EOS if* $\dim \boldsymbol{N}_1 = r$;
 (2c) $\mathcal{G}_p(\boldsymbol{N}_1)$ *is not EOS if* $\dim \boldsymbol{N}_1 > r$.

Proof. Note that if $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ is equivalent to $\boldsymbol{\Psi}$, then we have $|\boldsymbol{\Psi}'(x, y)| = |\boldsymbol{\Psi}(x, y)|$ for any $x, y \in T_p(M)$, where $|\boldsymbol{n}|$ denotes the norm of $\boldsymbol{n} \in \boldsymbol{N}$ with respect to $\langle\ ,\ \rangle$.

Now, suppose that the vectors $\boldsymbol{\Psi}(x, y)$ $(x, y \in T_p(M))$ do not span the whole space $\boldsymbol{N}$. Then, there is a non-zero vector $\boldsymbol{n} \in \boldsymbol{N}$ satisfying $\langle \boldsymbol{n}, \boldsymbol{\Psi}(x, y)\rangle = 0$ for any $x, y \in T_p(M)$. Define an element $\boldsymbol{\Psi}' \in S^2T_p^*(M)\otimes \boldsymbol{N}$ by

$$\boldsymbol{\Psi}' = \boldsymbol{\Psi} + (\xi^*)^2 \otimes \boldsymbol{n},$$

where ξ^* is a non-zero element of $T_p^*(M)$. Then, it is easy to see that $\boldsymbol{\Psi}' \in \mathcal{G}_p(\boldsymbol{N})$. However, by a simple calculation, we have $|\boldsymbol{\Psi}'(x, x)|^2 = |\boldsymbol{\Psi}(x, x)|^2 + |\boldsymbol{n}|^2\xi^*(x)^2$. Therefore, if we take $x \in T_p(M)$ such that $\xi^*(x) \neq 0$, then we have $|\boldsymbol{\Psi}'(x, x)| \neq |\boldsymbol{\Psi}(x, x)|$. This proves that $\boldsymbol{\Psi}'$ is not equivalent to $\boldsymbol{\Psi}$ and hence $\mathcal{G}_p(\boldsymbol{N})$ is not EOS. Thus, we obtain (1).

Next we prove (2). First assume $\dim \boldsymbol{N}_1 = r$. Let φ be an isometric linear isomorphism of $\boldsymbol{N}$ onto $\boldsymbol{N}_1$. Then we have $O(\boldsymbol{N}_1) = \varphi \cdot O(\boldsymbol{N}) \cdot \varphi^{-1}$. Moreover, by Lemma 3 we have $\widehat{\varphi}\,\mathcal{G}_p(\boldsymbol{N}) = \mathcal{G}_p(\boldsymbol{N}_1)$. Since $\mathcal{G}_p(\boldsymbol{N})$ is EOS, $O(\boldsymbol{N})$ acts transitively on $\mathcal{G}_p(\boldsymbol{N})$. Therefore, it is easily seen that $O(\boldsymbol{N}_1)$ acts transitively on $\mathcal{G}_p(\boldsymbol{N}_1)$. This proves that $\mathcal{G}_p(\boldsymbol{N}_1)$ is EOS.

We next consider the case $\dim \boldsymbol{N}_1 < r$. Suppose that $\mathcal{G}_p(\boldsymbol{N}_1) \neq \emptyset$ and $\boldsymbol{\Psi}_1 \in \mathcal{G}_p(\boldsymbol{N}_1)$. Let φ be an isometric linear mapping of $\boldsymbol{N}_1$ to $\boldsymbol{N}$. Then, we know that $\widehat{\varphi}\,\boldsymbol{\Psi}_1 \in \mathcal{G}_p(\boldsymbol{N})$ and the vectors $(\widehat{\varphi}\,\boldsymbol{\Psi}_1)(x, y)$ $(x, y \in T_p(M))$ are contained in the proper subspace $\varphi(\boldsymbol{N}_1)$ $(\subsetneq \boldsymbol{N})$. This contradicts (1). The case $\dim \boldsymbol{N}_1 > r$ is similarly dealt with. □

We say that a Riemannian manifold M is *formally rigid* in codimension r if there is a euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = r$ such that the Gaussian variety $\mathcal{G}_p(\boldsymbol{N})$ is EOS at each $p \in M$. By virtue of Proposition 4 (2), we know that if M is formally rigid in codimension r, then it is not formally rigid in any other codimension r' ($\neq r$).

Remark 1 It should be noted that there is a Riemannian manifold M that is not formally rigid in any codimension r. For example, assume that M is the space of negative constant curvature of dimension n. Let $\boldsymbol{N}$ be a euclidean vector space of dimension r. Then, by Ôtsuki's lemma we have $\mathcal{G}_p(\boldsymbol{N}) = \emptyset$ if $r < n-1$ (see Ôtsuki [20]). On the other hand, Kaneda [13] proved that if $r = n-1$, then $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ and around a suitable $\boldsymbol{\Psi}_0 \in \mathcal{G}_p(\boldsymbol{N})$, $\mathcal{G}_p(\boldsymbol{N})$ forms a submanifold of $S^2T_p^*(M) \otimes \boldsymbol{N}$ of dimension $n(n-1)$ (see Theorem 3.1 of [13]). Since $n(n-1) > \dim O(\boldsymbol{N})$, $\mathcal{G}_p(\boldsymbol{N})$ cannot be EOS. If $r \geq n$, then by Proposition 4 (2a) we know that $\mathcal{G}_p(\boldsymbol{N})$ is not EOS. Accordingly, the space of negative constant curvature M is not formally rigid in any codimension r.

Remark 2 For each Riemannian submanifold $M \subset \boldsymbol{R}^m$ listed below, $\mathcal{G}_p(\boldsymbol{N})$ is known to be EOS at each $p \in M$, where $\boldsymbol{N}$ is the normal vector space of M at p in $\boldsymbol{R}^m$:

(1) The sphere $S^n \subset \boldsymbol{R}^{n+1}$ ($n \geq 3$);

(2) The symplectic group $Sp(2) \subset \boldsymbol{R}^{16}$ (see Agaoka [1]);

(3) A submanifold $M \subset \boldsymbol{R}^m$ with type number ≥ 3 (see Allendoerfer [9], Kobayashi–Nomizu [18]).

Consequently, these submanifolds are formally rigid in our sense and it has been proved that they are actually rigid in $\boldsymbol{R}^m$ (see [1], [9]).

However, we note that the formal rigidness of M in codimension r does not imply the existence of an isometric immersion of M into $\boldsymbol{R}^{n+r}$ ($n = \dim M$). Indeed, Kaneda [14] gave an example of three dimensional Riemannian manifold M that is formally rigid in codimension 1 but cannot be locally isometrically immersed into $\boldsymbol{R}^4$.

We will prove in the next section that if a connected Riemannian manifold M is formally rigid in codimension r and if there is an isometric immersion $\boldsymbol{f}$ of M into $\boldsymbol{R}^{n+r}$ ($n = \dim M$), then M (precisely, $\boldsymbol{f}(M)$) is actually rigid in $\boldsymbol{R}^{n+r}$ (see Theorem 5).

3. Rigidity theorem

In this section, we will prove the following rigidity theorem:

Theorem 5 *Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the euclidean space $\boldsymbol{R}^m$. Assume*:

(1) *M is connected;*
(2) *M is formally rigid in codimension* $r = m - n$.
Then, any isometric immersion $\boldsymbol{f}_1$ *of* M *into the euclidean space* $\boldsymbol{R}^m$ *coincides with* $\boldsymbol{f}_0$ *up to a euclidean transformation of* $\boldsymbol{R}^m$, *i.e., there exists a euclidean transformation* a *of* $\boldsymbol{R}^m$ *such that* $\boldsymbol{f}_1 = a\boldsymbol{f}_0$.

Before proceeding to the proof of Theorem 5, we make some preparations. Let $M(m, m')$ be the space of real matrices of degree $m \times m'$, where m and m' are non-negative integers. In what follows we identify $M(m, 1)$ with the m-dimensional euclidean space $\boldsymbol{R}^m$ in a natural way. Then, we note that the canonical inner product $\langle\ ,\ \rangle$ of $\boldsymbol{R}^m$ is given by $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = {}^t\boldsymbol{v} \cdot \boldsymbol{w}$ for $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{R}^m$.

Let us define an operation of $M(m, m)$ on $\boldsymbol{R}^m$ by

$$M(m, m) \times \boldsymbol{R}^m \ni (H, \boldsymbol{v}) \longmapsto H \cdot \boldsymbol{v} \in \boldsymbol{R}^m,$$

where $\cdot$ means the usual matrix multiplication.

Let ∇ be the Riemannian connection associated with M. Let $\boldsymbol{f} = {}^t(f^1, \ldots, f^m)$ be a differentiable map of M into the euclidean space $\boldsymbol{R}^m$. By $\overbrace{\nabla \cdots \nabla}^{k} \boldsymbol{f}$ we denote the k-th order covariant derivative of $\boldsymbol{f}$, which is defined as follows:

$$\overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^{k} \boldsymbol{f} = {}^t(\ldots, \overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^{k} f^i, \ldots) \in \boldsymbol{R}^m,$$

where $p \in M$; $x_1, \ldots, x_k \in T_p(M)$. (Precisely, see Tanaka [23], Kaneda–Tanaka [16] or Kaneda [14].) It is known that $\nabla\nabla\boldsymbol{f}$ and $\nabla\nabla\nabla\boldsymbol{f}$ satisfy the following integrability conditions:

$$\nabla_x \nabla_y \boldsymbol{f} = \nabla_y \nabla_x \boldsymbol{f}, \tag{3.1}$$

$$\nabla_z \nabla_x \nabla_y \boldsymbol{f} = \nabla_x \nabla_z \nabla_y \boldsymbol{f} - \nabla_{R(z,x)y} \boldsymbol{f}. \tag{3.2}$$

We say that a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^m$ is *2-generic* if at each $p \in M$, the whole space $\boldsymbol{R}^m$ is spanned by the vectors of the form $\nabla_x \boldsymbol{f}$ ($x \in T_p(M)$), $\nabla_y \nabla_z \boldsymbol{f}$ ($y, z \in T_p(M)$). It is clear that if $\boldsymbol{f}$ is 2-generic, then we have the inequality $m \leq (1/2)n(n+3)$. Note that a 2-generic map $\boldsymbol{f}$ is not necessarily an immersion.

We first show the following congruence theorem:

Theorem 6 *Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_i$ $(i = 0, 1)$ be two differentiable maps of M into the euclidean space $\boldsymbol{R}^m$. Assume:*
(1) *M is connected;*
(2) *$\boldsymbol{f}_0$ is 2-generic;*
(3) *At each $p \in M$ there is an element $H(p) \in O(m)$ satisfying*

$$\nabla_x \boldsymbol{f}_1 = H(p) \cdot (\nabla_x \boldsymbol{f}_0), \qquad \forall x \in T_p(M), \tag{3.3}$$

$$\nabla_y \nabla_z \boldsymbol{f}_1 = H(p) \cdot (\nabla_y \nabla_z \boldsymbol{f}_0), \quad \forall y, z \in T_p(M). \tag{3.4}$$

Then, $\boldsymbol{f}_1$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^m$. More precisely, $H(p)$ is identically equal to a constant value $H_0 \in O(m)$ everywhere on M and $\boldsymbol{f}_1$ can be written as $\boldsymbol{f}_1 = H_0 \boldsymbol{f}_0 + \boldsymbol{c}_0$, where $\boldsymbol{c}_0$ is a constant vector of $\boldsymbol{R}^m$.

Proof. We first note that, since $\boldsymbol{f}_0$ is 2-generic, $H(p)$ satisfying (3.3) and (3.4) is uniquely determined at each $p \in M$ and the map $H: M \ni p \longmapsto H(p) \in O(m)$ is differentiable. Via the canonical inclusion $O(m) \subset M(m, m)$, we can regard H as an $M(m, m)$-valued function on M satisfying

$${}^tHH = I_m, \tag{3.5}$$

where I_m denotes the identity matrix of degree m. Differentiate (3.5) covariantly. Then by Leibnitz' law we get

$$\nabla_x({}^tH)H(p) + {}^tH(p)(\nabla_x H) = 0, \quad \forall x \in T_p(M). \tag{3.6}$$

In this equality, the covariant derivative $\nabla_x H$ means the element of $M(m, m)$ given by $\nabla_x H = \left(\nabla_x h_i^j\right)$, where h_i^j denotes the (i, j)-component of H. By the very definition of $\nabla_x H$ we have $\nabla_x({}^tH) = {}^t(\nabla_x H)$.

Let us define an $M(m, m)$-valued 1-form L by

$$L(x) = {}^tH(p)(\nabla_x H), \quad x \in T_p(M). \tag{3.7}$$

Then, by (3.6) we have

$${}^tL(x) + L(x) = 0, \quad \forall x \in T_p(M), \tag{3.8}$$

implying that the matrix $L(x) \in M(m, m)$ is skew-symmetric.

We now show that the equality $L(x) = 0$ holds for any $x \in T_p(M)$. Since $\boldsymbol{f}_0$ is 2-generic, it suffices to prove

$$L(y) \cdot (\nabla_x \boldsymbol{f}_0) = 0, \quad \forall x, y \in T_p(M), \tag{3.9}$$

$$L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) = 0, \quad \forall x,\, y,\, z \in T_p(M). \tag{3.10}$$

Differentiating (3.3) and (3.4) covariantly, we have

$$\begin{aligned} \nabla_y \nabla_x \boldsymbol{f}_1 = \nabla_y H \cdot (\nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0), \\ \forall x,\, y \in T_p(M), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \nabla_z \nabla_y \nabla_x \boldsymbol{f}_1 = \nabla_z H \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_z \nabla_y \nabla_x \boldsymbol{f}_0), \\ \forall x,\, y,\, z \in T_p(M). \end{aligned} \tag{3.12}$$

Then by (3.4) and (3.11) we have $\nabla_y H \cdot (\nabla_x \boldsymbol{f}_0) = 0$ for each $x,\, y \in T_p(M)$. Consequently, multiplying ${}^t H(p)$ from the left, we have (3.9).

We now prove (3.10). Exchanging z and y in (3.12), we have

$$\begin{aligned} \nabla_y \nabla_z \nabla_x \boldsymbol{f}_1 = \nabla_y H \cdot (\nabla_z \nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_y \nabla_z \nabla_x \boldsymbol{f}_0), \\ \forall x,\, y,\, z \in T_p(M). \end{aligned} \tag{3.13}$$

Subtract (3.13) from (3.12). Then, using the integrability condition (3.2) and the equality (3.3), we have

$$\nabla_z H (\nabla_y \nabla_x \boldsymbol{f}_0) = \nabla_y H (\nabla_z \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.14}$$

Consequently, multiplying ${}^t H(p)$ from the left, we get

$$L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) = L(y) \cdot (\nabla_z \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.15}$$

Since $L(z)$ is a skew-symmetric matrix, we have

$$\big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_u \boldsymbol{f}_0 \big\rangle = -\big\langle \nabla_y \nabla_x \boldsymbol{f}_0,\, L(z) \cdot (\nabla_u \boldsymbol{f}_0) \big\rangle = 0.$$

Therefore, to prove (3.10), we have to show

$$\big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_v \nabla_w \boldsymbol{f}_0 \big\rangle = 0, \quad \forall x,\, y,\, z,\, v,\, w \in T_p(M). \tag{3.16}$$

Define an element $X \in \otimes^5 T_p^*(M)$ by

$$\begin{aligned} X(z,\, y,\, x,\, v,\, w) = \big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_v \nabla_w \boldsymbol{f}_0 \big\rangle, \\ x,\, y,\, z,\, v,\, w \in T_p(M). \end{aligned} \tag{3.17}$$

In the following, we will show $X(z,\, y,\, x,\, v,\, w) = 0$ for $x,\, y,\, z,\, v,\, w \in T_p(M)$. By the integrability condition (3.1) and by (3.15), we easily know that $X(z,\, y,\, x,\, v,\, w)$ is symmetric with respect to the pairs $\{x,\, y\}$, $\{v,\, w\}$ and $\{z,\, y\}$. Further, since $L(z)$ is a skew-symmetric endomorphism of $\boldsymbol{R}^m$

(see (3.8)), it follows that

$$X(z, y, x, v, w) = -X(z, v, w, y, x). \tag{3.18}$$

Therefore, $X(z, y, x, v, w)$ is anti-symmetric with respect to the pair $\{x, w\}$, because

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, v, w, y, x) = -X(v, z, w, y, x) \\ &= X(v, y, x, z, w) = X(y, v, x, z, w) \\ &= -X(y, z, w, v, x) = -X(z, y, w, v, x). \end{aligned}$$

Consequently, we get

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, y, w, v, x) = -X(z, w, y, x, v) \\ &= X(z, w, v, x, y) = X(z, v, w, y, x). \end{aligned}$$

This, together with (3.18), proves $X(z, y, x, v, w) = 0$. Thus we get (3.10).

By the above argument, we know that $L(x) = {}^tH(p)(\nabla_x H) = 0$ for any $x \in T_p(M)$. This implies that H is a locally constant function and hence H is identically equal to an element $H_0 \in O(m)$ on M, because M is connected. Consequently, the difference $\boldsymbol{c} = \boldsymbol{f}_1 - H_0 \cdot \boldsymbol{f}_0$ satisfies

$$\nabla_x \boldsymbol{c} = \nabla_x(\boldsymbol{f}_1 - H_0 \cdot \boldsymbol{f}_0) = \nabla_x \boldsymbol{f}_1 - H_0 \cdot (\nabla_x \boldsymbol{f}_0) = 0, \quad \forall x \in T_p(M).$$

Therefore, $\boldsymbol{c}$ is also identically equal to a constant vector $\boldsymbol{c}_0 \in \boldsymbol{R}^m$, completing the proof of the theorem. □

Remark 3 The argument in the proof of the equality $X = 0$ is essentially the same that is developed in the proof of the uniqueness of the metric connection of the normal bundle associated with an isometric imbedding (see the proof of Theorem 1 of [19]); It is almost the same that is used to calculate the third prolongation of the symbol of the operator L (see Proposition 2.2 of [16]). Here we remark that $X = 0$ can be proved without assuming the existence of (isometric) immersions.

We are now in a position to prove Theorem 5.

Proof of Theorem 5. We show that the map $\boldsymbol{f}_i$ $(i = 0, 1)$ is 2-generic and for each $p \in M$ there is an element $H(p) \in O(m)$ satisfying the equalities (3.3) and (3.4).

Let $i = 0$ or 1. Let $\boldsymbol{f}_{i*}T_p(M)$ (resp. $\boldsymbol{N}_i$) be the tangent vector space (resp. normal vector space) of $\boldsymbol{f}_i(M)$ at $\boldsymbol{f}_i(p) \in \boldsymbol{R}^m$. Then, we have

$\dim \boldsymbol{f}_{i*}T_p(M) = n$ and $\dim \boldsymbol{N}_i = m - n$. We regard $\boldsymbol{f}_{i*}T_p(M)$ and $\boldsymbol{N}_i$ as euclidean vector spaces endowed with the inner products induced from the inner product $\langle\ ,\ \rangle$ of $\boldsymbol{R}^m$. By a natural parallel displacement from $\boldsymbol{f}_i(p)$ to the origin $o \in \boldsymbol{R}^m$, we regard $\boldsymbol{f}_{i*}T_p(M)$ and $\boldsymbol{N}_i$ as linear subspaces of $\boldsymbol{R}^m$. Since $\boldsymbol{f}_i$ is an isometric immersion, $\boldsymbol{f}_{i*}T_p(M)$ is spanned by the vectors $\nabla_x \boldsymbol{f}_i$ $(x \in T_p(M))$ and

$$\langle \nabla_x \boldsymbol{f}_i, \nabla_y \boldsymbol{f}_i \rangle = g_p(x, y), \quad \forall x, y \in T_p(M). \tag{3.19}$$

The second order derivative $\nabla\nabla \boldsymbol{f}_i$, which is so called the *second fundamental form* of $\boldsymbol{f}_i$, satisfies $\nabla\nabla \boldsymbol{f}_i \in S^2T_p^*(M) \otimes \boldsymbol{N}_i$ and $\nabla\nabla \boldsymbol{f}_i \in \mathcal{G}_p(\boldsymbol{N}_i)$ (see [23], [16]). Since $\mathcal{G}_p(\boldsymbol{N}_i)$ is EOS, the vectors $\nabla_x\nabla_y \boldsymbol{f}_i$ $(x, y \in T_p(M))$ span $\boldsymbol{N}_i$, implying that $\boldsymbol{f}_i$ is 2-generic (see Proposition 4 (1)). Take an isometric linear isomorphism φ_2 of $\boldsymbol{N}_0$ onto $\boldsymbol{N}_1$. Since $\widehat{\varphi_2}\,\nabla\nabla \boldsymbol{f}_0 \in \mathcal{G}_p(\boldsymbol{N}_1)$ and since $\mathcal{G}_p(\boldsymbol{N}_1)$ is EOS (see Proposition 4 (2*b*)), there is an element $h_1 \in O(\boldsymbol{N}_1)$ such that $h_1(\widehat{\varphi_2}\nabla\nabla \boldsymbol{f}_0) = \nabla\nabla \boldsymbol{f}_1$. On the other hand, in view of (3.19) we also know that there is an isometric linear isomorphism φ_1 of $\boldsymbol{f}_{0*}T_p(M)$ onto $\boldsymbol{f}_{1*}T_p(M)$ satisfying $\varphi_1(\nabla_x \boldsymbol{f}_0) = \nabla_x \boldsymbol{f}_1$ $(x \in T_p(M))$. Define a linear endomorphism $H(p)$ of $\boldsymbol{R}^m$ satisfying $H(p)|_{\boldsymbol{f}_{0*}T_p(M)} = \varphi_1$ and $H(p)|_{\boldsymbol{N}_0} = h_1 \cdot \varphi_2$. Then, it is easily seen that $H(p) \in O(m)$ and the equalities (3.3) and (3.4) are satisfied.

Therefore, by Theorem 6 we know that $\boldsymbol{f}_1$ can be written as $\boldsymbol{f}_1 = a\boldsymbol{f}_0$, where a denotes the euclidean transformation of $\boldsymbol{R}^m$ defined by $\boldsymbol{R}^m \ni \boldsymbol{x} \longmapsto H_0 \cdot \boldsymbol{x} + \boldsymbol{c}_0 \in \boldsymbol{R}^m$. Thus, we obtain the theorem. □

4. The Cayley projective plane $P^2(Cay)$

Let $M = G/K$ be a compact Riemannian symmetric space. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the canonical decomposition of $\mathfrak{g}$ associated with the symmetric pair (G, K). We denote by $(\ ,\ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual, we can identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G-invariant Riemannian metric g of G/K satisfies

$$g_o(X, Y) = (X, Y), \quad \forall X, Y \in \mathfrak{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type (1, 3) is given by

$$R_o(X, Y)Z = -\big[[X, Y], Z\big], \quad X, Y, Z \in \mathfrak{m}.$$

Hereafter, we consider the case of the Cayley projective plane $P^2(\boldsymbol{Cay})$. As is well-known, $P^2(\boldsymbol{Cay})$ can be represented by $P^2(\boldsymbol{Cay}) = G/K$, where $G = F_4$ and $K = Spin(9)$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and fix it in the following discussions. We note that since $\mathrm{rank}(P^2(\boldsymbol{Cay})) = 1$, we have $\dim \mathfrak{a} = 1$.

For each element $\lambda \in \mathfrak{a}$ we define two subspaces $\mathfrak{k}(\lambda) \subset \mathfrak{k}$ and $\mathfrak{m}(\lambda) \subset \mathfrak{m}$ by

$$\mathfrak{k}(\lambda) = \big\{X \in \mathfrak{k} \mid [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a}\big\},$$
$$\mathfrak{m}(\lambda) = \big\{Y \in \mathfrak{m} \mid [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a}\big\}.$$

We call λ a *restricted root* if $\mathfrak{m}(\lambda) \neq 0$. Let Σ be the set of all non-zero restricted roots. In the case of $P^2(\boldsymbol{Cay})$, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ. Then we have $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ and

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)},$$
$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)}.$$

(For details, see [6], [7].) For simplicity, for each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ ($|i| \leq 2$), $\mathfrak{k}_i = \mathfrak{m}_i = 0$ ($|i| > 2$). Then we have

Proposition 7 ([7]) (1) *Let* $i, j = 0, 1, 2$. *Then:*

$$\begin{aligned} &[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \\ &[\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \end{aligned} \tag{4.1}$$

(2) $\dim \mathfrak{m} = 16$, $\dim \mathfrak{k}_1 = \dim \mathfrak{m}_1 = 8$, $\dim \mathfrak{k}_2 = \dim \mathfrak{m}_2 = 7$.

In what follows, we recall the results obtained in [7], which will be needed in the proof of Theorem 2. Let V be a subspace of $\mathfrak{m}$. V is called *pseudo-abelian* if it satisfies $[V, V] \subset \mathfrak{k}_0$ (or equivalently $[[V, V], \mathfrak{a}] = 0$). (Precisely, see [6].) As is easily seen, $\mathfrak{m}_2$ is a pseudo-abelian subspace of $\mathfrak{m}$, because $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}_0$ (see (4.1)).

On the contrary, we have

Proposition 8 *Let* $G/K = P^2(\boldsymbol{Cay})$. *Then, any pseudo-abelian subspace* V *of* $\mathfrak{m}$ *with* $\dim V > 2$ *must be contained in* $\mathfrak{m}_2$.

For the proof, see Lemma 6 of [7]. The following proposition summarizes the results of [7] (see Proposition 7, Proposition 10 and Lemma 17 of [7]).

Proposition 9 (1) *Let* $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ *and* $Y_1 \in \mathfrak{m}_1$. *Assume that* $Y_0 \neq 0$, $Y_1 \neq 0$. *Then, there are elements* $k_0, k_1 \in K$ *satisfying*

$$\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0, \quad \mathrm{Ad}(k_0)\mathfrak{m}_2 = \{Y_0' \in \mathfrak{a}+\mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}, \tag{4.2}$$

$$\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1, \quad \mathrm{Ad}(k_1)\mathfrak{m}_2 = \{Y_1' \in \mathfrak{m}_1 \mid (Y_1', Y_1) = 0\}. \tag{4.3}$$

(2) *Let* $Y_0, Y_0' \in \mathfrak{a}+\mathfrak{m}_2$, $Y_1, Y_1' \in \mathfrak{m}_1$ *and* $X_1 \in \mathfrak{k}_1$. *Then:*

$$[Y_0, [Y_0, Y_0']] = \begin{cases} -4(\mu,\mu)(Y_0, Y_0)Y_0', & \text{if } (Y_0, Y_0') = 0, \\ 0, & \text{if } Y_0' \in \boldsymbol{R}Y_0, \end{cases} \tag{4.4}$$

$$[Y_0, [Y_0, Y_1]] = -(\mu,\mu)(Y_0, Y_0)Y_1, \tag{4.5}$$

$$[Y_1, [Y_1, Y_0]] = -(\mu,\mu)(Y_1, Y_1)Y_0, \tag{4.6}$$

$$[Y_1, [Y_1, Y_1']] = \begin{cases} -4(\mu,\mu)(Y_1, Y_1)Y_1', & \text{if } (Y_1, Y_1') = 0, \\ 0, & \text{if } Y_1' \in \boldsymbol{R}Y_1, \end{cases} \tag{4.7}$$

$$[X_1, [X_1, Y_0]] = -(\mu,\mu)(X_1, X_1)Y_0. \tag{4.8}$$

5. Solutions of the Gauss equation

In this and the next sections, we prove

Theorem 10 *The projective plane* $P^2(\boldsymbol{Cay})$ *is formally rigid in codimension* 10 $(= 26 - \dim P^2(\boldsymbol{Cay}))$.

If this theorem is established, then Theorem 2 immediately follows from Theorem 5.

On account of homogeneity of $P^2(\boldsymbol{Cay})$, in order to show Theorem 10 we have only to prove that the Gaussian variety $\mathcal{G}_o(\boldsymbol{N})$ is EOS at the origin o for any euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = 10$.

In what follows we assume that $M = P^2(\boldsymbol{Cay})$ and that $\boldsymbol{N}$ is a euclidean vector space with $\dim \boldsymbol{N} = 10$. We will prove the following theorem:

Theorem 11 *Let* $\Psi \in \mathcal{G}_o(\boldsymbol{N})$. *Then:*

(1) *There are linearly independent vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *satisfying*

(1*a*) $\langle \mathbf{A}, \mathbf{A}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu,\mu)$ *and* $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$;

(1*b*) $\Psi(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0, Y_0' \in \mathfrak{a}+\mathfrak{m}_2$;

(1c) $\boldsymbol{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1, Y_1' \in \mathfrak{m}_1$;
(1d) $\langle \mathbf{A}, \boldsymbol{\Psi}(\mu, \mathfrak{m}_1)\rangle = \langle \mathbf{B}, \boldsymbol{\Psi}(\mu, \mathfrak{m}_1)\rangle = 0$.
(2) $\boldsymbol{\Psi}(Y_1, Y_2) + \big(1/(\mu, \mu)^2\big)\boldsymbol{\Psi}\big(\mu, [[\mu, Y_1], Y_2]\big) = 0, \quad \forall Y_1 \in \mathfrak{m}_1, \forall Y_2 \in \mathfrak{m}_2$.
(3) $\langle \boldsymbol{\Psi}(\mu, Y_1), \boldsymbol{\Psi}(\mu, Y_1')\rangle = (\mu, \mu)^2 (Y_1, Y_1'), \quad \forall Y_1, Y_1' \in \mathfrak{m}_1$.

Before proceeding to the proof of Theorem 11 we make a somewhat lengthy preparation. Let $\boldsymbol{N}$ be a euclidean vector space and let $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ and $Y \in \mathfrak{m}$. We define a linear map $\boldsymbol{\Psi}_Y$ of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\boldsymbol{\Psi}_Y : \mathfrak{m} \ni Y' \longmapsto \boldsymbol{\Psi}(Y, Y') \in \boldsymbol{N}$$

and denote by $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ the kernel of $\boldsymbol{\Psi}_Y$. We say that an element $Y \in \mathfrak{m}$ is *singular* (resp. *non-singular*) with respect to $\boldsymbol{\Psi}$ if $\boldsymbol{\Psi}_Y(\mathfrak{m}) \neq \boldsymbol{N}$ (resp. $\boldsymbol{\Psi}_Y(\mathfrak{m}) = \boldsymbol{N}$). Apparently, 0 ($\in \mathfrak{m}$) is a singular element for any $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$.

Proposition 12 *Let $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Let $Y \in \mathfrak{m}$ $(Y \neq 0)$ and let k be an element of K satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then:*
(1) $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$. *Consequently,* $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \leq 7$.
(2) *Assume that Y is non-singular with respect to $\boldsymbol{\Psi}$. Then, it holds that* $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 6$ *and* $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneqq \mathrm{Ad}(k)\mathfrak{m}_2$.
(3) *Assume that Y is singular with respect to $\boldsymbol{\Psi}$. Then, it holds that* $\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$, $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 7$ *and* $\dim \boldsymbol{\Psi}_Y(\mathfrak{m}) = 9$.

Proof. First, note that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq \dim \mathfrak{m} - \dim \boldsymbol{N} = 6$. Consequently, it is easy to see that Y is singular (resp. non-singular) with respect to $\boldsymbol{\Psi}$ if and only if $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) > 6$ (resp. $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 6$).

Multiplying Y by a non-zero scalar if necessary, we may assume that $Y = \mathrm{Ad}(k)\mu$. From the Gauss equation (2.1) it follows that

$$R_o\big(\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathbf{Ker}(\boldsymbol{\Psi}_Y)\big)Y = 0.$$

In our terminology we have

$$\big[[\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathbf{Ker}(\boldsymbol{\Psi}_Y)], Y\big] = 0.$$

Applying $\mathrm{Ad}(k^{-1})$ to the both sides of the above equality, we have

$$\big[[\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y), \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y)], \mu\big] = 0.$$

Since $\mathfrak{a} = \boldsymbol{R}\mu$, it follows that $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$. By Proposition 8 and by the fact $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq 6$, we have $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathfrak{m}_2$ and hence $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$, proving (1).

Assume that Y is non-singular with respect to $\boldsymbol{\Psi}$. Then, as we have stated above, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 6$. Since $\dim \mathfrak{m}_2 = 7$ (see Proposition 7 (2)), it follows that $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$, proving (2).

Finally, we assume Y is singular with respect to $\boldsymbol{\Psi}$. Then, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) > 6$. Since $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ and since $\dim \mathfrak{m}_2 = 7$, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 7$ and $\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$. This proves (3). □

Corollary 13 *Let* $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. *Let* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$ *and* $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$. *Then:*

(1) $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$. *In particular, if* Y_0 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$.

(2) $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) \subset \{Y_1' \in \mathfrak{m}_1 \mid (Y_1', Y_1) = 0\}$. *In particular, if* Y_1 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) = \{Y_1' \in \mathfrak{m}_1 \mid (Y_1', Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ $(Y_0 \neq 0)$. By Proposition 9 (1), we know that there is an element $k_0 \in K$ satisfying (4.2). Applying Proposition 12 to Y_0, we easily get $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$. Assume that Y_0 is singular with respect to $\boldsymbol{\Psi}$. Then, by the equality $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \mathrm{Ad}(k_0)\mathfrak{m}_2$, we get (1).

The assertion (2) is similarly dealt with. □

Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. A subspace U of $\mathfrak{m}$ is called *singular* with respect to $\boldsymbol{\Psi}$ if each element of U is singular with respect to $\boldsymbol{\Psi}$.

Proposition 14 *Let* $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. *Let* $Y \in \mathfrak{m}$ $(Y \neq 0)$ *and let* $k \in K$ *satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Assume that* Y *is non-singular with respect to* $\boldsymbol{\Psi}$. *Then:*

(1) $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ *is a singular subspace with respect to* $\boldsymbol{\Psi}$.

(2) *There is an element* $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ *satisfying* $\boldsymbol{\Psi}(Y, Y') \neq 0$ *and*

$$\boldsymbol{N} = \boldsymbol{R}\boldsymbol{\Psi}(Y, Y') + \boldsymbol{\Psi}_{Y''}(\mathfrak{m}) \quad (\textit{orthogonal direct sum}), \tag{5.1}$$

where Y'' *is an arbitrary non-zero element of* $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$.

Proof. Since Y is non-singular with respect to $\boldsymbol{\Psi}$, we have $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$ (see Proposition 12). Take a non-zero element $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ such that $\big(Y', \mathbf{Ker}(\boldsymbol{\Psi}_Y)\big) = 0$. Then, since $Y' \notin \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, we have $\boldsymbol{\Psi}(Y, Y') \neq 0$.

Let $Y'' \in \mathbf{Ker}(\Psi_Y)$ ($Y'' \neq 0$). Then, by the Gauss equation (2.1) we have

$$\begin{aligned}&([[Y', Y''], Y], W)\\ &= \langle \Psi(Y', Y), \Psi(Y'', W)\rangle - \langle \Psi(Y', W), \Psi(Y'', Y)\rangle,\end{aligned} \tag{5.2}$$

where W is an arbitrary element of $\mathfrak{m}$. Note that the left hand side of (5.2) vanishes, because

$$\begin{aligned}[[Y', Y''], Y] &\in [[\mathrm{Ad}(k)\mathfrak{m}_2, \mathrm{Ad}(k)\mathfrak{m}_2], \mathrm{Ad}(k)\mu]\\ &= \mathrm{Ad}(k)[[\mathfrak{m}_2, \mathfrak{m}_2], \mu] = 0.\end{aligned}$$

We also note that $\Psi(Y'', Y) = 0$, because $Y'' \in \mathbf{Ker}(\Psi_Y)$. Consequently, we have $\langle \Psi(Y', Y), \Psi(Y'', W)\rangle = 0$. This implies that each element of $\Psi_{Y''}(\mathfrak{m})$ is orthogonal to $\Psi(Y', Y)$. Therefore, $\Psi_{Y''}(\mathfrak{m}) \neq N$, implying that Y'' is singular with respect to Ψ. Hence, by Proposition 12 (3) we have $\dim \Psi_{Y''}(\mathfrak{m}) = 9$, which proves (5.1). □

The following lemma assures that for each $\Psi \in \mathcal{G}_o(N)$ there are many high dimensional singular subspaces with respect to Ψ.

Lemma 15 *Let $\Psi \in \mathcal{G}_o(N)$. Then, there are singular subspaces U and V with respect to Ψ satisfying $U \subset \mathfrak{a}+\mathfrak{m}_2$, $V \subset \mathfrak{m}_1$, $\dim U \geq 6$ and $\dim V \geq 6$.*

Proof. If $\mathfrak{a} + \mathfrak{m}_2$ contains no non-singular element with respect to Ψ, then we can take $U = \mathfrak{a} + \mathfrak{m}_2$. (Note that $\dim(\mathfrak{a} + \mathfrak{m}_2) = 8$.) On the contrary, if $\mathfrak{a} + \mathfrak{m}_2$ contains a non-singular element Y_0, then we set $U = \mathbf{Ker}(\Psi_{Y_0})$. Then, we know that $U \subset \mathfrak{a} + \mathfrak{m}_2$, $\dim U = 6$ (see Proposition 12 (2) and Corollary 13 (1)) and that U is a singular subspace with respect to Ψ (see Proposition 14 (1)). Similarly, we can select a singular subspace $V \subset \mathfrak{m}_1$ with $\dim V \geq 6$. □

Proposition 16 *Let $\Psi \in \mathcal{G}_o(N)$. Let U and V be arbitrary singular subspaces with respect to Ψ satisfying $U \subset \mathfrak{a} + \mathfrak{m}_2$, $V \subset \mathfrak{m}_1$, $\dim U \geq 6$ and $\dim V \geq 6$. Then there are two vectors $\mathbf{A}$ and $\mathbf{B} \in N$ satisfying:*

(1) $\langle \mathbf{A}, \mathbf{A}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu, \mu)$;
(2) $\Psi(\xi, Y_0) = (\xi, Y_0)\mathbf{A}, \quad \forall \xi \in U, \ \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$;
(3) $\Psi(\eta, Y_1) = (\eta, Y_1)\mathbf{B}, \quad \forall \eta \in V, \ \forall Y_1 \in \mathfrak{m}_1$;
(4) $\langle \mathbf{A}, \Psi_{Y_0}(\mathfrak{m}_1)\rangle = \langle \mathbf{B}, \Psi_{Y_0}(\mathfrak{m}_1)\rangle = 0, \quad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.

Proof. Let $\xi \in U$ ($\xi \neq 0$). Since ξ is singular with respect to $\boldsymbol{\Psi}$, $\mathbf{Ker}(\boldsymbol{\Psi}_\xi)$ coincides with the orthogonal complement of $\boldsymbol{R}\xi$ in $\mathfrak{a} + \mathfrak{m}_2$ (see Corollary 13 (1)). Hence, the equality $\boldsymbol{\Psi}(\xi, Y_0) = 0$ holds for each $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfying $(\xi, Y_0) = 0$. In particular, we have

$$\boldsymbol{\Psi}(\xi, \xi') = 0, \quad \forall \xi, \xi' \in U \text{ with } (\xi, \xi') = 0.$$

Then, applying the same argument as in the proof of Proposition 9 of [7], we can prove that there is a vector $\mathbf{A} \in \boldsymbol{N}$ satisfying

$$\boldsymbol{\Psi}(\xi, \xi') = (\xi, \xi')\mathbf{A}, \quad \forall \xi, \xi' \in U. \tag{5.3}$$

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ satisfy $(Y_0, U) = 0$. Then, since $(\xi, Y_0) = 0$, we have $\boldsymbol{\Psi}(\xi, Y_0) = 0$ and $(\xi, Y_0)\mathbf{A} = 0$. This, together with (5.3), proves (2). The assertion (3) can be proved in the same way.

We now prove (1). Let $\xi, \xi' \in U$ satisfy $(\xi, \xi') = 0$ and $(\xi, \xi) = (\xi', \xi') = 1$. Put $X = Z = \xi$ and $Y = W = \xi'$ into the Gauss equation (2.1). Then, we have

$$\big([[\xi, \xi'], \xi], \xi'\big) = \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(\xi', \xi')\rangle - \langle \boldsymbol{\Psi}(\xi, \xi'), \boldsymbol{\Psi}(\xi', \xi)\rangle.$$

Since $\big[[\xi, \xi'], \xi\big] = 4(\mu, \mu)\xi'$ (see (4.4)), $\boldsymbol{\Psi}(\xi, \xi) = \boldsymbol{\Psi}(\xi', \xi') = \mathbf{A}$ and $\boldsymbol{\Psi}(\xi, \xi') = 0$, we have $\langle \mathbf{A}, \mathbf{A}\rangle = 4(\mu, \mu)$. Similarly, by (4.7) we can prove $\langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu, \mu)$, proving (1).

Finally, we prove (4). Let $Y_1 \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. Take an element $\xi \in U$ satisfying $(\xi, Y_0) = 0$ and $(\xi, \xi) = 1$. Such ξ can exist, because $\dim U \geq 6$. Put $X = Z = \xi$, $Y = Y_0$ and $W = Y_1$ into the Gauss equation (2.1). Then we have

$$\big([[\xi, Y_0], \xi], Y_1\big) = \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(Y_0, Y_1)\rangle - \langle \boldsymbol{\Psi}(\xi, Y_1), \boldsymbol{\Psi}(Y_0, \xi)\rangle.$$

Since $(\xi, Y_0) = 0$, we have $\boldsymbol{\Psi}(\xi, Y_0) = 0$ and $\big[[\xi, Y_0], \xi\big] = 4(\mu, \mu)Y_0$ (see (4.4)). Moreover, since $\boldsymbol{\Psi}(\xi, \xi) = \mathbf{A}$ and $(Y_0, Y_1) = 0$, we have

$$\begin{aligned}\langle \mathbf{A}, \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle &= \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(Y_0, Y_1)\rangle \\ &= \langle \boldsymbol{\Psi}(\xi, Y_1), \boldsymbol{\Psi}(Y_0, \xi)\rangle + 4(\mu, \mu)(Y_0, Y_1) \\ &= 0.\end{aligned}$$

Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\langle \mathbf{A}, \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = 0$. In a similar way, the equality $\langle \mathbf{B}, \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = 0$ can be proved. □

Remark 4 As seen in the proof of Lemma 15, singular subspaces U and V may not be uniquely determined. However, it is noted that the vectors $\mathbf{A}$ and $\mathbf{B}$ in Proposition 16 do not depend on the choice of U and V. In fact, let U' and V' be different singular subspaces with respect to $\boldsymbol{\Psi}$ satisfying $U' \subset \mathfrak{a} + \mathfrak{m}_2$ and $V' \subset \mathfrak{m}_1$ with $\dim U' \geq 6$, $\dim V' \geq 6$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be vectors of $\boldsymbol{N}$ satisfying (1) $\sim$ (4) of Proposition 16. Then, since $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1 = 8$, we have $U \cap U' \neq 0$, $V \cap V' \neq 0$. Take $\xi \in U \cap U'$ and $\eta \in V \cap V'$ such that $(\xi, \xi) = (\eta, \eta) = 1$. Then we have $\mathbf{A} = \boldsymbol{\Psi}(\xi, \xi) = \mathbf{A}'$ and $\mathbf{B} = \boldsymbol{\Psi}(\eta, \eta) = \mathbf{B}'$, showing our assertion.

In the following discussions, we fix an element $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$, singular subspaces U, V and vectors $\mathbf{A}$, $\mathbf{B}$ stated in Proposition 16 and prove several lemmas which are indispensable to the proof of Theorem 11.

Lemma 17 *Let $\xi \in U$, $\eta \in V$, $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Set $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$. Then $C > 0$ and:*
(1) $\langle \boldsymbol{\Psi}_{Y_0}(\eta), \boldsymbol{\Psi}_{Y_0}(Y_1) \rangle = \{\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B} \rangle - (\mu, \mu)(Y_0, Y_0)\}(\eta, Y_1)$;
(2) $\langle \boldsymbol{\Psi}_\xi(\eta), \boldsymbol{\Psi}_\xi(Y_1) \rangle = C(\xi, \xi)(\eta, Y_1)$.

Proof. Putting $X = Z = Y_0$, $Y = Y_1$ and $W = \eta$ into (2.1), we have

$$\big([[Y_0, Y_1], Y_0], \eta\big) = \langle \boldsymbol{\Psi}(Y_0, Y_0), \boldsymbol{\Psi}(Y_1, \eta) \rangle - \langle \boldsymbol{\Psi}(Y_0, \eta), \boldsymbol{\Psi}(Y_1, Y_0) \rangle.$$

Since $\big[[Y_0, Y_1], Y_0\big] = (\mu, \mu)(Y_0, Y_0)Y_1$ (see (4.5)) and $\boldsymbol{\Psi}(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we easily get (1). Putting $Y_0 = \xi \in U$ into (1), we easily have (2). If we set $Y_1 = \eta \in V$ in (2), we have $\langle \boldsymbol{\Psi}_\xi(\eta), \boldsymbol{\Psi}_\xi(\eta) \rangle = C(\xi, \xi)(\eta, \eta)$. Since $\mathbf{Ker}(\boldsymbol{\Psi}_\xi) \cap \mathfrak{m}_1 = 0$ (see Corollary 13 (1)), we have $\boldsymbol{\Psi}_\xi(\eta) \neq 0$ if $\eta \neq 0$. Consequently, we have $C > 0$. □

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Let ξ^0 be a non-zero element of U satisfying $(\xi^0, Y_0) = 0$. (Such ξ^0 exists, because $\dim U \geq 6$.) We define a linear mapping $\boldsymbol{\Theta}_{Y_0, \xi^0} \colon V \longrightarrow \boldsymbol{N}$ by

$$\boldsymbol{\Theta}_{Y_0, \xi^0}(\eta) = \boldsymbol{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \boldsymbol{\Psi}_{\xi^0}\big([[\xi^0, \eta], Y_0]\big), \quad \eta \in V.$$

Then we have

Lemma 18 $\langle \mathbf{A}, \boldsymbol{\Theta}_{Y_0, \xi^0}(V) \rangle = \langle \boldsymbol{\Psi}_{\xi^0}(V), \boldsymbol{\Theta}_{Y_0, \xi^0}(V) \rangle = 0$.

Proof. We first note that $\big[[\xi^0, \eta], Y_0\big] \in \mathfrak{m}_1$ for $\eta \in V$ and note that $\boldsymbol{\Theta}_{Y_0, \xi^0}(V) \subset \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) + \boldsymbol{\Psi}_{\xi^0}(\mathfrak{m}_1)$. By Proposition 16 (4), we have

$\langle \mathbf{A}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = \langle \mathbf{A}, \mathbf{\Psi}_{\xi^0}(\mathfrak{m}_1)\rangle = 0$ and hence $\langle \mathbf{A}, \mathbf{\Theta}_{Y_0,\xi^0}(V)\rangle = 0$.

Let η, $\eta' \in V$. Then by putting $X = Y_0$, $Y = \eta'$, $Z = \eta$ and $W = \xi^0$ into the Gauss equation (2.1), we have

$$\begin{aligned}\big([[Y_0, \eta'], \eta], \xi^0\big) &= \langle \mathbf{\Psi}(Y_0, \eta), \mathbf{\Psi}(\eta', \xi^0)\rangle - \langle \mathbf{\Psi}(Y_0, \xi^0), \mathbf{\Psi}(\eta', \eta)\rangle \\ &= \langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle - \langle \mathbf{A}, \mathbf{B}\rangle(Y_0, \xi^0)(\eta', \eta).\end{aligned}$$

Since $(Y_0, \xi^0) = 0$, we have

$$\langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle = \big([[Y_0, \eta'], \eta], \xi^0\big). \tag{5.4}$$

On the other hand, we have

$$\big\langle \mathbf{\Psi}_{\xi^0}\big([[\xi^0, \eta], Y_0]\big), \mathbf{\Psi}_{\xi^0}(\eta')\big\rangle = C(\xi^0, \xi^0)\big([[\xi^0, \eta], Y_0], \eta'\big)$$

(see Lemma 17 (2)). Therefore,

$$\begin{aligned}&\langle \mathbf{\Theta}_{Y_0,\xi^0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle \\ &= \left\langle \mathbf{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)}\mathbf{\Psi}_{\xi^0}\big([[\xi^0, \eta], Y_0]\big), \mathbf{\Psi}_{\xi^0}(\eta')\right\rangle \\ &= \big([[Y_0, \eta'], \eta], \xi^0\big) + \big([[\xi^0, \eta], Y_0], \eta'\big) \\ &= -([Y_0, \eta'], [\xi^0, \eta]) + ([\xi^0, \eta], [Y_0, \eta']) \\ &= 0.\end{aligned}$$

This completes the proof. □

We can further show

Lemma 19 *Let $\eta \in V$. Assume that $\big[[\xi^0, \eta], Y_0\big] \in V$. Then:*

$$\begin{aligned}&|\mathbf{\Theta}_{Y_0,\xi^0}(\eta)|^2 \\ &= \left[\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{B}\rangle - (\mu, \mu)(Y_0, Y_0)\left\{1 + \frac{(\mu, \mu)}{C}\right\}\right](\eta, \eta).\end{aligned} \tag{5.5}$$

Proof. Set $\eta' = \big[[\xi^0, \eta], Y_0\big]$. By Lemma 18, Lemma 17 and the equality (5.4) we have

$$\begin{aligned}&\langle \mathbf{\Theta}_{Y_0,\xi^0}(\eta), \mathbf{\Theta}_{Y_0,\xi^0}(\eta)\rangle \\ &= \left\langle \mathbf{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)}\mathbf{\Psi}_{\xi^0}(\eta'), \mathbf{\Theta}_{Y_0,\xi^0}(\eta)\right\rangle \\ &= \langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Theta}_{Y_0,\xi^0}(\eta)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \Psi_{Y_0}(\eta),\, \Psi_{Y_0}(\eta)\rangle + \frac{1}{C(\xi^0,\, \xi^0)}\langle \Psi_{Y_0}(\eta),\, \Psi_{\xi^0}(\eta')\rangle \\
&= \{\langle \Psi(Y_0,\, Y_0),\, \mathbf{B}\rangle - (\mu,\, \mu)(Y_0,\, Y_0)\}(\eta,\, \eta) \\
&\qquad\qquad + \frac{1}{C(\xi^0,\, \xi^0)}\big([[Y_0,\, \eta'],\, \eta],\, \xi^0\big).
\end{aligned}
$$

Since $[\xi^0,\, \eta] \in \mathfrak{k}_1$, by (4.8) and (4.5) we have

$$
\begin{aligned}
\big([[Y_0,\, \eta'],\, \eta],\, \xi^0\big) &= -([Y_0,\, \eta'],\, [\xi^0,\, \eta]) \\
&= \big(Y_0,\, [[\xi^0,\, \eta],\, \eta']\big) \\
&= \big(Y_0,\, [[\xi^0,\, \eta],\, [[\xi^0,\, \eta],\, Y_0]]\big) \\
&= -(\mu,\, \mu)([\xi^0,\, \eta],\, [\xi^0,\, \eta])(Y_0,\, Y_0) \\
&= (\mu,\, \mu)\big([\xi^0,\, [\xi^0,\, \eta]],\, \eta\big)(Y_0,\, Y_0) \\
&= -(\mu,\, \mu)^2(\xi^0,\, \xi^0)(\eta,\, \eta)(Y_0,\, Y_0).
\end{aligned}
$$

Therefore, we obtain (5.5). □

Lemma 20 *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then:*
(1) $\langle \Psi(Y_0,\, Y_0),\, \mathbf{B}\rangle = (\mu,\, \mu)(Y_0,\, Y_0)\{1 + (\mu,\, \mu)/C\}$.
(2) *Let ξ^0 be a non-zero element of U satisfying $(Y_0,\, \xi^0) = 0$. Then, $\Theta_{Y_0,\xi^0}(\eta) = 0$, i.e., the equality*

$$
\Psi(Y_0,\, \eta) + \frac{1}{C(\xi^0,\, \xi^0)}\Psi\big(\xi^0,\, [[\xi^0,\, \eta],\, Y_0]\big) = 0 \tag{5.6}
$$

holds for each $\eta \in V$ satisfying $\big[[\xi^0,\, \eta],\, Y_0\big] \in V$.

Proof. We first show that there is a non-zero element $\eta^0 \in V$ satisfying $\Theta_{Y_0,\xi^0}(\eta^0) = 0$ and $\big[[\xi^0,\, \eta^0],\, Y_0\big] \in V$. Let $\boldsymbol{D}$ be the orthogonal complement of $\boldsymbol{R}\mathbf{A} + \Psi_{\xi^0}(V)$ in $\boldsymbol{N}$ and let V' be the orthogonal complement of V in $\mathfrak{m}_1$. By Lemma 18, we easily have $\Theta_{Y_0,\xi^0}(V) \subset \boldsymbol{D}$. Therefore, to obtain η^0 satisfying the above condition, it suffices to find a non-zero solution $\eta = \eta^0 \in V$ of the system of linear homogeneous equations

$$
\langle \Theta_{Y_0,\xi^0}(\eta),\, \boldsymbol{D}\rangle = \big([[\xi^0,\, \eta],\, Y_0],\, V'\big) = 0. \tag{5.7}
$$

Since $\mathbf{Ker}(\Psi_{\xi^0}) \cap \mathfrak{m}_1 = 0$ (see Corollary13 (1)) and $\langle \mathbf{A},\, \Psi_{\xi^0}(\mathfrak{m}_1)\rangle = 0$ (see Proposition 16 (4)), we have $\dim(\boldsymbol{R}\mathbf{A} + \Psi_{\xi^0}(V)) = 1 + \dim V \geq 7$. (Recall that we are assuming $V \subset \mathfrak{m}_1$ and $\dim V \geq 6$.) Hence, we have $\dim \boldsymbol{D} \leq \dim \boldsymbol{N} - 7 = 3$. Moreover, we have $\dim V' = 8 - \dim V \leq 2$. Consequently, the rank of the system (5.7) is less than or equal to 5. Therefore, we can find a non-zero solution $\eta^0 \in V$ of (5.7). Putting $\eta = \eta^0$ into (5.5), we obtain the equality (1). Further, putting (1) into (5.5), we have $\boldsymbol{\Theta}_{Y_0, \xi^0}(\eta) = 0$ for any $\eta \in V$ satisfying $\big[[\xi^0, \eta], Y_0\big] \in V$. □

Lemma 21 *The vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *are linearly independent and* $\langle \mathbf{A}, \mathbf{B} \rangle = 2(\mu, \mu)$, $C = (\mu, \mu)$.

Proof. Let $\xi \in U$ with $(\xi, \xi) = 1$. Since $\boldsymbol{\Psi}(\xi, \xi) = \mathbf{A}$ (see (5.3)), by putting $Y_0 = \xi$ into the equality in Lemma 20 (1), we easily have $\langle \mathbf{A}, \mathbf{B} \rangle = (\mu, \mu)\{1 + (\mu, \mu)/C\}$. Since $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$, it immediately follows that $C^2 = (\mu, \mu)^2$. Since $C > 0$, we get $C = (\mu, \mu)$ and hence $\langle \mathbf{A}, \mathbf{B} \rangle = 2(\mu, \mu)$. This, together with Proposition 16 (1), proves that $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. □

These being prepared, we show Theorem 11.

Proof of Theorem 11. First we show that μ is singular with respect to any element $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Suppose that there is an element $\boldsymbol{\Psi}_0 \in \mathcal{G}_o(\boldsymbol{N})$ such that μ is non-singular with respect to $\boldsymbol{\Psi}_0$. Then, $\mathbf{Ker}((\boldsymbol{\Psi}_0)_\mu)$ is a singular subspace with respect to $\boldsymbol{\Psi}_0$ and it satisfies $\dim \mathbf{Ker}((\boldsymbol{\Psi}_0)_\mu) = 6$ and $\mathbf{Ker}((\boldsymbol{\Psi}_0)_\mu) \subset \mathfrak{m}_2$ (see Proposition 12 and Proposition 14).

Now, set $\boldsymbol{\Psi} = \boldsymbol{\Psi}_0$ and $U = \mathbf{Ker}((\boldsymbol{\Psi}_0)_\mu)$ in Proposition 16. Let $\mathbf{A}$, $\mathbf{B}$ be the vectors of $\boldsymbol{N}$ satisfying (1)–(4) of Proposition 16. Let $\xi \in U = \mathbf{Ker}((\boldsymbol{\Psi}_0)_\mu)$ with $\xi \neq 0$. First, we show $\mathbf{B} \in (\boldsymbol{\Psi}_0)_\xi(\mathfrak{m})$. In fact, there is a non-zero element $Y_2^0 \in \mathfrak{m}_2$ satisfying $\boldsymbol{\Psi}_0(\mu, Y_2^0) \neq 0$ and $\boldsymbol{N} = \boldsymbol{R}\boldsymbol{\Psi}_0(\mu, Y_2^0) + (\boldsymbol{\Psi}_0)_\xi(\mathfrak{m})$ (orthogonal direct sum) (see Proposition 14). By Lemma 20 (1) and by the relation

$$\boldsymbol{\Psi}_0(\mu, Y_2^0) = \frac{1}{2}\Big(\boldsymbol{\Psi}_0(\mu + Y_2^0, \mu + Y_2^0) - \boldsymbol{\Psi}_0(\mu, \mu) - \boldsymbol{\Psi}_0(Y_2^0, Y_2^0)\Big),$$

we easily have $\langle \boldsymbol{\Psi}_0(\mu, Y_2^0), \mathbf{B} \rangle = 0$, which proves $\mathbf{B} \in (\boldsymbol{\Psi}_0)_\xi(\mathfrak{m})$. Since $(\boldsymbol{\Psi}_0)_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + (\boldsymbol{\Psi}_0)_\xi(\mathfrak{m}_1)$ (orthogonal direct sum) and $\langle \mathbf{B}, (\boldsymbol{\Psi}_0)_\xi(\mathfrak{m}_1) \rangle = 0$ (see Proposition 16 (2), (4)), we have $\mathbf{B} \in \boldsymbol{R}\mathbf{A}$. This contradicts Lemma 21. Accordingly, we can conclude that μ is singular with respect to any element $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$.

Now we show that any element of $\mathfrak{m}$ is singular with respect to any $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Let Y be a non-zero element of $\mathfrak{m}$. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$ and define $\boldsymbol{\Psi}' \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ by

$$\Psi'(Y', Y'') = \Psi(\mathrm{Ad}(k)Y', \mathrm{Ad}(k)Y''), \quad Y', Y'' \in \mathfrak{m}.$$

Then, it is easily seen that $\Psi' \in \mathcal{G}_o(N)$. Applying the arguments developed above, we know that μ is also singular with respect to Ψ'. Note that $\Psi'_\mu(\mathfrak{m}) = \Psi_{\mathrm{Ad}(k)\mu}(\mathrm{Ad}(k)\mathfrak{m}) = \Psi_Y(\mathfrak{m})$. Then, since $\Psi'_\mu(\mathfrak{m}) \neq N$, we have $\Psi_Y(\mathfrak{m}) \neq N$, implying that Y is singular with respect to Ψ.

Accordingly, in Proposition 16 and in the discussion after it, we may allow to put $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Therefore, by Proposition 16 and Lemma 21, we get (1) of Theorem 11. Further, putting $Y_0 = Y_2 \in \mathfrak{m}_2$, $\xi^0 = \mu$ and $\eta = Y_1$ into (5.6), we get (2) of Theorem 11. The assertion (3) of Theorem 11 follows from Lemma 17 (2) and Lemma 21. This completes the proof of the theorem. □

6. Proof of Theorem 10

Let $\{E_i \ (1 \leq i \leq 8)\}$ be an orthonormal basis of $\mathfrak{m}_1$. (Note that $\dim \mathfrak{m}_1 = 8$.) Let $\Psi \in \mathcal{G}_o(N)$ and let $\mathbf{A}$, $\mathbf{B}$ be the vectors of N stated in Theorem 11. We define vectors $\{\mathbf{F}_i \ (1 \leq i \leq 10)\}$ of N by setting $\mathbf{F}_i = \Psi(\mu, E_i)/(\mu, \mu)$ $(1 \leq i \leq 8)$, $\mathbf{F}_9 = (\mathbf{A}+\mathbf{B})/2\sqrt{3}|\mu|$ and $\mathbf{F}_{10} = (\mathbf{A}-\mathbf{B})/2|\mu|$. We now show that $\{\mathbf{F}_i \ (1 \leq i \leq 10)\}$ forms an orthonormal basis of N. By Theorem 11 (3) we have $\langle \mathbf{F}_i, \mathbf{F}_j \rangle = \delta_{ij}$ $(1 \leq i, j \leq 8)$, where δ_{ij} denotes Kronecker's delta. Moreover, since $\langle \mathbf{A}, \mathbf{F}_i \rangle = \langle \mathbf{B}, \mathbf{F}_i \rangle = 0$ $(1 \leq i \leq 8)$ (see Theorem 11 (1d)), we have $\langle \mathbf{F}_9, \mathbf{F}_i \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_i \rangle = 0$ $(1 \leq i \leq 8)$. The equalities $\langle \mathbf{F}_9, \mathbf{F}_9 \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_{10} \rangle = 1$ and $\langle \mathbf{F}_9, \mathbf{F}_{10} \rangle = 0$ immediately follow from Theorem 11 (1a).

Now let Ψ' be another element of $\mathcal{G}_o(N)$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be the vectors stated in Theorem 11 for Ψ'. As in the case of Ψ we can also define an orthonormal basis $\{\mathbf{F}'_i \ (1 \leq i \leq 10)\}$ of N. Then, there is an element $h \in O(10)$ satisfying $\mathbf{F}'_i = h\mathbf{F}_i$ $(1 \leq i \leq 10)$. Here we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\Psi'(\mu, E_i) = h\Psi(\mu, E_i)$ $(1 \leq i \leq 8)$. Set $\Phi = \Psi' - h\Psi \in S^2\mathfrak{m}^* \otimes N$. Then, by Theorem 11 (1) we have

$$\Phi(\mathfrak{a} + \mathfrak{m}_2, \mathfrak{a} + \mathfrak{m}_2) = \Phi(\mathfrak{m}_1, \mathfrak{m}_1) = \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0.$$

By the fact $\big[[\mu, \mathfrak{m}_1], \mathfrak{m}_2\big] \subset \mathfrak{m}_1$ and Theorem 11 (2), we have

$$\Phi(\mathfrak{m}_2, \mathfrak{m}_1) \subset \Phi\big(\mu, \big[[\mu, \mathfrak{m}_1], \mathfrak{m}_2\big]\big) \subset \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0,$$

which proves that $\Phi(\mathfrak{m}_2, \mathfrak{m}_1) = 0$. Therefore, we have $\Phi = 0$, i.e., $\Psi' = h\Psi$. This implies that the Gaussian variety $\mathcal{G}_o(N)$ is EOS. This completes the proof of Theorem 10. □

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Rigidity of the canonical isometric imbedding of the quaternion projective plane $P^2(\boldsymbol{H})$

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Abstract. In this paper, we investigate isometric immersions of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ and prove that the canonical isometric imbedding $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, which is defined in Kobayashi [11], is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set U ($\subset P^2(\boldsymbol{H})$) into $\boldsymbol{R}^{14}$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{14}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

Key words: Curvature invariant, isometric immersion, quaternion projective plane, rigidity, root space decomposition.

1. Introduction

In our previous paper [8], we proved the rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(\boldsymbol{Cay})$. The purpose of this paper is to investigate a similar problem for (local) isometric immersions of the quaternion projective plane $P^2(\boldsymbol{H})$. As we have proved in [7], any open set of the quaternion projective plane $P^2(\boldsymbol{H})$ cannot be isometrically immersed into $\boldsymbol{R}^{13}$. On the other hand, there is an isometric immersion $\boldsymbol{f}_0$ of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$, which is called the canonical isometric imbedding of $P^2(\boldsymbol{H})$ (see Kobayashi [11]). Therefore, it follows that $\boldsymbol{R}^{14}$ is the least dimensional euclidean space into which $P^2(\boldsymbol{H})$ can be (locally) isometrically immersed.

In the present paper, we will show that the canonical isometric imbedding $\boldsymbol{f}_0$ is rigid in the following strongest sense:

Theorem 1 *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $P^2(\boldsymbol{H})$ into the euclidean space $\boldsymbol{R}^{14}$. Then, for any isometric immersion $\boldsymbol{f}_1$ defined on a connected open set U of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$, there exists a euclidean transformation a of $\boldsymbol{R}^{14}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

The proof of this theorem will be given by solving the Gauss equation

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associated with the isometric imbeddings (immersions) of $P^2(\boldsymbol{H})$ into $\boldsymbol{R}^{14}$ in the same line of [8] (see Theorem 7). We use the same notations and terminology as those of the previous papers [6], [7] and [8].

2. The quaternion projective plane $P^2(H)$

In this section we review the structure of the quaternion projective plane $P^2(\boldsymbol{H})$ and prepare several formulas concerning the bracket operation.

As is well-known, $P^2(\boldsymbol{H})$ can be represented by $P^2(\boldsymbol{H}) = G/K$, where $G = Sp(3)$ and $K = Sp(2) \times Sp(1)$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the symmetric pair (G, K). We denote by $(\ ,\ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual, we can identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G-invariant Riemannian metric g of G/K satisfies

$$g_o(X, Y) = (X, Y), \quad X, Y \in \mathfrak{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type $(1, 3)$ is given by

$$R_o(X, Y)Z = -\big[[X, Y], Z\big], \qquad \forall X,\ Y,\ Z \in \mathfrak{m}.$$

We now take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and fix it in the following discussions. We note that since $\mathrm{rank}(P^2(\boldsymbol{H})) = 1$, we have $\dim \mathfrak{a} = 1$.

For each element $\lambda \in \mathfrak{a}$ we define two subspaces $\mathfrak{k}(\lambda)$ $(\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)$ $(\subset \mathfrak{m})$ by

$$\mathfrak{k}(\lambda) = \Big\{ X \in \mathfrak{k} \ \Big|\ [H, [H, X]] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a} \Big\},$$
$$\mathfrak{m}(\lambda) = \Big\{ Y \in \mathfrak{m} \ \Big|\ [H, [H, Y]] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a} \Big\}.$$

Let Σ be the set of all non-zero restricted roots. (An element $\lambda \in \mathfrak{a}$ is called a *restricted root* if $\mathfrak{m}(\lambda) \neq 0$.) As is known, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ. For each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ $(|i| \leq 2)$, $\mathfrak{k}_i = \mathfrak{m}_i = 0$ $(|i| > 2)$. Then, we have $\mathfrak{m}_0 = \mathfrak{a} = \boldsymbol{R}\mu$ and

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_1 + \mathfrak{k}_2 \quad (\textit{orthogonal direct sum}),$$
$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \mathfrak{m}_2 \quad (\textit{orthogonal direct sum}).$$

The dimensions of the factors are given by $\dim \mathfrak{k}_0 = 6$, $\dim \mathfrak{k}_1 = \dim \mathfrak{m}_1 = 4$ and $\dim \mathfrak{k}_2 = \dim \mathfrak{m}_2 = 3$ (precisely, see [7]).

We now show several formulas concerning the bracket operation of $\mathfrak{g}$. By the definition of the subspaces $\mathfrak{k}_i$ and $\mathfrak{m}_i$ we easily have

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}. \tag{2.1}$$

Moreover, we have

Proposition 2 *Let $Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2$, $Y_1, Y_1' \in \mathfrak{m}_1$. Then:*

$$[Y_i, [Y_i, Y_j']] = -(1 + 3\delta_{ij})(\mu, \mu)\{(Y_i, Y_i)Y_j' - (Y_i, Y_j')Y_i\}, \quad (i, j = 0, 1), \tag{2.2}$$

$$[Y_i, [Y_i', Y_j]] + [Y_i', [Y_i, Y_j]] = -2(\mu, \mu)(Y_i, Y_i')Y_j, \quad (i, j = 0, 1,\ i \neq j), \tag{2.3}$$

$$[Y_i, [Y_i, X_1]] = -(\mu, \mu)(Y_i, Y_i)X_1, \quad \forall X_1 \in \mathfrak{k}_1 \quad (i = 0, 1), \tag{2.4}$$

where δ_{ij} denotes the Kronecker delta.

Proof. We first prove (2.2). Assume that $i = j$ and $Y_i \neq 0$. Set $Y_i'' = Y_i' - (Y_i', Y_i)/(Y_i, Y_i) \cdot Y_i$. Then, we know that $(Y_i, Y_i'') = 0$ and that $Y_i'' \in \mathfrak{a} + \mathfrak{m}_2$ if $i = 0$ and $Y_i'' \in \mathfrak{m}_1$ if $i = 1$. Hence, by Proposition 10 of [7], we have $[Y_i, [Y_i, Y_i'']] = -4(\mu, \mu)(Y_i, Y_i)Y_i''$. Therefore, we can easily obtain (2.2) in the case $i = j$. In the case $i \neq j$, (2.2) directly follows from Proposition 10 of [7].

We next prove (2.3). Since $i \neq j$, it follows that $(Y_i, Y_j) = (Y_i', Y_j) = 0$. Hence, by (2.2) we have $[Y_i + Y_i', [Y_i + Y_i', Y_j]] = -(\mu, \mu)(Y_i + Y_i', Y_i + Y_i')Y_j$. This, together with $[Y_i, [Y_i, Y_j]] = -(\mu, \mu)(Y_i, Y_i)Y_j$ and $[Y_i', [Y_i', Y_j]] = -(\mu, \mu)(Y_i', Y_i')Y_j$, proves (2.3).

We finally prove (2.4). We note that $[Y_1, \mathfrak{a} + \mathfrak{m}_2] = \mathfrak{k}_1$ holds for any $Y_1 \in \mathfrak{m}_1$ ($\neq 0$). In fact, it is easy to see $[Y_1, \mathfrak{a} + \mathfrak{m}_2] \subset \mathfrak{k}_1$ (see (2.1)). Moreover, the map $\mathfrak{a} + \mathfrak{m}_2 \ni Y_0' \longmapsto [Y_1, Y_0'] \in \mathfrak{k}_1$ is bijective, because $[Y_1, Y_0'] \neq 0$ if $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0' \neq 0$) (recall that $\operatorname{rank}(P^2(\boldsymbol{H})) = 1$) and because $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{k}_1$. Let $X_1 \in \mathfrak{k}_1$. Then, by $[Y_1, \mathfrak{a} + \mathfrak{m}_2] = \mathfrak{k}_1$ we can take an element $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$ such that $[Y_1, Y_0'] = X_1$. Now, applying $\operatorname{ad} Y_1$ to the equality $[Y_1, [Y_1, Y_0']] = -(\mu, \mu)(Y_1, Y_1)Y_0'$ (see (2.2)), we have $[Y_1, [Y_1, X_1]] = -(\mu, \mu)(Y_1, Y_1)X_1$, proving (2.4) for the case $i = 1$. Similarly, we can prove (2.4) for the case $i = 0$. □

Let Y_0, $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$. Define a linear mapping $L(Y_0, Y_0')$ of $\mathfrak{m}_1$ to $\mathfrak{m}$ by

$$L(Y_0, Y_0')Y_1 = \big[Y_0, [Y_0', Y_1]\big], \qquad Y_1 \in \mathfrak{m}_1.$$

Then, we have

Proposition 3 *Let Y_0, $Y_0' \in \mathfrak{a} + \mathfrak{m}_2$. Then:*
(1) *$L(Y_0, Y_0')\mathfrak{m}_1 \subset \mathfrak{m}_1$. The transpose of $L(Y_0, Y_0')$ with respect to $(\ ,\ )$ is given by $L(Y_0', Y_0)$, i.e., ${}^tL(Y_0, Y_0') = L(Y_0', Y_0)$.*
(2) *Let $\mathbf{1}_{\mathfrak{m}_1}$ be the identity map of $\mathfrak{m}_1$. Then:*

(2*a*) $L(Y_0, Y_0') + L(Y_0', Y_0) = -2(\mu, \mu)(Y_0, Y_0')\,\mathbf{1}_{\mathfrak{m}_1}$;
(2*b*) $L(Y_0, Y_0') \cdot L(Y_0', Y_0) = (\mu, \mu)^2(Y_0, Y_0)(Y_0', Y_0')\,\mathbf{1}_{\mathfrak{m}_1}$.

Proof. The assertion (1) is clear from (2.1) and the $\operatorname{ad}\mathfrak{g}$-invariance of $(\ ,\ )$. Let $Y_1 \in \mathfrak{m}_1$. Since $[Y_0, Y_1] \in \mathfrak{k}_1$, we have $\big[Y_0', [Y_0', [Y_0, Y_1]]\big] = -(\mu, \mu)(Y_0', Y_0')[Y_0, Y_1]$ (see (2.4)). Hence, by applying $\operatorname{ad} Y_0$ to this equality, we easily have (2*b*). The equality (2*a*) directly follows from (2.3). □

Here, we recall the notion of pseudo-abelian subspace of $\mathfrak{m}$. Let Q be a subspace of $\mathfrak{m}$. Q is called *pseudo-abelian* if it satisfies $[Q, Q] \subset \mathfrak{k}_0$ (see [6]).

Proposition 4 (1) *Any subspace Q of $\mathfrak{m}_2$ is pseudo-abelian.*
(2) *Let Q be a pseudo-abelian subspace satisfying $Q \not\subset \mathfrak{m}_2$. Then,* $\dim Q \leq 2$.

Accordingly, the inequality $\dim Q \leq 3$ *holds for any pseudo-abelian subspace Q, and the equality holds when and only when $Q = \mathfrak{m}_2$.*

Proof. Since $[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}_0$ (see (2.1)), it follows that any subspace of $\mathfrak{m}_2$ is pseudo-abelian. On the contrary, we already proved in Lemma 5.4 of [6] that for a pseudo-abelian subspace Q with $Q \not\subset \mathfrak{m}_2$ it holds $\dim Q \leq 1 + n(\mu)$, where $n(\mu)$ means the local pseudo-nullity of the restricted root μ. (For the definition of the local pseudo-nullity, see §3 of [6].) In the case $G/K = P^2(\boldsymbol{H})$, we have $n(\mu) = 1$ (see Theorem 3.2 and Table 3 of [6]). Hence, we have $\dim Q \leq 2$. □

For later use, we obtain the normal form of a 2-dimensional pseudo-abelian subspace Q with $Q \not\subset \mathfrak{m}_2$.

Proposition 5 *Let ξ_1 and η_1 be elements of $\mathfrak{m}_1$ satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$ and $(\xi_1, \eta_1) = 0$. Then, the 2-dimensional subspace Q $(\subset \mathfrak{m})$ defined by*

$$Q = \boldsymbol{R}(\mu+\xi_1)+\boldsymbol{R}\left(\eta_1+\frac{1}{4(\mu,\mu)^2}\big[\mu,\,[\xi_1,\eta_1]\big]\right) \tag{2.5}$$

is pseudo-abelian and $Q \not\subset \mathfrak{m}_2$.

Conversely, if Q *is a pseudo-abelian subspace of* $\mathfrak{m}$ *with* $Q \not\subset \mathfrak{m}_2$ *and* $\dim Q = 2$, *then* Q *can be written in the form* (2.5) *by utilizing suitable elements* ξ_1 *and* $\eta_1 \in \mathfrak{m}_1$ *satisfying* $(\xi_1,\xi_1) = 2(\mu,\mu)$, $\eta_1 \neq 0$ *and* $(\xi_1,\eta_1) = 0$.

Proof. Let ξ_1 and η_1 be elements of $\mathfrak{m}_1$ satisfying $(\xi_1,\xi_1) = 2(\mu,\mu)$, $\eta_1 \neq 0$ and $(\xi_1,\eta_1) = 0$. Then, the subspace Q defined by (2.5) satisfies $Q \not\subset \mathfrak{m}_2$ and $\dim Q = 2$. Set $\eta_2 = (1/4(\mu,\mu)^2)\big[\mu,\,[\xi_1,\eta_1]\big]$. Then, it is easily verified that $\eta_2 \in \mathfrak{m}_2$. We now show that Q is pseudo-abelian. By (2.3) and $(\xi_1,\eta_1) = 0$, we have $\big[\xi_1,\,[\eta_1,\mu]\big] = -\big[\eta_1,\,[\xi_1,\mu]\big]$. Hence, by the Jacobi identity we have

$$\big[\mu,\,[\xi_1,\eta_1]\big] = \big[[\mu,\xi_1],\eta_1\big]+\big[\xi_1,\,[\mu,\eta_1]\big] = -2\big[\xi_1,\,[\eta_1,\mu]\big].$$

Consequently, we have $\eta_2 = -(1/2(\mu,\mu)^2)\big[\xi_1,\,[\eta_1,\mu]\big]$. Note that $[\eta_1,\mu] \in \mathfrak{k}_1$. Then, by the formula (2.4) and the assumption $(\xi_1,\xi_1) = 2(\mu,\mu)$ we have

$$[\xi_1,\eta_2] = -\frac{1}{2(\mu,\mu)^2}\big[\xi_1,\,[\xi_1,\,[\eta_1,\mu]]\big] = \frac{(\xi_1,\xi_1)}{2(\mu,\mu)}[\eta_1,\mu] = -[\mu,\eta_1].$$

Moreover, since $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}$ and since

$$\big[\mu,\,[\mu,\eta_2]+[\xi_1,\eta_1]\big] = -4(\mu,\mu)^2\eta_2+\big[\mu,\,[\xi_1,\eta_1]\big] = 0,$$

it follows that $[\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}_0$. (Note that an element $X \in \mathfrak{k}$ belongs to $\mathfrak{k}_0$ if and only if $[\mu,X] = 0$.) By these relations we have

$$\begin{aligned}[\mu+\xi_1,\eta_1+\eta_2] &= [\mu,\eta_1] + [\xi_1,\eta_2] + [\mu,\eta_2] + [\xi_1,\eta_1]\\ &= 0 + [\mu,\eta_2] + [\xi_1,\eta_1] \in \mathfrak{k}_0.\end{aligned}$$

Since $Q = \boldsymbol{R}(\mu+\xi_1) + \boldsymbol{R}(\eta_1+\eta_2)$, this implies that Q is a pseudo-abelian subspace.

We next prove the converse. Let Q be a pseudo-abelian subspace with $Q \not\subset \mathfrak{m}_2$ and $\dim Q = 2$. Then, viewing the proof of Lemma 5.4 of [6], we know that $Q \cap \mathfrak{m}_2 = 0$ and $\dim(Q \cap (\mathfrak{m}_1+\mathfrak{m}_2)) \leq n(\mu) = 1$. Consequently, we have $Q \not\subset \mathfrak{m}_1+\mathfrak{m}_2$, because $\dim Q = 2$. Therefore, there is a basis $\{\xi,\eta\}$ of Q written in the form $\xi = \mu+\xi_1+\xi_2$, $\eta = \eta_1+\eta_2$, where ξ_1, $\eta_1 \in \mathfrak{m}_1$, ξ_2, $\eta_2 \in \mathfrak{m}_2$. Here, we note that $\eta_1 \neq 0$, because $Q \cap \mathfrak{m}_2 = 0$. Subtracting a constant multiple of η from ξ if necessary, we may assume that $(\xi_1,\eta_1) = 0$.

Since

$$[\xi,\eta]=[\mu+\xi_2,\eta_1]+[\xi_1,\eta_2]+[\mu+\xi_2,\eta_2]+[\xi_1,\eta_1]\in\mathfrak{k}_0$$

and since $[\mu+\xi_2,\eta_1]+[\xi_1,\eta_2]\in\mathfrak{k}_1$, $[\mu+\xi_2,\eta_2]+[\xi_1,\eta_1]\in\mathfrak{k}_0+\mathfrak{k}_2$ and $[\xi_2,\eta_2]\in\mathfrak{k}_0$, it follows that

$$[\mu+\xi_2,\eta_1]+[\xi_1,\eta_2]=0, \tag{2.6}$$

$$[\mu,\eta_2]+[\xi_1,\eta_1]\in\mathfrak{k}_0. \tag{2.7}$$

Applying $\mathrm{ad}\,\mu$ to (2.7), we have $\eta_2=(1/4(\mu,\mu)^2)[\mu,[\xi_1,\eta_1]]$. By this equality and the assumption $(\xi_1,\eta_1)=0$, we can deduce $[\xi_1,\eta_2]=((\xi_1,\xi_1)/2(\mu,\mu))[\eta_1,\mu]$ (see the arguments stated above). Putting this into (2.6), we have

$$\left[\left(1-\frac{(\xi_1,\xi_1)}{2(\mu,\mu)}\right)\mu+\xi_2,\eta_1\right]=0.$$

Since $\eta_1\neq 0$ and $\mathrm{rank}(P^2(\boldsymbol{H}))=1$, we have $\left(1-(\xi_1,\xi_1)/2(\mu,\mu)\right)\mu+\xi_2=0$. This proves $(\xi_1,\xi_1)=2(\mu,\mu)$ and $\xi_2=0$, completing the proof of the converse. □

3. The Gauss equation

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\boldsymbol{R}$ endowed with an inner product $\langle\,,\,\rangle$. Let $S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. We call the following equation on $\Psi\in S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ the *Gauss equation* associated with $\boldsymbol{N}$:

$$([[X,Y],Z],W)=\langle\Psi(X,Z),\Psi(Y,W)\rangle-\langle\Psi(X,W),\Psi(Y,Z)\rangle, \tag{3.1}$$

where X, Y, Z, $W\in\mathfrak{m}$. We denote by $\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$ the set of all solutions of (3.1), which is called the *Gaussian variety* associated with $\boldsymbol{N}$.

As in the case of $P^2(\boldsymbol{Cay})$ (Theorem 11 of [8]), we can prove the following

Theorem 6 *Let $\boldsymbol{N}$ be a euclidean vector space with* $\dim\boldsymbol{N}=6$. *Let $\Psi\in S^2\mathfrak{m}^*\otimes\boldsymbol{N}$ be a solution of the Gauss equation* (3.1), *i.e., $\Psi\in\mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. Then:*

(1) *There are linearly independent vectors $\mathbf{A}$ and $\mathbf{B}\in\boldsymbol{N}$ satisfying*

(i) $\langle\mathbf{A},\mathbf{A}\rangle=\langle\mathbf{B},\mathbf{B}\rangle=4(\mu,\mu)$ *and* $\langle\mathbf{A},\mathbf{B}\rangle=2(\mu,\mu)$;

(ii) $\Psi(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0,\ Y_0' \in \mathfrak{a} + \mathfrak{m}_2;$
(iii) $\Psi(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1,\ Y_1' \in \mathfrak{m}_1;$
(iv) $\langle \mathbf{A}, \Psi(\mu, \mathfrak{m}_1)\rangle = \langle \mathbf{B}, \Psi(\mu, \mathfrak{m}_1)\rangle = 0.$

(2) $\Psi(Y_1, Y_2) = -\dfrac{1}{(\mu,\mu)^2}\Psi(\mu, L(\mu, Y_2)Y_1), \qquad \forall Y_1 \in \mathfrak{m}_1,\ \forall Y_2 \in \mathfrak{m}_2.$

(3) $\langle \Psi(\mu, Y_1), \Psi(\mu, Y_1')\rangle = (\mu,\mu)^2 (Y_1, Y_1'), \qquad \forall Y_1,\ Y_1' \in \mathfrak{m}_1.$

Let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ by

$$(h\Psi)(X, Y) = h(\Psi(X, Y)),$$

where $\Psi \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$. It is easily seen that $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is invariant under this action, i.e., $h\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) = \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ for any $h \in O(\boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is *EOS* if $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) \neq \emptyset$ and if $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is consisting of essentially one solution, i.e., for any solutions Ψ and $\Psi' \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$, there is an element $h \in O(\boldsymbol{N})$ satisfying $\Psi' = h\Psi$ (see [8]).

By Theorem 6 we can show

Theorem 7 *Let $\boldsymbol{N}$ be a euclidean vector space with* $\dim \boldsymbol{N} = 6$. *Then, $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ is EOS.*

Proof. The proof of this theorem is quite similar to that of Theorem 10 in [8].

First we note that $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N}) \neq \emptyset$, because the second fundamental form of the canonical isometric imbedding $\boldsymbol{f}_0$ at the origin $o \in P^2(\boldsymbol{H})$ satisfies (3.1).

Let $\{E_i\ (1 \leq i \leq 4)\}$ be an orthonormal basis of $\mathfrak{m}_1$. (Note that $\dim \mathfrak{m}_1 = 4$.) Let $\Psi \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $\mathbf{A}, \mathbf{B}$ be the vectors of $\boldsymbol{N}$ stated in Theorem 6. We define vectors $\{\mathbf{F}_i\ (1 \leq i \leq 6)\}$ of $\boldsymbol{N}$ by setting $\mathbf{F}_i = \Psi(\mu, E_i)/(\mu,\mu)\ (1 \leq i \leq 4)$, $\mathbf{F}_5 = (\mathbf{A}+\mathbf{B})/2\sqrt{3}\,|\mu|$ and $\mathbf{F}_6 = (\mathbf{A}-\mathbf{B})/2|\mu|$. By Theorem 6 we can show that $\{\mathbf{F}_i\ (1 \leq i \leq 6)\}$ forms an orthonormal basis of $\boldsymbol{N}$. Now let Ψ' be another element of $\mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be the vectors stated in Theorem 6 for Ψ'. As in the case of Ψ we can also define an orthonormal basis $\{\mathbf{F}_i'\ (1 \leq i \leq 6)\}$ of $\boldsymbol{N}$. Then, there is an element $h \in O(6)$ satisfying $\mathbf{F}_i' = h\mathbf{F}_i\ (1 \leq i \leq 6)$. Here, we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\Psi'(\mu, E_i) = h\Psi(\mu, E_i)\ (1 \leq i \leq 4)$. Set $\Phi =$

$\Psi' - h\Psi \in S^2\mathfrak{m}^* \otimes N$. Then, by Theorem 6 (1) we have

$$\Phi(\mathfrak{a}+\mathfrak{m}_2, \mathfrak{a}+\mathfrak{m}_2) = \Phi(\mathfrak{m}_1, \mathfrak{m}_1) = \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0.$$

By Theorem 6 (2) and by the fact $L(\mu, \mathfrak{m}_2)\mathfrak{m}_1 \subset \mathfrak{m}_1$ we have

$$\Phi(\mathfrak{m}_2, \mathfrak{m}_1) \subset \Phi(\mu, L(\mu, \mathfrak{m}_2)\mathfrak{m}_1) \subset \Phi(\mathfrak{a}, \mathfrak{m}_1) = 0,$$

which proves $\Phi(\mathfrak{m}_2, \mathfrak{m}_1) = 0$. Therefore, we have $\Phi = 0$, i.e., $\Psi' = h\Psi$, completing the proof of Theorem 7. □

By Theorem 7 we know that $P^2(\boldsymbol{H})$ is formally rigid in codimension 6 in the sense of Agaoka-Kaneda [8]. Therefore, Theorem 1 can be obtained by Theorem 7 and the rigidity theorem (Theorem 5 of [8]).

Before proceeding to the proof of Theorem 6, we make several preparations.

Let $\boldsymbol{N}$ be a euclidean vector space. In what follows we assume $\dim \boldsymbol{N} = 6$. Let $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. Let $\Psi \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ and $Y \in \mathfrak{m}$. We define a linear map Ψ_Y of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\Psi_Y : \mathfrak{m} \ni Y' \longmapsto \Psi(Y, Y') \in \boldsymbol{N},$$

and denote by $\mathbf{Ker}(\Psi_Y)$ the kernel of Ψ_Y. We call an element $Y \in \mathfrak{m}$ *singular* (resp. *non-singular*) with respect to Ψ if $\Psi_Y(\mathfrak{m}) \neq \boldsymbol{N}$ (resp. $\Psi_Y(\mathfrak{m}) = \boldsymbol{N}$).

Let $\Psi \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ and let $Y \in \mathfrak{m}$ ($Y \neq 0$). Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, as shown in the proof of Proposition 5 of [7], the subspace $Q_Y = \mathrm{Ad}(k)^{-1}\,\mathbf{Ker}(\Psi_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$.

Proposition 8 *Let* $\Psi \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$ *and let* $Y \in \mathfrak{m}$ ($Y \neq 0$). *Then:*
(1) $\dim \mathbf{Ker}(\Psi_Y) = 2$ *or* 3. *Moreover, Y is non-singular (resp. singular) with respect to Ψ if and only if* $\dim \mathbf{Ker}(\Psi_Y) = 2$ (*resp.* $\dim \mathbf{Ker}(\Psi_Y) = 3$).
(2) *Let $k \in K$ satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Then,* $\mathbf{Ker}(\Psi_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$. *Consequently, Y is non-singular (resp. singular) with respect to Ψ if and only if* $\mathbf{Ker}(\Psi_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$ (*resp.* $\mathbf{Ker}(\Psi_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$).

Remark 1 Recall that in the case of the Cayley projective plane $P^2(\boldsymbol{Cay})$ the inclusion $\mathbf{Ker}(\Psi_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ in Proposition 8 (2) can be proved by a simple discussion. There, the inclusion automatically follows from the fact that any high-dimensional pseudo-abelian subspace must be contained in $\mathfrak{m}_2$ (see Propositions 8 and 12 of [8]). In contrast, it is not a simple task to show the inclusion $\mathbf{Ker}(\Psi_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ in our case $P^2(\boldsymbol{H})$. We will prove this inclusion by making use of the normal form of the pseudo-abelian subspaces not contained in $\mathfrak{m}_2$ (see Proposition 5).

Proof of Proposition 8. Let $Y \in \mathfrak{m}$ ($Y \neq 0$). Set $Q_Y = \mathrm{Ad}(k)^{-1}\mathbf{Ker}(\Psi_Y)$, where $k \in K$ is an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Since Q_Y is pseudo-abelian, it follows that $\dim Q_Y \leq 3$ (see Proposition 4). Hence, $\dim \mathbf{Ker}(\Psi_Y) \leq 3$. On the other hand, since $\dim \boldsymbol{N} = 6$ and $\dim \mathfrak{m} = 8$, it follows that $\dim \mathbf{Ker}(\Psi_Y) \geq 2$. Therefore, Y is non-singular (resp. singular) with respect to Ψ if and only if $\dim \mathbf{Ker}(\Psi_Y) = 2$ (resp. $\dim \mathbf{Ker}(\Psi_Y) = 3$). This proves (1).

To show the first statement of (2) it suffices to prove $Q_Y \subset \mathfrak{m}_2$. Now, let us suppose the contrary, i.e., $Q_Y \not\subset \mathfrak{m}_2$. Then, we have $\dim Q_Y = 2$ (see (1) and Proposition 4 (2)). Hence, there is a basis $\{\xi, \eta\}$ of Q_Y written in the form $\xi = \mu + \xi_1$, $\eta = \eta_1 + (1/4(\mu,\mu)^2)\big[\mu, [\xi_1, \eta_1]\big]$, where ξ_1 and η_1 are elements of $\mathfrak{m}_1$ satisfying $(\xi_1, \xi_1) = 2(\mu, \mu)$, $\eta_1 \neq 0$, $(\xi_1, \eta_1) = 0$ (see Proposition 5). Let $\{\zeta_1^1, \zeta_1^2\}$ be a basis of the orthogonal complement of $\boldsymbol{R}\xi_1 + \boldsymbol{R}\eta_1$ in $\mathfrak{m}_1$. Set $\zeta^i = \zeta_1^i + (1/4(\mu,\mu)^2)\big[\mu, [\xi_1, \zeta_1^i]\big]$ ($i = 1, 2$). Since $\big[\mu, [\xi_1, \zeta_1^i]\big] \in \mathfrak{m}_2$ ($i = 1, 2$), we know that the vectors ζ^1 and ζ^2 are linearly independent. More strongly, they are linearly independent modulo Q_Y, i.e., $Q_Y \cap (\boldsymbol{R}\zeta^1 + \boldsymbol{R}\zeta^2) = 0$. Moreover, by Proposition 5 we know that the subspace $Q^i = \boldsymbol{R}\xi + \boldsymbol{R}\zeta^i$ ($i = 1, 2$) is also pseudo-abelian, because $(\xi_1, \zeta_1^i) = 0$. Consequently, we have $\big[[\xi, \zeta^i], \mu\big] = 0$ ($i = 1, 2$).

Set $X = \mathrm{Ad}(k)\xi$, $Z^i = \mathrm{Ad}(k)\zeta^i$ ($i=1, 2$). Then, we have $X \in \mathbf{Ker}(\Psi_Y)$ ($X \neq 0$), $\mathbf{Ker}(\Psi_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$ and $\big[[X, Z^i], Y\big] = 0$ ($i = 1, 2$). By the Gauss equation (3.1) we have

$$\begin{aligned} 0 &= \big(\big[[X, Z^i], Y\big], W\big) \\ &= \big\langle \Psi(X,Y), \Psi(Z^i, W)\big\rangle - \big\langle \Psi(X, W), \Psi(Z^i, Y)\big\rangle, \quad (i = 1, 2), \end{aligned}$$

where W is an arbitrary element of $\mathfrak{m}$. Since $\Psi_Y(X) = 0$, we obtain by this equality $\big\langle \Psi_X(W), \Psi(Z^i, Y)\big\rangle = 0$, i.e., $\big\langle \Psi_X(\mathfrak{m}), \Psi(Z^i, Y)\big\rangle = 0$ ($i = 1, 2$). We note that the vectors $\Psi(Z^1, Y)$ and $\Psi(Z^2, Y)$ are linearly independent, because $\mathbf{Ker}(\Psi_Y) \cap (\boldsymbol{R}Z^1 + \boldsymbol{R}Z^2) = 0$. Hence, we have $\dim \Psi_X(\mathfrak{m}) \leq \dim \boldsymbol{N} - 2 = 4$, implying $\dim \mathbf{Ker}(\Psi_X) \geq 4$. This contradicts the assertion (1). Thus, we have $Q_Y \subset \mathfrak{m}_2$, proving the first statement of (2). The last statement of (2) is now clear. □

As a corollary of Proposition 8 we obtain

Proposition 9 *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Then:*
(1) *Let* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$. *Then,* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \mid (\xi, Y_0) = 0\}$. *If* Y_0 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \mid (\xi, Y_0) = 0\}$.
(2) *Let* $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$. *Then,* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) \subset \{\eta \in \mathfrak{m}_1 \mid (\eta, Y_1) = 0\}$. *If* Y_1 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) = \{\eta \in \mathfrak{m}_1 \mid (\eta, Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$. Then, we can take an element $k_0 \in K$ such that $\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0$ and $\mathrm{Ad}(k_0)(\mathfrak{m}_2) = \{\xi \in \mathfrak{a} + \mathfrak{m}_2 \mid (\xi, Y_0) = 0\}$ (see Proposition 7 of [7]). This proves (1). Similarly, for $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$, we can easily show (2). □

Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. We call a subspace U of $\mathfrak{m}$ *singular* with respect to $\boldsymbol{\Psi}$ if each element of U is singular with respect to $\boldsymbol{\Psi}$.

Proposition 10 *Let* $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. *Assume that* $Y \in \mathfrak{m}$ $(Y \neq 0)$ *is non-singular with respect to* $\boldsymbol{\Psi}$. *Then, there is a non-zero vector* $\mathbf{E} \in \boldsymbol{N}$ *such that*

$$\boldsymbol{N} = \boldsymbol{R}\mathbf{E} \dot{+} \boldsymbol{\Psi}_\xi(\mathfrak{m}) \qquad (\textit{orthogonal direct sum}) \tag{3.2}$$

holds for any $\xi \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)$ $(\xi \neq 0)$. *Consequently,* $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ *is a singular subspace with respect to* $\boldsymbol{\Psi}$.

Proof. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. Then, since Y is non-singular, we have $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$. Take a non-zero element satisfying $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ and $Y' \notin \mathbf{Ker}(\boldsymbol{\Psi}_Y)$ and set $\mathbf{E} = \boldsymbol{\Psi}(Y, Y')$ $(\neq 0)$. Let $\xi \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)$ $(\xi \neq 0)$. Then, by the Gauss equation (3.1) we have

$$\big([[\xi, Y'], Y], W\big) = \big\langle \boldsymbol{\Psi}(\xi, Y), \boldsymbol{\Psi}(Y', W)\big\rangle - \big\langle \boldsymbol{\Psi}(\xi, W), \boldsymbol{\Psi}(Y', Y)\big\rangle,$$

where W is an arbitrary element of $\mathfrak{m}$. Here, we note that $\big[[\xi, Y'], Y\big] = 0$, because $\big[[\xi, Y'], Y\big] \in \mathrm{Ad}(k)\big[[\mathfrak{m}_2, \mathfrak{m}_2], \mu\big] = 0$. Since $\boldsymbol{\Psi}(\xi, Y) = 0$, we obtain by the above equality $\big\langle \mathbf{E}, \boldsymbol{\Psi}(\xi, W)\big\rangle = 0$. This shows $\big\langle \mathbf{E}, \boldsymbol{\Psi}_\xi(\mathfrak{m})\big\rangle = 0$ and hence $\boldsymbol{\Psi}_\xi(\mathfrak{m}) \neq \boldsymbol{N}$. Consequently, ξ is singular with respect to $\boldsymbol{\Psi}$. Since $\dim \mathbf{Ker}(\boldsymbol{\Psi}_\xi) = 3$ (see Proposition 8), we have $\dim \boldsymbol{\Psi}_\xi(\mathfrak{m}) = 5$, which proves the decomposition (3.2). □

4. Proof of Theorem 6

In this section, with the preparations in the previous sections, we will prove Theorem 6. We first show

Proposition 11 *Let $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Then, there are singular subspaces U $(\subset \mathfrak{a}+\mathfrak{m}_2)$ and V $(\subset \mathfrak{m}_1)$ with respect to $\mathbf{\Psi}$ satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$.

Proof. If $\mathfrak{a}+\mathfrak{m}_2$ contains no non-singular element with respect to $\mathbf{\Psi}$, then set $U = \mathfrak{a}+\mathfrak{m}_2$. On the contrary, if there is a non-singular element $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$, then set $U = \mathbf{Ker}(\mathbf{\Psi}_{Y_0})$. In this case we know that $\dim U = 2$, $U \subset \mathfrak{a}+\mathfrak{m}_2$ and that U is a singular subspace with respect to $\mathbf{\Psi}$ (see Proposition 8, Proposition 9 and Proposition 10).

Similarly, we can show that there is a singular subspace V of $\mathfrak{m}_1$ with respect to $\mathbf{\Psi}$ satisfying the desired properties. □

Proposition 12 *Let $\mathbf{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}), \boldsymbol{N})$. Let U $(\subset \mathfrak{a}+\mathfrak{m}_2)$ and V $(\subset \mathfrak{m}_1)$ be singular subspaces with respect to $\mathbf{\Psi}$ satisfying* $\dim U \geq 2$ *and* $\dim V \geq 2$. *Then, there are vectors $\mathbf{A}$, $\mathbf{B} \in \boldsymbol{N}$ such that:*

(1) $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 4(\mu, \mu)$.

(2) *Let $\xi \in U$ and $\eta \in V$. Then:*

(2*a*) $\mathbf{\Psi}(\xi, Y_0) = (\xi, Y_0)\mathbf{A}, \quad \forall Y_0 \in \mathfrak{a}+\mathfrak{m}_2;$

(2*b*) $\mathbf{\Psi}(\eta, Y_1) = (\eta, Y_1)\mathbf{B}, \quad \forall Y_1 \in \mathfrak{m}_1.$

(3) *Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

(3*a*) $\langle \mathbf{A}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1) \rangle = \langle \mathbf{B}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1) \rangle = 0;$

(3*b*) $\langle \mathbf{A}, \mathbf{\Psi}_{Y_1}(\mathfrak{a}+\mathfrak{m}_2) \rangle = \langle \mathbf{B}, \mathbf{\Psi}_{Y_1}(\mathfrak{a}+\mathfrak{m}_2) \rangle = 0.$

(4) *Let $\xi \in U$ $(\xi \neq 0)$ and $\eta \in V$ $(\eta \neq 0)$. Then:*

(4*a*) $\mathbf{\Psi}_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + \mathbf{\Psi}_\xi(\mathfrak{m}_1)$ (*orthogonal direct sum*);

(4*b*) $\mathbf{\Psi}_\eta(\mathfrak{m}) = \boldsymbol{R}\mathbf{B} + \mathbf{\Psi}_\eta(\mathfrak{a}+\mathfrak{m}_2)$ (*orthogonal direct sum*).

(5) *Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then:*

(5*a*) $\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{A} \rangle = 4(\mu, \mu)(Y_0, Y_0);$

(5*b*) $\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{B} \rangle = 4(\mu, \mu)(Y_1, Y_1).$

(6) *Let $\xi \in U$, $\eta \in V$, $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Assume that $(\xi, Y_0) = (\eta, Y_1) = 0$. Then:*

(6*a*) $\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{\Psi}_\xi(\mathfrak{m}_1) \rangle = 0;$

(6*b*) $\langle \mathbf{\Psi}(Y_1, Y_1), \mathbf{\Psi}_\eta(\mathfrak{a}+\mathfrak{m}_2) \rangle = 0.$

Proof. The assertions (1), (2) and (3) can be proved in the same manner as in the proof of Proposition 16 of [8]. Hence, we omit their proofs.

Let $\xi \in U$ ($\xi \neq 0$). By (2a) we easily get $\boldsymbol{\Psi}_\xi(\mathfrak{a}+\mathfrak{m}_2) = \boldsymbol{R}\mathbf{A}$ and hence $\boldsymbol{\Psi}_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$. Since $\langle \mathbf{A}, \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)\rangle = 0$ (see (3a)), we have the decomposition (4a). Similarly, we can show (4b).

The assertions (5a) and (6a) are proved as follows: Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. Take $\xi \in U$ ($\xi \neq 0$) such that $(\xi, Y_0) = 0$. Then, we have $\big[[Y_0,\xi],Y_0\big] = 4(\mu,\mu)(Y_0,Y_0)\xi$ (see (2.2)) and $\boldsymbol{\Psi}(\xi,Y_0) = 0$ (see (2a)). By the Gauss equation (3.1) we have

$$\big(\big[[Y_0,\xi],Y_0\big],\xi\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,\xi)\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,\xi), \boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$
$$\big(\big[[Y_0,\xi],Y_0\big],Y_1'\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,Y_1')\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,Y_1'), \boldsymbol{\Psi}(\xi,Y_0)\big\rangle,$$

where Y_1' is an arbitrary element of $\mathfrak{m}_1$. By these equalities we have $\big\langle \boldsymbol{\Psi}(Y_0,Y_0), \mathbf{A}\big\rangle = 4(\mu,\mu)(Y_0,Y_0)$ and $\big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(\xi,Y_1')\big\rangle = 0$. Therefore, we obtain (5a) and (6a). The assertions (5b) and (6b) can be proved in a similar way. □

Remark 2 As seen in the proof of Proposition 11, singular subspaces U and V may not be uniquely determined. However, the vectors $\mathbf{A}$ and $\mathbf{B}$ in Proposition 8 do not depend on the choice of singular subspaces U and V, which will be clarified at the last part of this section (see Lemma 20).

In the following argument, we take and fix an element $\boldsymbol{\Psi} \in \mathcal{G}(P^2(\boldsymbol{H}),\boldsymbol{N})$. We denote by U and V singular subspaces with respect to $\boldsymbol{\Psi}$ satisfying U ($\subset \mathfrak{a}+\mathfrak{m}_2$), V ($\subset \mathfrak{m}_1$), $\dim U \geq 2$ and $\dim V \geq 2$. We also denote by $\mathbf{A}$, $\mathbf{B}$ the vectors of $\boldsymbol{N}$ obtained by applying Proposition 12 to the pair of singular subspaces U and V.

Lemma 13 (1) *Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. Then:*

$$\begin{aligned}&\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1')\big\rangle \\ &= \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1')\big\rangle - (\mu,\mu)(Y_0,Y_0)(Y_1,Y_1'), \quad \forall Y_1,\, Y_1' \in \mathfrak{m}_1.\end{aligned}$$

(2) *Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ and $\xi \in U$ satisfy $(\xi,Y_0) = 0$. Then:*

$$\big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_\xi(Y_1')\big\rangle = (L(Y_0,\xi)Y_1, Y_1'), \quad \forall Y_1,\, Y_1' \in \mathfrak{m}_1.$$

Proof. Putting $X = Y_0$, $Y = Y_1$, $Z = Y_0$, $W = Y_1'$ into (3.1), we have

$$\big(\big[[Y_0,Y_1],Y_0\big],Y_1'\big) = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1')\big\rangle - \big\langle \boldsymbol{\Psi}(Y_0,Y_1'), \boldsymbol{\Psi}(Y_1,Y_0)\big\rangle.$$

Since $\big[Y_0, [Y_0,Y_1]\big] = -(\mu,\mu)(Y_0,Y_0)Y_1$ (see (2.2)), we easily get (1).

Similarly, putting $X = \xi$, $Y = Y_1$, $Z = Y_0$ and $W = Y_1'$ into (3.1), we have

$$\begin{aligned}([[\xi, Y_1], Y_0], Y_1') &= \langle \boldsymbol{\Psi}(\xi, Y_0), \boldsymbol{\Psi}(Y_1, Y_1')\rangle - \langle \boldsymbol{\Psi}(\xi, Y_1'), \boldsymbol{\Psi}(Y_1, Y_0)\rangle \\ &= \langle \mathbf{A}, \boldsymbol{\Psi}(Y_1, Y_1')\rangle(\xi, Y_0) - \langle \boldsymbol{\Psi}_\xi(Y_1'), \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle.\end{aligned}$$

Since $(\xi, Y_0) = 0$, we have

$$\langle \boldsymbol{\Psi}_\xi(Y_1'), \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle = -([[\xi, Y_1], Y_0], Y_1') = (L(Y_0, \xi)Y_1, Y_1'),$$

proving (2). □

Let $\xi \in U$ $(\xi \neq 0)$. Since $\dim \mathbf{Ker}(\boldsymbol{\Psi}_\xi) = 3$ (see Proposition 8) and since $\dim \mathfrak{m} = 8$, we have $\dim \boldsymbol{\Psi}_\xi(\mathfrak{m}) = 5$. Let us denote by $\boldsymbol{E}_\xi$ the one dimensional orthogonal complement of $\boldsymbol{\Psi}_\xi(\mathfrak{m})$ in $\boldsymbol{N}$.

Proposition 14 *Set $C = \langle \mathbf{A}, \mathbf{B}\rangle - (\mu, \mu)$. Then:*
(1) *Let $\xi \in U$. Then:*

$$\langle \boldsymbol{\Psi}_\xi(Y_1), \boldsymbol{\Psi}_\xi(\eta)\rangle = C(\xi, \xi)(Y_1, \eta), \qquad \forall Y_1 \in \mathfrak{m}_1,\ \forall \eta \in V. \tag{4.1}$$

(2) *The inequality $0 < C \leq 3(\mu, \mu)$ holds. The vectors $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if $C \neq 3(\mu, \mu)$ and $\mathbf{A} = \mathbf{B}$ if $C = 3(\mu, \mu)$.*
(3) *Let $\xi \in U$ $(\xi \neq 0)$. Then, $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$, $\quad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.*
(4) *If $C \neq 3(\mu, \mu)$, then:*

$$\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) = \boldsymbol{\Psi}_\xi(\mathfrak{m}_1), \qquad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2\ (Y_0 \neq 0),\ \forall \xi \in U\ (\xi \neq 0); \tag{4.2}$$

$$\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2; \tag{4.3}$$

$$\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}, \qquad \forall Y_1 \in \mathfrak{m}_1. \tag{4.4}$$

Proof. Put $Y_0 = \xi$ and $Y_1' = \eta$ into Lemma 13 (1). Then, since $\boldsymbol{\Psi}(\xi, \xi) = (\xi, \xi)\mathbf{A}$ and $\boldsymbol{\Psi}(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we get (4.1).

In view of Proposition 12 (1), we easily have $\langle \mathbf{A}, \mathbf{B}\rangle \leq 4(\mu, \mu)$ and hence $C \leq 3(\mu, \mu)$. Further, by putting $Y_1 = \eta$ $(\neq 0)$ into (4.1) we know $C > 0$, because $\boldsymbol{\Psi}_\xi(\eta) \neq 0$ (see Proposition 9). This shows $\langle \mathbf{A}, \mathbf{B}\rangle > (\mu, \mu)$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ are linearly independent if $\langle \mathbf{A}, \mathbf{B}\rangle \neq 4(\mu, \mu)$, i.e., $C \neq 3(\mu, \mu)$. It is easy to see that if $C = 3(\mu, \mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 4(\mu, \mu)$, then $\mathbf{A} = \mathbf{B}$.

We next prove (3). Let $\xi \in U$ $(\xi \neq 0)$. By Proposition 12 (4*a*) we know that the orthogonal complement of $\boldsymbol{R}\mathbf{A}$ in $\boldsymbol{N}$ is given by $\boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$.

Hence, by Proposition 12 ($3a$), we have $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) \subset \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.

Finally, we prove (4). Since $C \neq 3(\mu, \mu)$, the subspace $\boldsymbol{R}\mathrm{A} + \boldsymbol{R}\mathrm{B}$ forms a 2-dimensional subspace of $\boldsymbol{N}$. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0 \neq 0$). Then, by Proposition 12 ($3a$) we know that $\boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)$ coincides with the orthogonal complement of $\boldsymbol{R}\mathrm{A}+\boldsymbol{R}\mathrm{B}$ in $\boldsymbol{N}$. (Recall that $\dim \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) = 4$ and $\dim \boldsymbol{N} = 6$.) Let $\xi \in U$ ($\xi \neq 0$). Since $\boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ is also an orthogonal complement of $\boldsymbol{R}\mathrm{A} + \boldsymbol{R}\mathrm{B}$, it follows that $\boldsymbol{\Psi}_\xi(\mathfrak{m}_1) = \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)$. If we take $\xi \in U$ ($\xi \neq 0$) satisfying $(\xi, Y_0) = 0$, then by Proposition 12 ($6a$) we obtain $\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathrm{A} + \boldsymbol{R}\mathrm{B}$. Similarly, we can prove $\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathrm{A} + \boldsymbol{R}\mathrm{B}$ for any $Y_1 \in \mathfrak{m}_1$, completing the proof of (4). □

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $\xi \in U$ ($\xi \neq 0$). Define a linear mapping $\boldsymbol{\Theta}_{Y_0,\xi}$: $\mathfrak{m}_1 \longrightarrow \boldsymbol{N}$ by

$$\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = \boldsymbol{\Psi}_{Y_0}(Y_1) + \frac{1}{C(\xi, \xi)} \boldsymbol{\Psi}_\xi(L(\xi, Y_0)Y_1), \quad Y_1 \in \mathfrak{m}_1. \tag{4.5}$$

Then, we have

Proposition 15 *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$, $\xi \in U$ ($\xi \neq 0$) and $Y_1 \in \mathfrak{m}_1$. Assume that $(\xi, Y_0) = 0$ and $L(\xi, Y_0)Y_1 \in V$. Then:*
(1) $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. *More strongly, if $C \neq 3(\mu, \mu)$, then $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.*
(2) $|\boldsymbol{\Theta}_{Y_0,\xi}(Y_1)|^2 = \langle \boldsymbol{\Psi}(Y_0, Y_0), \boldsymbol{\Psi}(Y_1, Y_1) \rangle - (\mu, \mu)\{1 + (\mu, \mu)/C\}(Y_0, Y_0)(Y_1, Y_1)$.

Proof. By Proposition 14 (3) we know that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$. Here, we note that $\langle \boldsymbol{E}_\xi, \boldsymbol{\Psi}_\xi(\mathfrak{m}_1) \rangle = 0$, because $\boldsymbol{E}_\xi$ is orthogonal to $\boldsymbol{\Psi}_\xi(\mathfrak{m})$. Let $Y_1' \in \mathfrak{m}_1$. Then, by Lemma 13 (2), Proposition 14 (1) and Proposition 3 (2) we have

$$\begin{aligned}
&\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(Y_1') \rangle \\
&= \langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_\xi(Y_1') \rangle + \frac{1}{C(\xi, \xi)} \langle \boldsymbol{\Psi}_\xi(L(\xi, Y_0)Y_1), \boldsymbol{\Psi}_\xi(Y_1') \rangle \\
&= (L(Y_0, \xi)Y_1, Y_1') + (L(\xi, Y_0)Y_1, Y_1') \\
&= 0,
\end{aligned}$$

proving $\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathfrak{m}_1) \rangle = 0$. This implies that $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{E}_\xi$. In the case where $C \neq 3(\mu, \mu)$, we have $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) \in \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1) + \boldsymbol{\Psi}_\xi(\mathfrak{m}_1) = \boldsymbol{\Psi}_\xi(\mathfrak{m}_1)$ (see (4.2)), which proves $\boldsymbol{\Theta}_{Y_0,\xi}(Y_1) = 0$.

Next, we show (2). By Lemma 13 and by the equality $\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_\xi(\mathfrak{m}_1) \rangle$

$= 0$, we have

$$
\begin{aligned}
&\big\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Theta}_{Y_0,\xi}(Y_1)\big\rangle \\
&\quad = \big\langle \boldsymbol{\Theta}_{Y_0,\xi}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle \\
&\quad = \big\langle \boldsymbol{\Psi}_{Y_0}(Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle + \frac{1}{C(\xi,\xi)}\big\langle \boldsymbol{\Psi}_{\xi}(L(\xi,Y_0)Y_1), \boldsymbol{\Psi}_{Y_0}(Y_1)\big\rangle \\
&\quad = \big\langle \boldsymbol{\Psi}(Y_0,Y_0), \boldsymbol{\Psi}(Y_1,Y_1)\big\rangle - (\mu,\mu)(Y_0,Y_0)(Y_1,Y_1) \\
&\qquad\qquad + \frac{1}{C(\xi,\xi)}\big(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y\big).
\end{aligned}
$$

On the other hand, by Proposition 3 we have

$$
\begin{aligned}
\big(L(\xi,Y_0)Y_1, L(Y_0,\xi)Y_1\big) &= \big(L(\xi,Y_0)L(\xi,Y_0)Y_1, Y_1\big) \\
&= -\big(L(Y_0,\xi)L(\xi,Y_0)Y_1, Y_1\big) \\
&= -(\mu,\mu)^2(\xi,\xi)(Y_0,Y_0)(Y_1,Y_1).
\end{aligned}
$$

Therefore, we get the assertion (2). □

With these preparations we begin with the proof Theorem 6. First, we consider the case $\dim V = 2$.

Lemma 16 *Assume that* $\dim V = 2$. *Then,* $C \neq 3(\mu,\mu)$. *Accordingly, the vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *are linearly independent.*

Proof. Take non-zero elements ξ, $\xi' \in U$ satisfying $(\xi,\xi') = 0$. Then, by Proposition 3 (2) it follows that $L(\xi,\xi') = -L(\xi',\xi)$ and $L(\xi,\xi')$ gives an isomorphism of $\mathfrak{m}_1$ onto itself. Let $Y_1 \in L(\xi,\xi')V$. Then, by Proposition 3 ($2b$) we have $L(\xi,\xi')Y_1 \in V$. Hence, by Proposition 15 (1) we have $\boldsymbol{\Theta}_{\xi',\xi}(Y_1) \in \boldsymbol{E}_\xi$. Since $\dim L(\xi,\xi')V = \dim V = 2$ and $\dim \boldsymbol{E}_\xi = 1$, it is possible to take a non-zero element $Y_1 \in L(\xi,\xi')V$ satisfying $\boldsymbol{\Theta}_{\xi',\xi}(Y_1) = 0$. Therefore, by Proposition 15 (2) and Proposition 12 ($2a$) we have

$$
\begin{aligned}
0 &= |\boldsymbol{\Theta}_{\xi',\xi}(Y_1)|^2 \\
&= \big[\big\langle \boldsymbol{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle - (\mu,\mu)\{1 + (\mu,\mu)/C\}(Y_1,Y_1)\big](\xi',\xi').
\end{aligned}
$$

Since $(\xi',\xi') \neq 0$, we have

$$
\big\langle \boldsymbol{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_1,Y_1). \tag{4.6}
$$

Now, we suppose the case $C = 3(\mu,\mu)$. Then, by (4.6) we have $\big\langle \boldsymbol{\Psi}(Y_1,Y_1), \mathbf{A}\big\rangle = \frac{4}{3}(\mu,\mu)(Y_1,Y_1)$. On the other hand, by Proposition 12 ($5b$)

we have $\langle \Psi(Y_1, Y_1), \mathbf{A} \rangle = 4(\mu, \mu)(Y_1, Y_1)$, because $\mathbf{A} = \mathbf{B}$ in case $C = 3(\mu, \mu)$ (see Proposition 14 (2)). Hence, we have $(Y_1, Y_1) = 0$, which contradicts the assumption $Y_1 \neq 0$. Therefore, we have $C \neq 3(\mu, \mu)$ and hence $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. □

Lemma 17 *Assume that* $\dim V = 2$. *Then,* V *can be extended to a 3-dimensional singular subspace contained in* $\mathfrak{m}_1$*, i.e., there is a singular subspace* $\widehat{V}$ $(\subset \mathfrak{m}_1)$ *such that* $V \subset \widehat{V}$ *and* $\dim \widehat{V} = 3$.

Proof. Let $\mathbf{F} \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ be a unit vector which is orthogonal to $\mathbf{B}$. Then, for any $\eta \in V$ we have $\langle \mathbf{F}, \Psi_\eta(\mathfrak{m}) \rangle = 0$, because $\langle \mathbf{F}, \Psi_\eta(\mathfrak{m}) \rangle = \langle \mathbf{F}, \boldsymbol{R}\mathbf{B} + \Psi_\eta(\mathfrak{a} + \mathfrak{m}_2) \rangle = 0$ (see Proposition 12 (4*b*) and (3*b*)).

Now, define a symmetric bilinear form χ on $\mathfrak{m}_1$ by setting

$$\chi(Y_1, Y_1') = \langle \Psi(Y_1, Y_1'), \mathbf{F} \rangle, \qquad Y_1,\ Y_1' \in \mathfrak{m}_1.$$

Since $\Psi(Y_1, Y_1') \in \boldsymbol{R}\mathbf{B} + \boldsymbol{R}\mathbf{F}$ (see Proposition 14 (4)) and $\langle \Psi(Y_1, Y_1'), \mathbf{B} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1')$ for $Y_1,\ Y_1' \in \mathfrak{m}_1$ (see Proposition 12 (5)), we have

$$\Psi(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B} + \chi(Y_1, Y_1')\mathbf{F}, \qquad Y_1,\ Y_1' \in \mathfrak{m}_1. \tag{4.7}$$

Let $V^\perp$ be the orthogonal complement of V in $\mathfrak{m}_1$. Then, we have $\dim V^\perp = 2$. (Recall that $\dim \mathfrak{m}_1 = 4$ and $\dim V = 2$.) Let $\{Y_1, Y_1'\}$ be an orthonormal basis of $V^\perp$. Then, putting $X = Z = Y_1$ and $Y = W = Y_1'$ into the Gauss equation (3.1), we have

$$\begin{aligned}([[Y_1, Y_1'], Y_1], Y_1') = {} & \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1)(Y_1', Y_1') \\ & + \chi(Y_1, Y_1)\chi(Y_1', Y_1') - \chi(Y_1, Y_1')\chi(Y_1', Y_1).\end{aligned}$$

Since $([[Y_1, Y_1'], Y_1], Y_1') = \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1)(Y_1', Y_1')$ (see (2.2)), we have

$$\chi(Y_1, Y_1)\chi(Y_1', Y_1') - \chi(Y_1, Y_1')\chi(Y_1', Y_1) = 0.$$

This implies that χ is degenerate on $V^\perp$. Therefore, there is a non-zero vector $\zeta \in V^\perp$ such that $\chi(\zeta, V^\perp) = 0$, i.e., $\langle \mathbf{F}, \Psi_\zeta(V^\perp) \rangle = 0$.

Let us show that the subspace $\widehat{V} = \boldsymbol{R}\zeta + V$ $(\subset \mathfrak{m}_1)$ is singular with respect to Ψ. Note that $\langle \mathbf{F}, \Psi_\zeta(\mathfrak{a} + \mathfrak{m}_2) \rangle = 0$ (see Proposition 12 (3*b*)). Then, since $\mathfrak{m} = \mathfrak{a} + \mathfrak{m}_2 + V + V^\perp$ and $\Psi_\zeta(V) \subset \boldsymbol{R}\mathbf{B}$, it follows that

$$\begin{aligned}\langle \mathbf{F}, \Psi_\zeta(\mathfrak{m}) \rangle &= \langle \mathbf{F}, \Psi_\zeta(\mathfrak{a} + \mathfrak{m}_2) + \Psi_\zeta(V) + \Psi_\zeta(V^\perp) \rangle \\ &\subset 0 + \langle \mathbf{F}, \boldsymbol{R}\mathbf{B} \rangle + 0 = 0.\end{aligned}$$

Hence, we have $\langle \mathbf{F}, \Psi_{a\zeta+\eta}(\mathfrak{m})\rangle = 0$ for any $a \in \boldsymbol{R}$ and $\eta \in V$. Consequently, $\Psi_{a\zeta+\eta}(\mathfrak{m}) \neq \boldsymbol{N}$, which implies that $a\zeta+\eta \in \widehat{V}$ is singular with respect to Ψ. □

Now, we assume that $\dim V = 2$ and denote by $\widehat{V}$ be the singular subspace stated in the above lemma. Let $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ be the vectors obtained by applying Proposition 12 to the pair of singular subspaces U and $\widehat{V}$. Then, by Proposition 12 (2) we can easily see that $\widehat{\mathbf{A}} = \mathbf{A}$ and $\widehat{\mathbf{B}} = \mathbf{B}$. Therefore, we know that all the statements in Proposition 12 and hence the arguments developed after Proposition 12 are also true if we simply replace V by $\widehat{V}$. Accordingly, without loss of generality we can assume that $\dim V \geq 3$.

Lemma 18 $\langle \Psi(Y_0, Y_0), \mathbf{B}\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_0, Y_0), \quad \forall Y_0 \in \mathfrak{a}+\mathfrak{m}_2.$

Proof. As in the proof of Lemma 16, we can prove that $C \neq 3(\mu,\mu)$. Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ ($Y_0 \neq 0$). Take $\xi \in U$ ($\xi \neq 0$) such that $(\xi, Y_0) = 0$, which is possible because $\dim U \geq 2$. Then, by Proposition 3 (2) it follows that $L(\xi, Y_0) = -L(Y_0, \xi)$ and that the map $L(\xi, Y_0)$ gives an isomorphism of $\mathfrak{m}_1$ onto itself. Now, take $\eta \in V$ ($\eta \neq 0$) such that $L(\xi, Y_0)\eta \in V$. This is also possible because $\dim L(\xi, Y_0)V = \dim V \geq 3$ and $\dim(V \cap L(\xi, Y_0)V) \geq 2$. (Note that $\dim \mathfrak{m}_1 = 4$.) Then, by Proposition 15 and Proposition 12 (2*b*) we have

$$\begin{aligned} 0 &= |\Theta_{Y_0,\xi}(\eta)|^2 \\ &= \big[\langle \Psi(Y_0, Y_0), \mathbf{B}\rangle - (\mu,\mu)\{1+(\mu,\mu)/C\}(Y_0, Y_0)\big](\eta,\eta). \end{aligned}$$

Since $(\eta,\eta) \neq 0$, we get the lemma. □

Lemma 19 $C = (\mu,\mu)$, *i.e.*, $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$.

Proof. Take $\xi \in U$ ($\xi \neq 0$). Then, by Lemma 18 and $\Psi(\xi,\xi) = (\xi,\xi)\mathbf{A}$ (see Proposition 12 (2*a*)), we have $\langle \mathbf{A}, \mathbf{B}\rangle = (\mu,\mu)\{1+(\mu,\mu)/C\}$. Since $C = \langle \mathbf{A}, \mathbf{B}\rangle - (\mu,\mu)$, we easily have $C^2 = (\mu,\mu)^2$. Moreover, since $C > 0$ (see Proposition 14 (2)), it follows that $C = (\mu,\mu)$, i.e., $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu,\mu)$. □

Now, we show

Lemma 20 (1) $\Psi(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0,\ Y_0' \in \mathfrak{a}+\mathfrak{m}_2.$
(2) $\Psi(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1,\ Y_1' \in \mathfrak{m}_1.$

Proof. On account of an elementary fact concerning symmetric bilinear forms, we have only to show $\boldsymbol{\Psi}(Y_0, Y_0) = (Y_0, Y_0)\mathbf{A}$ and $\boldsymbol{\Psi}(Y_1, Y_1) = (Y_1, Y_1)\mathbf{B}$ for any $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$.

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then, by Lemma 18 and Lemma 19 we have $\big\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{B} \big\rangle = \langle \mathbf{A}, \mathbf{B} \rangle (Y_0, Y_0)$. Moreover, by Proposition 12 (1) and (5a) we have $\big\langle \boldsymbol{\Psi}(Y_0, Y_0), \mathbf{A} \big\rangle = \langle \mathbf{A}, \mathbf{A} \rangle (Y_0, Y_0)$. Since $\boldsymbol{\Psi}(Y_0, Y_0) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ (see (4.3)), it follows that $\boldsymbol{\Psi}(Y_0, Y_0) = (Y_0, Y_0)\mathbf{A}$, which proves (1).

We next prove (2). Let $Y_1 \in \mathfrak{m}_1$ ($Y_1 \neq 0$). Take elements $\xi \in U$ ($\xi \neq 0$) and $\eta \in V$ ($\eta \neq 0$) such that $(\eta, Y_1) = 0$. Set $Y_0 = \big[Y_1, [\xi, \eta]\big]$. Then, it is easy to see that $[\xi, \eta] \in \mathfrak{k}_1$ and $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ (see (2.1)). Further, we have $(\xi, Y_0) = 0$ and $L(\xi, Y_0)Y_1 \in V$, because

$$\begin{aligned} (\xi, Y_0) &= \big(\xi, \big[Y_1, [\xi, \eta]\big]\big) = -\big(\big[\xi, [\xi, \eta]\big], Y_1\big) \\ &= (\mu, \mu)(\xi, \xi)(\eta, Y_1) = 0, \\ L(\xi, Y_0)Y_1 &= \big[\xi, \big[\big[Y_1, [\xi, \eta]\big], Y_1\big]\big] = (\mu, \mu)(Y_1, Y_1)\big[\xi, [\xi, \eta]\big] \\ &= -(\mu, \mu)^2 (\xi, \xi)(Y_1, Y_1)\eta \in V \end{aligned}$$

(see (2.2) and (2.4)). Thus, by Proposition 15 (2), Lemma 19 and $\boldsymbol{\Psi}(Y_0, Y_0) = (Y_0, Y_0)\mathbf{A}$ (see (1)), we have

$$0 = |\boldsymbol{\Theta}_{Y_0, \xi}(Y_1)|^2 = \big[\big\langle \mathbf{A}, \boldsymbol{\Psi}(Y_1, Y_1) \big\rangle - 2(\mu, \mu)(Y_1, Y_1)\big](Y_0, Y_0).$$

Here, we note that $Y_0 \neq 0$, because $L(\xi, Y_0)Y_1 \neq 0$. Hence, by the above equality and Lemma 19, we have $\big\langle \boldsymbol{\Psi}(Y_1, Y_1), \mathbf{A} \big\rangle = \langle \mathbf{B}, \mathbf{A} \rangle (Y_1, Y_1)$. On the other hand, by Proposition 12 (1) and (5b) we have $\big\langle \boldsymbol{\Psi}(Y_1, Y_1), \mathbf{B} \big\rangle = \langle \mathbf{B}, \mathbf{B} \rangle (Y_1, Y_1)$. Consequently, it follows that $\boldsymbol{\Psi}(Y_1, Y_1) = (Y_1, Y_1)\mathbf{B}$, because $\boldsymbol{\Psi}(Y_1, Y_1) \in \boldsymbol{R}\mathbf{A} + \boldsymbol{R}\mathbf{B}$ (see (4.4)). This proves (2). □

We are now in a final position of the proof of Theorem 6. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ ($Y_0 \neq 0$). Then, by Lemma 20 (1) we have $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \supset \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \,|\, (Y_0, Y_0') = 0\}$. This shows $\dim \mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \geq 3$ and hence Y_0 is singular with respect to $\boldsymbol{\Psi}$ (see Proposition 9 (1)). Accordingly, $\mathfrak{a} + \mathfrak{m}_2$ is a singular subspace. Similarly, by Lemma 20 (2) we can show that $\mathfrak{m}_1$ is also a singular subspace.

Now, let us put into Proposition 12 $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Then, by Lemma 20 we know that the vectors $\mathbf{A}$ and $\mathbf{B}$ are not altered by this change of singular subspaces. Therefore, all the statements in Proposition 12 and the arguments developed after Proposition 12 are also true under our setting $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Consequently, by Proposition 12 (1), (2), (3) and Lemma 19 we get the assertion (1) of Theorem 6. We also obtain by Proposition 14 and $C = (\mu, \mu)$ (see Lemma 19) the assertion (3) of Theorem 6.

Finally, we prove the assertion (2) of Theorem 6. Let $Y_2 \in \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Then, since $C \neq 3(\mu, \mu)$ and $(\mu, Y_2) = 0$, we have

$$\boldsymbol{\Theta}_{Y_2,\mu}(Y_1) = \boldsymbol{\Psi}_{Y_2}(Y_1) + \frac{1}{(\mu, \mu)^2}\boldsymbol{\Psi}_{\mu}(L(\mu, Y_2)Y_1) = 0$$

(see Proposition 15). Here we note that the conditions $\mu \in U$ and $L(\mu, Y_2)Y_1 \in V$ in Proposition 15 have no significance, because $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Accordingly, we obtain the assertion (2). This completes the proof of Theorem 6. □

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Rigidity of the canonical isometric imbedding of the symplectic group $Sp(n)$

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Abstract. In this paper, we discuss the rigidity of $Sp(n)$ as a Riemannian submanifold of $M(n, n; \mathbb{H})$. We prove that the inclusion map $\boldsymbol{f}_0$, which is called the canonical isometric imbedding of $Sp(n)$, is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set $U(\subset Sp(n))$ into $\boldsymbol{R}^{4n^2}(\cong M(n, n; \mathbb{H}))$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{4n^2}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{4n^2}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

Key words: curvature invariant, isometric imbedding, rigidity, symplectic group.

Introduction

The subject of this paper is to prove the rigidity of the symplectic group $Sp(n)$ as a Riemannian submanifold of the space of matrices over the field of quaternion numbers.

Let $M(n, n; \mathbb{H})$ be the space of $n \times n$-matrices over the field $\mathbb{H}$ of quaternion numbers. Considering $M(n, n; \mathbb{H})$ as a real vector space, we define a bilinear form ν on $M(n, n; \mathbb{H})$ by setting

$$\nu(X, Y) = \mathrm{Re}(\mathrm{Trace}({}^t\bar{X}Y)), \quad X, Y \in M(n, n; \mathbb{H}).$$

It is easily seen that ν defines an inner product on $M(n, n; \mathbb{H})$. With this inner product ν we can regard $M(n, n; \mathbb{H})$ as the euclidean space $\mathbb{R}^{4n^2}$. The *symplectic group* $Sp(n)$ is given by a submanifold of $M(n, n; \mathbb{H})$ consisting of all matrices $g \in M(n, n; \mathbb{H})$ satisfying $g\,{}^t\bar{g} = {}^t\bar{g}g = I_n$, where I_n is the identity matrix of degree n. The induced metric on $Sp(n)$, which is denoted by the same symbol ν, is bi-invariant on $Sp(n)$. The inclusion map $\boldsymbol{f}_0 \colon Sp(n) \longrightarrow M(n, n; \mathbb{H}) \cong \mathbb{R}^{4n^2}$ gives an isometric imbedding of the Riemannian manifold $(Sp(n), \nu)$ into $\mathbb{R}^{4n^2}$ and is called the *canonical isometric imbedding* of $Sp(n)$ into $\mathbb{R}^{4n^2}$ (cf. Kobayashi [17]). In this paper we will discuss the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$.

Let M be a Riemannian manifold and let $\boldsymbol{f}$ be an isometric imbedding of

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M into the euclidean space $\mathbb{R}^N$. By definition $\boldsymbol{f}$ is called *strongly rigid* when $\boldsymbol{f}$ is rigid even if we restrict $\boldsymbol{f}$ to any connected open set of M, i.e., for any isometric immersion $\boldsymbol{f}'$ of a connected open set $U(\subset M)$ into $\mathbb{R}^N$ there exists a euclidean transformation a of $\mathbb{R}^N$ satisfying $\boldsymbol{f}' = a\boldsymbol{f}$ on U. In [8] and [9] we showed that the canonical isometric imbeddings of the quaternion projective plane $P^2(\mathbb{H})$ and the Cayley projective plane $P^2(\mathbb{CAY})$ are strongly rigid.

Concerning the canonical isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)$ into $\mathbb{R}^{4n^2}$, the following results are known:

(1) In the case where $n = 1$, $\boldsymbol{f}_0$ is just the standard isometric imbedding of $S^3(\cong Sp(1))$ into $\mathbb{R}^4$ with radius 1, which is a typical example of isometric imbeddings with type number 3. Accordingly, by Allendoerfer [12] $\boldsymbol{f}_0$ is known to be strongly rigid.

(2) By investigating the Gauss equation of $Sp(2)$ in codimension 6 (for the definition, see §2 below), Agaoka [1] showed that the set of solutions of the Gauss equation is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of $\boldsymbol{f}_0$. Utilizing this fact, Agaoka proved that $\boldsymbol{f}_0$ is strongly rigid when $n = 2$.

(3) Kaneda [15] proved that $\boldsymbol{f}_0(n \geq 1)$ is globally rigid in the sense of Tanaka [19], i.e., if two differentiable maps $\boldsymbol{f}_i(i = 1, 2)$ of $Sp(n)$ into $\mathbb{R}^{4n^2}$ lie both near to $\boldsymbol{f}_0$ with respect to C^3-topology, and if they induce the same Riemannian metric on $Sp(n)$, then there is a euclidean transformation a of $\mathbb{R}^{4n^2}$ such that $\boldsymbol{f}_2 = a\boldsymbol{f}_1$.

(4) By determining the pseudo-nullity of $Sp(n)(n \geq 1)$, Agaoka-Kaneda [4] proved that $\mathbb{R}^{4n^2}$ is the least dimensional euclidean space into which $Sp(n)$ can be locally isometrically immersed. (For the definition of the pseudo-nullity, see §1.) In other words, $Sp(n)(n \geq 1)$ cannot be isometrically immersed into $\mathbb{R}^{4n^2-1}$ even locally.

In this paper, we will extend these results (1) $\sim$ (4) in the following strongest sense:

Theorem 1 *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of the symplectic group $Sp(n)$ into the euclidean space $\mathbb{R}^{4n^2}$. Then $\boldsymbol{f}_0$ is strongly rigid, i.e., for any isometric immersion $\boldsymbol{f}$ of a connected open set $U(\subset Sp(n))$ into $\mathbb{R}^{4n^2}$ there is a euclidean transformation a of $\mathbb{R}^{4n^2}$ satisfying $\boldsymbol{f} = a\boldsymbol{f}_0$ on U.*

It should be noted that $Sp(n)(n \geq 1)$ are the first examples such that the canonical isometric imbeddings of a series of Riemannian symmetric spaces parametrized by rank are strongly rigid. We note that Theorem 1 for the cases $n \geq 2$ cannot be proved by applying the theory of type number in [12]. In fact, the type number of the canonical isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)$ is less than 2 in case $n \geq 2$ (precisely, see Remark 11 in §2). The method of our proof is quite similar to the methods adopted in [8] and [9]. We first make a preparatory study on pseudo-abelian subspaces of $\mathfrak{sp}(n)$, which is the Lie algebra of $Sp(n)$. Utilizing the knowledge about the pseudo-abelian subspaces of maximum dimension, we determine the set of all solutions of the Gauss equation of $Sp(n)$ in codimension $2n^2 - n (= 4n^2 - \dim Sp(n))$. Under this situation, it will be shown that the set of solutions is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of $\boldsymbol{f}_0$. Therefore by the theorem of coincidence (Theorem 5 of [8, pp. 335–336]) we can establish our rigidity theorem of $Sp(n)$ (Theorem 1).

Throughout this paper we will assume the differentiability of class C^∞. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [14]. For the quaternion numbers and the symplectic group $Sp(n)$, see Chevalley [13].

1. The pseudo-nullity of $Sp(n)$

In this section we study the pseudo-nullity of $Sp(n)$. We first recall the notion of a pseudo-abelian subspace (precisely, see [3]). Let G be a compact simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A subspace $W \subset \mathfrak{g}$ is called *pseudo-abelian with respect to* $\mathfrak{h}$ (or simply, *pseudo-abelian*) if it satisfies $[W, W] \subset \mathfrak{h}$. The maximum dimension of pseudo-abelian subspaces, which does not depend on the choice of a Cartan subalgebra $\mathfrak{h}$, is called the *pseudo-nullity* of G and is denoted by p_G. The pseudo-nullity of the symplectic group $Sp(n)$ has been already determined:

Theorem 2 (see [4]) *For the symplectic group $G = Sp(n) (n \geq 1)$, the pseudo-nullity is equal to $2n$, i.e., $p_{Sp(n)} = 2n$.*

In what follows we determine the pseudo-abelian subspace W of $\mathfrak{sp}(n)$ which attains the maximum dimension, i.e., $\dim W = p_{Sp(n)} = 2n$. First recall the field of quaternion numbers: Let $\mathbb{R}$ be the field of real numbers. The field $\mathbb{H}$ of quaternion numbers is an algebra over $\mathbb{R}$ generated by the elements e^0, e^1, e^2 and e^3 satisfying

(1) $e^0 e^i = e^i e^0 = e^i$ $(i = 0, 1, 2, 3)$;
(2) $(e^i)^2 = -e^0$ $(i = 1, 2, 3)$;
(3) For each permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ it holds $e^i e^j = \varepsilon(ijk) e^k$, where $\varepsilon(ijk) = 1$ (resp. $\varepsilon(ijk) = -1$) if $\{i, j, k\}$ is an even (resp. odd) permutation.

From (1) we can see that e^0 is a unit element of $\mathbb{H}$. Let us simply express the element ae^0 $(a \in \mathbb{R})$ as a. In this meaning $\mathbb{R}$ is contained in $\mathbb{H}$ and forms a subfield of $\mathbb{H}$.

Let $f \in \mathbb{H}$. Then f may be written in the form $f = f_0 + \sum_{i=1}^3 f_i e^i$, where $f_0, f_1, f_2, f_3 \in \mathbb{R}$. As usual we define the real part and the conjugate of f as follows: $\mathrm{Re}(f) = f_0$; $\bar{f} = f_0 - \sum_{i=1}^3 f_i e^i$. Then we have $\mathrm{Re}(f) = \mathrm{Re}(\bar{f})$, $f\bar{f} = \bar{f}f = \sum_{i=0}^3 f_i^2$. Moreover:

$$\mathrm{Re}(fh) = \mathrm{Re}(hf), \quad \overline{fh} = \bar{h}\bar{f}, \qquad f, h \in \mathbb{H}.$$

Let $i = 1, 2$ or 3. Define a subset $\mathbb{C}^i$ of $\mathbb{H}$ by $\mathbb{C}^i = \mathbb{R} + \mathbb{R}e^i$. It is easily seen that $\mathbb{C}^i$ forms a subfield of $\mathbb{H}$ and is isomorphic to the field $\mathbb{C}$ of complex numbers. We also define a subset $\mathbb{D}^i$ of $\mathbb{H}$ by $\mathbb{D}^i = \mathbb{R}e^j + \mathbb{R}e^k$, where j and k are so chosen that $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$. Then it is clear that

$$\mathbb{C}^i \mathbb{D}^i = \mathbb{D}^i \mathbb{C}^i = \mathbb{D}^i; \quad \mathbb{D}^i \mathbb{D}^i = \mathbb{C}^i.$$

In the following we denote by $M(p, q; \mathbb{H})$ the space of $p \times q$-matrices over $\mathbb{H}$. As stated in Introduction, the symplectic group $Sp(n)$ is considered as a submanifold of $M(n, n; \mathbb{H}) \cong \mathbb{R}^{4n^2}$. As usual, we identify the tangent space of $Sp(n)$ at the identity $I_n \in Sp(n)$ with the Lie algebra $\mathfrak{sp}(n)$, which is consisting of all matrices $X \in M(n, n; \mathbb{H})$ satisfying ${}^t\bar{X} = -X$. Let us denote by E_{st} $(1 \leq s, t \leq n)$ the matrix of $M(n, n; \mathbb{H})$ such that the (s, t)-component is 1 and the others are 0. We define subspaces $\mathfrak{h}(n)^i$ and $\mathfrak{p}(n)^i$ of $\mathfrak{sp}(n)$ by

$$\mathfrak{h}(n)^i = \sum_{s=1}^n \mathbb{R}e^i E_{ss}; \quad \mathfrak{p}(n)^i = \sum_{s=1}^n \mathbb{D}^i E_{ss}.$$

As is well-known, $\mathfrak{h}(n)^i$ is a Cartan subalgebra of $\mathfrak{sp}(n)$. Moreover:

Proposition 3 *Let $i = 1, 2$ or 3. Then, $\mathfrak{p}(n)^i$ is pseudo-abelian with respect to $\mathfrak{h}(n)^i$ with* $\dim \mathfrak{p}(n)^i = p_{Sp(n)}$.

Proof. It is clear that $\dim \mathfrak{p}(n)^i = 2n$. Let $X = \sum_s u_s E_{ss}$, $Y = \sum_s v_s E_{ss} \in \mathfrak{p}(n)^i$, where $u_s, v_s \in \mathbb{D}^i$. Then, since $E_{ss}E_{ss} = E_{ss}$ and $E_{ss}E_{s's'} = 0$ $(s \neq s')$, we have $[X, Y] = \sum_s (u_s v_s - v_s u_s)E_{ss}$. Since $u_s, v_s \in \mathbb{D}^i$, it follows that $u_s v_s, v_s u_s \in \mathbb{C}^i$ and $u_s v_s - v_s u_s \in \mathbb{R}e^i$. Hence $[X, Y] \in \mathfrak{h}(n)^i$, proving $[\mathfrak{p}(n)^i, \mathfrak{p}(n)^i] \subset \mathfrak{h}(n)^i$. □

Further, the space $\mathfrak{p}(n)^i$ is the only pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ of dimension $p_{Sp(n)}$. In fact, we have

Theorem 4 *Let $i = 1, 2$ or 3. Let W be a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying $\dim W = p_{Sp(n)}$. Then $W = \mathfrak{p}(n)^i$.*

In the rest of this section we prove this theorem. Let $X = \sum_{st} \xi_{st} E_{st} \in M(n, n; \mathbb{H})$. We denote by $x_p = (\xi_{p1}, \ldots, \xi_{pn}) \in M(1, n; \mathbb{H})$ the p-th row of X and by $x^q = {}^t(\xi_{1q}, \ldots, \xi_{nq}) \in M(n, 1; \mathbb{H})$ the q-th column of X. Then we may write

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x^1, \ldots, x^n).$$

As is easily seen, $X \in \mathfrak{sp}(n)$ if and only if

$${}^t\bar{x}_p + x^p = 0 \quad (1 \leq p \leq n). \tag{1.1}$$

Let $X = (x^1, \ldots, x^n)$, $Y = (y^1, \ldots, y^n) \in \mathfrak{sp}(n)$. Then $[X, Y] \in \mathfrak{h}(n)^i$ if and only if the following conditions are satisfied:

$$(x^p, y^q) = (y^p, x^q) \quad (1 \leq p < q \leq n), \tag{1.2}$$

$$(x^r, y^r) \in \mathbb{C}^i \quad (1 \leq r \leq n), \tag{1.3}$$

where $(\ ,\ )$ denotes the inner product of $M(n, 1; \mathbb{H})$ defined by $(\xi, \eta) = {}^t\bar{\xi}\eta$ for $\xi, \eta \in M(n, 1; \mathbb{H})$. Then we note the following formula:

$$\overline{(\xi, \eta)} = (\eta, \xi), \quad (\xi f, \eta) = \bar{f}(\xi, \eta), \quad (\xi, \eta f) = (\xi, \eta) f, \quad f \in \mathbb{H}. \tag{1.4}$$

Now we start the proof of Theorem 4 by induction on n. First consider the case $n = 1$. In a natural way we identify $M(1, 1; \mathbb{H})$ with $\mathbb{H}$. Then by (1.1) we know that $w = a_0 + \sum_{j=1}^3 a_j e^j \in \mathbb{H}$ belongs to $\mathfrak{sp}(1)$ if and only if $a_0 = 0$. Let W be a pseudo-abelian subspace of $\mathfrak{sp}(1)$ with respect to $\mathfrak{h}(1)^i$ with $\dim W = 2$. Suppose that $W \neq \mathbb{D}^i$. Take a basis $\{w, w'\}$ of W

such that $w \notin \mathbb{D}^i$, i.e., w is an element written in the form $w = \sum_{j=1}^{3} a_j e^j$, where $a_i \neq 0$. By subtracting a scalar multiple of w from w' if necessary, we may assume that $w' \in \mathbb{D}^i$. Then we have $ww' = (\sum_{j \neq i} a_j e^j)w' + a_i e^i w'$, $(\sum_{j \neq i} a_j e^j)w' \in \mathbb{C}^i$ and $a_i e^i w' \in \mathbb{D}^i$. On the other hand, by (1.3) we have $ww' = -\bar{w}w' \in \mathbb{C}^i$. This is impossible because $a_i e^i w' \neq 0$. Hence we have $W = \mathbb{D}^i = \mathfrak{p}(1)^i$, showing that Theorem 4 is true when $n = 1$.

We now assume that $n \geq 2$ and Theorem 4 is true for any n' ($1 \leq n' < n$). For simplicity, we regard $\mathfrak{sp}(s)$ ($1 \leq s < n$) as a subalgebra of $\mathfrak{sp}(n)$ in the following manner:

$$\mathfrak{sp}(s) \ni X \longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sp}(n).$$

Let W be a pseudo-abelian subspace of $\mathfrak{sp}(n)$ with respect to $\mathfrak{h}(n)^i$. As in [4] we define an ascending chain of subspaces

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = W$$

by setting $W_r = \mathfrak{sp}(r) \cap W$ ($1 \leq r \leq n$). (Note that the numbering of the above chain is the reverse order of that in [4, p. 79].) It is obvious that W_r is a pseudo-abelian subspace of $\mathfrak{sp}(r)$ with respect to $\mathfrak{h}(r)^i$. Put

$$C_r = \{x^r \in M(n, 1; \mathbb{H}) \mid (x^1, \ldots, x^r, \overbrace{0, \ldots, 0}^{n-r}) \in W_r\} \qquad (r = 1, \ldots, n).$$

Then it is clear that $C_r \cong W_r / W_{r-1}$ ($1 \leq r \leq n$) and $\dim W = c_1 + \cdots + c_n$, where we set $c_r = \dim C_r$ ($1 \leq r \leq n$). Moreover, by (1.2) and (1.3) we have

$$(C_p, C_q) = 0 \quad (1 \leq p < q \leq n), \tag{1.5}$$

$$(C_r, C_r) \subset \mathbb{C}^i \quad (1 \leq r \leq n). \tag{1.6}$$

The above equalities (1.5) and (1.6) will play decisive roles in the proof of Theorem 4.

By $C_r^{\mathbb{H}}$ ($1 \leq r \leq n$) we denote the right $\mathbb{H}$-subspace of $M(n, 1; \mathbb{H})$ generated by C_r. Set $k_r = \dim_{\mathbb{H}} C_r^{\mathbb{H}}$ ($1 \leq r \leq n$). Then, in view of (1.5) and (1.4) we have

$$(C_p^{\mathbb{H}}, C_q^{\mathbb{H}}) = 0 \quad (1 \leq p < q \leq n). \tag{1.7}$$

Utilizing (1.6) and (1.7), we have proved in [4] the following

Lemma 5 (see [4]) *Under the setting stated above the following* (1) *and* (2) *hold:*
(1) $k_1 + \cdots + k_n \leq n$.
(2) $c_r \leq 2k_r \quad (1 \leq r \leq n)$.
In particular, if $\dim W = p_{Sp(n)}$ $(= 2n)$, *then* $k_1 + \cdots + k_n = n$ *and* $c_r = 2k_r$ $(1 \leq r \leq n)$.

In what follows we assume that W is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying $\dim W = p_{Sp(n)}$. Let us define an $\mathbb{R}$-linear endomorphism $\xi \longmapsto \widetilde{\xi}$ of $M(n, 1; \mathbb{H})$ by setting $\widetilde{\xi} = {}^t(\xi_1, \ldots, \xi_{n-1}, 0)$ for $\xi = {}^t(\xi_1, \ldots, \xi_n) \in M(n, 1; \mathbb{H})$. Let $\widetilde{C_n}$ be the image of C_n by this endomorphism. We first prove

Lemma 6 $k_n \geq 1$ *and* $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$.

Proof. Suppose that $k_n = 0$. Then we have $C_n = 0$ and hence $W = W_{n-1}$. Therefore, in a natural way W may be regarded as a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. This implies $\dim W \leq p_{Sp(n-1)} = 2(n-1)$, contradicting the assumption $\dim W = 2n$. Consequently, we have $k_n \geq 1$. Let $\xi \in C_n$ and $\eta \in C_1 + \cdots + C_{n-1}$. Since η is written as $\eta = {}^t(\eta_1, \ldots, \eta_{n-1}, 0)$, we have $(\widetilde{\xi}, \eta) = (\xi, \eta) = 0$ (see (1.5)). Hence we have $(\widetilde{C_n}, C_1 + \cdots + C_{n-1}) = 0$. Viewing (1.4), we have $(\widetilde{C_n}^{\mathbb{H}}, C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}) = 0$. Since both $\widetilde{C_n}^{\mathbb{H}}$ and $C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}$ may be regarded as subspaces of $M(n-1, 1; \mathbb{H})$, we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq n - 1 - (k_1 + \cdots + k_{n-1})$ (see (1.7)). Therefore by Lemma 5 we obtain $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$. □

Let C'_n be the subset of C_n consisting of all ${}^t(\xi_1, \ldots, \xi_n) \in C_n$ such that the n-th component $\xi_n \in \mathbb{D}^i$, i.e., $C'_n = \{{}^t(\xi_1, \ldots, \xi_n) \in C_n \mid \xi_n \in \mathbb{D}^i\}$. Clearly, C'_n is a subspace of C_n. We denote by $\widetilde{C'_n}$ the image of C'_n by the endomorphism $\xi \longmapsto \widetilde{\xi}$. Then we can show

Lemma 7 $\dim C'_n \geq 2k_n - 1$ *and* $\dim \widetilde{C'_n} \leq 2(k_n - 1)$.

Proof. First we note that $\xi_n \in \mathbb{R}e^i + \mathbb{D}^i$ holds for any $\xi = {}^t(\xi_1, \ldots, \xi_n) \in C_n$. Indeed, ξ_n is the (n, n)-component of a certain matrix $X \in \mathfrak{sp}(n)$ (recall the definition of C_n). Consequently, we have $\dim C'_n \geq \dim C_n - 1 = c_n - 1 = 2k_n - 1$.

We next prove the second inequality. Let $\xi = {}^t(\xi_1, \ldots, \xi_n) \in C'_n$ and $\eta = {}^t(\eta_1, \ldots, \eta_n) \in C'_n$. Then we easily have $(\widetilde{\xi}, \widetilde{\eta}) = (\xi, \eta) - \overline{\xi_n}\eta_n$. Since $(\xi, \eta) \in \mathbb{C}^i$ (see (1.6)) and $\overline{\xi_n}\eta_n \in \mathbb{D}^i\mathbb{D}^i = \mathbb{C}^i$, it follows that $(\widetilde{\xi}, \widetilde{\eta}) \in \mathbb{C}^i$. This proves $(\widetilde{C'_n}, \widetilde{C'_n}) \subset \mathbb{C}^i$. By this fact we can deduce that $\widetilde{C'_n} \cap \widetilde{C'_n}e^j = 0$ for any j $(= 1, 2, 3)$ such that $j \neq i$. In fact, if there is an element $\widetilde{\xi} \in \widetilde{C'_n}$ such that $\widetilde{\xi}e^j \in \widetilde{C'_n}$, then we have $\mathbb{C}^i \ni (\widetilde{\xi}, \widetilde{\xi}e^j) = (\widetilde{\xi}, \widetilde{\xi})e^j \in \mathbb{C}^ie^j = \mathbb{D}^i$. Since $\mathbb{C}^i \cap \mathbb{D}^i = 0$, it follows that $(\widetilde{\xi}, \widetilde{\xi}) = 0$, i.e., $\widetilde{\xi} = 0$. Thus, we know that $\widetilde{C'_n} + \widetilde{C'_n}e^j$ $(\subset \widetilde{C_n}^{\mathbb{H}})$ is a direct sum if $j \neq i$. Consequently, we have $2\dim\widetilde{C'_n} \leq 4\dim_{\mathbb{H}}\widetilde{C_n}^{\mathbb{H}} \leq 4(k_n - 1)$, i.e., $\dim\widetilde{C'_n} \leq 2(k_n - 1)$ (see Lemma 6). This completes the proof of the lemma. □

With the basis of Lemma 7 we can show

Lemma 8 *Let D_n be the kernel of the linear mapping $C_n \ni \xi \longmapsto \widetilde{\xi} \in \widetilde{C_n}$. Then:*

(1) $D_n = \{{}^t(0, \ldots, 0, w) \in M(n, 1; \mathbb{H}) \mid w \in \mathbb{D}^i\}$.
(2) $\widetilde{C_n} \subset C_n$.
(3) $C_n = D_n + \widetilde{C_n}$ *(direct sum)*; $\dim\widetilde{C_n} = c_n - 2$.

Proof. By Lemma 7 we have $\dim C'_n - \dim\widetilde{C'_n} \geq 2k_n - 1 - 2(k_n - 1) > 0$. This implies that $D_n \cap C'_n \neq 0$. Let ξ be a non-trivial element of $D_n \cap C'_n$. Then, by the definitions of D_n and C'_n, we know that ξ may be written as $\xi = {}^t(0, \ldots, 0, w)$, where $w \in \mathbb{D}^i$ $(w \neq 0)$. Let $\eta = {}^t(\eta_1, \ldots, \eta_n)$ be an arbitrary element of C_n. Then by (1.6) we have $(\xi, \eta) = \bar{w}\eta_n \in \mathbb{C}^i$. Hence we can easily show that $\eta_n \in \mathbb{D}^i$ (see the proof for the case $n = 1$). Accordingly, $\eta \in C'_n$ and hence $C'_n = C_n$. Therefore, we have

$$\dim D_n = \dim C_n - \dim\widetilde{C_n} = \dim C_n - \dim\widetilde{C'_n} \geq c_n - 2(k_n - 1) = 2.$$

On the other hand, since $D_n \subset C_n = C'_n$, we have $D_n \subset \{{}^t(0, \ldots, 0, w) \mid w \in \mathbb{D}^i\}$ and hence $\dim D_n \leq \dim\mathbb{D}^i = 2$. This, together with the above inequality, proves $\dim D_n = 2$ and $D_n = \{{}^t(0, \ldots, 0, w) \mid w \in \mathbb{D}^i\}$. Thus we obtain (1).

Let $\zeta = {}^t(\zeta_1, \ldots, \zeta_n) \in M(n, 1; \mathbb{H})$ be an arbitrary element of C_n. Since $C_n = C'_n$, we have $\zeta_n \in \mathbb{D}^i$ and hence $\zeta' = {}^t(0, \ldots, 0, \zeta_n) \in D_n \subset C_n$. Consequently, $\widetilde{\zeta} = {}^t(\zeta_1, \ldots, \zeta_{n-1}, 0) = \zeta - \zeta' \in C_n$, showing (2). The assertion (3) immediately follows from (1) and (2). □

With these preparations we can show

Lemma 9 $\widetilde{C_n} = 0$. *Accordingly,* $C_n = D_n$.

Proof. We first prove

$$\widetilde{C_n} \cap \widetilde{C_n} e^i = 0. \tag{1.8}$$

Suppose that there is an element $\tilde{\xi} = {}^t(\xi_1, \ldots, \xi_{n-1}, 0) \in \widetilde{C_n}$ such that $\tilde{\xi} e^i \in \widetilde{C_n}$. Note that $\widetilde{C_n} \subset C_n$ (see Lemma 8 (2)). By the definition of C_n we know that there are matrices X and $Y \in W$ written in the form

$$X = \begin{pmatrix} X' & \xi' \\ -{}^t\bar{\xi}' & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & \xi' e^i \\ e^i\, {}^t\bar{\xi}' & 0 \end{pmatrix},$$

where $X', Y' \in \mathfrak{sp}(n-1)$ and $\xi' = {}^t(\xi_1, \ldots, \xi_{n-1}) \in M(n-1, 1; \mathbb{H})$. Take an integer j $(= 1, 2, 3)$ such that $j \neq i$. Since ${}^t(0, \ldots, 0, e^j) \in D_n \subset C_n$, we know that there is an element $Z \in W$ of the form

$$Z = \begin{pmatrix} Z' & 0 \\ 0 & e^j \end{pmatrix},$$

where $Z' \in \mathfrak{sp}(n-1)$. Since W is a pseudo-abelian with respect to $\mathfrak{h}(n)^i$, we have $[X, Z] \in \mathfrak{h}(n)^i$ and $[Y, Z] \in \mathfrak{h}(n)^i$. Hence by a direct calculation we can show

$$Z'\xi' = \xi' e^j; \quad Z'(\xi' e^i) = (\xi' e^i) e^j. \tag{1.9}$$

By the second equality of (1.9) we have $(Z'\xi')e^i = \xi'(e^i e^j) = -\xi'(e^j e^i) = -(\xi' e^j)e^i$ and hence $Z'\xi' = -\xi' e^j$. This, together with the first equality of (1.9), proves $Z'\xi' = \xi' e^j = 0$. Hence we have $\xi' = 0$, i.e., $\tilde{\xi} = 0$. This implies (1.8). As a result of (1.8), the subspace $\widetilde{C_n} + \widetilde{C_n} e^i$ $(\subset \widetilde{C_n}^{\mathbb{H}})$ is a direct sum. Since $\dim \widetilde{C_n} = c_n - 2 = 2(k_n - 1)$ (see Lemma 8 (3) and Lemma 5), it follows that $\dim_{\mathbb{R}} \widetilde{C_n}^{\mathbb{H}} \geq 2 \dim \widetilde{C_n} = 4(k_n - 1)$. Hence we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} = (1/4) \dim_{\mathbb{R}} \widetilde{C_n}^{\mathbb{H}} \geq k_n - 1$. On the other hand, we have $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} \leq k_n - 1$ (see Lemma 6). Therefore, we obtain $\dim_{\mathbb{H}} \widetilde{C_n}^{\mathbb{H}} = k_n - 1$ and $\widetilde{C_n}^{\mathbb{H}} = \widetilde{C_n} + \widetilde{C_n} e^i$. More strongly, we can prove $\widetilde{C_n} = 0$. In fact, since $\widetilde{C_n}^{\mathbb{H}} = \widetilde{C_n} + \widetilde{C_n} e^i$, it follows that

$$(\widetilde{C_n}^{\mathbb{H}}, \widetilde{C_n}^{\mathbb{H}}) \subset (\widetilde{C_n}, \widetilde{C_n}) + (\widetilde{C_n} e^i, \widetilde{C_n}) + (\widetilde{C_n}, \widetilde{C_n} e^i) + (\widetilde{C_n} e^i, \widetilde{C_n} e^i).$$

If $\widetilde{C_n} \neq 0$, then it is easy to see that $(\widetilde{C_n}^{\mathbb{H}}, \widetilde{C_n}^{\mathbb{H}}) = \mathbb{H}$. However, the right side of the above inclusion is contained in $\mathbb{C}^i$, because $(\widetilde{C_n}, \widetilde{C_n}) \subset (C_n, C_n) \subset \mathbb{C}^i$ (see Lemma 8 (2) and (1.6)), $(\widetilde{C_n}e^i, \widetilde{C_n}) \subset e^i\mathbb{C}^i = \mathbb{C}^i$, $(\widetilde{C_n}, \widetilde{C_n}e^i) \subset \mathbb{C}^i e^i = \mathbb{C}^i$ and $(\widetilde{C_n}e^i, \widetilde{C_n}e^i) \subset e^i\mathbb{C}^i e^i = \mathbb{C}^i$ (see (1.4)). This is a contradiction. Hence we have $\widetilde{C_n} = 0$. The equality $C_n = D_n$ now follows immediately. □

Proof of Theorem 4. By Lemma 9 and Lemma 8 (3) we have $c_n = 2k_n = 2$. Hence, W_{n-1}, which is a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$, satisfies $\dim W_{n-1} = c_1 + \cdots + c_{n-1} = 2(n-1) = p_{Sp(n-1)}$. Therefore, by the hypothesis of our induction we know that $W_{n-1} = \mathfrak{p}(n-1)^i$. From this fact we can deduce $W = \mathfrak{p}(n)^i$. In fact, let X be an arbitrary element of W. Then X may be written as $X = \begin{pmatrix} X' & 0 \\ 0 & w \end{pmatrix}$, where $X' \in \mathfrak{sp}(n-1)$, $w \in \mathbb{D}^i$ (see Lemma 9 and Lemma 8 (1)). Since $[X, W_{n-1}] \subset \mathfrak{h}(n)^i$, it follows that $[X', \mathfrak{p}(n-1)^i] \subset \mathfrak{h}(n-1)^i$. Hence we have $X' \in \mathfrak{p}(n-1)^i$, because $\mathfrak{p}(n-1)^i$ is a maximal pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. Consequently, we have $X \in \mathfrak{p}(n)^i$ and $W = \mathfrak{p}(n)^i$, which completes the proof of Theorem 4. □

2. The Gauss equation of $Sp(n)$

Let M be a Riemannian manifold. We denote by g the Riemannian metric of M and by R the Riemannian curvature tensor of type (1, 3) with respect to g. Let $x \in M$ and let $T_x(M)$ (resp. $T_x^*(M)$) be the tangent (resp. cotangent) vector space of M at x. Let r be a non-negative integer. We define a quadratic equation with respect to an unknown $\Psi \in S^2T_x^*(M) \otimes \mathbb{R}^r$ by

$$\begin{aligned} &-g(R(X, Y)Z, W) \\ &\qquad = \langle \Psi(X, Z), \Psi(Y, W) \rangle - \langle \Psi(X, W), \Psi(Y, Z) \rangle, \quad (2.1) \end{aligned}$$

where $X, Y, Z, W \in T_x(M)$ and $\langle\ ,\ \rangle$ is the standard inner product of $\mathbb{R}^r$. We call (2.1) the *Gauss equation* in codimension r at x. The set of solutions of (2.1) is called the *Gaussian variety* in codimension r at x and is denoted by $\mathcal{G}_x(M, \mathbb{R}^r)$.

Let $O(r)$ be the orthogonal group of $\mathbb{R}^r$. We define an action of $O(r)$ on $S^2T_x^*(M) \otimes \mathbb{R}^r$ by

$$(\rho\Psi)(X, Y) = \rho(\Psi(X, Y)), \quad X, Y \in T_x(M),\ \rho \in O(r). \qquad (2.2)$$

As is easily seen, if Ψ is a solution of (2.1), then $\rho\Psi$ is also a solution of (2.1) for any $\rho \in O(r)$. We say that $\mathcal{G}_x(M, \mathbb{R}^r)$ is *EOS* if $\mathcal{G}_x(M, \mathbb{R}^r) \neq \emptyset$ and if $\mathcal{G}_x(M, \mathbb{R}^r)$ is composed of essentially one solution, i.e., for any solutions Ψ_1 and $\Psi_2 \in \mathcal{G}_x(M, \mathbb{R}^r)$ there is an element $\rho \in O(r)$ such that $\Psi_2 = \rho\Psi_1$.

In the following we consider the case where M is the symplectic group $Sp(n)$ endowed with the bi-invariant metric ν, which is induced from the inclusion $Sp(n) \subset M(n, n; \mathbb{H})$. As usual we identify the tangent space of $Sp(n)$ at the identity I_n with the Lie algebra $\mathfrak{sp}(n)$. We denote by $(\, , \,)$ the inner product of $\mathfrak{sp}(n)$ induced from ν at I_n. The curvature transformation $R_0(X, Y)$ $(X, Y \in \mathfrak{sp}(n))$ of $Sp(n)$ at I_n is given by $R_0(X, Y) = -(1/4)\,\mathrm{ad}([X, Y])$ (see [14]). Hence at I_n the Gauss equation (2.1) is written as

$$\begin{aligned}&\frac{1}{4}([[X, Y], Z], W)\\ &\qquad = \langle \Psi(X, Z), \Psi(Y, W)\rangle - \langle \Psi(X, W), \Psi(Y, Z)\rangle, \quad (2.3)\end{aligned}$$

where $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathbb{R}^r$ and $X, Y, Z, W \in \mathfrak{sp}(n)$. We simply denote by $\mathcal{G}(Sp(n), \mathbb{R}^r)$ the Gaussian variety in codimension r at I_n. The main aim of this and the subsequent sections is to prove

Theorem 10 *For any positive integer n the Gaussian variety $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ in codimension $2n^2 - n$ is EOS.*

By homogeneity, we know that the Gaussian variety $\mathcal{G}_x(Sp(n), \mathbb{R}^{2n^2-n})$ in codimension $2n^2-n$ is *EOS* at each $x \in Sp(n)$. By this result we conclude that $Sp(n)$ is formally rigid in codimension $2n^2 - n$. (For the definition of formal rigidness, see [8].) Accordingly, by Theorem 5 of [8] we can establish the rigidity theorem of $Sp(n)$ (Theorem 1).

In the following we will prove Theorem 10 by induction on n. As we have stated in the introduction, if $n = 1$, then $Sp(1) \cong S^3$ and the canonical isometric imbedding $\boldsymbol{f}_0$ is the inclusion map of the standard sphere S^3 with radius 1 into $\mathbb{R}^4$. The second fundamental form Ψ_0 of $\boldsymbol{f}_0$ at $\boldsymbol{x} \in S^3$ is given by $\Psi_0 = -\nu\boldsymbol{x}$. Hence $\boldsymbol{f}_0$ is a typical example of an isometric imbedding with type number 3. By applying the theory of type number in [12] or by a direct calculation we know that any solution Ψ of the Gauss equation of S^3 in codimension 1 can be represented by $\Psi = \pm\Psi_0$. Therefore we get Theorem 10 for the case $n = 1$. For this reason we may assume $n \geq 2$ in the following discussion.

Remark 11 It should be noted that in case $n \geq 2$ the theory of type number in [12] is not applicable to the canonical isometric imbedding $\boldsymbol{f}_0$ of $Sp(n)$. In fact, for an isometric imbedding $\boldsymbol{f}$ of a Riemannian manifold M into the euclidean space $\mathbb{R}^m$, the type number k of $\boldsymbol{f}$ must satisfy the inequality $k \leq \dim M/(m - \dim M)$ (see [18] or [16]). Consequently, in the case of $\boldsymbol{f}_0$ we can easily show that $k < 2$ when $n \geq 2$.

Now let $\mathfrak{N}(n)$ be the subspace of $M(n, n; \mathbb{H})$ composed of all $X \in M(n, n; \mathbb{H})$ satisfying ${}^t\bar{X} = X$. Clearly, we have $\dim \mathfrak{N}(n) = 2n^2 - n$ and

$$M(n, n; \mathbb{H}) = \mathfrak{sp}(n) \dotplus \mathfrak{N}(n) \quad \text{(orthogonal direct sum).}$$

As is easily seen, $\mathfrak{N}(n)$ is the normal vector space of the canonical isometric imbedding $\boldsymbol{f}_0$ at I_n. The second fundamental form $\boldsymbol{\Psi}_0$ of $\boldsymbol{f}_0$ at I_n is an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ given by

$$\boldsymbol{\Psi}_0(X, Y) = \frac{1}{2}(XY + YX), \quad X, Y \in \mathfrak{sp}(n) \tag{2.4}$$

(see [15, p. 370]). Under a natural identification $(\mathfrak{N}(n), \nu) \cong (\mathbb{R}^{2n^2-n}, \langle\ ,\ \rangle)$ as euclidean vector spaces we can regard the unknown $\boldsymbol{\Psi}$ in the Gauss equation (2.3) in codimension $2n^2 - n$ as an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. (In what follows, the inner product ν of $\mathfrak{N}(n)$ will be denoted by $\langle\ ,\ \rangle$.) Therefore the Gaussian variety $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ may be considered as a subset of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. In this meaning we write $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ as $\mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then $\boldsymbol{\Psi}_0$ may be considered as an element of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$, which is called the *canonical solution* of the Gauss equation (2.3) in codimension $2n^2 - n$. Now Theorem 10 may be stated in the following way: Any solution $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ of the Gauss equation (2.3) is equivalent to $\boldsymbol{\Psi}_0$, i.e., there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\boldsymbol{\Psi} = \rho\boldsymbol{\Psi}_0$, where $O(\mathfrak{N}(n))$ stands for the orthogonal group of $\mathfrak{N}(n)$.

3. The space $\boldsymbol{K}_{\Psi}(X)$

In this section we assume that $n \geq 2$. Let $\boldsymbol{\Psi} \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $X \in \mathfrak{sp}(n)$. We define a linear mapping $\boldsymbol{\Psi}_X : \mathfrak{sp}(n) \longrightarrow \mathfrak{N}(n)$ by setting $\boldsymbol{\Psi}_X(Y) = \boldsymbol{\Psi}(X, Y)$ $(Y \in \mathfrak{sp}(n))$. By $\boldsymbol{K}_{\Psi}(X)$ $(\subset \mathfrak{sp}(n))$ we denote the kernel of $\boldsymbol{\Psi}_X$. In this section we investigate the kernel $\boldsymbol{K}_{\Psi}(X)$ for a solution $\boldsymbol{\Psi}$ of the Gauss equation (2.3), i.e., $\boldsymbol{\Psi} \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. As in the case of $P^2(\mathbb{H})$ or $P^2(\mathbb{CAY})$, the knowledge about $\boldsymbol{K}_{\Psi}(X)$ will play an important role to determine the solutions of the Gauss equation (2.3) (cf. [8] and [9]).

Let $X \in \mathfrak{sp}(n)$. By $C(X)$ we denote the centralizer of X in $\mathfrak{sp}(n)$. Then we have

Lemma 12 *Let* $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ *and* $X \in \mathfrak{sp}(n)$. *Then:*
(1) $\dim \boldsymbol{K}_\Psi(X) \geq 2n$.
(2) *If* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, *then* $[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)] \subset C(X)$.

Proof. Since

$$\dim \boldsymbol{K}_\Psi(X) \geq \dim Sp(n) - \dim \mathfrak{N}(n) = (2n^2+n)-(2n^2-n) = 2n,$$

we get (1). Assume that $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then by (2.3) for each $Y \in \mathfrak{sp}(n)$ we have

$$\begin{aligned}\big([[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], X], Y]\big) &\subset \big\langle \Psi(\boldsymbol{K}_\Psi(X), X), \Psi(\boldsymbol{K}_\Psi(X), Y)\big\rangle \\ &= 0.\end{aligned}$$

Consequently, we have $[[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], X] = 0$. The assertion (2) immediately follows from this equality (cf. [10, Lemma 3]). □

Let $X \in \mathfrak{sp}(n)$. Since $\mathfrak{sp}(n)$ is a compact simple Lie algebra, we know that $\dim C(X) \geq \operatorname{rank}(\mathfrak{sp}(n)) = n$. We recall that an element $X \in \mathfrak{sp}(n)$ is called *regular* (resp. *singular*) if $\dim C(X) = n$ (resp. $\dim C(X) > n$).

Lemma 13 *Let* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ *and* $H \in \mathfrak{h}(n)^i$ $(i = 1, 2, 3)$. *Then* $\boldsymbol{K}_\Psi(H) \supset \mathfrak{p}(n)^i$. *If* H *is regular, then the equality* $\boldsymbol{K}_\Psi(H) = \mathfrak{p}(n)^i$ *holds.*

Proof. Let $H \in \mathfrak{h}(n)^i$. Then by Lemma 12 (2) we have $[\boldsymbol{K}_\Psi(H), \boldsymbol{K}_\Psi(H)] \subset C(H)$. Assume that H is regular. Then, since $C(H) = \mathfrak{h}(n)^i$, we have $[\boldsymbol{K}_\Psi(H), \boldsymbol{K}_\Psi(H)] \subset \mathfrak{h}(n)^i$. This implies that $\boldsymbol{K}_\Psi(H)$ is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$. Therefore we have $\dim \boldsymbol{K}_\Psi(H) \leq p_{Sp(n)} = 2n$ (see Theorem 2). On the other hand, since $\dim \boldsymbol{K}_\Psi(H) \geq 2n$ (see Lemma 12 (1)), it follows that $\dim \boldsymbol{K}_\Psi(H) = 2n$. Hence $\boldsymbol{K}_\Psi(H) = \mathfrak{p}(n)^i$ (see Theorem 4). Let $H' \in \mathfrak{h}(n)^i$ be an arbitrary element. Note that regular elements are dense in $\mathfrak{h}(n)^i$ and, as we have shown, $\Psi(H, \mathfrak{p}(n)^i) = 0$ holds for any regular element $H \in \mathfrak{h}(n)^i$. Because of the continuity of Ψ we have $\Psi(H', \mathfrak{p}(n)^i) = 0$. This shows that $\boldsymbol{K}_\Psi(H') \supset \mathfrak{p}(n)^i$. □

Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. We define an element

$\Psi^g \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ by

$$(\Psi^g)(X, Y) = \Psi(\mathrm{Ad}(g^{-1})X, \mathrm{Ad}(g^{-1})Y), \quad X, Y \in \mathfrak{sp}(n). \tag{3.1}$$

Then we can easily see the following

Lemma 14 *Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. Then:*
(1) $\boldsymbol{K}_{\Psi^g}(X) = \mathrm{Ad}(g)\boldsymbol{K}_\Psi(\mathrm{Ad}(g^{-1})X), \quad X \in \mathfrak{sp}(n)$.
(2) $\Psi^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ *if and only if* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

Combining Lemma 13 with Lemma 14, we have

Proposition 15 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, $X \in \mathfrak{sp}(n)$ and $g \in Sp(n)$. Assume that $\mathrm{Ad}(g)X \in \mathfrak{h}(n)^i$ for some i $(= 1, 2, 3)$. Then $\boldsymbol{K}_\Psi(X) \supset \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. Further, if X is regular, then $\boldsymbol{K}_\Psi(X) = \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$.*

Proof. Note that $\Psi^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ (see Lemma 14 (2)). Applying Lemma 13 to Ψ^g, we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) \supset \mathfrak{p}(n)^i$. Therefore by Lemma 14 (1) we have $\mathfrak{p}(n)^i \subset \boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_\Psi(X)$. Consequently, $\boldsymbol{K}_\Psi(X) \supset \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. If X is regular, then $\mathrm{Ad}(g)X$ is also regular. Accordingly, we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \mathfrak{p}(n)^i$ and hence $\boldsymbol{K}_\Psi(X) = \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. □

Remark 16 Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. It is well-known that any element of $\mathfrak{sp}(n)$ is conjugate to an element of a Cartan subalgebra $\mathfrak{h}(n)^i$. Therefore, for a regular element $X \in \mathfrak{sp}(n)$ the space $\boldsymbol{K}_\Psi(X)$ is determined by Proposition 15. Here we note that if X is regular, then $\boldsymbol{K}_\Psi(X)$ does not depend on the choice of the solution $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, i.e., $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_{\Psi'}(X)$ holds for any Ψ, $\Psi' \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

In the following discussion, we will determine $\boldsymbol{K}_\Psi(X)$ for singular elements $X \in \mathfrak{sp}(n)$ of special type. By Proposition 15 we immediately obtain

Proposition 17 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Let $i = 1$, 2 or 3 and $X \in \mathfrak{sp}(n)$. Denote by G_X^i the subset of $Sp(n)$ consisting of all $g \in Sp(n)$ such that $\mathrm{Ad}(g)X \in \mathfrak{h}(n)^i$. Then:*

$$\boldsymbol{K}_\Psi(X) \supset \sum_{g \in G_X^i} \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i. \tag{3.2}$$

Let a, b and i are integers satisfying $1 \leq a \neq b \leq n$, $1 \leq i \leq 3$. Define elements H^i_a, P_{ab} and $Q^i_{ab} \in M(n, n; \mathbb{H})$ by

$$H^i_a = E_{aa}e^i; \quad P_{ab} = -P_{ba} = E_{ab} - E_{ba}; \quad Q^i_{ab} = Q^i_{ba} = (E_{ab} + E_{ba})e^i.$$

Then it is easily seen that H^i_a, P_{ab}, $Q^i_{ab} \in \mathfrak{sp}(n)$ and

$$\begin{aligned} &(H^i_a, H^j_b) = \delta_{ab}\delta_{ij}; \quad (H^i_a, P_{cd}) = (H^i_a, Q^j_{cd}) = 0; \\ &(P_{ab}, P_{cd}) = 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}); \quad (P_{ab}, Q^i_{cd}) = 0; \\ &(Q^i_{ab}, Q^j_{cd}) = 2(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\delta_{ij}. \end{aligned} \tag{3.3}$$

Therefore the set $\{H^i_a \ (1 \le a \le n)\}$ forms an orthonormal basis of $\mathfrak{h}(n)^i$ $(1 \le i \le 3)$ and the set $\{H^i_a \ (1 \le a \le n,\ 1 \le i \le 3),\ (1/\sqrt{2})P_{ab} \ (1 \le a < b \le n),\ (1/\sqrt{2})Q^i_{ab} \ (1 \le a < b \le n,\ 1 \le i \le 3)\}$ forms an orthonormal basis of $\mathfrak{sp}(n)$.

Let a, b and i are integers satisfying $1 \le a \ne b \le n$, $1 \le i \le 3$. Define a subspace $\mathfrak{s}^i_{ab}$ by $\mathfrak{s}^i_{ab} = \mathbb{R}(H^i_a - H^i_b) + \mathbb{R}P_{ab} + \mathbb{R}Q^i_{ab}$. By an easy calculation we have

$$\begin{aligned} &[H^i_a - H^i_b, P_{ab}] = 2Q^i_{ab}; \quad [H^i_a - H^i_b, Q^i_{ab}] = -2P_{ab}; \\ &[P_{ab}, Q^i_{ab}] = 2(H^i_a - H^i_b). \end{aligned}$$

This indicates that $\mathfrak{s}^i_{ab}$ forms a three-dimensional subalgebra of $\mathfrak{sp}(n)$ and is not abelian. Now we note the following lemma, which holds for any compact Lie algebra:

Lemma 18 *Let $\mathfrak{s}$ be a three-dimensional subalgebra of a compact Lie algebra $\mathfrak{g}$. Assume that $\mathfrak{s}$ is not abelian. Then, for any linearly independent elements Z, $Z' \in \mathfrak{s}$, there is an element $g \in \exp(\mathbb{R}[Z, Z'])$ $(\subset \exp(\mathfrak{g}))$ such that $\mathrm{Ad}(g)Z \in \mathbb{R}Z'$.*

Proof. Since $\mathfrak{g}$ is compact, $\mathfrak{s}$ is also a compact Lie algebra. Hence $\mathfrak{s}$ may be represented by a direct sum of its center and its semi-simple part. Note that any simple Lie algebra is of dimension ≥ 3. Under the assumption that $\mathfrak{s}$ is not abelian and $\dim \mathfrak{s} = 3$, we know that the center of $\mathfrak{s}$ is trivial and that $\mathfrak{s}$ is simple. Hence, $\mathfrak{s}$ is isomorphic to the simple Lie algebra $\mathfrak{su}(2)$.

Let B be an $\mathrm{ad}(\mathfrak{g})$-invariant inner product of $\mathfrak{g}$. Let Z, $Z' \in \mathfrak{s}$. If Z and Z' are linearly independent, then it follows that $[Z, Z'] \ne 0$, because $\mathrm{rank}(\mathfrak{s}) = 1$. Set $\mathfrak{s}' = \mathbb{R}Z + \mathbb{R}Z'$. Then we have $B(\mathfrak{s}', \mathbb{R}[Z, Z']) = 0$, i.e., $\mathbb{R}[Z, Z']$ is the orthogonal complement of $\mathfrak{s}'$ in $\mathfrak{s}$ with respect to B. Indeed, we have

$$B(Z, [Z, Z']) = B([Z, Z], Z') = 0;$$
$$B(Z', [Z, Z']) = -B([Z', Z'], Z) = 0.$$

Similarly, we can prove $B(\mathrm{ad}[Z, Z'](Z), [Z, Z']) = B(\mathrm{ad}[Z, Z'](Z'), [Z, Z']) = 0$. This means that $\mathfrak{s}'$ is invariant by $\mathrm{ad}[Z, Z']$. Moreover, we have $\mathrm{ad}([Z, Z'])Z'' \neq 0$ for any $Z'' \in \mathfrak{s}'$ with $Z'' \neq 0$. Therefore, $\mathrm{Ad}(\exp(\mathbb{R}[Z, Z']))$ forms a non-trivial subgroup of rotations of $\mathfrak{s}'$ with respect to B. From this fact the lemma follows immediately. □

In the following, we say a subalgebra $\mathfrak{s}$ of $\mathfrak{sp}(n)$ is *NAT* if $\mathfrak{s}$ is non-abelian and $\dim \mathfrak{s} = 3$. As we have seen, $\mathfrak{s}^i_{ab} = \mathbb{R}(H^i_a - H^i_b) + \mathbb{R}P_{ab} + \mathbb{R}Q^i_{ab}$ is *NAT*. For non-zero elements X and $Y \in \mathfrak{sp}(n)$ we write $X \sim Y$ if there is an element $g \in Sp(n)$ such that $\mathrm{Ad}(g)X \in \mathbb{R}Y$. Apparently, $\sim$ defines an equivalence relation in $\mathfrak{sp}(n) \setminus \{0\}$. According to Lemma 18 if $\mathfrak{s}$ is *NAT*, then $Z \sim Z'$ for any $Z, Z' \in \mathfrak{s} \setminus \{0\}$. For example, we have $(H^i_a - H^i_b) \sim P_{ab} \sim Q^i_{ab}$.

For simplicity in the following discussion we set $\boldsymbol{K}_0(X) = \boldsymbol{K}_{\Psi_0}(X)$. As in the previous section we regard $\mathfrak{sp}(s)$ $(0 \leq s < n)$ as a subalgebra of $\mathfrak{sp}(n)$. Then by easy calculations we have

$$\begin{aligned} \boldsymbol{K}_0(H^i_n) &= \mathfrak{sp}(n-1) + \sum_{j \neq i} \mathbb{R}H^j_n; \\ \boldsymbol{K}_0(H^i_{n-1} + H^i_n) &= \mathfrak{sp}(n-2) + \sum_{j \neq i} \mathbb{R}H^j_{n-1} \\ &\quad + \sum_{j \neq i} \mathbb{R}H^j_n + \sum_{j \neq i} \mathbb{R}Q^j_{n-1,n}. \end{aligned} \tag{3.4}$$

Let Ψ be an arbitrary solution of the Gauss equation (2.3). By Remark 16 we know that $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_0(X)$ holds for a regular element $X \in \mathfrak{sp}(n)$. We now extend this relation to singular elements:

Proposition 19 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then for each i $(= 1, 2, 3)$ it holds:*
(1) $\boldsymbol{K}_\Psi(H^i_n) = \boldsymbol{K}_0(H^i_n)$.
(2) $\boldsymbol{K}_\Psi(H^i_{n-1} + H^i_n) = \boldsymbol{K}_0(H^i_{n-1} + H^i_n)$.

Proof. Let $Sp(n-1)$ be the analytic subgroup of $Sp(n)$ corresponding to the subalgebra $\mathfrak{sp}(n-1)$. Let $g \in Sp(n-1)$. Then it is easy to see that $\mathrm{Ad}(g)H_n^i = H_n^i$. Hence by Proposition 17 we have $\boldsymbol{K}_\Psi(H_n^i) \supset \sum_{g\in Sp(n-1)} \mathrm{Ad}(g^{-1})\mathfrak{p}(n)^i$. Since $\mathfrak{h}(n-1)^j$ $(j \neq i)$ is a Cartan subalgebra of $\mathfrak{sp}(n-1)$, any element of $\mathfrak{sp}(n-1)$ is conjugate to an element of $\mathfrak{h}(n-1)^j$ under the action of $Sp(n-1)$. Hence we have $\bigcup_{g\in Sp(n-1)} \mathrm{Ad}(g^{-1})\mathfrak{h}(n-1)^j = \mathfrak{sp}(n-1)$. Since $\mathfrak{p}(n)^i \supset \mathfrak{h}(n-1)^j$, we have $\boldsymbol{K}_\Psi(H_n^i) \supset \mathfrak{sp}(n-1)$. This, together with $\boldsymbol{K}_\Psi(H_n^i) \supset \mathfrak{p}(n)^i$, shows $\boldsymbol{K}_\Psi(H_n^i) \supset \mathfrak{sp}(n-1)+\mathfrak{p}(n)^i = \boldsymbol{K}_0(H_n^i)$. We now show the equality $\boldsymbol{K}_\Psi(H_n^i) = \boldsymbol{K}_0(H_n^i)$. Take an element $X \in \boldsymbol{K}_\Psi(H_n^i) \cap \boldsymbol{K}_0(H_n^i)^\perp$, where $\boldsymbol{K}_0(H_n^i)^\perp$ is the orthogonal complement of $\boldsymbol{K}_0(H_n^i)$ in $\mathfrak{sp}(n)$. Then X can be expressed as

$$X = \begin{pmatrix} 0 & \xi \\ -{}^t\bar{\xi} & ce^i \end{pmatrix}, \quad \xi \in M(n-1,\, 1; \mathbb{H}),\ c \in \mathbb{R}.$$

Take $j,\, k\ (= 1,\, 2,\, 3)$ so that $\{i,\, j,\, k\}$ is an even permutation of $\{1,\, 2,\, 3\}$. Then since $X \in \boldsymbol{K}_\Psi(H_n^i)$ and $H_n^j \in \boldsymbol{K}_\Psi(H_n^i)$, we obtain by Lemma 12 the following

$$0 = [[X,\, H_n^j],\, H_n^i] = \begin{pmatrix} 0 & -\xi e^k \\ -e^k\, {}^t\bar{\xi} & 4ce^j \end{pmatrix}.$$

Hence we have $\xi = 0$ and $c = 0$, i.e., $X = 0$. This proves $\boldsymbol{K}_\Psi(H_n^i) \cap \boldsymbol{K}_0(H_n^i)^\perp = 0$, i.e., $\boldsymbol{K}_\Psi(H_n^i) = \boldsymbol{K}_0(H_n^i)$.

Next we prove $\boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i) = \boldsymbol{K}_0(H_{n-1}^i + H_n^i)$. As in the case of $\boldsymbol{K}_\Psi(H_n^i)$, we can easily show that $\boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i) \supset \mathfrak{sp}(n-2) + \sum_{j\neq i} \mathbb{R}H_{n-1}^j + \sum_{j\neq i}\mathbb{R}H_n^j$. Take an element $Y \in \boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i)$ such that $\left(Y,\, \mathfrak{sp}(n-2) + \sum_{j\neq i} \mathbb{R}H_{n-1}^j + \sum_{j\neq i}\mathbb{R}H_n^j\right) = 0$. Then Y can be expressed as

$$Y = \begin{pmatrix} 0 & \xi & \eta \\ -{}^t\bar{\xi} & \alpha & \beta \\ -{}^t\bar{\eta} & -\bar{\beta} & \gamma \end{pmatrix}, \quad \xi,\, \eta \in M(n-2,\, 1; \mathbb{H}),\ \alpha,\, \gamma \in \mathbb{R}e^i,\ \beta \in \mathbb{H}.$$

Take $j,\, k\ (= 1,\, 2,\, 3)$ so that $\{i,\, j,\, k\}$ is an even permutation of $\{1,\, 2,\, 3\}$. Then by a direct calculation have

$$[[Y,\, H_{n-1}^j \pm H_n^j],\, H_{n-1}^i + H_n^i] = \begin{pmatrix} 0 & -\xi e^k & \mp\eta e^k \\ -e^k\, {}^t\bar{\xi} & -4\alpha e^k & \beta'' \\ \mp e^k\, {}^t\bar{\eta} & -\overline{\beta''} & \mp 4\gamma e^k \end{pmatrix},$$

where $\beta' = \pm\beta e^j - e^j\beta$, $\beta'' = \beta' e^i - e^i\beta'$. (Note that $e^j\alpha = -\alpha e^j$, $e^j\gamma = -\gamma e^j$, $e^i\alpha = \alpha e^i$, $e^i\gamma = \gamma e^i$, because $\alpha,\, \gamma \in \mathbb{R}e^i$.) Since $Y \in \boldsymbol{K}_\Psi(H_{n-1}^i +$

H_n^i) and $H_{n-1}^j \pm H_n^j \in \boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i)$, we have $[[Y, H_{n-1}^j \pm H_n^j], H_{n-1}^i + H_n^i] = 0$ (see Lemma 12). Hence we conclude that $\xi = \eta = 0$ and $\alpha = \gamma = 0$ and $\beta'' = 0$. From the equality $\beta'' = 0$, we immediately have $\beta' \in \mathbb{C}^i$. Further, from $\beta' \in \mathbb{C}^i$ we can easily conclude that $\beta \in \mathbb{D}^i$. Thus we have $Y \in \sum_{j\neq i} \mathbb{R}Q_{n-1,n}^j$ and hence $\boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i) \subset \boldsymbol{K}_0(H_{n-1}^i + H_n^i)$.

To complete the proof of (2) we have to show $\boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i) \supset \sum_{j\neq i} \mathbb{R}Q_{n-1,n}^j$. Take j $(1 \leq j \leq 3)$ such that $j \neq i$. Since $\mathfrak{s}_{n-1,n}^j = \mathbb{R}(H_{n-1}^j - H_n^j) + \mathbb{R}P_{n-1,n} + \mathbb{R}Q_{n-1,n}^j$ is *NAT*, there is an element $g \in \exp(\mathbb{R}P_{n-1,n})$ such that $\mathrm{Ad}(g)Q_{n-1,n}^j \in \mathbb{R}(H_{n-1}^j - H_n^j)$ $(\subset \mathfrak{p}(n)^i)$ (see Lemma 18). Moreover, since $[P_{n-1,n}, H_{n-1}^i + H_n^i] = 0$, we have $\mathrm{Ad}(g)(H_{n-1}^i + H_n^i) = H_{n-1}^i + H_n^i \in \mathfrak{h}(n)^i$, i.e., $g \in G^i_{(H_{n-1}^i + H_n^i)}$. Therefore, by Proposition 17 we have $Q_{n-1,n}^j \in \boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i)$. Accordingly, it follows that $\boldsymbol{K}_\Psi(H_{n-1}^i + H_n^i) \supset \sum_{j\neq i} \mathbb{R}Q_{n-1,n}^j$, completing the proof of (2). □

By $\mathcal{S}$ we denote the subset of $\mathfrak{sp}(n)$ consisting of all non-zero elements $X \in \mathfrak{sp}(n)$ such that $X \sim H_n^i$ or $X \sim H_{n-1}^i + H_n^i$ for some i $(= 1, 2, 3)$. We note that each element $X \in \mathcal{S}$ is a singular element of $\mathfrak{sp}(n)$, because H_n^i and $H_{n-1}^i + H_n^i$ are singular elements of $\mathfrak{sp}(n)$.

By use of Proposition 19 we can prove

Proposition 20 *Let* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. *Assume* $X \in \mathcal{S}$. *Then* $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_0(X)$.

Proof. Let $g \in Sp(n)$. Then we have Ψ^g and $\Psi_0^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ (see Lemma 14 (2)). By applying Proposition 19 to Ψ^g and Ψ_0^g, we have

$$\boldsymbol{K}_{\Psi^g}(H_n^i) = \boldsymbol{K}_0(H_n^i) = \boldsymbol{K}_{\Psi_0^g}(H_n^i);$$
$$\boldsymbol{K}_{\Psi^g}(H_{n-1}^i + H_n^i) = \boldsymbol{K}_0(H_{n-1}^i + H_n^i) = \boldsymbol{K}_{\Psi_0^g}(H_{n-1}^i + H_n^i)$$

for any i $(= 1, 2, 3)$. Now assume that $X \in \mathcal{S}$ and that g is an element of $Sp(n)$ such that $\mathrm{Ad}(g)X \in \mathbb{R}H_n^i$ or $\mathrm{Ad}(g)X \in \mathbb{R}(H_{n-1}^i + H_n^i)$. Then by the above equalities we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \boldsymbol{K}_{\Psi_0^g}(\mathrm{Ad}(g)X)$. (Note that $\boldsymbol{K}_\Psi(cZ) = \boldsymbol{K}_\Psi(Z)$ holds for any $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$, $Z \in \mathfrak{sp}(n)$ and $c \in \mathbb{R}$ $(c \neq 0)$.) On account of Lemma 14 (1) we have $\boldsymbol{K}_{\Psi^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_\Psi(X)$ and $\boldsymbol{K}_{\Psi_0^g}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\boldsymbol{K}_{\Psi_0}(X) = \mathrm{Ad}(g)\boldsymbol{K}_0(X)$. Therefore $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_0(X)$ follows immediately. □

As a consequence of Proposition 20 we can show

Proposition 21 *Let $i = 1, 2$ or 3. Then*
(1) $H_a^i \in \mathcal{S} \quad (1 \leq a \leq n)$;
(2) $H_a^i \pm H_b^i \in \mathcal{S} \quad (1 \leq a < b \leq n)$;
(3) $P_{ab} \in \mathcal{S},\ Q_{ab}^i \in \mathcal{S} \quad (1 \leq a < b \leq n)$.
Consequently, for any $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ the following equalities hold:

$$\begin{aligned} &\boldsymbol{K}_\Psi(H_a^i) = \boldsymbol{K}_0(H_a^i); \quad \boldsymbol{K}_\Psi(H_a^i \pm H_b^i) = \boldsymbol{K}_0(H_a^i \pm H_b^i); \\ &\boldsymbol{K}_\Psi(P_{ab}) = \boldsymbol{K}_0(P_{ab}); \quad \boldsymbol{K}_\Psi(Q_{ab}^i) = \boldsymbol{K}_0(Q_{ab}^i). \end{aligned} \tag{3.5}$$

Proof. Let $i = 1, 2$ or 3. It is easily shown that under the action of $Sp(n)$, H_a^i $(1 \leq a \leq n-1)$ is conjugate to H_n^i. This implies that $H_a^i \in \mathcal{S}$ $(1 \leq a \leq n)$. It is also known that $H_a^i + H_b^i$ $(1 \leq a < b \leq n)$ (resp. $H_a^i - H_b^i$ $(1 \leq a < b \leq n)$) is conjugate to $H_{n-1}^i + H_n^i$ (resp. $H_{n-1}^i - H_n^i$). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then we easily have $[H_n^i, H_n^j] = 2\varepsilon(ijk)H_n^k$. This proves that $\mathfrak{s} = \sum_{i=1}^3 \mathbb{R}H_n^i$ is *NAT*. In view of the proof of Lemma 18 $\exp(\mathbb{R}H_n^k)$ acts on $\mathfrak{s}' = \mathbb{R}H_n^i + \mathbb{R}H_n^j$ as a non-trivial subgroup of rotations of $\mathfrak{s}'$. Hence, we can find an element $h \in \exp(\mathbb{R}H_n^k)$ such that $\mathrm{Ad}(h)H_n^i = -H_n^i$. Since $[H_n^k, H_{n-1}^i] = 0$, we have $\mathrm{Ad}(h)H_{n-1}^i = H_{n-1}^i$ and hence $\mathrm{Ad}(h)(H_{n-1}^i - H_n^i) = H_{n-1}^i + H_n^i$. Therefore, we have $H_a^i \pm H_b^i \in \mathcal{S}$ $(1 \leq a < b \leq n)$. As we have pointed out, $P_{ab} \sim Q_{ab}^i \sim (H_a^i - H_b^i)$. Since $H_a^i - H_b^i \in \mathcal{S}$, it follows that $P_{ab} \in \mathcal{S}$ and $Q_{ab}^i \in \mathcal{S}$. This completes the proof. □

Remark 22 In the next section, after the proof of Theorem 10 we will know that $\boldsymbol{K}_\Psi(X) = \boldsymbol{K}_0(X)$ holds for any $X \in \mathfrak{sp}(n)$ (see Remark 36).

4. Solutions of the Gauss equation

In this section we will prove Theorem 10. We assume that $n \geq 2$ and that the Gaussian variety $\mathcal{G}(Sp(n'), \mathfrak{N}(n'))$ is *EOS* for any n' such that $n' < n$.

We now regard $\mathfrak{N}(n-1)$ as a subspace of $\mathfrak{N}(n)$ by the assignment

$$\mathfrak{N}(n-1) \ni Z \longmapsto \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{N}(n).$$

Let $\mathfrak{M}$ be the orthogonal complement of $\mathfrak{N}(n-1)$ in $\mathfrak{N}(n)$. Then we easily have $\dim \mathfrak{M} = 4n - 3$ and

$$\mathfrak{M} = \mathbb{R}E_{nn} + \sum_{a=1}^{n-1}\left\{\mathbb{R}(E_{an}+E_{na}) + \sum_{j=1}^{3}\mathbb{R}(E_{an}-E_{na})e^j\right\}$$

(orthogonal direct sum).

As in the previous section, we denote by Ψ_0 the canonical solution (2.4). By a simple calculation we can easily verify that $\Psi_0(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$ and $\mathfrak{M} = (\Psi_0)_{H_n^i}(\mathfrak{sp}(n))$ $(i = 1, 2, 3)$. In a natural manner, the restriction $\Psi_0|_{\mathfrak{sp}(n-1)}$ of Ψ_0 to $\mathfrak{sp}(n-1)$ may be regarded as an element $\mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$. Therefore, by the hypothesis of our induction we have:

Lemma 23 *For any $\Psi' \in \mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$ there is an element $\rho' \in O(\mathfrak{N}(n-1))$ such that $\rho'\Psi' = \Psi_0|_{\mathfrak{sp}(n-1)}$.*

Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. By $V_\Psi(X)$ $(\subset \mathfrak{N}(n))$ we denote the image of $\mathfrak{sp}(n)$ by the map Ψ_X. We call Ψ a *normal solution* if Ψ satisfies:
(1) $V_\Psi(H_n^i) = \mathfrak{M}$ $(i = 1, 2, 3)$;
(2) $\Psi|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$,
where $\Psi|_{\mathfrak{sp}(n-1)}$ means the restriction of Ψ to $\mathfrak{sp}(n-1)$. By $\mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ we mean the subset of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$ consisting of all normal solutions.

Proposition 24 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\rho\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$.*

Proof. Since $\dim K_\Psi(H_n^i) = \dim K_0(H_n^i)$ (see Proposition 19), we have $\dim V_\Psi(H_n^i) = \dim V_{\Psi_0}(H_n^i)$. Hence we have $\dim V_\Psi(H_n^i) = \dim \mathfrak{M}$ for any i $(= 1, 2, 3)$. Let $X, Y \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we get

$$\frac{1}{4}([[X, H_n^i], Y], Z) = \langle \Psi(X, Y), \Psi(H_n^i, Z)\rangle - \langle \Psi(X, Z), \Psi(H_n^i, Y)\rangle$$

for any $Z \in \mathfrak{sp}(n)$ and i $(= 1, 2, 3)$. Since $[X, H_n^i] = 0$ and $K_\Psi(H_n^i) = K_0(H_n^i) \supset \mathfrak{sp}(n-1)$ (see (3.4) and Proposition 19), we have $\Psi(H_n^i, Y) = 0$. Consequently, we have $\langle \Psi(X, Y), \Psi(H_n^i, Z)\rangle = 0$, which proves

$$\langle \Psi(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)), V_\Psi(H_n^i)\rangle = 0. \qquad (4.1)$$

Take an element $\rho_1 \in O(\mathfrak{N}(n))$ such that $\rho_1(V_\Psi(H_n^1)) = \mathfrak{M}$. Then by (4.1) we have $(\rho_1\Psi)(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \rho_1(\Psi(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1))) \subset$

$\mathfrak{N}(n-1)$. Hence, in a natural manner, $(\rho_1\Psi)|_{\mathfrak{sp}(n-1)}$ may be regarded as an element of $\mathcal{G}(Sp(n-1),\ \mathfrak{N}(n-1))$. Hence there is an element $\rho_2' \in O(\mathfrak{N}(n-1))$ such that $\rho_2'((\rho_1\Psi)|_{\mathfrak{sp}(n-1)}) = \Psi_0|_{\mathfrak{sp}(n-1)}$ (see Lemma 23). Take $\rho_2 \in O(\mathfrak{N}(n))$ such that $\rho_2|_{\mathfrak{M}} = \mathbf{1}_{\mathfrak{M}}$ and $\rho_2|_{\mathfrak{N}(n-1)} = \rho_2'$. Put $\rho = \rho_2\rho_1$. Then we have $\boldsymbol{V}_{\rho\Psi}(H_n^1) = \rho(\boldsymbol{V}_{\Psi}(H_n^1)) = \mathfrak{M}$ and $(\rho\Psi)|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$. We finally prove $\boldsymbol{V}_{\rho\Psi}(H_n^i) = \mathfrak{M}$ $(i = 2,\ 3)$. As is easily seen, we have $\Psi(\mathfrak{sp}(n-1),\ \mathfrak{sp}(n-1)) = \rho^{-1}(\mathfrak{N}(n-1))$. Hence by (4.1) we have $\boldsymbol{V}_{\Psi}(H_n^i) \subset \rho^{-1}(\mathfrak{M})$. Therefore, $\boldsymbol{V}_{\rho\Psi}(H_n^i) = \rho(\boldsymbol{V}_{\Psi}(H_n^i)) \subset \mathfrak{M}$. Since $\dim \boldsymbol{V}_{\rho\Psi}(H_n^i) = \dim \mathfrak{M}$, we have $\boldsymbol{V}_{\rho\Psi}(H_n^i) = \mathfrak{M}$, implying $\rho\Psi \in \mathcal{G}^0(Sp(n),\ \mathfrak{N}(n))$. This completes the proof. □

By virtue of Proposition 24 to show Theorem 10 it suffices to prove that any element of $\mathcal{G}^0(Sp(n),\ \mathfrak{N}(n))$ is equivalent to Ψ_0.

By $\mathfrak{m}$ we denote the orthogonal complement of $\mathfrak{sp}(n-1)$ in $\mathfrak{sp}(n)$. For simplicity, we set $P_a = P_{an}$, $Q_a^i = Q_{an}^i$ and $H^i = H_n^i$ for integers a $(1 \leq a \leq n-1)$ and i $(1 \leq i \leq 3)$. Set

$$\mathfrak{m}_a = \mathbb{R}P_a + \sum_{i=1}^{3} \mathbb{R}Q_a^i \ (1 \leq a \leq n-1), \quad \mathfrak{m}_n = \sum_{i=1}^{3} \mathbb{R}H^i.$$

Since $\big(\mathfrak{m}_a,\ \mathfrak{m}_b\big) = 0$ $(a \neq b)$, we have

$$\mathfrak{m} = \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n \quad \text{(orthogonal direct sum)}.$$

Lemma 25 *Let* $\Psi \in \mathcal{G}^0(Sp(n),\ \mathfrak{N}(n))$ *and let* $i = 1,\ 2$ *or* 3. *Then:*

$$\mathfrak{M} = \sum_{a=1}^{n-1} \Psi(H^i,\ \mathfrak{m}_a) + \mathbb{R}\Psi(H^i,\ H^i) \quad (\textit{direct sum}).$$

Proof. Since $\boldsymbol{K}_{\Psi}(H^i) = \mathfrak{sp}(n-1) + \sum_{j \neq i} \mathbb{R}H^j$ and $\boldsymbol{V}_{\Psi}(H^i) = \Psi(H^i,\ \mathfrak{m}) = \mathfrak{M}$, we have the lemma. □

In what follows we will observe the value $\Psi(X,\ Y)$ $(X,\ Y \in \mathfrak{sp}(n))$ for the following four cases:

(I) $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
(II) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_n$;
(III) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a$ $(1 \leq a \leq n-1)$;
(IV) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_a$ $(1 \leq a \leq n-1)$.

We first observe Case (I):

Proposition 26 *Let* $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Then:*
(1) $\Psi(\mathfrak{m}, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$.
(2) *Let* $X, Y \in \mathfrak{m}$ *and* $Z \in \mathfrak{sp}(n-1)$. *Then:*

$$\big\langle \Psi(X, Z), \Psi(H^i, Y)\big\rangle = \frac{1}{4}\big([[X, Z], H^i], Y\big). \tag{4.2}$$

Proof. We first note that $\Psi(H^i, \mathfrak{sp}(n-1)) = 0$ $(1 \leq i \leq 3)$, because $\boldsymbol{K}_{\Psi}(H^i) \supset \mathfrak{sp}(n-1)$. This proves $\Psi(\mathfrak{m}_n, \mathfrak{sp}(n-1)) = 0$. We now prove $\Psi(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$ for any a $(1 \leq a \leq n-1)$. To show this we prove

$$\Psi(P_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}; \quad \Psi(Q_a^i, \mathfrak{sp}(n-1)) \subset \mathfrak{M} \quad (i = 1, 2, 3). \tag{4.3}$$

Define an element $Z_0^i \in \mathfrak{sp}(n-1)$ $(1 \leq i \leq 3)$ by $Z_0^i = (\sum_{s=1}^{n-1} sE_{ss})e^i$. Then it is well-known that Z_0^i is a regular element of $\mathfrak{sp}(n-1)$. Moreover, since $\Psi|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$, it follows that $\Psi(Z_0^i, \mathfrak{sp}(n-1)) \subset \mathfrak{N}(n-1)$. Here we note that the equality $\Psi(Z_0^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$ holds. Indeed, since $\dim \mathbf{Ker}((\Psi_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) = 2(n-1)$ (see Proposition 15), we have

$$\begin{aligned}\dim \Psi(Z_0^i, \mathfrak{sp}(n-1)) &= \dim \mathfrak{sp}(n-1) - \dim \mathbf{Ker}((\Psi_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) \\ &= \dim \mathfrak{N}(n-1).\end{aligned}$$

Now let us set $W_a^i = Z_0^i - aH^i \in \mathfrak{sp}(n)$ $(1 \leq a \leq n-1)$. By a direct calculation we can verify $\Psi_0(P_a, W_a^i) = \Psi_0(Q_a^i, W_a^i) = 0$. Hence by (3.5) we have $\Psi(P_a, W_a^i) = \Psi(Q_a^i, W_a^i) = 0$. Moreover, since $\Psi(H^i, \mathfrak{sp}(n-1)) = 0$, we have $\Psi(W_a^i, \mathfrak{sp}(n-1)) = \Psi(Z_0^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$. Let $Z, Z' \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\begin{aligned}&\frac{1}{4}([[W_a^i, Z], Z'], P_a) \\ &\quad = \big\langle \Psi(W_a^i, Z'), \Psi(Z, P_a)\big\rangle - \big\langle \Psi(W_a^i, P_a), \Psi(Z, Z')\big\rangle, \qquad (4.4)\\ &\frac{1}{4}([[W_a^i, Z], Z'], Q_a^i) \\ &\quad = \big\langle \Psi(W_a^i, Z'), \Psi(Z, Q_a^i)\big\rangle - \big\langle \Psi(W_a^i, Q_a^i), \Psi(Z, Z')\big\rangle. \qquad (4.5)\end{aligned}$$

Since $[H^i, Z] = 0$, we have $[[W_a^i, Z], Z'] = [[Z_0^i, Z], Z'] \in \mathfrak{sp}(n-1)$. Hence, the left sides of (4.4) and (4.5) vanish. Further, since $\Psi(P_a, W_a^i) = \Psi(Q_a^i, W_a^i) = 0$, we have $\big\langle \Psi(W_a^i, Z'), \Psi(Z, P_a)\big\rangle = \big\langle \Psi(W_a^i, Z'), \Psi(Z, Q_a^i)\big\rangle = 0$. Since Z and Z' are arbitrary elements of $\mathfrak{sp}(n-1)$ and since

$\Psi(W_a^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$, we have

$$\langle \mathfrak{N}(n-1), \Psi(\mathfrak{sp}(n-1), P_a) \rangle = \langle \mathfrak{N}(n-1), \Psi(\mathfrak{sp}(n-1), Q_a^i) \rangle = 0,$$

showing (4.3). Consequently, we have $\Psi(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$, which completes the proof of (1).

Next we show (2). Let $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}([[X, H^i], Z], Y) = \langle \Psi(X, Z), \Psi(H^i, Y) \rangle - \langle \Psi(X, Y), \Psi(H^i, Z) \rangle.$$

Note that $\Psi(H^i, Z) = 0$ and $[Z, H^i] = 0$. The latter equality, together with the Jacobi identity, shows $[[X, H^i], Z] = [[X, Z], H^i]$. Thus we obtain (4.2). □

Remark 27 Here we state a remark on the value $\Psi(X, Z)$ $(X \in \mathfrak{m}, Z \in \mathfrak{sp}(n-1))$. Note that the right side of (4.2) is an intrinsic quantity. Since $\Psi(H^i, \mathfrak{m}) = \mathfrak{M}$, we know that $\Psi(X, Z) \in \mathfrak{M}$ is uniquely determined if the values $\Psi(H^i, Y)$ $(Y \in \mathfrak{m})$ are given. Therefore, if $\Psi(H^i, Y) = \Psi_0(H^i, Y)$ holds for any $Y \in \mathfrak{m}$, then we may conclude that $\Psi(X, Z) = \Psi_0(X, Z)$ $(X \in \mathfrak{m}, Z \in \mathfrak{sp}(n-1))$. See Case (c) below in the proof of Theorem 10.

We next observe Case (II):

Proposition 28 *Let* $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Then:*
(1) $\Psi(H^1, H^1) = \Psi(H^2, H^2) = \Psi(H^3, H^3)$.
(2) $\Psi(H^1, H^2) = \Psi(H^2, H^3) = \Psi(H^3, H^1) = 0$.
(3) $\langle \Psi(H^i, H^i), \Psi(H^i, H^i) \rangle = 1 \quad (1 \leq i \leq 3)$.
(4) $\langle \Psi(H^i, H^i), \Psi(H^i, \mathfrak{m}_a) \rangle = 0 \quad (1 \leq i \leq 3,\ 1 \leq a \leq n-1)$.

To prove the proposition we prepare

Lemma 29 *Let* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. *Let* X *and* $Y \in \mathfrak{sp}(n)$. *Assume:*
(i) $\Psi_0(X, X) = \Psi_0(Y, Y)$.
(ii) $X + Y \in \mathcal{S}$.
Then $\Psi(X, X) = \Psi(Y, Y)$.

Proof. By (i) we easily have $\Psi_0(X+Y, X-Y) = 0$, i.e., $X - Y \in \boldsymbol{K}_0(X+Y)$. Since $X + Y \in \mathcal{S}$, we have $\boldsymbol{K}_0(X+Y) = \boldsymbol{K}_\Psi(X+Y)$ (see Proposition 20). Consequently, it follows that $X - Y \in \boldsymbol{K}_\Psi(X+Y)$, i.e., $\Psi(X+Y, X-Y) = 0$. This implies $\Psi(X, X) = \Psi(Y, Y)$. □

Proof of Proposition 28. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. As shown in the proof of Proposition 21, $\mathfrak{s} = \sum_{i=1}^{3} \mathbb{R}H^i$ is NAT. Consequently, $H^i + H^j \in \mathcal{S}$, because $(H^i + H^j) \sim H^i$. On the other hand, it is easily checked that $\Psi_0(H^i, H^i) = \Psi_0(H^j, H^j) = -E_{nn}$. Hence by Lemma 29 we have $\Psi(H^i, H^i) = \Psi(H^j, H^j)$. Similarly, we have $\Psi(H^j, H^j) = \Psi(H^k, H^k)$, proving (1). The assertion (2) is clear from Lemma 13. Finally we prove (3) and (4). Let k be an integer such that $1 \leq k \leq 3$, $k \neq i$ and $X \in \mathfrak{sp}(n)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}\big([[H^i, H^k], H^k], X\big) = \big\langle \Psi(H^i, H^k), \Psi(H^k, X)\big\rangle - \big\langle \Psi(H^i, X), \Psi(H^k, H^k)\big\rangle.$$

By a simple calculation we have $[[H^i, H^k], H^k] = -4H^i$. Moreover, by the results obtained in (1) and (2) we have $\Psi(H^i, H^k) = 0$ and $\Psi(H^k, H^k) = \Psi(H^i, H^i)$. Consequently, we have

$$\big\langle \Psi(H^i, X), \Psi(H^i, H^i)\big\rangle = \big(H^i, X\big).$$

Therefore, we obtain (3) and (4), because $\big(H^i, H^i\big) = 1$ and $\big(H^i, \mathfrak{m}_a\big) = 0$ (see (3.3)). □

In Case (III) the value $\Psi(X, Y)$ $(X, Y \in \mathfrak{m}_a)$ $(1 \leq a \leq n-1)$ are determined by

Proposition 30 *Let* $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ *and let* a *be an integer such that* $1 \leq a \leq n-1$. *Then:*
(1) $\Psi(P_a, Q_a^i) = 0 \quad (1 \leq i \leq 3)$.
(2) $\Psi(Q_a^i, Q_a^j) = 0 \quad (1 \leq i \neq j \leq 3)$.
(3) $\Psi(P_a, P_a) = \Psi(Q_a^i, Q_a^i) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i) \quad (1 \leq i \leq 3)$.

Proof. Since $\Psi_0(P_a, Q_a^i) = 0$ and $\Psi_0(Q_a^i, Q_a^j) = 0$ $(i \neq j)$, we obtain (1) and (2) (see (3.5)). We now prove (3). Since $\mathfrak{s}_{an}^i = \mathbb{R}(H^i - H_a^i) + \mathbb{R}P_a + \mathbb{R}Q_a^i$ is NAT, it follows that $Q_a^i + (H^i - H_a^i) \in \mathcal{S}$. Indeed, $Q_a^i + (H^i - H_a^i) \sim (H^i - H_a^i)$. By Lemma 29 we have $\Psi(Q_a^i, Q_a^i) = \Psi(H^i - H_a^i, H^i - H_a^i)$, because $\Psi_0(Q_a^i, Q_a^i) = \Psi_0(H^i - H_a^i, H^i - H_a^i) = -(E_{aa} + E_{nn})$. Since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\Psi(H^i, H_a^i) = 0$. Consequently, $\Psi(Q_a^i, Q_a^i) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$. Similarly, we can prove $\Psi(P_a, P_a) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$. □

Before proceeding to Case (IV) we extend Lemma 29 to the following form:

Lemma 31 *Let* $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. *Let* X, X', Y *and* $Y' \in \mathfrak{sp}(n)$. *Assume*:
(i) $\Psi_0(X, Y') = \Psi_0(Y, X') = 0$.
(ii) $\Psi_0(X, X') = \Psi_0(Y, Y')$.
(iii) $X \in \mathcal{S}$, $Y \in \mathcal{S}$ *and* $X + Y \in \mathcal{S}$.
Then $\Psi(X, X') = \Psi(Y, Y')$.

Proof. By (i) and (ii) we have $Y' \in \boldsymbol{K}_0(X)$, $X' \in \boldsymbol{K}_0(Y)$ and $\Psi_0(X + Y, X' - Y') = 0$. The last equality implies that $X' - Y' \in \boldsymbol{K}_0(X + Y)$. Hence by (iii) we have $Y' \in \boldsymbol{K}_\Psi(X)$, $X' \in \boldsymbol{K}_\Psi(Y)$ and $X' - Y' \in \boldsymbol{K}_\Psi(X + Y)$. Consequently, we have $\Psi(Y', X) = \Psi(X', Y) = \Psi(X + Y, X' - Y') = 0$. Hence $\Psi(X, X') = \Psi(Y, Y')$. □

With this preparation we observe Case (IV).

Proposition 32 *Let* $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. *Let a be an integer such that* $1 \leq a \leq n - 1$. *Then*:
(1) $\Psi(H^1, Q^1_a) = \Psi(H^2, Q^2_a) = \Psi(H^3, Q^3_a)$.
(2) $\Psi(H^i, Q^j_a) = -\varepsilon(ijk)\Psi(H^k, P_a)$, *where* $\{i, j, k\}$ *is a permutation of* $\{1, 2, 3\}$.
(3) $\Psi(H^1, \mathfrak{m}_a) = \Psi(H^2, \mathfrak{m}_a) = \Psi(H^3, \mathfrak{m}_a)$.
(4) *For each* i $(1 \leq i \leq 3)$ *the set* $\{\sqrt{2}\Psi(H^i, P_a), \sqrt{2}\Psi(H^i, Q^j_a)$ $(1 \leq j \leq 3)\}$ *forms an orthonormal basis of* $\Psi(H^i, \mathfrak{m}_a)$.

Proof. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. We note that the subspace $\mathfrak{s} = \mathbb{R}(H^i_a + H^i) + \mathbb{R}Q^j_a + \mathbb{R}Q^k_a$ forms a subalgebra of $\mathfrak{sp}(n)$ and is *NAT*. In fact, by simple calculations we have

$$[H^i_a + H^i, Q^j_a] = 2\varepsilon(ijk)Q^k_a; \quad [H^i_a + H^i, Q^k_a] = -2\varepsilon(ijk)Q^j_a;$$
$$[Q^j_a, Q^k_a] = 2\varepsilon(ijk)(H^i_a + H^i).$$

Hence we have $H^i_a + H^i + Q^j_a \in \mathcal{S}$ and $H^i_a + H^i + Q^k_a \in \mathcal{S}$, because $H^i_a + H^i + Q^j_a \sim H^i_a + H^i + Q^k_a \sim H^i_a + H^i \in \mathcal{S}$.

Now we prove (1). By direct calculations we can show $\Psi_0(H^1_a + H^1, Q^1_a) = \Psi_0(H^2_a + H^2, Q^2_a) = \Psi_0(H^3_a + H^3, Q^3_a) = -(E_{an} + E_{na})$. Moreover we have $\Psi_0(H^i_a + H^i, H^j_a + H^j) = \Psi_0(Q^i_a, Q^j_a) = 0$ if $i \neq j$ (see Lemma 13 and Proposition 30). Therefore by Lemma 31 we have

$$\Psi(H^1_a + H^1, Q^1_a) = \Psi(H^2_a + H^2, Q^2_a) = \Psi(H^3_a + H^3, Q^3_a). \tag{4.6}$$

Here we show $\Psi(H_a^1, Q_a^1) = \Psi(H_a^2, Q_a^2) = \Psi(H_a^3, Q_a^3)$. Let $i = 1, 2$ or 3. Since $H_a^i \in \mathfrak{sp}(n-1)$ and $Q_a^i \in \mathfrak{m}$, it follows from Proposition 26 (1) that $\Psi(H_a^i, Q_a^i) \in \mathfrak{M}$. Moreover, by Proposition 26 (2) we have

$$\langle \Psi(Q_a^i, H_a^i), \Psi(H^1, Y)\rangle = \frac{1}{4}([[Q_a^i, H_a^i], H^1], Y)$$

for any $Y \in \mathfrak{m}$. Since $[Q_a^i, H_a^i] = P_a$, the right side of the above equality does not depend on the choice of i. This implies that $\Psi(H_a^1, Q_a^1) = \Psi(H_a^2, Q_a^2) = \Psi(H_a^3, Q_a^3)$, because $\Psi(H^1, \mathfrak{m}) = \mathfrak{M}$. This, together with (4.6), proves (1).

We next prove (2). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then by direct calculations we have $\Psi_0(H_a^i - H^i, Q_a^j) = \varepsilon(ijk)\Psi_0(H_a^k + H^k, P_a) = \varepsilon(ijk)(E_{an} - E_{na})e^k$. Moreover, $\Psi_0(H_a^i - H^i, H_a^k + H^k) = \Psi_0(Q_a^j, P_a) = 0$ (see Lemma 13 and Proposition 30). Since $H_a^k + H^k + Q_a^j \in \mathcal{S}$, we obtain by Lemma 31 the following

$$\Psi(H_a^i - H^i, Q_a^j) = \varepsilon(ijk)\Psi(H_a^k + H^k, P_a). \tag{4.7}$$

Note that $H_a^i, H_a^k \in \mathfrak{sp}(n-1)$, $Q_a^j, P_a \in \mathfrak{m}$ and $[Q_a^j, H_a^i] = \varepsilon(ijk)[P_a, H_a^k] = -\varepsilon(ijk)Q_a^k$. As in the proof of (1) we have $\Psi(H_a^i, Q_a^j) = \varepsilon(ijk)\Psi(H_a^k, P_a)$. Accordingly, from (4.7) we have $\Psi(H^i, Q_a^j) = -\varepsilon(ijk)\Psi(H^k, P_a)$. This completes the proof of (2).

By (1) and (2) we have

$$\begin{aligned} &\Psi(H^1, P_a) = -\Psi(H^2, Q_a^3) = \Psi(H^3, Q_a^2); \\ &\Psi(H^1, Q_a^1) = \Psi(H^2, Q_a^2) = \Psi(H^3, Q_a^3); \\ &\Psi(H^1, Q_a^2) = -\Psi(H^2, Q_a^1) = -\Psi(H^3, P_a); \\ &\Psi(H^1, Q_a^3) = \Psi(H^2, P_a) = -\Psi(H^3, Q_a^1). \end{aligned} \tag{4.8}$$

By these equalities we clearly obtain (3).

Finally, we prove (4). Let X and Y are one of P_a and Q_a^j $(1 \leq j \leq 3)$, i.e., $X, Y \in \{P_a, Q_a^j \ (1 \leq j \leq 3)\}$. By the Gauss equation (2.3) we have

$$\begin{aligned} &\frac{1}{4}([[H^i, X], H^i], Y) \\ &\qquad = \langle \Psi(H^i, H^i), \Psi(X, Y)\rangle - \langle \Psi(H^i, Y), \Psi(X, H^i)\rangle. \end{aligned}$$

By direct calculations we can verify $[[H^i, X], H^i] = X$. Hence the left side of the above equality becomes $(1/4)(X, Y)$. First assume that $X = Y$. Then we have $\Psi(X, X) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$ (see Proposition 30 (3)). Since $\langle \Psi(H^i, H^i), \Psi(H^i, H^i)\rangle = 1$ (see Proposition 28), $\Psi(H^i, H^i) \in \mathfrak{M}$

and $\Psi(H_a^i, H_a^i) \in \mathfrak{N}(n-1)$, we have

$$\langle \Psi(H^i, H^i), \Psi(X, X) \rangle = \langle \Psi(H^i, H^i), \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i) \rangle$$
$$= 1.$$

Since $(X, X) = 2$ (see (3.3)), we have $\langle \Psi(H^i, X), \Psi(H^i, X) \rangle = 1/2$. We next consider the case $X \neq Y$. Then we have $(X, Y) = 0$ and $\Psi(X, Y) = 0$ (see (3.3) and Proposition 30 (1), (2)). Hence it follows that $\langle \Psi(H^i, X), \Psi(H^i, Y) \rangle = 0$. This completes the proof of (4). □

We are now in a position to prove Theorem 10.

Proof of Theorem 10. Let $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. Set $\boldsymbol{H} = \Psi(H^1, H^1)$, $\boldsymbol{P}_a = \sqrt{2}\Psi(H^1, P_a)$ $(1 \leq a \leq n-1)$, $\boldsymbol{Q}_a^i = \sqrt{2}\Psi(H^1, Q_a^i)$ $(1 \leq a \leq n-1, 1 \leq i \leq 3)$. Then we have

Lemma 33 *The set* $\mathfrak{O} = \{\boldsymbol{H}, \boldsymbol{P}_a \ (1 \leq a \leq n-1), \boldsymbol{Q}_a^i \ (1 \leq a \leq n-1, 1 \leq i \leq 3)\}$ *forms an orthonormal basis of* $\mathfrak{M}$.

Proof. By virtue of Proposition 28 (3), (4) and Proposition 32 (4) we have only to prove

$$\langle \Psi(H^1, \mathfrak{m}_a), \Psi(H^1, \mathfrak{m}_b) \rangle = 0 \quad (1 \leq a \neq b \leq n-1). \tag{4.9}$$

Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. By the Gauss equation (2.3) we have

$$\frac{1}{4}([[H^1, X], H^2], Y)$$
$$= \langle \Psi(H^1, H^2), \Psi(X, Y) \rangle - \langle \Psi(H^1, Y), \Psi(X, H^2) \rangle.$$

As is easily seen, $[[H^1, X], H^2] \in \mathfrak{m}_a$. Hence the left side of the above equality vanishes. On the other hand, since $\Psi(H^1, H^2) = 0$ (see Proposition 28), it follows that $\langle \Psi(H^1, Y), \Psi(X, H^2) \rangle = 0$. This proves that $\langle \Psi(H^1, \mathfrak{m}_b), \Psi(H^2, \mathfrak{m}_a) \rangle = 0$. Therefore, we obtain (4.9), because $\Psi(H^2, \mathfrak{m}_a) = \Psi(H^1, \mathfrak{m}_a)$ (see Proposition 32 (3)). This completes the proof. □

Let $\mathfrak{O}_0 = \{\boldsymbol{H}_0, (\boldsymbol{P}_a)_0 \ (1 \leq a \leq n-1), (\boldsymbol{Q}_a^i)_0 \ (1 \leq a \leq n-1, 1 \leq i \leq 3)\}$ be the orthonormal basis of $\mathfrak{M}$ corresponding to Ψ_0, i.e., $\boldsymbol{H}_0 = \Psi_0(H^1, H^1)$, $(\boldsymbol{P}_a)_0 = \sqrt{2}\Psi_0(H^1, P_a)$ and $(\boldsymbol{Q}_a^i)_0 = \sqrt{2}\Psi_0(H^1, Q_a^i)$. Then, there is an orthogonal transformation ρ' of $\mathfrak{M}$ such that $\boldsymbol{H}_0 = \rho'(\boldsymbol{H})$, $(\boldsymbol{P}_a)_0 = \rho'(\boldsymbol{P}_a)$ and $(\boldsymbol{Q}_a^i)_0 = \rho'(\boldsymbol{Q}_a^i)$. Extend ρ' to the orthogonal transformation ρ of $\mathfrak{N}(n)$

satisfying $\rho|_{\mathfrak{M}} = \rho'$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$. Then, it is easy to see that $\rho\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. For simplicity, set $\Psi_1 = \rho\Psi$. In the following we will prove $\Psi_1 = \Psi_0$. In view of Lemma 25 and the decomposition $\mathfrak{sp}(n) = \mathfrak{m} + \mathfrak{sp}(n-1)$, we may conclude $\Psi_1 = \Psi_0$ if $\Psi_1(X, Y) = \Psi_0(X, Y)$ holds for any pairs X and Y listed in the following (a) $\sim$ (e):

(a) $X \in \mathfrak{sp}(n-1)$ and $Y \in \mathfrak{sp}(n-1)$;
(b) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}$;
(c) $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
(d) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a$ $(1 \leq a \leq n-1)$;
(e) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$ $(1 \leq a \neq b \leq n-1)$.

Case (a): Let $X, Y \in \mathfrak{sp}(n-1)$. Since $\Psi(X, Y) = \Psi_0(X, Y) \in \mathfrak{N}(n-1)$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$, we have $\Psi_1(X, Y) = \rho(\Psi(X, Y)) = \rho(\Psi_0(X, Y)) = \Psi_0(X, Y)$.

Case (b): By the very definition of ρ we have $\Psi_1(H^1, Y) = \Psi_0(H^1, Y)$ for $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathbb{R}H^1$. Applying Proposition 32 to both Ψ_1 and Ψ_0, we have $\Psi_1(H^i, Y) = \Psi_0(H^i, Y)$ for $i = 2, 3$, $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a$ (see (1), (2) and (4.8)). Further, since $\Psi_1(H^1, H^1) = \Psi_0(H^1, H^1)$, we have $\Psi_1(H^i, H^j) = \Psi_0(H^i, H^j)$ $(1 \leq i, j \leq 3)$ (see Proposition 28 (1), (2)). Thus we obtain $\Psi_1(X, Y) = \Psi_0(X, Y)$ for any $X \in \mathfrak{m}_n$ and $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n = \mathfrak{m}$.

Case (c): By Case (b) we have $\Psi_1(H^i, Y) = \Psi_0(H^i, Y)$ $(i = 1, 2, 3; Y \in \mathfrak{m})$. As we have remarked (see Remark 27), we obtain $\Psi_1(X, Y) = \Psi_0(X, Y)$ for $X \in \mathfrak{m}$, $Y \in \mathfrak{sp}(n-1)$.

Case (d): As seen in Case (b), we have $\Psi_1(H^i, H^i) = \Psi_0(H^i, H^i)$. Moreover, since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\Psi_1(H_a^i, H_a^i) = \Psi_0(H_a^i, H_a^i)$ $(i = 1, 2, 3)$. Hence by applying Proposition 30 to Ψ_1 and Ψ_0, we easily have $\Psi_1(X, Y) = \Psi_0(X, Y)$ for $X, Y \in \mathfrak{m}_a$.

Case (e): We note that this case occurs when $n \geq 3$. We first show

Lemma 34 *Assume that $n \geq 3$. Let a and c be integers such that $1 \leq a \neq c \leq n-1$. Then $P_a \pm P_{ac} \in \mathcal{S}$; $Q_a^i \pm Q_{ac}^i \in \mathcal{S}$ $(i = 1, 2, 3)$.*

Proof. By easy calculations we have

$$[H_c^i - H^i, P_a \pm P_{ac}] = Q_a^i \mp Q_{ac}^i;$$
$$[H_c^i - H^i, Q_a^i \mp Q_{ac}^i] = -(P_a \pm P_{ac});$$

$$[P_a \pm P_{ac},\, Q^i_a \mp Q^i_{ac}] = 2(H^i_c - H^i).$$

Consequently, both the subspaces $\mathfrak{s}_+ = \mathbb{R}(H^i_c - H^i) + \mathbb{R}(P_a + P_{ac}) + \mathbb{R}(Q^i_a - Q^i_{ac})$ and $\mathfrak{s}_- = \mathbb{R}(H^i_c - H^i) + \mathbb{R}(P_a - P_{ac}) + \mathbb{R}(Q^i_a + Q^i_{ac})$ are *NAT*. Therefore, we have $P_a \pm P_{ac} \sim H^i_c - H^i \sim Q^i_a \pm Q^i_{ac}$. Since $H^i_c - H^i \in \mathcal{S}$, it follows that $P_a \pm P_{ac} \in \mathcal{S}$ and $Q^i_a \pm Q^i_{ac} \in \mathcal{S}$. □

First assume $n \geq 4$. Let us consider the case $X = P_a$ and $Y = P_b$. Take an integer c $(1 \leq c \leq n-1)$ such that $c \neq a$ and $c \neq b$. By easy calculations we have $\boldsymbol{\Psi}_0(P_a, P_b) = \boldsymbol{\Psi}_0(P_{ac}, P_{bc}) = -(1/2)(E_{ab} + E_{ba})$ and $\boldsymbol{\Psi}_0(P_a, P_{bc}) = \boldsymbol{\Psi}_0(P_{ac}, P_b) = 0$. Since P_a, P_{ac} and $P_a + P_{ac} \in \mathcal{S}$, it follows that $\boldsymbol{\Psi}_1(P_a, P_b) = \boldsymbol{\Psi}_1(P_{ac}, P_{bc})$ (see Lemma 31). Since $P_{ac}, P_{bc} \in \mathfrak{sp}(n-1)$, we have $\boldsymbol{\Psi}_1(P_{ac}, P_{bc}) = \boldsymbol{\Psi}_0(P_{ac}, P_{bc})$ (see the Case (a)). Hence we have $\boldsymbol{\Psi}_1(P_a, P_b) = \boldsymbol{\Psi}_0(P_a, P_b)$. In a similar manner we can prove $\boldsymbol{\Psi}_1(P_a, Q^i_b) = \boldsymbol{\Psi}_0(P_a, Q^i_b)$ $(i = 1, 2, 3)$ and $\boldsymbol{\Psi}_1(Q^i_a, Q^j_b) = \boldsymbol{\Psi}_0(Q^i_a, Q^j_b)$ $(i, j = 1, 2, 3)$. By these facts we obtain the equality $\boldsymbol{\Psi}_1(X, Y) = \boldsymbol{\Psi}_0(X, Y)$ $(X \in \mathfrak{m}_a, Y \in \mathfrak{m}_b)$ when $n \geq 4$.

Next we assume $n = 3$. Apparently, the method used in the case $n \geq 4$ cannot be applied to this case. We prove

Lemma 35 *Assume that $n = 3$. Then $\boldsymbol{\Psi}_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$.*

Proof. Set $\mathfrak{B}_a = \{P_a, Q^1_a, Q^2_a, Q^3_a\}$ $(a = 1, 2)$. Let $X \in \mathfrak{B}_1$ and $Y \in \mathfrak{B}_2$. We first show

$$\langle \boldsymbol{\Psi}_1(X, Y),\, \boldsymbol{\Psi}_1(H^1, H^1) \rangle = \langle \boldsymbol{\Psi}_1(X, Y),\, \boldsymbol{\Psi}_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2) \rangle = 0. \tag{4.10}$$

If this is true, then we have $\boldsymbol{\Psi}_1(X, Y) \in \mathfrak{N}(2)$, because $\mathfrak{M} = \mathbb{R}\boldsymbol{\Psi}_1(H^1, H^1) + \boldsymbol{\Psi}_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2)$ (see Lemma 25) and because $\mathfrak{N}(2)$ is the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{N}(3)$.

By the Gauss equation (2.3) we have

$$\begin{aligned}&\frac{1}{4}([[H^1, X], H^1], Y)\\ &\qquad = \langle \boldsymbol{\Psi}_1(H^1, H^1),\, \boldsymbol{\Psi}_1(X, Y) \rangle - \langle \boldsymbol{\Psi}_1(H^1, Y),\, \boldsymbol{\Psi}_1(X, H^1) \rangle.\end{aligned}$$

As observed in the proof of Proposition 32, we have $[[H^1, X], H^1] = X$. Since $(X, Y) = 0$, the left side of the above equality vanishes. Moreover, in view of (4.9) we have $\langle \boldsymbol{\Psi}_1(H^1, Y),\, \boldsymbol{\Psi}_1(X, H^1) \rangle = 0$. Consequently, we have $\langle \boldsymbol{\Psi}_1(X, Y),\, \boldsymbol{\Psi}_1(H^1, H^1) \rangle = 0$. Let Z be an arbitrary element of $\mathfrak{B}_1$.

Then by the Gauss equation (2.3) we have

$$\frac{1}{4}([[X, H^1], Y], Z) = \langle \Psi_1(X, Y), \Psi_1(H^1, Z)\rangle - \langle \Psi_1(X, Z), \Psi_1(H^1, Y)\rangle.$$

Here we can easily verify that $[[X, H^1], Y] \in \mathfrak{sp}(2)$ and hence the left side of the above equality vanishes. By Proposition 30 (1), (2) we have $\Psi_1(X, Z) = 0$ if $X \neq Z$. Hence $\langle \Psi_1(X, Y), \Psi_1(H^1, Z)\rangle = 0$. On the other hand, if $X = Z$, then we have $\Psi_1(X, Z) = \Psi_1(X, X) = \Psi_1(H^1, H^1) + \Psi_1(H_1^1, H_1^1)$ (see Proposition 30). Hence by Proposition 28 (4) and the fact $\Psi_1(H_1^1, H_1^1) \in \mathfrak{N}(2)$ we have $\langle \Psi_1(X, Z), \Psi_1(H^1, Y)\rangle = 0$. Therefore, in this case, we also obtain $\langle \Psi_1(X, Y), \Psi_1(H^1, Z)\rangle = 0$. Since Z is an arbitrary element of $\mathfrak{B}_1$, we have $\langle \Psi_1(X, Y), \Psi_1(H^1, \mathfrak{m}_1)\rangle = 0$. In a similar way we can prove $\langle \Psi_1(X, Y), \Psi_1(H^1, \mathfrak{m}_2)\rangle = 0$, showing (4.10). Accordingly, we get $\Psi_1(X, Y) \in \mathfrak{N}(2)$ and hence $\Psi_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$. □

Now let $X \in \mathfrak{m}_1$, $Y \in \mathfrak{m}_2$. Take arbitrary elements $Z_1, Z_2 \in \mathfrak{sp}(2)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}([[X, Z_1], Y], Z_2) = \langle \Psi_1(X, Y), \Psi_1(Z_1, Z_2)\rangle - \langle \Psi_1(X, Z_2), \Psi_1(Z_1, Y)\rangle.$$

By the results of Case (a) and Case (c) we have $\Psi_1(Z_1, Z_2) = \Psi_0(Z_1, Z_2)$, $\Psi_1(X, Z_2) = \Psi_0(X, Z_2)$ and $\Psi_1(Y, Z_1) = \Psi_0(Y, Z_1)$. Therefore we have

$$\langle \Psi_1(X, Y), \Psi_0(Z_1, Z_2)\rangle = \frac{1}{4}([[X, Z_1], Y], Z_2) + \langle \Psi_0(X, Z_2), \Psi_0(Z_1, Y)\rangle.$$

Since Ψ_0 is a solution of the Gauss equation (2.3), we have

$$\langle \Psi_0(X, Y), \Psi_0(Z_1, Z_2)\rangle = \frac{1}{4}([[X, Z_1], Y], Z_2) + \langle \Psi_0(X, Z_2), \Psi_0(Z_1, Y)\rangle.$$

Hence, by subtraction, we have $\langle \Psi_1(X, Y) - \Psi_0(X, Y), \Psi_0(Z_1, Z_2)\rangle = 0$. Here we note that $\Psi_1(X, Y) - \Psi_0(X, Y) \in \mathfrak{N}(2)$. Indeed, we have $\Psi_1(X, Y) \in \mathfrak{N}(2)$ (see Lemma 35) and have $\Psi_0(X, Y) \in \mathfrak{N}(2)$ by a simple calculation. Since $\Psi_0(\mathfrak{sp}(2), \mathfrak{sp}(2)) = \mathfrak{N}(2)$, the above equality implies that

$\Psi_1(X, Y) - \Psi_0(X, Y) = 0$, i.e., $\Psi_1(X, Y) = \Psi_0(X, Y)$. This completes the proof of (e) in the case where $n = 3$.

Thus by the above case studies (a) $\sim$ (e) we get $\Psi_1 = \Psi_0$, i.e., $\rho\Psi = \Psi_0$. This completes the proof of Theorem 10. □

Remark 36 As seen in the above discussion, we have proved Theorem 10 by utilizing the equality $\boldsymbol{K}_{\Psi}(X) = \boldsymbol{K}_0(X)$ for regular elements X or for elements $X \in \mathcal{S}$. After we have established Theorem 10, we easily conclude that $\boldsymbol{K}_{\Psi}(X) = \boldsymbol{K}_0(X)$ holds for any element $X \in \mathfrak{sp}(n)$.

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RIGIDITY OF THE CANONICAL ISOMETRIC IMBEDDING OF THE HERMITIAN SYMMETRIC SPACE $Sp(n)/U(n)$

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ABSTRACT. In this paper we discuss the rigidity of the canonical isometric imbedding f_0 of the Hermitian symmetric space $Sp(n)/U(n)$ into the Lie algebra $\mathfrak{sp}(n)$. We will show that if $n \geq 2$, then f_0 is strongly rigid, i.e., for any isometric immersion f_1 of a connected open set U of $Sp(n)/U(n)$ into $\mathfrak{sp}(n)$ there is a euclidean transformation a of $\mathfrak{sp}(n)$ satisfying $f_1 = af_0$ on U.

1. INTRODUCTION

In a series of our work [4], [5] and [7] we showed the strong rigidity of the canonical isometric imbeddings of the projective planes $P^2(\mathbb{CAY})$, $P^2(\mathbb{H})$ and the symplectic group $Sp(n)$. In this paper we will investigate the canonical isometric imbedding f_0 of the Hermitian symmetric space $Sp(n)/U(n)$ $(n \geq 2)$ and establish the strong rigidity theorem for f_0.

As is known, any Hermitian symmetric space M of compact type is isometrically imbedded into the Lie algebra $\mathfrak{g}$ of the holomorphic isometry group of M (see Lichnérowicz [15]). Thus, $Sp(n)/U(n)$ can be isometrically imbedded into $\mathfrak{sp}(n)$, which is the Lie algebra of the symplectic group $Sp(n)$. Identifying $\mathfrak{sp}(n)$ with the euclidean space $\mathbb{R}^{2n^2+n}$, we obtain an isometric imbedding f_0 of $Sp(n)/U(n)$ into $\mathbb{R}^{2n^2+n}$, which is called the *canonical isometric imbedding* of $Sp(n)/U(n)$. In [2] we proved that any open set of $Sp(n)/U(n)$ cannot be isometrically immersed into the euclidean space $\mathbb{R}^N$ with $N \leq \dim \mathfrak{sp}(n) - 1$. Accordingly, the canonical isometric imbedding f_0 gives the least dimensional (local) isometric imbedding of $Sp(n)/U(n)$ into the euclidean space (see Corollary 2.5 of [2]).

In this paper we will prove

Theorem 1. *Let f_0 be the canonical isometric imbedding of $Sp(n)/U(n)$ $(n \geq 2)$ into the euclidean space $\mathfrak{sp}(n)$ $(= \mathbb{R}^{2n^2+n})$. Then f_0 is strongly rigid, i.e., for any isometric immersion f_1 of a connected open set U of $Sp(n)/U(n)$ into $\mathfrak{sp}(n)$ there is a euclidean transformation a of $\mathfrak{sp}(n)$ satisfying $f_1 = af_0$ on U.*

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As for the rigidity on the canonical isometric imbeddings of connected irreducible Hermitian symmetric spaces M of compact type, the following results are known:

(1) f_0 is globally rigid in the following sense: Let f_1 be an isometric imbedding of M into $\mathfrak{g}$. If f_1 is sufficiently close to f_0 with respect to C^3-topology, then there is a euclidean transformation a of $\mathfrak{g}$ such that $f_1 = af_0$ (see Tanaka [17]).

(2) If M is not isomorphic to any complex projective space $P^n(\mathbb{C})$, then f_0 is locally rigid in the following sense: Let U be a connected open set of M and let f_1 be an isometric imbedding of U into $\mathfrak{g}$. If f_1 is sufficiently close to f_0 with respect to C^2-topology on U, then there is a euclidean transformation a of $\mathfrak{g}$ such that $f_1 = af_0$ holds on U (see Kaneda-Tanaka [12]).

We note that the topological condition on the mappings are removed in the statement of Theorem 1. In this sense Theorem 1 strengthens the rigidity theorem in [17] and [12] for the Hermitian symmetric space $Sp(n)/U(n)$ $(n \geq 2)$.

The method of our proof is quite similar to the methods adopted in [4], [5] and [7]. We will solve the Gauss equation on $Sp(n)/U(n)$ in codimension $n^2 (= \dim \mathfrak{sp}(n) - \dim Sp(n)/U(n))$ and prove that any solution Ψ of the Gauss equation is Hermitian, i.e., $\Psi(IX, IY) = \Psi(X, Y)$. This fact, together with the criterion on the isometric imbeddings of almost Hermitian manifolds (Theorem 5), indicates that any solution of the Gauss equation is equivalent to the second fundamental form of f_0. Therefore by the congruence theorem obtained in [4] (see Theorem 3 below) we can establish Theorem 1.

Throughout this paper we will assume the differentiability of class C^∞. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [11]. For the quaternion numbers and the symplectic group $Sp(n)$, see Chevalley [9].

2. The Gauss equation and rigidity of isometric imbeddings

Let M be a Riemannian manifold and let $T(M)$ be the tangent bundle of M. We denote by ν the Riemannian metric of M and by R the Riemannian curvature tensor of type $(1,3)$ with respect to ν. We also denote by C the Riemannian curvature tensor of type $(0,4)$, which is, at each point $p \in M$, given by

$$C(x, y, z, w) = -\nu\big(R(x, y)z, w\big), \qquad x, y, z, w \in T_p(M).$$

Let N be a euclidean vector space, i.e., N is a vector space over $\mathbb{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $S^2T_p^*(M) \otimes N$ be the space of N-valued symmetric bilinear forms on $T_p(M)$. We call the following equation on $\Psi \in S^2T_p^*(M) \otimes N$ the *Gauss equation* at $p \in M$ modeled on N:

$$C(x, y, z, w) = \big\langle \Psi(x, z), \Psi(y, w) \big\rangle - \big\langle \Psi(x, w), \Psi(y, z) \big\rangle, \tag{2.1}$$

where $x, y, z, w \in T_p(M)$. We denote by $\mathcal{G}_p(M, \boldsymbol{N})$ the set of all solutions of (2.1), which is called the *Gaussian variety* at $p \in M$ modeled on $\boldsymbol{N}$. Let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2T_p^*(M) \otimes \boldsymbol{N}$ by

$$(h\Psi)(x, y) = h(\Psi(x, y)),$$

where $\Psi \in S^2T_p^*(M) \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$, $x, y \in T_p(M)$. We say that two elements Ψ and $\Psi' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are *equivalent* if there is an element $h \in O(\boldsymbol{N})$ such that $\Psi' = h\Psi$. It is easily seen that if Ψ and $\Psi' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are equivalent and $\Psi \in \mathcal{G}_p(M, \boldsymbol{N})$, then $\Psi' \in \mathcal{G}_p(M, \boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}_p(M, \boldsymbol{N})$ is *EOS* if $\mathcal{G}_p(M, \boldsymbol{N}) \neq \emptyset$ and if it consists of essentially one solution, i.e., any solutions of the Gauss equation (2.1) are equivalent to each other under the action of $O(\boldsymbol{N})$. We proved

Proposition 2 ([4], p. 334). *Let M be a Riemannian manifold and $p \in M$. Let $\boldsymbol{N}$ be a euclidean vector space such that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. Then:*

- (1) *Let Ψ be an arbitrary element of $\mathcal{G}_p(M, \boldsymbol{N})$. Then, the vectors $\Psi(x, y)$ $(x, y \in T_p(M))$ span the whole space $\boldsymbol{N}$.*
- (2) *Let $\boldsymbol{N}_1$ be a euclidean vector space. Then:*
  - (2a) *$\mathcal{G}_p(M, \boldsymbol{N}_1) = \emptyset$ if $\dim \boldsymbol{N}_1 < \dim \boldsymbol{N}$;*
  - (2b) *$\mathcal{G}_p(M, \boldsymbol{N}_1)$ is EOS if $\dim \boldsymbol{N}_1 = \dim \boldsymbol{N}$;*
  - (2c) *$\mathcal{G}_p(M, \boldsymbol{N}_1)$ is not EOS if $\dim \boldsymbol{N}_1 > \dim \boldsymbol{N}$.*

We say that a Riemannian manifold M is *formally rigid* in codimension r if there is a euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = r$ such that the Gaussian variety $\mathcal{G}_p(M, \boldsymbol{N})$ modeled on $\boldsymbol{N}$ is EOS at each $p \in M$. In [4] we have obtained the following rigidity theorem for formally rigid Riemannian manifolds:

Theorem 3 ([4], pp. 335–336). *Let M be an m-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the euclidean space $\mathbb{R}^N$. Assume:*

- (1) *M is connected;*
- (2) *M is formally rigid in codimension $r = N - m$.*

Then, any isometric immersion $\boldsymbol{f}_1$ of M into the euclidean space $\mathbb{R}^N$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\mathbb{R}^N$, i.e., there exists a euclidean transformation a of $\mathbb{R}^N$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$.

In the subsequent sections we will prove

Theorem 4. *The Hermitian symmetric space $Sp(n)/U(n)$ $(n \geq 2)$ is formally rigid in codimension n^2 $(= \dim \mathfrak{u}(n))$.*

If Theorem 4 is true, then it is easily seen that Theorem 1 immediately follows from Theorem 3.

Remark 1. We note that, in the case $n = 1$, Theorem 4 is not true. In this case we have $Sp(1)/U(1) \cong S^2$ and the canonical isometric imbedding $\boldsymbol{f}_0$ coincides with the standard isometric imbedding of S^2 into $\mathbb{R}^3$. Consequently, $\boldsymbol{f}_0$ is globally rigid (remember the rigidity theorem for ovaloids by Cohn-Vossen[10]). However, it is not locally rigid, i.e., there are infinitely many non-equivalent surfaces of revolution possessing constant positive curvature. Therefore, the Gauss equation in codimension 1 admits infinitely many non-equivalent solutions corresponding to the second fundamental forms of these surfaces. For details, see Spivak [16].

3. The Gauss equation on almost Hermitian manifolds

For the proof of Theorem 4 we start from a general setting. Let M be an even dimensional Riemannian manifold with Riemannian metric ν. Assume that there is an almost complex structure I on M such that $\nu(Ix, Iy) = \nu(x,y)$ $(x, y \in T_p(M))$ at each $p \in M$. Then M is called an *almost Hermitian manifold.*

Let M be an almost Hermitian manifold and $p \in M$. Let $\boldsymbol{N}$ be a euclidean vector space. An element $\boldsymbol{\Psi} \in S^2T_p^* \otimes \boldsymbol{N}$ is called *Hermitian* if $\boldsymbol{\Psi}(IX, IY) = \boldsymbol{\Psi}(X, Y)$ holds for any X, $Y \in T_p(M)$. In what follows we will consider the case where the Gauss equation (2.1) admits a Hermitian solution. We will prove

Theorem 5. *Let M be an almost Hermitian manifold and $\boldsymbol{N}$ a euclidean vector space. Let $\mathcal{G}_p(M, \boldsymbol{N})$ be the Gaussian variety at $p \in M$ modeled on $\boldsymbol{N}$. Assume:*

(1) *$\mathcal{G}_p(M, \boldsymbol{N}) \neq \emptyset$;*

(2) *Any solution $\boldsymbol{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N})$ is Hermitian.*

Then, $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS.

Let M be a $2m$-dimensional almost Hermitian manifold and let $p \in M$. For simplicity, we set $\boldsymbol{T} = T_p(M)$. Let $\boldsymbol{T}^{\mathbb{C}} = \boldsymbol{T} + \sqrt{-1}\boldsymbol{T}$ be the complexification of $\boldsymbol{T}$. By $\overline{X}$ we denote the complex conjugate of $X \in \boldsymbol{T}^{\mathbb{C}}$ with respect to $\boldsymbol{T}$. The almost complex structure I is extended to a $\mathbb{C}$-linear endomorphism of $\boldsymbol{T}^{\mathbb{C}}$, which is also denoted by I. Set

$$\boldsymbol{T}^{1,0} = \{Z \in \boldsymbol{T}^{\mathbb{C}} \,|\, IZ = \sqrt{-1}Z\}, \qquad \boldsymbol{T}^{0,1} = \{Z \in \boldsymbol{T}^{\mathbb{C}} \,|\, IZ = -\sqrt{-1}Z\}.$$

Then, as is known, $\boldsymbol{T}^{\mathbb{C}} = \boldsymbol{T}^{1,0} + \boldsymbol{T}^{0,1}$ (direct sum) and $\boldsymbol{T}^{0,1} = \overline{\boldsymbol{T}^{1,0}}$; $\boldsymbol{T}^{1,0} = \overline{\boldsymbol{T}^{0,1}}$. Take a basis $\{Z_1, \cdots, Z_m\}$ of $\boldsymbol{T}^{1,0}$ and put $Z_{\bar{i}} = \overline{Z_i}$ $(1 \leq i \leq m)$. Then the set $\{Z_i, Z_{\bar{i}}\, (1 \leq i \leq m)\}$ forms a basis of $\boldsymbol{T}^{\mathbb{C}}$. In the following we will fix such a basis $\{Z_i, Z_{\bar{i}}\, (1 \leq i \leq m)\}$ and rewrite the Gauss equation (2.1).

As usual, the Riemannian curvature is extended to a tensor of type $(0,4)$ on $T^{\mathbb{C}}$. Define $C_{abcd} \in \mathbb{C}$ by setting

$$C_{abcd} = C(Z_a, Z_b, Z_c, Z_d),$$

where the suffices a, b, c, d run through the range $1, \ldots, m, \bar{1}, \ldots, \bar{m}$. We also extend an element $\Psi \in S^2 T^* \otimes N$ to an element of $S^2 T^{\mathbb{C}*} \otimes N^{\mathbb{C}}$, where $N^{\mathbb{C}} = N + \sqrt{-1}N$ denotes the complexification of a euclidean vector space N. Define vectors $\Psi_{ab} \in N^{\mathbb{C}}$ by setting

$$\Psi_{ab} = \Psi(Z_a, Z_b), \qquad a, b = 1, \ldots, m, \bar{1}, \ldots, \bar{m}.$$

Then we easily have

$$\overline{\Psi_{ab}} = \Psi_{\bar{a}\bar{b}}, \qquad a, b = 1, \ldots, m, \bar{1}, \ldots, \bar{m},$$

where for an element $v \in N^{\mathbb{C}}$ we mean by $\bar{v}$ the complex conjugate of v with respect to N and we promise $\bar{\bar{i}} = i$ for $i = 1, \ldots, m$.

By use of C_{abcd} and Ψ_{ab} we can rewrite the Gauss equation (2.1) as follows:

$$C_{abcd} = \langle \Psi_{ac}, \Psi_{bd} \rangle - \langle \Psi_{ad}, \Psi_{bc} \rangle, \qquad a, b, c, d = 1, \ldots, m, \bar{1}, \ldots, \bar{m}, \tag{3.1}$$

where $\langle\ ,\ \rangle$ means the symmetric bilinear form on $N^{\mathbb{C}}$ which is a natural extension of the inner product of N. We now prove

Lemma 6. *Let $\Psi \in \mathcal{G}_p(M, N)$. Assume that Ψ is Hermitian. Then*

$$C_{i\bar{k}\bar{j}l} = \langle \Psi_{i\bar{j}}, \Psi_{\bar{k}l} \rangle, \qquad 1 \le i, j, k, l \le m. \tag{3.2}$$

Proof. Let i, j, k and l be integers such that $1 \le i, j, k, l \le m$. Putting $a = i$, $b = \bar{k}$, $c = \bar{j}$ and $d = l$ into (3.1), we have

$$C_{i\bar{k}\bar{j}l} = \langle \Psi_{i\bar{j}}, \Psi_{\bar{k}l} \rangle - \langle \Psi_{il}, \Psi_{\bar{k}\bar{j}} \rangle.$$

Since Ψ is Hermitian, we have $\Psi_{il} = \Psi_{\bar{k}\bar{j}} = 0$. Hence we get (3.2). □

Let us define a Hermitian inner product $(\ ,\ )$ of $N^{\mathbb{C}}$ by setting

$$(Y, Y') = \langle Y, \overline{Y'} \rangle, \qquad Y, Y' \in N^{\mathbb{C}}.$$

Then $N^{\mathbb{C}}$ is considered as a Hermitian vector space.

We now define a quadratic form $\widehat{C(p)}$ on $T^{1,0} \otimes \overline{T^{1,0}}$ by

$$\widehat{C(p)}(X \otimes \overline{Y}, Z \otimes \overline{W}) = C(X, \overline{Z}, \overline{Y}, W), \qquad X \otimes \overline{Y},\ Z \otimes \overline{W} \in T^{1,0} \otimes \overline{T^{1,0}}.$$

By $C(p)$ we denote the matrix corresponding to $\widehat{C(p)}$ with respect to the basis $\{Z_i \otimes Z_{\bar{j}}\ (1 \le i, j \le m)\}$ of $T^{1,0} \otimes \overline{T^{1,0}}$. As is easily seen, $C(p) = (C(p)_{\alpha\beta})$ is a complex square matrix of degree m^2, where Greek letters $\alpha, \beta, \ldots$ run over the pairs of indices $\{i\bar{j}\}$ $(i, j = 1, \ldots, m)$ and

$$C(p)_{\alpha\beta} = C_{i\bar{k}\bar{j}l}, \qquad \alpha = \{i\bar{j}\},\ \beta = \{k\bar{l}\}.$$

It is easily checked that $\mathcal{C}(p)$ is a Hermitian matrix, i.e., ${}^t\mathcal{C}(p) = \overline{\mathcal{C}(p)}$. Moreover, the rank of $\mathcal{C}(p)$ and the cardinal number of positive or negative eignevalues of $\mathcal{C}(p)$ do not depend on the choice of the basis $\{Z_i\}$ of $\boldsymbol{T}^{1,0}$.

Now, let $\Psi \in \mathcal{G}_p(M,\boldsymbol{N})$. Assume that Ψ is Hermitian. Then by (3.2) we have

$$\mathcal{C}(p)_{\alpha\beta} = (\Psi_\alpha, \Psi_\beta), \qquad \alpha, \beta = \{1\bar{1}\}, \ldots, \{i\bar{j}\}, \ldots, \{m\bar{m}\}, \tag{3.3}$$

where we write $\Psi_\alpha = \Psi_{i\bar{j}}$ when $\alpha = \{i\bar{j}\}$. The equality (3.3) indicates that $\mathcal{C}(p)$ is nothing but the Gram matrix of the vectors $\{\Psi_\alpha\}_\alpha$ with respect to $(\, , \,)$. Therefore, $\mathcal{C}(p)$ must be positive semi-definite and $\operatorname{rank}(\mathcal{C}(p)) = \dim_{\mathbb{C}}(\sum_\alpha \mathbb{C}\Psi_\alpha)$, where α runs through the indices $\{1\bar{1}\}, \ldots, \{m\bar{m}\}$. Let $\boldsymbol{N}_\Psi$ be the subspace of $\boldsymbol{N}$ spanned by the vectors $\Psi(X,Y)$, where $X, Y \in \boldsymbol{T}$. Then we easily have $\boldsymbol{N}_\Psi^{\mathbb{C}} = \sum_\alpha \mathbb{C}\Psi_\alpha$. Hence $\dim \boldsymbol{N}_\Psi = \dim_{\mathbb{C}}(\sum_\alpha \mathbb{C}\Psi_\alpha)$. Therefore, we get

Lemma 7. *Let $\Psi \in \mathcal{G}_p(M,\boldsymbol{N})$. Assume that Ψ is Hermitian. Then $\mathcal{C}(p)$ is positive semi-definite and*

$$\dim \boldsymbol{N}_\Psi = \operatorname{rank}(\mathcal{C}(p)).$$

Consequently, $\mathcal{G}_p(M,\boldsymbol{N})$ does not contain any Hermitian element if one of the following conditions are satisfied:

(1) *$\mathcal{C}(p)$ has at least one negative eigenvalue;*
(2) $\dim \boldsymbol{N} < \operatorname{rank}(\mathcal{C}(p))$.

Example 1. Let M be a Kähler manifold of constant holomorphic sectional curvature $c\,(\neq 0)$ and $p \in M$. Let $(z_1, \ldots, z_m)$ be a complex local coordinate system of M around p. Put $Z_i = \partial/\partial z_i \,(1 \leq i \leq m)$. Then we get a basis $\{Z_i\}_{1\leq i\leq m}$ of $\boldsymbol{T}^{1,0}$. By use of the basis $\{Z_i, Z_{\bar{i}}\,(1 \leq i \leq m)\}$ the curvature tensor C of M can be written as

$$C_{i\bar{k}\bar{j}l} = \frac{1}{2}c\,(\nu_{i\bar{k}}\nu_{\bar{j}l} + \nu_{i\bar{j}}\nu_{\bar{k}l}), \qquad 1 \leq i, j, k, l \leq m,$$

where we set $\nu_{i\bar{k}} = \nu(Z_i, Z_{\bar{k}})\,(1 \leq i, k \leq m)$ (see Kobayashi-Nomizu [14]). By a suitable change of the coordinate $(z_1, \ldots, z_m)$ we may assume that $\nu_{i\bar{k}} = \delta_{ik}\,(1 \leq i, k \leq m)$ at p, where δ means the Kronecker delta. Consequently, the component $\mathcal{C}(p)_{\alpha\beta}$ of the matrix $\mathcal{C}(p)$ is given by

$$\mathcal{C}(p)_{\alpha\beta} = \frac{1}{2}c\,(\delta_{\alpha\beta} + \delta_{\alpha\bar{\alpha}}\delta_{\beta\bar{\beta}}), \qquad \alpha, \beta = \{1\bar{1}\}, \ldots, \{i\bar{j}\}, \ldots, \{m\bar{m}\},$$

where $\{i\bar{j}\} = \{k\bar{l}\}$ means $i = k$ and $j = l$; $\overline{\{i\bar{j}\}} = \{j\bar{i}\}$, $\overline{\{k\bar{l}\}} = \{l\bar{k}\}$. Therefore, we have $\operatorname{rank}(\mathcal{C}(p)) = m^2$, because $c \neq 0$. Further, if $c < 0$ (resp. $c > 0$), then $\mathcal{C}(p)$ is negative (resp. positive) definite. Accordingly, $\mathcal{G}_p(M,\boldsymbol{N})$ does not contain any Hermitian element in case $c < 0$ or $\dim \boldsymbol{N} < m^2$.

With these preparations we prove Theorem 5.

Proof of Theorem 5. First suppose that $\dim \boldsymbol{N} > \operatorname{rank}(\mathcal{C}(p))$. Let Ψ be an arbitrary element of $\mathcal{G}_p(M, \boldsymbol{N})$. Then by Lemma 7 we have $\dim \boldsymbol{N}_\Psi < \dim \boldsymbol{N}$. Take a non-zero vector $\boldsymbol{n} \in \boldsymbol{N}$ such that $\langle \boldsymbol{N}_\Psi, \boldsymbol{n} \rangle = 0$ and take a non-zero covector $\xi \in \boldsymbol{T}^*$. Set $\Psi_1 = \Psi + \xi^2 \otimes \boldsymbol{n}$. Then it is easily verified that $\Psi_1 \in \mathcal{G}_p(M, \boldsymbol{N})$ and that Ψ_1 is not Hermitian. This contradicts the assumption (2). Consequently, we have $\dim \boldsymbol{N} = \operatorname{rank}(\mathcal{C}(p))$. Moreover, we have $\boldsymbol{N}_\Psi = \boldsymbol{N}$ for any $\Psi \in \mathcal{G}_p(M, \boldsymbol{N})$.

We now prove that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. Let Ψ and $\Psi' \in \mathcal{G}_p(M, \boldsymbol{N})$. Since Ψ and Ψ' are Hermitian, they satisfy the equality (3.3). Hence we have

$$(\Psi_\alpha, \Psi_\beta) = (\Psi'_\alpha, \Psi'_\beta), \qquad \alpha, \beta = \{1\bar{1}\}, \ldots, \{i\bar{j}\}, \ldots, \{m\bar{m}\}.$$

Since $\boldsymbol{N}_\Psi = \boldsymbol{N}$, the vectors $\{\Psi_\alpha\}_\alpha$ span the whole $\boldsymbol{N}^{\mathbb{C}}$. By an elementary linear algebra we know that there is a unitary transformation h of $\boldsymbol{N}^{\mathbb{C}}$ satisfying $\Psi'_\alpha = h(\Psi_\alpha)$ ($\alpha = \{1\bar{1}\}, \ldots, \{i\bar{j}\}, \ldots, \{m, \bar{m}\}$). Let $\alpha = \{i\bar{j}\}$. Then we have

$$h(\overline{\Psi_\alpha}) = h(\Psi_{j\bar{i}}) = \Psi'_{j\bar{i}} = \overline{\Psi'_\alpha} = \overline{h(\Psi_\alpha)}.$$

Consequently, we have $h(\boldsymbol{N}) \subset \boldsymbol{N}$. Hence h is an orthogonal transformation of $\boldsymbol{N}$ and satisfies $\Psi' = h\Psi$. This shows that $\mathcal{G}_p(M, \boldsymbol{N})$ is EOS. □

Remark 2. Let $\boldsymbol{N}$ be a euclidean vector space. Assume that $\mathcal{G}_p(M, \boldsymbol{N})$ satisfies the conditions (1) and (2) in Theorem 5. Let $\boldsymbol{N}'$ be another euclidean vector space such that $\dim \boldsymbol{N}' = \dim \boldsymbol{N}$. Then, $\mathcal{G}_p(M, \boldsymbol{N}')$ also satisfies the conditions (1) and (2) in Theorem 5. To observe this, take an isometric isomorphism $\varphi \colon \boldsymbol{N} \longrightarrow \boldsymbol{N}'$ and define a linear mapping $S^2T_p^* \otimes N \ni \Psi \longmapsto \widehat{\Psi} \in S^2T_p^* \otimes \boldsymbol{N}'$ by $\widehat{\Psi}(X, Y) = \varphi(\Psi(X, Y))$ $(X, Y \in T_p(M))$. Then the following assertions can be easily verified:

(1) $\widehat{\Psi}$ is Hermitian if and only if Ψ is Hermitian;
(2) $\widehat{\Psi} \in \mathcal{G}_p(M, \boldsymbol{N}')$ if and only if $\Psi \in \mathcal{G}_p(M, \boldsymbol{N})$.

Thus, we note that the conditions (1) and (2) in Theorem 5 are the conditions only related to the dimension of the eulidean space $\boldsymbol{N}$. As seen in the proof of Theorem 5, $\dim \boldsymbol{N}$ equals $\operatorname{rank}(\mathcal{C}(p))$, which is uniquely determined by the curvature of M at p.

4. The canonical isometric imbeddings of Hermitian symmetric spaces of compact type

We now review the canonical isometric imbeddings of Hermitian symmetric spaces of compact type defined by Lichnérowicz [15]. Let M be an almost Hermitian manifold. A mapping g of M into itself is called *holomorphic* if g satisfies $g_* \circ I = I \circ g_*$. A connected almost Hermitian manifold M is called a *Hermitian symmetric space* if each

$p \in M$ is an isolated fixed point of an involutive holomorphic isometry of M. Utilizing the identity component G of the holomorphic isometry group of M, we can represent M as a Riemannian symmetric space G/K, where K is an isotropy group at a suitable point $o \in M$; usually o is called the *origin* of G/K (see Helgason [11]). We say a Hermitian symmetric space G/K is *of compact type* if the Lie algebra $\mathfrak{g}$ of G is compact and semisimple.

Let $M = G/K$ be a Hermitian symmetric space of compact type. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ with respect to the Riemannian symmetric pair (G, K). As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(M)$. It is known that there is an element $I_0 \in \mathfrak{k}$ such that: (i) I_0 belongs to the center of $\mathfrak{k}$; (ii) $\mathrm{ad}\,(I_0)|_{\mathfrak{m}}$ coincides with the almost complex structure I at o (see [14], [11]). Consider the $\mathrm{Ad}(G)$-orbit in $\mathfrak{g}$ passing through I_0, i.e., $\mathrm{Ad}(G)I_0 \subset \mathfrak{g}$. Since $\mathrm{Ad}(K)I_0 = I_0$, we get a differential mapping

$$\boldsymbol{f}_0 \colon G/K \ni gK \longmapsto \mathrm{Ad}(g)I_0 \in \mathfrak{g}.$$

We may regard $\mathfrak{g}$ as a euclidean vector space with a suitable $\mathrm{Ad}(G)$-invariant inner product. The induced Riemannian metric ν of G/K via $\boldsymbol{f}_0$ is easily understood to be G-invariant. The mapping $\boldsymbol{f}_0$ is called the *canonical isometric imbedding* of $M = G/K$.

Let ∇ be the Riemannian connection on $M = G/K$ associated with ν. By $\nabla \boldsymbol{f}_0$ (resp. $\nabla\nabla \boldsymbol{f}_0$) we denote the first order (resp. second order) covariant derivative of the canonical isometric imbedding $\boldsymbol{f}_0$. The second order covariant derivative $\nabla\nabla \boldsymbol{f}_0$ is called the *second fundamental form* of the canonical isometric imbedding $\boldsymbol{f}_0$. In view of Tanaka [17], Kaneda-Tanaka [12] we know that at the origin o, $\nabla \boldsymbol{f}_0$ and $\nabla\nabla \boldsymbol{f}_0$ are given as follows:

$$\nabla_X \boldsymbol{f}_0 = [X, I_0] = -IX, \quad X \in \mathfrak{m}; \tag{4.1}$$

$$\nabla_X \nabla_Y \boldsymbol{f}_0 = [X, [Y, I_0]] = -[X, IY], \quad X, Y \in \mathfrak{m}. \tag{4.2}$$

Let $\boldsymbol{T}$ (resp. $\boldsymbol{N}$) be the tangent (resp. normal) vector space of $\boldsymbol{f}_0(G/K)$ at I_0 $(= \boldsymbol{f}_0(o) \in \mathfrak{g})$. By (4.1) we know that the tangent space $\boldsymbol{T}$, which is generated by the first order covariant derivatives of $\boldsymbol{f}_0$ at o, coincides with $\mathfrak{m}$. Consequently, the normal vector space $\boldsymbol{N}$ at o is given by $\boldsymbol{N} = \mathfrak{k}$. Similarly, by (4.2) we know that the value of the second order covariant derivative $\nabla_X \nabla_Y \boldsymbol{f}_0\,(X, Y \in \mathfrak{m})$ belongs to the normal vector space $\boldsymbol{N} = \mathfrak{k}$. As for the second fundamental form we have

Proposition 8 ([17], [12]). *Let G/K be a Hermitian symmetric space of compact type and let $\boldsymbol{f}_0$ be the canonical isometric imbedding of G/K into $\mathfrak{g}$. Then the second fundamental form $\Psi_0 \in S^2\mathfrak{m}^* \otimes \mathfrak{k}$ of $\boldsymbol{f}_0$ at the origin o satisfies the following*

(1) $\Psi_0 \in \mathcal{G}_o(G/K, \mathfrak{k})$;

(2) *The vectors $\Psi_0(X, Y)\,(X, Y \in \mathfrak{m})$ span the whole $\mathfrak{k}$;*

(3) *Ψ_0 is Hermitian, i.e., $\Psi_0(IX, IY) = \Psi_0(X, Y)$ for $X, Y \in \mathfrak{m}$.*

By this proposition we have

Proposition 9. *Let G/K be a Hermitian symmetric space of compact type. Then for each $p \in G/K$ the following assertions hold:*

(1) $\operatorname{rank}(\mathcal{C}(p)) = \dim \mathfrak{k}$;

(2) *Let N be a euclidean vector space with $\dim N = \dim \mathfrak{k}$. Then, $\mathcal{G}_p(G/K, N)$ is EOS if and only if any element $\Psi \in \mathcal{G}_o(G/K, \mathfrak{k})$ is Hermitian.*

Proof. By Proposition 8 and Lemma 7 we immediately know that $\operatorname{rank}(\mathcal{C}(o)) = \dim \mathfrak{k}$. Then, by homogeneity of G/K, we have (1). Also, by homogeneity, we easily see that $\mathcal{G}_p(G/K, N)$ is EOS if and only if $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS. Note that $\mathcal{G}_o(G/K, \mathfrak{k})$ contains a Hermitian element Ψ_0. Hence, if $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS, then any element $\Psi \in \mathcal{G}_o(G/K, \mathfrak{k})$ is Hermitian. The converse part follows from Theorem 5. □

Remark 3. Let G/K be a Hermitian symmetric space of compact type and let $p \in G/K$. Then, the equality $\operatorname{rank}(\mathcal{C}(p)) = \dim \mathfrak{k}$ in Proposition 9 indicates that $\dim \mathfrak{k}$ is the least dimension of a euclidean vector space N such that $\mathcal{G}_p(G/K, N)$ contains a Hermitian element. In fact, if N_1 is a euclidean vector space with $\dim N_1 < \dim \mathfrak{k}$, then $\mathcal{G}_p(G/K, N_1)$ does not contain any Hermitian element (see Lemma 7). However, we note that this fact does not imply $\mathcal{G}_p(G/K, N_1) = \emptyset$. Agaoka [1] proved that for the complex projective space $P^n(\mathbb{C})$ $(n \geq 2)$, $\mathcal{G}_o(P^n(\mathbb{C}), N_1) \neq \emptyset$ when $\dim N_1 = n^2 - 1$. We note that in this case we have $\dim \mathfrak{k} = \dim \mathfrak{u}(n) = n^2$ and hence $\mathcal{G}_o(P^n(\mathbb{C}), \mathfrak{k})$ is not EOS. It seems to the authors that this is a special case. For the other irreducible Hermitian symmetric space G/K except $P^n(\mathbb{C})$ $(n \geq 1)$, such as the complex Grassmann manifold $G^{p,q}(\mathbb{C})$ $(p \geq q \geq 2)$, the complex quadric $Q^n(\mathbb{C})$ $(n \geq 4)$, etc., we conjecture that the Gaussian variety $\mathcal{G}_o(G/K, \mathfrak{k})$ is EOS. As will be seen in the following sections, our conjecture is true for the Hermitian symmetric space $Sp(n)/U(n)$ $(n \geq 2)$. In TABLE 1 we show all *irreducible* Hermitian symmetric spaces G/K of compact type and related data:

TABLE 1. Irreducible Hermitian symmetric spaces of compact type

G/K	$\operatorname{rank}(G/K)$	$\dim G/K$	$\dim \mathfrak{g}$	$\dim \mathfrak{k}$
$P^n(\mathbb{C})$ $(n \geq 1)$	1	$2n$	$n^2 + 2n$	n^2
$G^{p,q}(\mathbb{C})$ $(p \geq q \geq 2)$	q	$2pq$	$(p+q)^2 - 1$	$p^2 + q^2 - 1$
$Q^n(\mathbb{C})$ $(n \geq 5)$	2	$2n$	$\frac{1}{2}(n+1)(n+2)$	$\frac{1}{2}n(n-1) + 1$
$SO(2n)/U(n)$ $(n \geq 5)$	$[n/2]$	$n^2 - n$	$2n^2 - n$	n^2
$Sp(n)/U(n)$ $(n \geq 1)$	n	$n^2 + n$	$2n^2 + n$	n^2
$E_6/Spin(10) \cdot SO(2)$	2	32	78	46
$E_7/E_6 \cdot SO(2)$	3	54	133	79

5. The Hermitian symmetric space $Sp(n)/U(n)$

Let $\mathbb{H}$ be the field of quaternion numbers. As is well-known, $\mathbb{H}$ is an associative algebra over the field $\mathbb{R}$ of real numbers generated by $\mathbf{1}$ ($\in \mathbb{R}$) and three elements $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ satisfying the following multiplication rule:

(1) $\mathbf{1}\boldsymbol{i} = \boldsymbol{i}\mathbf{1} = \boldsymbol{i}, \quad \mathbf{1}\boldsymbol{j} = \boldsymbol{j}\mathbf{1} = \boldsymbol{j}, \quad \mathbf{1}\boldsymbol{k} = \boldsymbol{k}\mathbf{1} = \boldsymbol{k}$;
(2) $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathbf{1}$;
(3) $\boldsymbol{i}\boldsymbol{j} = -\boldsymbol{j}\boldsymbol{i} = \boldsymbol{k}, \quad \boldsymbol{j}\boldsymbol{k} = -\boldsymbol{k}\boldsymbol{j} = \boldsymbol{i}, \quad \boldsymbol{k}\boldsymbol{i} = -\boldsymbol{i}\boldsymbol{k} = \boldsymbol{j}$.

Let $\boldsymbol{q} \in \mathbb{H}$. Then $\boldsymbol{q}$ is written as $\boldsymbol{q} = q_0\mathbf{1} + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. We define the norm $|\boldsymbol{q}|$, the real part $\mathrm{Re}(\boldsymbol{q})$ and the conjugate $\overline{\boldsymbol{q}}$ by

$$|\boldsymbol{q}|^2 = \sum_{i=0}^{3} q_i^2; \qquad \mathrm{Re}(\boldsymbol{q}) = q_0; \qquad \overline{\boldsymbol{q}} = q_0\mathbf{1} - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}.$$

Then we easily have $|\overline{\boldsymbol{q}}| = |\boldsymbol{q}|$, $\mathrm{Re}(\boldsymbol{q}) = \mathrm{Re}(\overline{\boldsymbol{q}})$, $\overline{\overline{\boldsymbol{q}}} = \boldsymbol{q}$ and $\boldsymbol{q}\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}\boldsymbol{q} = |\boldsymbol{q}|^2$. Further, we have

$$|\boldsymbol{q}\boldsymbol{q}'| = |\boldsymbol{q}||\boldsymbol{q}'|; \qquad \mathrm{Re}(\boldsymbol{q}\boldsymbol{q}') = \mathrm{Re}(\boldsymbol{q}'\boldsymbol{q}); \qquad \overline{\boldsymbol{q}\boldsymbol{q}'} = \overline{\boldsymbol{q}'}\,\overline{\boldsymbol{q}}, \qquad \forall \boldsymbol{q}, \boldsymbol{q}' \in \mathbb{H}.$$

By the identification $a \in \mathbb{R}$ with $a\mathbf{1} \in \mathbb{H}$ the field $\mathbb{R}$ is canonically considered as a subfield of $\mathbb{H}$. By the identification $a + b\sqrt{-1} \longmapsto a + b\boldsymbol{i}\,(a, b \in \mathbb{R})$ we may regard the field $\mathbb{C}$ of complex numbers as the subfield $\mathbb{R} + \mathbb{R}\boldsymbol{i}$ of $\mathbb{H}$. In this meaning we will write $\mathbb{C} = \mathbb{R} + \mathbb{R}\boldsymbol{i}$. The real part and the conjugate of a quaternion number defined above are compatible with the usual one defined on $\mathbb{C}$.

For later convenience we set $\mathbb{D} = \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$. Then we easily have

$$\mathbb{H} = \mathbb{C} + \mathbb{D} \text{ (direct sum)}; \quad \mathbb{C}\mathbb{C} = \mathbb{D}\mathbb{D} = \mathbb{C}, \quad \mathbb{C}\mathbb{D} = \mathbb{D}\mathbb{C} = \mathbb{D}.$$

We now define a bracket in $\mathbb{H}$ by $[\boldsymbol{q}, \boldsymbol{q}'] = \boldsymbol{q}\boldsymbol{q}' - \boldsymbol{q}'\boldsymbol{q}$. Then it is known that $\mathbb{H}$ endowed with $[\,,\,]$ is a Lie algebra over $\mathbb{R}$. Moreover, it is easily verified

(1) $[\mathbf{1}, \boldsymbol{q}] = [\boldsymbol{q}, \mathbf{1}] = 0, \qquad \boldsymbol{q} \in \mathbb{H}$;
(2) $[\boldsymbol{i}, \boldsymbol{i}] = [\boldsymbol{j}, \boldsymbol{j}] = [\boldsymbol{k}, \boldsymbol{k}] = 0$;
(3) $[\boldsymbol{i}, \boldsymbol{j}] = -[\boldsymbol{j}, \boldsymbol{i}] = 2\boldsymbol{k}$, $[\boldsymbol{j}, \boldsymbol{k}] = -[\boldsymbol{k}, \boldsymbol{j}] = 2\boldsymbol{i}$, $[\boldsymbol{k}, \boldsymbol{i}] = -[\boldsymbol{i}, \boldsymbol{k}] = 2\boldsymbol{j}$.

Consequently, we have

$$[\mathbb{C}, \mathbb{C}] = 0, \quad [\mathbb{D}, \mathbb{D}] = \mathbb{R}\boldsymbol{i}, \quad [\mathbb{C}, \mathbb{D}] = [\mathbb{D}, \mathbb{C}] = \mathbb{D}.$$

Let n be a positive integer. By $M(n; \mathbb{H})$ we denote the space of square matrices of degree n over the field $\mathbb{H}$. We will regard $M(n; \mathbb{H})$ as a $4n^2$-dimensional vector space over $\mathbb{R}$. Define a bracket in $M(n; \mathbb{H})$ by $[X, Y] = XY - YX$ $(X, Y \in M(n; \mathbb{H}))$. Then $M(n; \mathbb{H})$ endowed with $[\,,\,]$, which is a natural extension of the bracket $[\,,\,]$ defined in $\mathbb{H} = M(1; \mathbb{H})$, forms a Lie algebra over $\mathbb{R}$. For an element $X = (X_i^j)_{1\leq i,j\leq n} \in M(n;\mathbb{H})$ we mean by $\overline{X}$ the conjugate matrix $\overline{X} = (\overline{X_i^j})_{1\leq i,j\leq n}$. Then we have $\overline{\overline{X}} = X$ and

$$^t\overline{XY} = {}^t\overline{Y}\,{}^t\overline{X}, \qquad X, Y \in M(n;\mathbb{H}).$$

Now define a real bilinear form $\langle\, ,\, \rangle$ of $M(n;\mathbb{H})$ by

$$\langle X, Y\rangle = \mathrm{Re}(\mathrm{Trace}(X\,{}^t\overline{Y})), \qquad X, Y \in M(n;\mathbb{H}).$$

It can be easily verified that $\langle\, ,\, \rangle$ is symmetric and positive definite on $M(n;\mathbb{H})$, i.e., $\langle\, ,\, \rangle$ is an inner product of $M(n;\mathbb{H})$. With this inner product $\langle\, ,\, \rangle$ we may regard $M(n;\mathbb{H})$ as the euclidean space $\mathbb{R}^{4n^2}$.

Let $Sp(n)$ denote the symplectic group of degree n, i.e., $Sp(n)$ is the subset of $M(n;\mathbb{H})$ consisting of all $g \in M(n;\mathbb{H})$ such that

$$g\,{}^t\overline{g} = {}^t\overline{g}g = 1_n,$$

where 1_n is the identity matrix of order n. Let $\mathfrak{sp}(n)$ be the Lie algebra of $Sp(n)$. As is known, $\mathfrak{sp}(n)$ is a real subspace of $M(n;\mathbb{H})$ consisting of all $X \in M(n;\mathbb{H})$ such that

$$X + {}^t\overline{X} = 0.$$

As is easily seen, $\dim \mathfrak{sp}(n) = 2n^2 + n$ and the inner product $\langle\, ,\, \rangle$ is invariant under the actions of $\mathrm{Ad}(Sp(n))$ and $\mathrm{ad}\,(\mathfrak{sp}(n))$:

$$\begin{aligned} &\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle, && g \in Sp(n),\ X, Y \in M(n;\mathbb{H}); \\ &\langle \mathrm{ad}\,(Z)X, Y\rangle + \langle X, \mathrm{ad}\,(Z)Y\rangle = 0, && Z \in \mathfrak{sp}(n), X, Y \in M(n;\mathbb{H}). \end{aligned}$$

In the following we regard $\mathfrak{sp}(n)$ with the inner product $\langle\, ,\, \rangle$ as the euclidean space $\mathbb{R}^{2n^2+n}$. By $M(n;\mathbb{C})$ (resp. $M(n;\mathbb{D})$) we denote the subspace of $M(n;\mathbb{H})$ consisting of all matrices $X \in M(n;\mathbb{H})$ whose components are all contained in $\mathbb{C}$ (resp. $\mathbb{D}$). Then the unitary group $U(n)$ of degree n and its Lie algebra $\mathfrak{u}(n)$ are represented by $U(n) = Sp(n) \cap M(n;\mathbb{C})$ and $\mathfrak{u}(n) = \mathfrak{sp}(n) \cap M(n;\mathbb{C})$.

Lemma 10. *Let $\mathfrak{m}(n)$ be the space of symmetric matrices of degree n whose components are all contained in $\mathbb{D}$. Then the sum $\mathfrak{sp}(n) = \mathfrak{u}(n) + \mathfrak{m}(n)$ is an orthogonal direct sum with respect to $\langle\, ,\, \rangle$ and*

$$[\mathfrak{u}(n), \mathfrak{u}(n)] \subset \mathfrak{u}(n); \ [\mathfrak{m}(n), \mathfrak{m}(n)] \subset \mathfrak{u}(n); \ [\mathfrak{u}(n), \mathfrak{m}(n)] \subset \mathfrak{m}(n).$$

In other words, $\mathfrak{sp}(n) = \mathfrak{u}(n) + \mathfrak{m}(n)$ gives the canonical decomposition of $\mathfrak{sp}(n)$ associated with the symmetric pair $(Sp(n), U(n))$.

Hereafter, we consider the symmetric space $M = Sp(n)/U(n)$. Identifying $\mathfrak{m}(n)$ with the tangent space $T_o(Sp(n)/U(n))$ at the origin o, we define an $Sp(n)$-invariant metric ν on $Sp(n)/U(n)$ by

$$\nu(X,Y) = \langle X,Y\rangle, \qquad X,Y \in \mathfrak{m}(n).$$

As is known, the Riemannian curvature R of type $(1,3)$ associated with ν is given as follows (see [14, Ch. XI]):

$$R(X,Y)Z = -[[X,Y],Z], \qquad X,Y,Z \in \mathfrak{m}(n).$$

Set $I_0 = (1/2)\boldsymbol{i}1_n$ $(\in M(n;\mathbb{C}))$. Then I_0 is included in the center of $\mathfrak{u}(n)$ and satisfies

$$\begin{aligned} &\mathrm{ad}\,(I_0)X = \boldsymbol{i}X; \quad \mathrm{ad}\,(I_0)^2X = -X, \qquad X \in \mathfrak{m}(n);\\ &\langle \mathrm{ad}\,(I_0)X, \mathrm{ad}\,(I_0)Y\rangle = \langle X,Y\rangle, \qquad X,Y \in \mathfrak{m}(n);\\ &\mathrm{Ad}(a)\cdot \mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)} = \mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)}\cdot \mathrm{Ad}(a), \qquad a \in U(n). \end{aligned}$$

Thus, it is easy to see that $\mathrm{ad}\,(I_0)|_{\mathfrak{m}(n)}$ can be extended to an $Sp(n)$-invariant almost complex structure I. Thus the symmetric space $Sp(n)/U(n)$ endowed with the Riemannian metric ν and the almost complex structure I becomes a Hermitian symmetric space of compact type.

6. The Gauss equation on $Sp(n)/U(n)$

In this section, we consider the Gauss equation (2.1) at o modeled on the space $\mathfrak{u}(n)$, which is written in the form

$$\langle[[X,Y],Z],W\rangle = \langle\Psi(X,Z),\Psi(Y,W)\rangle - \langle\Psi(X,W),\Psi(Y,Z)\rangle, \tag{6.1}$$

where $\Psi \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ and $X,Y,Z,W \in \mathfrak{m}(n)$. The inner product of $\mathfrak{u}(n)$ is taken to be the restriction of $\langle\ ,\ \rangle$ to the subspace $\mathfrak{u}(n)$ $(\subset M(n;\mathbb{H}))$. Notations are the same in the previous sections. For simplicity, we set $\mathcal{G}(n) = \mathcal{G}_o(Sp(n)/U(n),\mathfrak{u}(n))$. In the following we will prove

Theorem 11. *Assume that $n \geq 2$. Then any solution $\Psi \in \mathcal{G}(n)$ is Hermitian, i.e.,*

$$\Psi(IX,IY) = \Psi(X,Y), \qquad X,Y \in \mathfrak{m}(n).$$

If Theorem 11 is true, then by Proposition 9 we conclude that at each $p \in Sp(n)/U(n)$, $\mathcal{G}_p(Sp(n)/U(n),\boldsymbol{N})$ is EOS when $n \geq 2$ and $\dim \boldsymbol{N} = n^2$. This shows that $Sp(n)/U(n)$ $(n \geq 2)$ is formally rigid in codimension n^2, proving Theorem 4.

For the proof of Theorem 11 we make several preparations. Let $\Psi \in \mathcal{G}(n)$. For each $X \in \mathfrak{m}(n)$ we define a linear map Ψ_X of $\mathfrak{m}(n)$ to $\mathfrak{u}(n)$ by $\Psi_X(Y) = \Psi(X,Y)$ $(Y \in \mathfrak{m}(n))$. By $\boldsymbol{K}_\Psi(X)$ we denote the kernel of Ψ_X. Then we have

Proposition 12. *Let* $\Psi \in \mathcal{G}(n)$ *and* $X \in \mathfrak{m}(n)$. *Then*

(1) $\dim \boldsymbol{K}_\Psi(X) \geq n$.

(2) $[[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], X] = 0$.

Proof. (1) is clear from $\dim \boldsymbol{K}_\Psi(X) \geq \dim \mathfrak{m}(n) - \dim \mathfrak{u}(n) = n$. Let Y, $Z \in \boldsymbol{K}_\Psi(X)$ and let W be an arbitrary element of $\mathfrak{m}(n)$. Then by the Gauss equation (6.1) we have

$$\langle [[Y,Z],X],W \rangle = \langle \Psi(Y,X), \Psi(Z,W) \rangle - \langle \Psi(Y,W), \Psi(Z,X) \rangle = 0.$$

Since W is an arbitrary element, we have $[[Y,Z],X] = 0$, proving (2). □

Let $X \in \mathfrak{m}(n)$. We define a subspace $C(X) \subset \mathfrak{m}(n)$ by $C(X) = \{Y \in \mathfrak{m}(n) \mid [X,Y] = 0\}$. Then we have $\dim C(X) \geq \operatorname{rank}(Sp(n)/U(n)) = n$. We say an element $X \in \mathfrak{m}(n)$ is *regular* if $\dim C(X) = n$. It is obvious that for a regular element $X \in \mathfrak{m}(n)$, $C(X)$ is a unique maximal abelian subspace of $\mathfrak{m}(n)$ containing X. More strongly, since $\operatorname{rank}(Sp(n)) = n$, $C(X)$ is a unique maximal abelian subalgebra of $\mathfrak{sp}(n)$ containing X. We note that the set of regular elements forms an open dense subset of $\mathfrak{m}(n)$ and that any maximal abelian subspace $\mathfrak{a}$ contains regular elements as an open dense subset with respect to the induced topology of $\mathfrak{a}$.

Proposition 13. *Let* $\Psi \in \mathcal{G}(n)$ *and* $X \in \mathfrak{m}(n)$.

(1) *If* $X \neq 0$, *then* $X \notin \boldsymbol{K}_\Psi(X)$.

(2) *If* X *is regular, then* $\boldsymbol{K}_\Psi(X)$ *is a maximal abelian subspace of* $\mathfrak{m}(n)$. *Moreover,*

$$\Psi(\boldsymbol{K}_\Psi(X), C(X)) = 0. \tag{6.2}$$

(3) *If* X *is not regular, then* $\dim \boldsymbol{K}_\Psi(X) \geq \dim C(X)$ $(> n)$.

Proof. Let $X \in \mathfrak{m}(n)$. Putting $Y = W = IX$ and $Z = X$ into (6.1), we have

$$\langle [[X,IX],X],IX \rangle = \langle \Psi(X,X), \Psi(IX,IX) \rangle - \langle \Psi(X,IX), \Psi(IX,X) \rangle.$$

Assume that $X \in \boldsymbol{K}_\Psi(X)$, i.e., $\Psi(X,X) = 0$. Then the right side of the above equality becomes $-|\Psi(X,IX)|^2 \leq 0$, where $|\cdot|$ means the norm determined by $\langle\, , \rangle$. On the other hand, since $Sp(n)/U(n)$ has positive holomorphic sectional curvature, the left side of the above equality becomes > 0 when $X \neq 0$. This is a contradiction. Hence we have $X \notin \boldsymbol{K}_\Psi(X)$ if $X \neq 0$.

Next we show (2). Assume that $X \in \mathfrak{m}(n)$ is regular. Since $[\mathfrak{m}(n), \mathfrak{m}(n)] \subset \mathfrak{u}(n)$ and $[\mathfrak{u}(n), \mathfrak{m}(n)] \subset \mathfrak{m}(n)$, it follows that $[[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], C(X)] \subset \mathfrak{m}(n)$. In view of (2) of Proposition 12, we easily get $[X, [[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], C(X)]] = 0$. Consequently, $[[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], C(X)] \subset C(X)$. Since $C(X)$ is an abelian subspace, we have

$$\langle [[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], C(X)], C(X) \rangle = \langle [\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], [C(X), C(X)] \rangle = 0.$$

This implies that $[[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)], C(X)] = 0$. Let $W \in [\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)]$. Then the sum $C(X) + \mathbb{R}W$ forms an abelian subalgebra of $\mathfrak{sp}(n)$. Since $C(X)$ is a unique maximal abelian subalgebra of $\mathfrak{sp}(n)$ containing X, we have $C(X) = C(X) + \mathbb{R}W$. Therefore $W \in C(X)$. However, since $W \in \mathfrak{u}(n)$ and $C(X) \subset \mathfrak{m}(n)$, we have $W = 0$. This proves $[\boldsymbol{K}_\Psi(X), \boldsymbol{K}_\Psi(X)] = 0$, i.e., $\boldsymbol{K}_\Psi(X)$ is an abelian subspace of $\mathfrak{m}(n)$. Since $\dim \boldsymbol{K}_\Psi(X) \geq n$, it follows that $\dim \boldsymbol{K}_\Psi(X) = n$ and hence $\boldsymbol{K}_\Psi(X)$ is a maximal abelian subspace of $\mathfrak{m}(n)$.

Now take a regular element $Y \in \boldsymbol{K}_\Psi(X)$. Then it follows that $\Psi(Y, C(X)) = 0$. In fact, as we have shown, $\boldsymbol{K}_\Psi(Y)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ and satisfies $\Psi_Y(X) = \Psi(X,Y) = \Psi_X(Y) = 0$, i.e., $X \in \boldsymbol{K}_\Psi(Y)$. Since $C(X)$ is a unique maximal abelian subspace containing X, we have $\boldsymbol{K}_\Psi(Y) = C(X)$, which proves $\Psi(Y, C(X)) = 0$. Note that regular elements of $\boldsymbol{K}_\Psi(X)$ form an open dense subset of $\boldsymbol{K}_\Psi(X)$. Therefore by continuity of Ψ we have $\Psi(Y', C(X)) = 0$ for any $Y' \in \boldsymbol{K}_\Psi(X)$, i.e., $\Psi(\boldsymbol{K}_\Psi(X), C(X)) = 0$, completing the proof of (2).

Finally, assume that $X \in \mathfrak{m}(n)$ is not regular. Let $\mathfrak{a}$ be a maximal abelian subspace containing X. Since X is not regular, we have $C(X) \supsetneqq \mathfrak{a}$. Take a regular element $H \in \mathfrak{a}$. Then, since $\boldsymbol{K}_\Psi(H)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see (2)), we can take a regular element $Z \in \boldsymbol{K}_\Psi(H)$. We now show that the image of $C(X)$ via the map Ψ_Z is isomorphic to the quotient $C(X)/\mathfrak{a}$, i.e., $\Psi_Z(C(X)) \cong C(X)/\mathfrak{a}$. In fact, since $C(H) = \mathfrak{a}$, it follows that

$$\Psi_Z(\mathfrak{a}) = \Psi(Z, \mathfrak{a}) \subset \Psi(\boldsymbol{K}_\Psi(H), C(H)) = 0,$$

i.e., $\boldsymbol{K}_\Psi(Z) \supset \mathfrak{a}$ (see (6.2)). Since $\boldsymbol{K}_\Psi(Z)$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see (2)), we have $\boldsymbol{K}_\Psi(Z) = \mathfrak{a}$, proving our assertion.

Now let $Y \in C(X)$. Then by the Gauss equation (6.1) we have

$$\langle [[X,Y],Z], W \rangle = \langle \Psi(X,Z), \Psi(Y,W) \rangle - \langle \Psi(X,W), \Psi(Y,Z) \rangle, \quad W \in \mathfrak{m}(n).$$

Since $[X,Y] = 0$ and $\Psi(X,Z) = \Psi_Z(X) = 0$, we have

$$\langle \Psi(X,W), \Psi(Y,Z) \rangle = \langle \Psi_X(W), \Psi_Z(Y) \rangle = 0.$$

Consequently, the subspace $\Psi_X(\mathfrak{m}(n))$ of $\mathfrak{u}(n)$ is perpendicular to the subspace $\Psi_Z(C(X))$. Hence we have $\dim \Psi_X(\mathfrak{m}(n)) \leq \dim \mathfrak{u}(n) - \dim \Psi_Z(C(X))$. On the other hand, since $\Psi_Z(C(X)) \cong C(X)/\mathfrak{a}$, it follows that $\dim \Psi_Z(C(X)) = \dim C(X) - n$. Therefore,

$$\begin{aligned}
\dim \boldsymbol{K}_\Psi(X) &= \dim \mathfrak{m}(n) - \dim \Psi_X(\mathfrak{m}(n)) \\
&\geq \dim \mathfrak{m}(n) - (\dim \mathfrak{u}(n) - \dim \Psi_Z(C(X))) \\
&= (\dim \mathfrak{m}(n) - \dim \mathfrak{u}(n)) + \dim \Psi_Z(C(X)) \\
&= n + (\dim C(X) - n) \\
&= \dim C(X),
\end{aligned}$$

completing the proof of (3). □

Proposition 14. *Let $\Psi \in \mathcal{G}(n)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}(n)$. Then:*

(1) *There exists a unique maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{m}(n)$ such that*

$$\Psi(\mathfrak{a}, \mathfrak{a}') = 0. \tag{6.3}$$

(2) *Let $\{H_1, \cdots, H_n\}$ be a basis of $\mathfrak{a}$. Then the maximal abelian subspace $\mathfrak{a}'$ stated in (1) can be written as*

$$\mathfrak{a}' = \bigcap_{i=1}^{n} \boldsymbol{K}_\Psi(H_i).$$

Proof. First we prove the existence of $\mathfrak{a}'$ satisfying (6.3). Take a regular element $X \in \mathfrak{a}$ and set $\mathfrak{a}' = K_\Psi(X)$. Then, we know that $\mathfrak{a}'$ is a maximal abelian subspace of $\mathfrak{m}(n)$ (see Proposition 13 (2)). Since $C(X) = \mathfrak{a}$, by (6.2) we obtain $\Psi(\mathfrak{a}, \mathfrak{a}') = \Psi(C(X), \boldsymbol{K}_\Psi(X)) = 0$. Next, we prove the uniqueness of $\mathfrak{a}'$. Let $\mathfrak{a}'$ be a maximal abelian subspace satisfying (6.3). Take an arbitrary regular element X contained in $\mathfrak{a}$. Then by (6.3) it is clear that $\boldsymbol{K}_\Psi(X) \supset \mathfrak{a}'$. Since $\boldsymbol{K}_\Psi(X)$ is a maximal abelian subspace of $\mathfrak{m}(n)$, we have $\boldsymbol{K}_\Psi(X) = \mathfrak{a}'$. This proves the uniqueness of $\mathfrak{a}'$.

Let $\{H_1, \cdots, H_n\}$ be a basis of $\mathfrak{a}$. Then by (6.3) we have $\Psi(H_i, \mathfrak{a}') = 0$ and hence $\boldsymbol{K}_\Psi(H_i) \supset \mathfrak{a}'$. Therefore, $\mathfrak{a}' \subset \bigcap_{i=1}^n \boldsymbol{K}_\Psi(H_i)$. On the other hand, by linearity of Ψ, we have $\boldsymbol{K}_\Psi(X) \supset \bigcap_{i=1}^n \boldsymbol{K}_\Psi(H_i)$ for any $X \in \mathfrak{a}$. In particular, if X is regular, then we have $\mathfrak{a}' = \boldsymbol{K}_\Psi(X)$ and hence $\mathfrak{a}' \supset \bigcap_{i=1}^n \boldsymbol{K}_\Psi(H_i)$. This completes the proof of (2). □

Let $\Psi \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ and $a \in U(n)$. Define an element $\Psi^a \in S^2\mathfrak{m}(n)^* \otimes \mathfrak{u}(n)$ by

$$\Psi^a(X, Y) = \Psi(\mathrm{Ad}(a^{-1})X, \mathrm{Ad}(a^{-1})Y), \quad X, Y \in \mathfrak{m}(n).$$

Then, since $\mathrm{Ad}(a^{-1})$ preserves the curvature, we have $\Psi^a \in \mathcal{G}(n)$ if and only if $\Psi \in \mathcal{G}(n)$. We can easily show the following

Lemma 15. *Let $\Psi \in \mathcal{G}(n)$ and $a \in U(n)$. Then*

$$\boldsymbol{K}_{\Psi^a}(\mathrm{Ad}(a)X) = \mathrm{Ad}(a)(\boldsymbol{K}_\Psi(X)), \quad X \in \mathfrak{m}(n).$$

Proof. The proof is obtained by the following

$$\begin{aligned} Y \in \boldsymbol{K}_{\Psi^a}(\mathrm{Ad}(a)X) &\Longleftrightarrow \Psi^a(\mathrm{Ad}(a)X, Y) = 0 \\ &\Longleftrightarrow \Psi(X, \mathrm{Ad}(a^{-1})Y) = 0 \\ &\Longleftrightarrow \mathrm{Ad}(a^{-1})Y \in \boldsymbol{K}_\Psi(X) \\ &\Longleftrightarrow Y \in \mathrm{Ad}(a)(\boldsymbol{K}_\Psi(X)). \end{aligned}$$

□

Finally, we state Theorem 11 in a different form, which is somewhat easy to prove. Let E_{ij} $(1 \le i, j \le n)$ denote the matrix $\left(\delta_{is}\delta_{jt}\right)_{1\le s,t\le n} \in M(n; \mathbb{H})$. Then it is easily seen that the sum $\mathfrak{a}_0 = \sum_{i=1}^{n} \mathbb{R}\boldsymbol{j}E_{ii}$ forms a maximal abelian subspace of $\mathfrak{m}(n)$. Now consider the following:

Proposition 16. *Assume that $n \ge 2$. Let $\Psi \in \mathcal{G}(n)$. Then*

$$\Psi(\mathfrak{a}_0, I\mathfrak{a}_0) = 0.$$

We now show that Proposition 16 implies Theorem 11. Assume that Proposition 16 is true. Under this setting we will show that any element $\Psi \in \mathcal{G}(n)$ is Hermitian. Let X be an arbitrary element of $\mathfrak{m}(n)$. As is known, there is an element $a \in U(n)$ such that $H = \mathrm{Ad}(a)X \in \mathfrak{a}_0$. By Proposition 16 we have $\Psi^a(H, IH) = 0$, because $\Psi^a \in \mathcal{G}(n)$. Noticing the relation $\mathrm{Ad}(a^{-1})I = I\mathrm{Ad}(a^{-1})$, we have

$$0 = \Psi^a(H, IH) = \Psi(\mathrm{Ad}(a^{-1})H, \mathrm{Ad}(a^{-1})IH) = \Psi(X, IX).$$

Consequently, for any $X \in \mathfrak{m}(n)$ we have $\Psi(X, IX) = 0$, which means that Ψ is Hermitian. Thus we get Theorem 11.

7. Proof of Proposition 16

In this section we will prove Proposition 16. Let n' be a non-negative integer such that $n' < n$. By the assignment

$$\mathfrak{m}(n') \ni X \longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}(n)$$

we may regard $\mathfrak{m}(n')$ as a subspace of $\mathfrak{m}(n)$. In the special case $n' = n - 1$ we have the direct sum

$$\mathfrak{m}(n) = \mathfrak{m}(n-1) + \mathbb{D}E_{nn} + \sum_{i=1}^{n-1} \mathbb{D}(E_{in} + E_{ni}). \tag{7.1}$$

For simplicity we set $H_i = \boldsymbol{j}E_{ii}$ $(i = 1, \dots, n)$. Then we have $IH_i = \boldsymbol{i}H_i = \boldsymbol{k}E_{ii}$. Consequently, we have

$$\mathbb{C}H_i = \mathbb{R}H_i + \mathbb{R}IH_i = \mathbb{D}E_{ii}, \qquad i = 1, \dots, n.$$

As in the previous section we set $\mathfrak{a}_0 = \sum_{i=1}^{n} \mathbb{R}H_i$. In the following we will prove $\Psi(\mathfrak{a}_0, I\mathfrak{a}_0) = 0$ for any $\Psi \in \mathcal{G}(n)$. First we show

Lemma 17. *Let $\Psi \in \mathcal{G}(n)$. Then there exists a real number $a \in \mathbb{R}$ such that*

$$\boldsymbol{K}_\Psi(H_n) = \mathfrak{m}(n-1) + \mathbb{R}(IH_n - aH_n) \qquad (\textit{direct sum}). \tag{7.2}$$

Accordingly, $\dim \boldsymbol{K}_\Psi(H_n) = \dim \mathfrak{m}(n-1) + 1$.

Proof. First assume that the following inclusion holds:

$$\boldsymbol{K}_{\Psi}(H_n) \subset \mathfrak{m}(n-1) + \mathbb{C}H_n. \tag{7.3}$$

Then, since $H_n \notin \boldsymbol{K}_{\Psi}(H_n)$, we have $\dim \boldsymbol{K}_{\Psi}(H_n) \leq \dim \mathfrak{m}(n-1)+1$. On the other hand, by a simple calculation we can verify that $C(H_n) = \mathfrak{m}(n-1) + \mathbb{R}H_n$. On account of the relation $\dim \boldsymbol{K}_{\Psi}(H_n) \geq \dim C(H_n)$ we have $\dim \boldsymbol{K}_{\Psi}(H_n) = \dim \mathfrak{m}(n-1) + 1$. Moreover, we have

$$\boldsymbol{K}_{\Psi}(H_n) + \mathbb{R}H_n = \mathfrak{m}(n-1) + \mathbb{C}H_n. \tag{7.4}$$

Now we show $\mathfrak{m}(n-1) \subset \boldsymbol{K}_{\Psi}(H_n)$. By (7.4) it is known that there is a real number a such that $IH_n - aH_n \in \boldsymbol{K}_{\Psi}(H_n)$. Similarly, for any $X \in \mathfrak{m}(n-1)$, there is a real number $b \in \mathbb{R}$ such that $X - bH_n \in \boldsymbol{K}_{\Psi}(H_n)$. Consider the equality $[[IH_n - aH_n, X - bH_n], H_n] = 0$. By a direct calculation we have $[H_n, X] = [IH_n, X] = 0$ and $[[IH_n, H_n], H_n] = -4IH_n$. Consequently, $[[IH_n - aH_n, X - bH_n], H_n] = 4bIH_n = 0$, implying $b = 0$. Hence we have $X \in \boldsymbol{K}_{\Psi}(H_n)$, i.e., $\mathfrak{m}(n-1) \subset \boldsymbol{K}_{\Psi}(H_n)$. Thus if (7.3) is true, then we obtain the lemma.

Now we suppose that (7.3) is not true, i.e., $\boldsymbol{K}_{\Psi}(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$. Let r and s be non-negative integers. By $M(r, s; \mathbb{D})$ we denote the space of $\mathbb{D}$-valued $r \times s$-matrices. As is easily seen, each element $X \in \mathfrak{m}(n)$ can be written in the form

$$X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix}, \qquad X' \in \mathfrak{m}(n-1), \xi \in M(n-1, 1; \mathbb{D}), \boldsymbol{x} \in \mathbb{D}.$$

Under our assumption $\boldsymbol{K}_{\Psi}(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$ we know that there is an element $X = \left(\begin{smallmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{smallmatrix}\right) \in \boldsymbol{K}_{\Psi}(H_n)$ such that $\xi \neq 0$. Let $\varphi \colon \mathfrak{m}(n) \longrightarrow M(n-1, 1; \mathbb{D})$ be the natural projection defined by $\varphi(\left(\begin{smallmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{smallmatrix}\right)) = \xi$. By $\varphi(\boldsymbol{K}_{\Psi}(H_n))$ we denote the image of $\boldsymbol{K}_{\Psi}(H_n)$ by φ. Then we have $\varphi(\boldsymbol{K}_{\Psi}(H_n)) \neq 0$. As is easily seen, from the right multiplication $\xi \longmapsto \xi c$ of $c \in \mathbb{C}$, $M(n-1, 1; \mathbb{D})$ may be regarded as a complex vector space with $\dim_{\mathbb{C}} M(n-1, 1; \mathbb{D}) = n - 1$. By $\varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}}$ we mean the complex subspace of $M(n-1, 1; \mathbb{D})$ generated by $\varphi(\boldsymbol{K}_{\Psi}(H_n))$, i.e., $\varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}} = \varphi(\boldsymbol{K}_{\Psi}(H_n)) + \varphi(\boldsymbol{K}_{\Psi}(H_n))\boldsymbol{i}$. Set $s = \dim_{\mathbb{C}} \varphi(\boldsymbol{K}_{\Psi}(H_n))^{\mathbb{C}}$. Then, clearly we have $1 \leq s \leq n-1$, $\dim \varphi(\boldsymbol{K}_{\Psi}(H_n)) \leq 2s$ and

$$\dim \boldsymbol{K}_{\Psi}(H_n) = \dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_{\Psi}(H_n)) + \dim \varphi(\boldsymbol{K}_{\Psi}(H_n)). \tag{7.5}$$

Now, let us show

$$\dim(\mathfrak{m}(n-1) \cap \boldsymbol{K}_{\Psi}(H_n)) \leq (n-s-1)(n-s). \tag{7.6}$$

Let $\mathfrak{m}(n-1)'$ be the subspace of $\mathfrak{m}(n-1)$ consisting of all $Y' \in \mathfrak{m}(n-1)$ satisfying $Y = \left(\begin{smallmatrix} Y' & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_{\Psi}(H_n)$. To show (7.6) it suffices to prove $\dim \mathfrak{m}(n-1)' \leq (n-s-1)(n-s)$. For the proof we prepare the following formula:

$$[[X, Y], H_n] = \begin{pmatrix} 0 & (\xi \boldsymbol{y} - Y'\xi)\boldsymbol{j} \\ -\boldsymbol{j}({}^t\xi Y' - \boldsymbol{y}\,{}^t\xi) & [[\boldsymbol{x}, \boldsymbol{y}], \boldsymbol{j}] \end{pmatrix}, \tag{7.7}$$

where $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix} \in \mathfrak{m}(n)$ and $Y = \begin{pmatrix} Y' & 0 \\ 0 & \boldsymbol{y} \end{pmatrix} \in \mathfrak{m}(n-1) + \mathbb{C}H_n$. This formula can be easily obtained by a simple calculation. Utilizing (7.7), we show (7.6). Let $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix} \in \boldsymbol{K}_\Psi(H_n)$ and $Y = \begin{pmatrix} Y' & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n)$. Since $[[X,Y],H_n] = 0$, by (7.7) it follows that $Y'\xi = 0$. We note that this equality holds for any $\xi \in \varphi(\boldsymbol{K}_\Psi(H_n))$ and $Y' \in \mathfrak{m}(n-1)'$. Since $Y'(\xi \boldsymbol{i}) = (Y'\xi)\boldsymbol{i}$, we have

$$Y'\xi = 0, \qquad \forall Y' \in \mathfrak{m}(n-1)',\ \forall \xi \in \varphi(\boldsymbol{K}_\Psi(H_n))^{\mathbb{C}}. \tag{7.8}$$

Select a basis $\{\eta_1, \cdots, \eta_{n-s-1}, \xi_1, \cdots, \xi_s\}$ of the complex vector space $M(n-1,1;\mathbb{D})$ such that $\{\xi_1, \cdots, \xi_s\}$ forms a basis of $\varphi(\boldsymbol{K}_\Psi(H_n))^{\mathbb{C}}$. Define a matrix $U \in M(n-1;\mathbb{H})$ by $U = (\eta_1, \cdots, \eta_{n-s-1}, \xi_1, \cdots, \xi_s)$. Let $Y' \in \mathfrak{m}(n-1)'$. Since $Y'\xi_1 = \cdots = Y'\xi_s = 0$ and ${}^t\xi_1 Y' = \cdots = {}^t\xi_s Y' = 0$, we have ${}^tU \cdot Y' \cdot U \in \mathfrak{m}(n-s-1)$. This means that ${}^tU \cdot \mathfrak{m}(n-1)' \cdot U \subset \mathfrak{m}(n-s-1)$. Since U is a non-singular matrix, we have $\dim_{\mathbb{R}} \mathfrak{m}(n-1)' \leq \dim_{\mathbb{R}} \mathfrak{m}(n-s-1) = (n-s-1)(n-s)$, proving the desired inequality (7.6).

Next we consider the intersection $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n)$. Since $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n) \supset \mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n)$, the following two cases are possible:

(i) $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n) = \mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n)$.
(ii) $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n) \supsetneqq \mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n)$.

In the case (i), we have $\dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n)) \leq (n-s-1)(n-s)$. Since $\dim \varphi(\boldsymbol{K}_\Psi(H_n)) \leq 2s$, by (7.5) we have

$$\begin{aligned} \dim \boldsymbol{K}_\Psi(H_n) &\leq (n-s-1)(n-s) + 2s \\ &= s^2 - (2n-3)s + n(n-1). \end{aligned} \tag{7.9}$$

Since $1 \leq s \leq n-1$, the right side of (7.9) attains its maximum when $s = 1$. Consequently, we have $\dim \boldsymbol{K}_\Psi(H_n) \leq 4 - 2n + n(n-1) < 1 + \dim \mathfrak{m}(n-1)$, because $\dim \mathfrak{m}(n-1) = n(n-1)$ and $n \geq 2$. This contradicts the fact $\dim \boldsymbol{K}_\Psi(H_n) \geq 1 + \dim \mathfrak{m}(n-1)$. Therefore we know that the case (i) is impossible.

Next we show the case (ii) is also impossible. Let $Y = \begin{pmatrix} Y' & 0 \\ 0 & \boldsymbol{y} \end{pmatrix}$ be an element of $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n)$ satisfying $\boldsymbol{y} \neq 0$. Let $X = \begin{pmatrix} X' & \xi \\ {}^t\xi & \boldsymbol{x} \end{pmatrix}$ be an arbitrary element of $\boldsymbol{K}_\Psi(H_n)$. Then, since $[[X,Y],H_n] = 0$, we obtain by (7.7)

$$[[\boldsymbol{x},\boldsymbol{y}],\boldsymbol{j}] = 0; \qquad \xi\boldsymbol{y} - Y'\xi = 0. \tag{7.10}$$

From the first equality in (7.10) we have $\boldsymbol{x} \in \mathbb{R}\boldsymbol{y}$. In fact, since $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{D}$, we have $[\boldsymbol{x},\boldsymbol{y}] \in \mathbb{R}\boldsymbol{i}$. However, since $[\boldsymbol{i},\boldsymbol{j}] = 2\boldsymbol{k} \neq 0$, we have $[\boldsymbol{x},\boldsymbol{y}] = 0$, implying $\boldsymbol{x} \in \mathbb{R}\boldsymbol{y}$. This fact means that for any element $X \in (\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n)$ there is a real number c such that $X - cY \in \mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n)$. Consequently, we have $(\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n) \subset$

$\mathbb{R}Y + (\mathfrak{m}(n-1) \cap \boldsymbol{K}_\Psi(H_n))$ and hence

$$\dim((\mathfrak{m}(n-1) + \mathbb{C}H_n) \cap \boldsymbol{K}_\Psi(H_n)) \leq 1 + (n-s-1)(n-s). \tag{7.11}$$

Moreover, in this case we have $\dim \varphi(\boldsymbol{K}_\Psi(H_n)) = s$. In fact, we have

$$\varphi(\boldsymbol{K}_\Psi(H_n))^{\mathbb{C}} = \varphi(\boldsymbol{K}_\Psi(H_n)) + \varphi(\boldsymbol{K}_\Psi(H_n))\boldsymbol{i} \quad \text{(direct sum)}. \tag{7.12}$$

It is easily seen that to show (7.12) it suffices to prove $\varphi(\boldsymbol{K}_\Psi(H_n)) \cap \varphi(\boldsymbol{K}_\Psi(H_n))\boldsymbol{i} = 0$. Assume that $\xi \in \varphi(\boldsymbol{K}_\Psi(H_n))$ satisfies $\xi\boldsymbol{i} \in \varphi(\boldsymbol{K}_\Psi(H_n))$. Take elements X, $X_1 \in \boldsymbol{K}_\Psi(H_n)$ such that $\varphi(X) = \xi$, $\varphi(X_1) = \xi\boldsymbol{i}$. Then, since $[[X,Y],H_n] = [[X_1,Y],H_n] = 0$, by the second equality in (7.10) we have $\xi\boldsymbol{y} - Y'\xi = 0$ and $(\xi\boldsymbol{i})\boldsymbol{y} - Y'(\xi\boldsymbol{i}) = 0$. Since $\boldsymbol{i}\boldsymbol{y} = -\boldsymbol{y}\boldsymbol{i}$, the last equality becomes $(\xi\boldsymbol{i})\boldsymbol{y} - Y'(\xi\boldsymbol{i}) = -(\xi\boldsymbol{y} + Y'\xi)\boldsymbol{i} = 0$. Consequently, $\xi\boldsymbol{y} + Y'\xi = 0$, showing $\xi\boldsymbol{y} = 0$. Hence we get $\xi = 0$, because $\boldsymbol{y} \neq 0$.

Thus by (7.5) we have

$$\begin{aligned} \dim \boldsymbol{K}_\Psi(H_n) &\leq 1 + (n-s-1)(n-s) + s \\ &= s^2 - 2(n-1)s + \dim \mathfrak{m}(n-1) + 1. \end{aligned} \tag{7.13}$$

The right side of (7.13) attains its maximum when $s = 1$ and therefore $\dim \boldsymbol{K}_\Psi(H_n) \leq 4 - 2n + \dim \mathfrak{m}(n-1) < 1 + \dim \mathfrak{m}(n-1)$, which is also a contradiction.

Thus, assuming $\boldsymbol{K}_\Psi(H_n) \not\subset \mathfrak{m}(n-1) + \mathbb{C}H_n$, we meet a contradiction. Hence we have $\boldsymbol{K}_\Psi(H_n) \subset \mathfrak{m}(n-1) + \mathbb{C}H_n$, completing the proof of the lemma. □

In a similar manner we can prove the following lemma:

Lemma 18. *Let $\Psi \in \mathcal{G}(n)$. Then*

$$\dim \boldsymbol{K}_\Psi(H_i) = \dim \boldsymbol{K}_\Psi(IH_i) = \dim \mathfrak{m}(n-1) + 1, \quad i = 1, \ldots, n.$$

Moreover there exist real numbers a' and $b \in \mathbb{R}$ such that

$$\boldsymbol{K}_\Psi(IH_n) = \mathfrak{m}(n-1) + \mathbb{R}(H_n - a'IH_n) \quad (\textit{direct sum});$$

$$\begin{aligned} \boldsymbol{K}_\Psi(H_{n-1}) = \mathfrak{m}(n-2) &+ \sum_{i=1}^{n-2} \mathbb{D}(E_{in} + E_{ni}) + \mathbb{C}H_n \\ &+ \mathbb{R}(IH_{n-1} - bH_{n-1}) \quad (\textit{direct sum}). \end{aligned}$$

With the aid of Lemma 18 we can prove the refinement of Lemma 17.

Lemma 19. *Let $\Psi \in \mathcal{G}(n)$. Then $\boldsymbol{K}_\Psi(H_n) = \mathfrak{m}(n-1) + \mathbb{R}IH_n$ (direct sum).*

Proof. Let $\Psi \in \mathcal{G}(n)$. Take real numbers a, a' and $b \in \mathbb{R}$ stated in Lemma 17 and Lemma 18 and set $Y = IH_n - aH_n$, $Z = IH_{n-1} - bH_{n-1}$ and $W = H_n - a'IH_n$. Then clearly we have

$$Y = (\boldsymbol{k} - a\boldsymbol{j})E_{nn}, \; Z = (\boldsymbol{k} - b\boldsymbol{j})E_{n-1,n-1}, \; W = (\boldsymbol{j} - a'\boldsymbol{k})E_{nn}.$$

In the following we will show that $a = a' = b = 0$. If this can be done, the lemma follows immediately.

First we prove $a = b$. For this purpose we consider the space $\boldsymbol{K}_\Psi(H_{n-1} + H_n)$. By an easy calculation we can verify that $C(H_{n-1} + H_n) = \mathfrak{m}(n-2) + \mathbb{R}H_{n-1} + \mathbb{R}H_n + \mathbb{R}\boldsymbol{j}(E_{n-1,n} + E_{n,n-1})$. Therefore we have $\dim \boldsymbol{K}_\Psi(H_{n-1} + H_n) \geq \dim C(H_{n-1} + H_n) = \dim \mathfrak{m}(n-2) + 3$. Since $\boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(H_n) = \mathfrak{m}(n-2) + \mathbb{R}Y + \mathbb{R}Z$, it follows that $\dim(\boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(H_n)) = \dim \mathfrak{m}(n-2) + 2$. Consequently, we can take an element $X \in \boldsymbol{K}_\Psi(H_{n-1} + H_n)$ such that $X \notin \boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(H_n)$. Write

$$X = \begin{pmatrix} X'' & \eta & \xi \\ {}^t\eta & \boldsymbol{y} & \boldsymbol{z} \\ {}^t\xi & \boldsymbol{z} & \boldsymbol{x} \end{pmatrix}, \tag{7.14}$$

where $X'' \in \mathfrak{m}(n-2)$, ξ, $\eta \in M(n-2, 1; \mathbb{D})$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{D}$. Since X, Y, $Z \in \boldsymbol{K}_\Psi(H_{n-1}+H_n)$, we have $[[X, Y], H_{n-1} + H_n] = [[X, Z], H_{n-1} + H_n] = 0$. Set $\boldsymbol{h} = \boldsymbol{k} - a\boldsymbol{j}$ and $\boldsymbol{h}' = \boldsymbol{k} - b\boldsymbol{j}$. Then we have

$$\xi\boldsymbol{h}\boldsymbol{j} = \eta\boldsymbol{h}'\boldsymbol{j} = 0; \tag{7.15}$$

$$[[\boldsymbol{x}, \boldsymbol{h}], \boldsymbol{j}] = [[\boldsymbol{y}, \boldsymbol{h}'], \boldsymbol{j}] = 0; \tag{7.16}$$

$$[\boldsymbol{z}\boldsymbol{h}, \boldsymbol{j}] = [\boldsymbol{z}\boldsymbol{h}', \boldsymbol{j}] = 0. \tag{7.17}$$

By (7.15) and (7.16) we easily have $\xi = \eta = 0$, $\boldsymbol{x} \in \mathbb{R}\boldsymbol{h}$, and $\boldsymbol{y} \in \mathbb{R}\boldsymbol{h}'$. Thus, if $\boldsymbol{z} = 0$, then we have $X \in \mathfrak{m}(n-2) + \mathbb{R}Y + \mathbb{R}Z = \boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(H_n)$. This contradicts the assumption $X \notin \boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(H_n)$. Hence $\boldsymbol{z} \neq 0$. Now consider (7.17). First note that $\boldsymbol{z}\boldsymbol{h}$, $\boldsymbol{z}\boldsymbol{h}' \in \mathbb{C} = \mathbb{R} + \mathbb{R}\boldsymbol{i}$. Since $[\boldsymbol{i}, \boldsymbol{j}] \neq 0$, $[\boldsymbol{z}\boldsymbol{h}, \boldsymbol{j}] = [\boldsymbol{z}\boldsymbol{h}', \boldsymbol{j}] = 0$ holds if and only if $\boldsymbol{z}\boldsymbol{h}$, $\boldsymbol{z}\boldsymbol{h}' \in \mathbb{R}$. Since $\boldsymbol{z} \neq 0$, we have $\boldsymbol{h} \in \mathbb{R}\boldsymbol{z}^{-1}$ and $\boldsymbol{h}' \in \mathbb{R}\boldsymbol{z}^{-1}$. Consequently, we have $\mathbb{R}\boldsymbol{h} = \mathbb{R}\boldsymbol{z}^{-1}$ and hence $\boldsymbol{h}' \in \mathbb{R}\boldsymbol{h} = \mathbb{R}(\boldsymbol{k} - a\boldsymbol{j})$. Therefore we have $a = b$, because $\boldsymbol{h}' = \boldsymbol{k} - b\boldsymbol{j}$.

Next we prove $a' = -a$. For this purpose we consider the space $\boldsymbol{K}_\Psi(H_{n-1} + IH_n)$. We can easily see that $C(H_{n-1}+IH_n) = \mathfrak{m}(n-2)+\mathbb{R}H_{n-1}+\mathbb{R}IH_n+\mathbb{R}(\boldsymbol{j}+\boldsymbol{k})(E_{n-1,n}+E_{n,n-1})$. Hence $\dim \boldsymbol{K}_\Psi(H_{n-1}+IH_n) \geq \dim \mathfrak{m}(n-2)+3$. Since $\boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(IH_n) = \mathfrak{m}(n-2) + \mathbb{R}Z + \mathbb{R}W$, it follows that $\dim(\boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(IH_n)) = \dim \mathfrak{m}(n-2) + 2$. Consequently, we can take an element $X \in \boldsymbol{K}_\Psi(H_{n-1}+IH_n)$ such that $X \notin \boldsymbol{K}_\Psi(H_{n-1}) \cap \boldsymbol{K}_\Psi(IH_n)$. Since X, Z, $W \in \boldsymbol{K}_\Psi(H_{n-1} + IH_n)$, we have $[[X, Z], H_{n-1} + IH_n] = [[X, W], H_{n-1} + IH_n] = 0$. Writing X in the form (7.14), we have

$$\xi\boldsymbol{h}''\boldsymbol{k} = \eta\boldsymbol{h}\boldsymbol{j} = 0; \tag{7.18}$$

$$[[\boldsymbol{x}, \boldsymbol{h}''], \boldsymbol{k}] = [[\boldsymbol{y}, \boldsymbol{h}], \boldsymbol{j}] = 0; \tag{7.19}$$

$$\boldsymbol{z}\boldsymbol{h}''\boldsymbol{k} - \boldsymbol{j}\boldsymbol{z}\boldsymbol{h}'' = \boldsymbol{h}\boldsymbol{z}\boldsymbol{k} - \boldsymbol{j}\boldsymbol{h}\boldsymbol{z} = 0, \tag{7.20}$$

where we set $\boldsymbol{h} = \boldsymbol{k} - a\boldsymbol{j}$ and $\boldsymbol{h}'' = \boldsymbol{j} - a'\boldsymbol{k}$. By (7.18) and (7.19) we have $\xi = \eta = 0$, $\boldsymbol{x} \in \mathbb{R}\boldsymbol{h}''$ and $\boldsymbol{y} \in \mathbb{R}\boldsymbol{h}$. Hence if $\boldsymbol{z} = 0$, then $X \in \mathfrak{m}(n-2) + \mathbb{R}Z + \mathbb{R}W = \boldsymbol{K}_{\Psi}(H_{n-1}) \cap \boldsymbol{K}_{\Psi}(IH_n)$. This contradicts the assumption $X \notin \boldsymbol{K}_{\Psi}(H_{n-1}) \cap \boldsymbol{K}_{\Psi}(IH_n)$. Hence $\boldsymbol{z} \neq 0$. Now consider (7.20). It is easily verified that $\boldsymbol{z}\boldsymbol{h}''\boldsymbol{k} - \boldsymbol{j}\boldsymbol{z}\boldsymbol{h}'' = \boldsymbol{h}\boldsymbol{z}\boldsymbol{k} - \boldsymbol{j}\boldsymbol{h}\boldsymbol{z} = 0$ holds when and only when $\boldsymbol{z}\boldsymbol{h}'' \in \mathbb{R}(1-\boldsymbol{i})$ and $\boldsymbol{h}\boldsymbol{z} \in \mathbb{R}(1-\boldsymbol{i})$. Since $\boldsymbol{z} \neq 0$, we have $\boldsymbol{h}'' \in \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$ and $\boldsymbol{h} \in \mathbb{R}(1-\boldsymbol{i})\boldsymbol{z}^{-1}$. Therefore, $(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) \in \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$. Consequently, $\mathbb{R}(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) = \mathbb{R}\boldsymbol{z}^{-1}(1-\boldsymbol{i})$ and hence $\boldsymbol{h}'' \in \mathbb{R}(1+\boldsymbol{i})\boldsymbol{h}(1-\boldsymbol{i}) = \mathbb{R}(\boldsymbol{j} + a\boldsymbol{k})$. Accordingly, we have $a' = -a$, because $\boldsymbol{h}'' = \boldsymbol{j} - a'\boldsymbol{k}$.

Finally, we prove $a = 0$. By the definition we have $Y = IH_n - aH_n \in \boldsymbol{K}_{\Psi}(H_n)$. Moreover, by the above discussion we know $W = H_n + aIH_n \in \boldsymbol{K}_{\Psi}(IH_n)$. Hence

$$\Psi(H_n, IH_n - aH_n) = \Psi(IH_n, H_n + aIH_n) = 0.$$

Consequently, we have $\Psi(H_n, IH_n) = a\,\Psi(H_n, H_n)$ and $\Psi(H_n, IH_n) = -a\,\Psi(IH_n, IH_n)$. If $a \neq 0$, then we have $\Psi(IH_n, IH_n) = -\Psi(H_n, H_n)$. Putting $X = Z = H_n$ and $Y = W = IH_n$ into (6.1), we have

$$\begin{aligned}\big\langle [[H_n, IH_n], H_n], IH_n \big\rangle &= \big\langle \Psi(H_n, H_n), \Psi(IH_n, IH_n) \big\rangle - \big\langle \Psi(H_n, IH_n), \Psi(IH_n, H_n) \big\rangle \\ &= -(1+a^2)\big\langle \Psi(H_n, H_n), \Psi(H_n, H_n) \big\rangle \\ &\leq 0.\end{aligned}$$

On the other hand, the left side is > 0, which is a contradiction. Thus we have $a = 0$, completing the proof of the lemma. □

We now complete the proof of Proposition 16.

Proof of Proposition 16. Assume that $n \geq 2$. Let $\Psi \in \mathcal{G}(n)$. We will prove

$$\boldsymbol{K}_{\Psi}(H_i) \supset I\mathfrak{a}_0, \qquad i = 1, \ldots, n. \tag{7.21}$$

Let i be an integer such that $1 \leq i \leq n$. If $i = n$, then (7.21) follows from Lemma 19. Now assume that $i < n$. Set $a = E_{ni} - E_{in} + \sum_{j=1, j\neq i}^{n-1} E_{jj}$. Then it is easy to see that $a \in SO(n)\,(\subset U(n))$, $\mathrm{Ad}(a)H_i = H_n$ and $\mathrm{Ad}(a)\mathfrak{a}_0 = \mathfrak{a}_0$. Consequently, by Lemma 15 we have $\boldsymbol{K}_{\Psi^a}(H_n) = \mathrm{Ad}(a)(\boldsymbol{K}_{\Psi}(H_i))$. On the other hand, since $\Psi^a \in \mathcal{G}(n)$, we have $\boldsymbol{K}_{\Psi^a}(H_n) \supset I\mathfrak{a}_0$. This shows $\mathrm{Ad}(a)(\boldsymbol{K}_{\Psi}(H_i)) \supset I\mathfrak{a}_0$ and hence $\boldsymbol{K}_{\Psi}(H_i) \supset \mathrm{Ad}(a^{-1})(I\mathfrak{a}_0) = I\mathfrak{a}_0$. Consequently, we get (7.21), which implies $\bigcap_{i=1}^{n} \boldsymbol{K}_{\Psi}(H_i) \supset I\mathfrak{a}_0$. Therefore, in view of Proposition 14, we have $\Psi(\mathfrak{a}_0, I\mathfrak{a}_0) = 0$, proving the proposition. □

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A lower bound for the class number of $P^n(C)$ and $P^n(H)$

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Abstract. We obtain new lower bounds on the codimension of local isometric imbeddings of complex and quaternion projective spaces. We show that any open set of the complex projective space $P^n(\boldsymbol{C})$ (resp. quaternion projective space $P^n(\boldsymbol{H})$) cannot be locally isometrically imbedded into the euclidean space of dimension $4n-3$ (resp. $8n-4$). These estimates improve the previously known results obtained in [2] and [7].

Key words: curvature invariant, isometric imbedding, complex projective space, quaternion projective space, root space decomposition.

1. Introduction

Let M be a Riemannian manifold. As is known, M can be locally or globally isometrically imbedded into a euclidean space of sufficiently large dimension (see Janet [19], Cartan [14], Nash [24], Greene-Jacobowitz [16], Gromov-Rokhlin [17]). It is a natural and interesting question to ask the least dimension of euclidean spaces into which M can be locally or globally isometrically imbedded. In this paper we will investigate the problem of local isometric imbeddings of the projective spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ and give a new estimate on the least dimension of the ambient euclidean spaces.

Let $x \in M$. Assume that there is a neighborhood U of x in M such that U is isometrically imbedded into a euclidean space $\boldsymbol{R}^D$. If any neighborhood of x cannot be isometrically imbedded into $\boldsymbol{R}^{D-1}$, then the codimension $D-\dim M$ is called the *class number* of M at x and is denoted by $\mathrm{class}(M)_x$.

Let G/K be a Riemannian symmetric space. By homogeneity, the class number of G/K is constant everywhere on G/K, which is denoted by $\mathrm{class}(G/K)$. In Agaoka-Kaneda [4], [5], [7], [8], [9] and [10] we have tried to estimate $\mathrm{class}(G/K)$ from below. In doing this we mainly used the following inequality

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K),$$

where $p(G/K)$ is the pseudo-nullity of G/K (see §2 below or [4]). For

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the following Riemannian symmetric spaces G/K our estimates just hit class(G/K), i.e., class$(G/K) = \dim G/K - p(G/K)$:

a) The sphere S^n $(n \geq 2)$;
b) *CI*: $Sp(n)/U(n)$ $(n \geq 1)$ (see [4]);
c) The symplectic group $Sp(n)$ $(n \geq 1)$ (see [5]).

As for the class numbers of the projective spaces such as the complex projective space $P^n(\boldsymbol{C})$, the quaternion projective space $P^n(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$, the following are known:

(1) class$(P^n(\boldsymbol{C})) \geq \max\{n+1, [\frac{6}{5}n]\}$ $(n \geq 2)$ (see [2] and [7]);
(2) class$(P^n(\boldsymbol{H})) \geq \min\{4n-3, 3n+1\}$ $(n \geq 3)$ (see [7]);
(3) class$(P^n(\boldsymbol{C})) \leq n^2$ $(n \geq 2)$; class$(P^n(\boldsymbol{H})) \leq 2n^2 - n$ $(n \geq 2)$ (see [22]);
(4) class$(P^2(\boldsymbol{H})) = 6$; class$(P^2(\boldsymbol{Cay})) = 10$ (see [8] and [22]).

It should be noted that any local isometric imbedding of $P^2(\boldsymbol{H})$ (resp. $P^2(\boldsymbol{Cay})$) into the euclidean space $\boldsymbol{R}^{14}$ (resp. $\boldsymbol{R}^{26}$) is rigid in the strongest sense (see [9] and [10]).

In this paper we will propose a new type of estimate and by applying it we will prove

Theorem 1 *Let G/K denote the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 3)$ or the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 3)$. Define an integer $q(G/K)$ by*

$$q(G/K) = \begin{cases} 4n-2, & \text{if } G/K = P^n(\boldsymbol{C}) \ \ (n \geq 3); \\ 8n-3, & \text{if } G/K = P^n(\boldsymbol{H}) \ \ (n \geq 3). \end{cases}$$

Then, any open set of G/K cannot be isometrically imbedded into the euclidean space $\boldsymbol{R}^D$ with $D \leq q(G/K) - 1$. In other words,

$$\text{class}(P^n(\boldsymbol{C})) \geq 2n-2 \quad (n \geq 3);$$
$$\text{class}(P^n(\boldsymbol{H})) \geq 4n-3 \quad (n \geq 3).$$

It is clearly seen that Theorem 1 improves the estimates (1) and (2) stated above. However, we have to recognize a large gap between our estimate and the upper bound stated in (3), which cannot be filled at present.

Throughout this paper we will assume the differentiability of class C^∞. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [18].

2. The Gauss equation

Let M be a Riemannian manifold and g be the Riemannian metric of M. We denote by R the Riemannian curvature tensor of type $(1,3)$ with respect to g.

For each $x \in M$ we denote by $T_x(M)$ (resp. $T_x^*(M)$) the tangent (resp. cotangent) vector space of M at $x \in M$. Let r be a non-negative integer. We define a quadratic equation with respect to an unknown $\Psi \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ by

$$-g(R(X,Y)Z,W) = \big\langle \Psi(X,Z), \Psi(Y,W)\big\rangle - \big\langle \Psi(X,W), \Psi(Y,Z)\big\rangle, \tag{2.1}$$

where X, Y, Z, $W \in T_x(M)$ and $\langle \ , \ \rangle$ is the standard inner product of $\boldsymbol{R}^r$. We call (2.1) the *Gauss equation in codimension* r at x. It is well-known that for a sufficiently large r the Gauss equation (2.1) in codimension r admits a solution (see Berger [12], Berger-Bryant-Griffiths [13]). On the other hand, in general, for a small r (2.1) does not admit any solution. By $\mathrm{Crank}(M)_x$ we denote the least value of r with which (2.1) admits a solution and call it the *curvature rank* of M at x. It should be noted that $\mathrm{Crank}(M)_x$ is a curvature invariant, i.e., it can be determined only by the curvature R of M at x.

As is well-known, if there is an isometric immersion $\boldsymbol{f}$ of M into $\boldsymbol{R}^D$, then the second fundamental form of $\boldsymbol{f}$ at x satisfies the Gauss equation in codimension $r = D - \dim M$. Therefore, we have

Lemma 2 $\mathrm{class}(M)_x \geq \mathrm{Crank}(M)_x$ *holds for any* $x \in M$.

In the following, we assume that $\Psi \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ is a solution of the Gauss equation in codimension r. Let $X \in T_x(M)$. We define a linear mapping $\Psi_X \colon T_x(M) \longrightarrow \boldsymbol{R}^r$ by $\Psi_X(Y) = \Psi(X,Y)$ $(Y \in T_x(M))$. The kernel of this map Ψ_X is denoted by $\mathbf{Ker}(\Psi_X)$. Then we can easily show the following

Lemma 3 *Let* $X \in T_x(M)$. *Then* $R(\mathbf{Ker}(\Psi_X), \mathbf{Ker}(\Psi_X))X = 0$.

For the proof, see [4]. By this lemma we can get the following estimate for $\mathrm{Crank}(M)_x$: Let $X \in T_x(M)$. By $d(X)$ we denote the maximum value of the dimensions of those subspaces $V \subset T_x(M)$ such that $R(V,V)X = 0$. Then by Lemma 3 it is easily seen that $d(X) \geq \dim \mathbf{Ker}(\Psi_X) \geq \dim M - r$. Set $p_M(x) = \min\{d(X) \mid X \in T_x(M)\}$. Then $p_M(x) \geq \dim M - r$, i.e.,

$r \geq \dim M - p_M(x)$. The number $p_M(x)$ thus defined is also a curvature invariant, which is called the *pseudo-nullity* of M at x. By the above discussion we have

Lemma 4 $\mathrm{Crank}(M)_x \geq \dim M - p_M(x)$.

In the case where M is a Riemannian homogeneous space G/K, the class number, the curvature rank and the pseudo-nullity of G/K are constant everywhere on G/K, which are denoted by class(G/K), Crank(G/K) and $p(G/K)$, respectively. Combining Lemma 4 with Lemma 2, we obtain

Proposition 5 *Let G/K be a Riemannian homogeneous space. Then:*

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K).$$

This is a fundamental tool in our works [5] and [7] to estimate the class numbers of Riemannian symmetric spaces from below.

Now, we show a new type of estimate:

Theorem 6 *Let $\boldsymbol{\Psi} \in S^2 T_x^*(M) \otimes \boldsymbol{R}^r$ be a solution of the Gauss equation in codimension r. Assume that there are tangent vectors $X, Y \in T_x(M)$ and a subspace $\boldsymbol{U}$ of $T_x(M)$ satisfying*
(1) $\boldsymbol{\Psi}(X,Y) = 0$;
(2) $\boldsymbol{U} \supset \mathbf{Ker}(\boldsymbol{\Psi}_X)$ *and* $R(\boldsymbol{U},Y)X = 0$.
Then the following inequality holds:

$$r \geq \dim M + \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y). \tag{2.2}$$

Proof. Let Z be an arbitrary element of $T_x(M)$. Then by the Gauss equation (2.1) it follows that

$$\begin{aligned} 0 &= -g(R(\boldsymbol{U},Y)X,Z) \\ &= \langle \boldsymbol{\Psi}(\boldsymbol{U},X), \boldsymbol{\Psi}(Y,Z)\rangle - \langle \boldsymbol{\Psi}(\boldsymbol{U},Z), \boldsymbol{\Psi}(Y,X)\rangle \\ &= \langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z)\rangle - 0. \end{aligned}$$

Hence, we have $\langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z)\rangle = 0$. This implies that the image of $T_x(M)$ via the map $\boldsymbol{\Psi}_Y$ is included in the orthogonal complement of $\boldsymbol{\Psi}_X(\boldsymbol{U})$. Since $\dim \boldsymbol{\Psi}_X(\boldsymbol{U}) = \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$, we have $\dim \boldsymbol{\Psi}_Y(T_x(M)) \leq r - \dim \boldsymbol{U} + \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$. Moreover, since $\dim \boldsymbol{\Psi}_Y(T_x(M)) = \dim M - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, we immediately obtain the inequality (2.2). □

As is easily seen, the right side of the inequality (2.2) heavily depends on tangent vectors X, Y and Ψ. Accordingly, only by (2.2) we cannot obtain an estimate for $\mathrm{Crank}(M)_x$. In the following sections, by applying Theorem 6 to the complex and quaternion projective spaces we will show Theorem 1.

3. Projective spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$

In this section we make several preparations that are needed in the succeeding sections. Hereafter, G/K denotes one of the following projective spaces:

(1) The complex projective spaces $P^n(\boldsymbol{C}) = SU(n+1)/S(U(n) \times U(1))$ $(n \geq 2)$.

(2) The quaternion projective spaces $P^n(\boldsymbol{H}) = Sp(n+1)/Sp(n) \times Sp(1)$ $(n \geq 2)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair (G, K). Let $(\ ,\ )$ be the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. We define a G-invariant Riemannian metric g of G/K by $g(X,Y) = (X,Y)$ $(X,\ Y \in \mathfrak{m})$, where we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. Since the curvature at o is given by $R(X,Y)Z = -\big[[X,Y],Z\big]$ $(X,\ Y,\ Z \in \mathfrak{m})$ (see Helgason [18]), the Gauss equation (2.1) in codimension r at o can be written as follows:

$$\big(\big[[X,Y],Z\big],W\big) = \big\langle \Psi(X,Z), \Psi(Y,W)\big\rangle - \big\langle \Psi(X,W), \Psi(Y,Z)\big\rangle, \tag{3.1}$$

where $\Psi \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$, X, Y, Z and $W \in \mathfrak{m}$.

Let us take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then, since $\mathrm{rank}(G/K) = 1$, we have $\dim \mathfrak{a} = 1$. We call an element $\lambda \in \mathfrak{a}$ a *restricted root* when the subspaces $\mathfrak{k}(\lambda)$ $(\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)$ $(\subset \mathfrak{m})$ defined below are not non-trivial:

$$\begin{aligned}\mathfrak{k}(\lambda) &= \big\{X \in \mathfrak{k} \,\big|\, \big[H,[H,X]\big] = -(\lambda,H)^2 X, \quad \forall H \in \mathfrak{a}\big\},\\ \mathfrak{m}(\lambda) &= \big\{Y \in \mathfrak{m} \,\big|\, \big[H,[H,Y]\big] = -(\lambda,H)^2 Y, \quad \forall H \in \mathfrak{a}\big\}.\end{aligned}$$

As is known, by use of a non-zero restricted root μ the set of non-zero restricted roots Σ can be written as $\Sigma = \{\pm\mu, \pm 2\mu\}$. Further, we have the following orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)},$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)},$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see §5 of [7]).

For convenience, in the following we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ ($|i| \leq 2$) and $\mathfrak{k}_i = \mathfrak{m}_i = 0$ ($|i| > 2$) for any integer i. Then for i, $j = 0, 1, 2$ we have a formula:

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

We summarize in the following table the basic data for the spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ (see [18], [7]):

G/K	$\dim \mathfrak{m}_1$ $(= \dim \mathfrak{k}_1)$	$\dim \mathfrak{m}_2 (= \dim \mathfrak{k}_2)$
$P^n(\boldsymbol{C})$ $(n \geq 2)$	$2(n-1)$	1
$P^n(\boldsymbol{H})$ $(n \geq 2)$	$4(n-1)$	3

As is known, each non-zero element of $\mathfrak{m}$ is conjugate to a scalar multiple of μ under the action of the isotropy group $\mathrm{Ad}(K)$, because $\mathrm{rank}(P^n(\boldsymbol{C})) = \mathrm{rank}(P^n(\boldsymbol{H})) = 1$. More precisely we can show the following

Proposition 7 *Let $Y_i \in \mathfrak{m}_i$ $(i = 0, 1, 2)$. Assume that $Y_i \neq 0$. Then there is an element $k_i \in K$ such that $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$.*

Proof. In the case $i = 0$ we have only to set $k_0 = e$, where e is the identity element of K.

Now assume $i = 1$ or 2. Set $X_i = [\mu, Y_i]$. Then we have $X_i \in \mathfrak{k}_i$. Further, we have $[X_i, [X_i, \mu]] \in \mathfrak{a}$, because $[X_i, [X_i, \mu]] \in \mathfrak{m}$ and $[\mu, [X_i, [X_i, \mu]]] = [[\mu, X_i], [X_i, \mu]] + [X_i, [\mu, [X_i, \mu]]] = 0$. Since

$$\begin{aligned}(\mu, [X_i, [X_i, \mu]]) &= ([\mu, X_i], [X_i, \mu]) = ([\mu, [\mu, X_i]], X_i) \\ &= -i^2 (\mu, \mu)^2 (X_i, X_i),\end{aligned}$$

we have $[X_i, [X_i, \mu]] = -i^2(\mu, \mu)(X_i, X_i)\mu$. By this equality and the fact $[X_i, \mu] = [[\mu, Y_i], \mu] = i^2(\mu, \mu)^2 Y_i$ we have

$$\begin{aligned}\mathrm{Ad}(\exp(tX_i))\mu &= \cos(i|\mu|\,|X_i|t)\mu \\ &\quad + \frac{1}{i|\mu|\,|X_i|} \sin(i|\mu|\,|X_i|t)\,[X_i, \mu], \quad \forall t \in \boldsymbol{R}.\end{aligned}$$

Define $t_i \in \boldsymbol{R}$ by $i|\mu|\,|X_i|t_i = \pi/2$. Then, by setting $k_i = \exp(t_i X_i) \in K$, we easily get $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$. □

4. Pseudo-abelian subspaces

Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. We say that a subspace V of $\mathfrak{m}$ is *pseudo-abelian* if $[V, V] \subset \mathfrak{k}_0$. It is easily seen that a subspace V of $\mathfrak{m}$ is pseudo-abelian if and only if $[[V, V], \mu] = 0$, because $\operatorname{rank}(G/K) = 1$. We note that the pseudo-nullity $p(G/K)$ coincides with the maximum dimension of pseudo-abelian subspaces in $\mathfrak{m}$ (see [4]). In [7] we have determined the pseudo-nullities for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$: $p(P^n(\boldsymbol{C})) = \max\{n-1, 2\}$ $(n \geq 2)$; $p(P^n(\boldsymbol{H})) = \max\{n-1, 3\}$ $(n \geq 2)$ (see Theorem 5.1 of [7]).

For later use, we here study more detailed facts about pseudo-abelian subspaces in $\mathfrak{m}$ for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$. We first prove

Lemma 8 *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V \cap \mathfrak{m}_i \neq 0$ for some $\mathfrak{m}_i$ $(i = 0, 1, 2)$, then $V \subset \mathfrak{m}_i$.*

Proof. Assume that $V \cap \mathfrak{m}_1 \neq 0$. Take a non-zero element $Y_1^0 \in V \cap \mathfrak{m}_1$. Let $Y = Y_0 + Y_1$ be an arbitrary element of V, where $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$; $Y_1 \in \mathfrak{m}_1$. Then we have $[Y_1^0, Y_0 + Y_1] = [Y_1^0, Y_0] + [Y_1^0, Y_1] \in \mathfrak{k}_0$. However, since $[Y_1^0, Y_0] \in \mathfrak{k}_1$ and $[Y_1^0, Y_1] \in \mathfrak{k}_0 + \mathfrak{k}_2$, we have $[Y_1^0, Y_0] = 0$. Therefore we have $Y_0 = 0$, because $\operatorname{rank}(G/K) = 1$. This proves $V \subset \mathfrak{m}_1$. The other cases $V \cap \mathfrak{a} \neq 0$ and $V \cap \mathfrak{m}_2 \neq 0$ are similarly dealt with. □

We say that a pseudo-abelian subspace V is *categorical* if it can be decomposed into a direct sum $V = V \cap \mathfrak{a} + V \cap \mathfrak{m}_1 + V \cap \mathfrak{m}_2$. By Lemma 8 we immediately have

Proposition 9 *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If V is categorical and $V \neq 0$, then V is contained in one of $\mathfrak{a}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$.*

By this proposition, we can easily estimate $\dim V$ for a categorical pseudo-abelian subspace V in $\mathfrak{m}$: $\dim V \leq 1$ if $V \subset \mathfrak{a}$; $\dim V \leq \dim \mathfrak{m}_2$ if $V \subset \mathfrak{m}_2$. In the case $V \subset \mathfrak{m}_1$ we proved in [7] $\dim V \leq n-1$ (see Theorem 3.2 of [7]). For completeness, we review this proof and show an additional property of $V \subset \mathfrak{m}_1$.

Let $E(\mathfrak{m}_1)$ denote the space of all linear endomorphisms of $\mathfrak{m}_1$. Let $X \in \mathfrak{k}_2$. We define an element $X^\dagger \in E(\mathfrak{m}_1)$ by

$$X^\dagger(Y) = [X, Y], \quad Y \in \mathfrak{m}_1.$$

(Note that $[\mathfrak{k}_2, \mathfrak{m}_1] \subset \mathfrak{m}_1$.) It is easy to see that $X^\dagger$ is skew-symmetric with respect to the inner product $(\ ,\ )$. We denote by $\mathfrak{k}_2^\dagger$ the subspace of $E(\mathfrak{m}_1)$

consisting of all $X^\dagger$ ($X \in \mathfrak{k}_2$). Set $\mathfrak{F}^\dagger = \boldsymbol{R}\mathbf{1}_{\mathfrak{m}_1} + \mathfrak{k}_2^\dagger$ ($\subset E(\mathfrak{m}_1)$), where $\mathbf{1}_{\mathfrak{m}_1}$ denotes the identity mapping of $\mathfrak{m}_1$. We have proved in [7] (Theorem 3.5) the following

Proposition 10 *Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. Then, $\mathfrak{F}^\dagger$ forms a subalgebra of $E(\mathfrak{m}_1)$, i.e., $\mathfrak{F}^\dagger$ is closed under addition and multiplication of $E(\mathfrak{m}_1)$. Further, in the case $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$, $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{C}$ of complex numbers and in the case $G/K = P^n(\boldsymbol{H})$ $(n \geq 2)$, $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{H}$ of quaternion numbers.*

We now set $f = \dim_{\boldsymbol{R}} \mathfrak{F}^\dagger$, i.e., $f = 2$ if $G/K = P^n(\boldsymbol{C})$; $f = 4$ if $G/K = P^n(\boldsymbol{H})$. By the definition we have $\dim \mathfrak{m}_2 = f - 1$, $\dim \mathfrak{m}_1 = (n-1)f$ and $\dim G/K = \dim \mathfrak{m} = nf$. As seen in Proposition 10, $\mathfrak{m}_1$ can be regarded as a vector space over the field $\mathfrak{F}^\dagger$. For an element $Y_1 \in \mathfrak{m}_1$ we denote by $\mathfrak{F}^\dagger(Y_1)$ the subspace of $\mathfrak{m}_1$ spanned by Y_1 over $\mathfrak{F}^\dagger$. Then we easily have $\mathfrak{F}^\dagger(\mathfrak{F}^\dagger(Y_1)) = \mathfrak{F}^\dagger(Y_1)$ and $\dim_{\boldsymbol{R}} \mathfrak{F}^\dagger(Y_1) = f$ if $Y_1 \neq 0$.

Lemma 11 *Let $Y_1, Y_1' \in \mathfrak{m}_1$. Then $[Y_1, Y_1'] \in \mathfrak{k}_0$ if and only if $(\mathfrak{k}_2^\dagger(Y_1), Y_1') = 0$. Accordingly, a subspace $V \subset \mathfrak{m}_1$ is pseudo-abelian if and only if $(\mathfrak{k}_2^\dagger(V), V) = 0$.*

Proof. Since $[Y_1, Y_1'] \in \mathfrak{k}_0 + \mathfrak{k}_2$, $[Y_1, Y_1'] \in \mathfrak{k}_0$ holds if and only if $([Y_1, Y_1'], \mathfrak{k}_2) = 0$. Clearly, the last equality is equivalent to $(\mathfrak{k}_2^\dagger(Y_1), Y_1') = 0$. □

Utilizing the above lemma, we can show the following

Proposition 12 *Let V be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that $V \subset \mathfrak{m}_1$. Then:*

(1) $\dim \mathfrak{F}^\dagger(V) = f \dim V$. *Accordingly,* $\dim V \leq n - 1$.

(2) *Let $\xi \in V$ $(\xi \neq 0)$. Then there is a subspace U of $\mathfrak{m}_1$ satisfying $U \supset V$, $[\xi, U] \subset \mathfrak{k}_0$ and* $\dim U = (n-2)f + 1$.

Proof. Let $\{Y_1^1, \ldots, Y_1^s\}$ $(s = \dim V)$ be an orthonormal basis of V. Let i, j be integers such that $1 \leq i \neq j \leq s$. Then, since $(\mathfrak{k}_2^\dagger(Y_1^i), Y_1^j) = (Y_1^i, \mathfrak{k}_2^\dagger(Y_1^j)) = 0$ (see Lemma 11) and since $(\mathfrak{k}_2^\dagger)^2 \subset \mathfrak{F}^\dagger$, we have

$$\begin{aligned}(\mathfrak{F}^\dagger(Y_1^i), \mathfrak{F}^\dagger(Y_1^j)) &= (\boldsymbol{R}Y_1^i + \mathfrak{k}_2^\dagger(Y_1^i), \boldsymbol{R}Y_1^j + \mathfrak{k}_2^\dagger(Y_1^j)) \\ &\subset (Y_1^i, (\mathfrak{k}_2^\dagger)^2(Y_1^j)) = 0.\end{aligned}$$

This proves $\mathfrak{F}^\dagger(V) = \sum_{1 \leq i \leq s} \mathfrak{F}^\dagger(Y_1^i)$ (orthogonal direct sum) and hence

$\dim_{\boldsymbol{R}} \mathfrak{F}^{\dagger}(V) = sf$. Therefore we have $s \leq n-1$, because $\dim \mathfrak{m}_1 = (n-1)f$.

Next we prove (2). Since V is pseudo-abelian and $\xi \in V$, we have $(\mathfrak{k}_2^{\dagger}(\xi), V) = 0$. Let U be the orthogonal complement of $\mathfrak{k}_2^{\dagger}(\xi)$ in $\mathfrak{m}_1$. Then U satisfies $U \supset V$ and $[\xi, U] \subset \mathfrak{k}_0$ (see Lemma 11). Moreover, since $\dim \mathfrak{k}_2^{\dagger}(\xi) = f-1$ and $\dim \mathfrak{m}_1 = (n-1)f$, we immediately obtain the equality $\dim U = (n-2)f+1$. □

Finally, we refer to non-categorical pseudo-abelian subspaces. Let V be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that V is not categorical, i.e., V cannot be represented by a direct sum of subspaces $V \cap \mathfrak{a}$, $V \cap \mathfrak{m}_1$ and $V \cap \mathfrak{m}_2$. Then it is clear that $V \not\subset \mathfrak{a}$, $V \not\subset \mathfrak{m}_1$ and $V \not\subset \mathfrak{m}_2$. In view of Lemma 8, we know that $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$. Apparently, this condition is sufficient for a pseudo-abelian subspace V to be non-categorical. Hence we have

Proposition 13 *Let V be a pseudo-abelian subspace of $\mathfrak{m}$ such that $V \neq 0$.*
(1) *V is non-categorical if and only if $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$.*
(2) *If V is non-categorical, then $\dim V \leq 2$.*

For the proof of (2), see Proposition 5.2 (1) of [7].

5. Proof of Theorem 1

Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. In the following we assume that the Gauss equation in codimension r admits a solution $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$. We first prove

Lemma 14 *Let $X \in \mathfrak{m}$ $(X \neq 0)$ and let k be an element of K satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$.*

Proof. By Lemma 3 we have $\big[[\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathbf{Ker}(\boldsymbol{\Psi}_X)], X\big] = 0$. Applying $\mathrm{Ad}(k^{-1})$ to this equality, we have $\big[[\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)], \mu\big] = 0$. This proves that $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. □

Let $X \in \mathfrak{m}$ $(X \neq 0)$. If $\mathbf{Ker}(\boldsymbol{\Psi}_X) = 0$, then we say X is *of type P_{inj}*. Now assume $\mathbf{Ker}(\boldsymbol{\Psi}_X) \neq 0$. Let $k \in K$ be an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. As is shown in Lemma 14, $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. If $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is categorical and is contained in $\mathfrak{m}_i$ $(i = 0, 1, 2)$, then we say X is *of type P_i* $(i = 0, 1, 2)$. We ălso say X is *of* *type* P_{non} if $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\Psi_X)$ is non-categorical, i.e., $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\Psi_X)\cap \mathfrak{m}_i = 0$ $(i = 0, 1, 2)$.

The following lemma asserts that the type of X does not depend on the choice of $k \in K$ satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$.

Lemma 15 *Let $X \in \mathfrak{m}$ $(X \neq 0)$. Let $i = 0, 1$ or 2 and let k_j $(j = 1, 2)$ be elements of K satisfying $\mathrm{Ad}(k_j)\mu \in \boldsymbol{R}X$. Then:*
(1) $\mathrm{Ad}(k_1^{-1})\,\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$ *if and only if* $\mathrm{Ad}(k_2^{-1})\,\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$.
(2) $\mathrm{Ad}(k_1^{-1})\,\mathbf{Ker}(\Psi_X)\cap \mathfrak{m}_i = 0$ *if and only if* $\mathrm{Ad}(k_2^{-1})\,\mathbf{Ker}(\Psi_X)\cap \mathfrak{m}_i = 0$.

Proof. Set $k' = k_1^{-1}k_2 \in K$. By the assumption we have $\mathrm{Ad}(k')\mu = \pm\mu$. Therefore it is easily seen that $\mathrm{Ad}(k')\mathfrak{m}_i = \mathfrak{m}_i$ for any $i = 0, 1, 2$. Since $\mathrm{Ad}(k')\,\mathrm{Ad}(k_2^{-1}) = \mathrm{Ad}(k_1^{-1})$, the lemma follows immediately. □

Let us denote by $\mathfrak{p}_i$ $(i = 0, 1, 2, non, inj)$ the subset of $\mathfrak{m}$ consisting of all elements of type P_i. Then it is clear that

$$\mathfrak{m} \setminus \{0\} = \mathfrak{p}_0 \cup \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_{non} \cup \mathfrak{p}_{inj} \quad \text{(disjoint union).} \tag{5.1}$$

Proposition 16 *Let $X, Y \in \mathfrak{m}$ $(X \neq 0, Y \neq 0)$. Assume that $\Psi(X, Y) = 0$. Then $X \in \mathfrak{p}_i$ if and only if $Y \in \mathfrak{p}_i$ $(i = 0, 1, 2, non)$.*

Proof. We note that under the assumption $\Psi(X, Y) = 0$ we have $X \notin \mathfrak{p}_{inj}$ and $Y \notin \mathfrak{p}_{inj}$, because $Y \in \mathbf{Ker}(\Psi_X)$ and $X \in \mathbf{Ker}(\Psi_Y)$.

First consider the case $X \in \mathfrak{p}_i$ $(i = 0, 1, 2)$. Let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})Y \in \mathfrak{m}_i$, because $\mathrm{Ad}(k^{-1})Y \in \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\Psi_X) \subset \mathfrak{m}_i$. Take an element $k' \in K$ satisfying $\mathrm{Ad}(k'^{\pm 1})\mu \in \boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y$ and set $k'' = kk'$ (see Proposition 7). Then we have $\mathrm{Ad}(k'')\mu = \mathrm{Ad}(k)\,\mathrm{Ad}(k')\mu \in \mathrm{Ad}(k)\boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y = \boldsymbol{R}Y$ and $\mathrm{Ad}(k''^{-1})X = \mathrm{Ad}(k'^{-1})\,\mathrm{Ad}(k^{-1})X \in \boldsymbol{R}\,\mathrm{Ad}(k'^{-1})\mu = \boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y \subset \mathfrak{m}_i$. Since $X \in \mathbf{Ker}(\Psi_Y)$, it follows that $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\Psi_Y) \cap \mathfrak{m}_i \neq 0$. Hence $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\Psi_Y)$ is categorical (see Proposition 13) and $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\Psi_Y) \subset \mathfrak{m}_i$ (see Proposition 9). This means $Y \in \mathfrak{p}_i$. The converse can be proved in the same manner.

By these arguments we know that $X \in \mathfrak{p}_{non}$ if and only if $Y \in \mathfrak{p}_{non}$. □

Lemma 17 *Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. Then:*
(1) $\mathfrak{p}_0 = \emptyset$.
(2) *Let $X \in \mathfrak{m}$ $(X \neq 0)$. Then:*

$$\dim \mathbf{Ker}(\Psi_X) \leq \begin{cases} n-1, & \text{if } X \in \mathfrak{p}_1; \\ f-1, & \text{if } X \in \mathfrak{p}_2; \\ 2, & \text{if } X \in \mathfrak{p}_{non}. \end{cases} \tag{5.2}$$

Proof. Suppose that $\mathfrak{p}_0 \neq \emptyset$. Let $X \in \mathfrak{p}_0$ and let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\Psi_X) \subset \mathfrak{a} = \boldsymbol{R}\mu$. Hence we have $\mathbf{Ker}(\Psi_X) = \boldsymbol{R}\,\mathrm{Ad}(k)\mu = \boldsymbol{R}X$, i.e., $\Psi(X,X) = 0$. Let $Y \in \mathfrak{m}$ such that $Y \notin \boldsymbol{R}X$. By (3.1) we have

$$\begin{aligned}\big(\big[[X,Y],X\big],Y\big) &= \big\langle \Psi(X,X), \Psi(Y,Y)\big\rangle - \big\langle \Psi(X,Y), \Psi(Y,X)\big\rangle \\ &= -\big\langle \Psi_X(Y), \Psi_X(Y)\big\rangle.\end{aligned}$$

Since G/K is of positive curvature, the left side of the above equality is ≥ 0. Therefore we have $\Psi_X(Y) = 0$, which contradicts $Y \notin \boldsymbol{R}X$. Thus we have, $\mathfrak{p}_0 = \emptyset$.

The assertion (2) follows from Propositions 12, Proposition 13, $\dim \mathfrak{m}_2 = f-1$ and the discussions in the previous section. □

Proposition 18 *Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. Then:*
(1) $\mathfrak{p}_{inj} = \emptyset$ *if* $r \leq nf-1$;
(2) $\mathfrak{p}_1 = \emptyset$ *if* $r \leq 2(n-1)(f-1)$;
(3) $\mathfrak{p}_2 = \emptyset$ *if* $r \leq (n-1)f$;
(4) $\mathfrak{p}_{non} = \emptyset$ *if* $r \leq nf-3$.

Proof. We first note that $\dim \mathbf{Ker}(\Psi_X) \geq \dim G/K - r = nf - r$ holds for any $X \in \mathfrak{m}$. By this fact we can easily prove (1), (3) and (4). In fact, if $r \leq nf-1$, then it is clear that $\mathbf{Ker}(\Psi_X) \neq 0$ for any $X \in \mathfrak{m}$. Hence $X \notin \mathfrak{p}_{inj}$, which implies $\mathfrak{p}_{inj} = \emptyset$. Similarly, if $r \leq (n-1)f$ (resp. $r \leq nf-3$), then $\dim \mathbf{Ker}(\Psi_X) \geq f$ (resp. $\dim \mathbf{Ker}(\Psi_X) \geq 3$) holds for any $X \in \mathfrak{m}$ and hence $\mathfrak{p}_2 = \emptyset$ (resp. $\mathfrak{p}_{non} = \emptyset$) (see Lemma 17).

Next we prove (2). Suppose that $\mathfrak{p}_1 \neq \emptyset$. Let $X \in \mathfrak{p}_1$. Take $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$ and set $V = \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\Psi_X)$. Then V is a pseudo-abelian subspace such that $V \subset \mathfrak{m}_1$. Consequently, by Lemma 17 we have $\dim V \leq n-1$.

Now let us take a non-zero element $\xi \in V$ and a subspace $U \subset \mathfrak{m}_1$ satisfying $U \supset V$, $[\xi, U] \subset \mathfrak{k}_0$ and $\dim U = (n-2)f+1$ (see Proposition 12 (2)). Put $Y = \mathrm{Ad}(k)\xi$ $(\in \mathbf{Ker}(\Psi_X))$ and $\boldsymbol{U} = \mathrm{Ad}(k)U$ $(\subset \mathfrak{m})$. Then we have $\Psi(X,Y) = 0$ and $\boldsymbol{U} \supset \mathbf{Ker}(\Psi_X)$. Moreover, we have $\big[[\boldsymbol{U},Y],X\big] = 0$, because $\big[[\boldsymbol{U},Y],X\big] = \mathrm{Ad}(k)\big[[U,\xi],\mu\big] = 0$. Therefore, by Theorem 6 we have the following inequality:

$$r \geq nf + (n-2)f + 1 - \dim \mathbf{Ker}(\Psi_X) - \dim \mathbf{Ker}(\Psi_Y).$$

Since X and $Y \in \mathfrak{p}_1$ (see Proposition 16), it follows that $\dim \mathbf{Ker}(\Psi_X) \leq n-1$ and $\dim \mathbf{Ker}(\Psi_Y) \leq n-1$ (see Lemma 17). Consequently, we have $r \geq 2(n-1)(f-1)+1$, which proves (2). □

We are now in a position to prove Theorem 1. If there is a solution Ψ of the Gauss equation in codimension r, then at least one of the sets $\mathfrak{p}_{inj}$, $\mathfrak{p}_0$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_{non}$ is not empty (see (5.1)). Therefore, in view of Lemma 17 (1) and Proposition 18, we have $r \geq 1+\min\{nf-1, 2(n-1)(f-1), (n-1)f, nf-3\}$. Accordingly, we have $r \geq 2n-2$ if $G/K = P^n(\boldsymbol{C})$ and $r \geq 4n-3$ if $G/K = P^n(\boldsymbol{H})$. Hence, $\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-2$ and $\mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-3$. This, together with Lemma 2, shows Theorem 1. □

Remark 1 The proof of Theorem 1 stated above is effective in the case $n = 2$. We thereby have $\mathrm{Crank}(P^2(\boldsymbol{C})) \geq 2$ and $\mathrm{Crank}(P^2(\boldsymbol{H})) \geq 5$. However, for the spaces $P^2(\boldsymbol{C})$ and $P^2(\boldsymbol{H})$, we have already known the best results: $\mathrm{Crank}(P^2(\boldsymbol{C})) = 3$ (see [1]) and $\mathrm{class}(P^2(\boldsymbol{H})) = \mathrm{Crank}(P^2(\boldsymbol{H})) = 6$ (see [8]).

As for the class number of $P^2(\boldsymbol{C})$ we have $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or 4 (see Lemma 2 and Introduction). It is still an open question whether $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or not (cf. [20]).

Remark 2 Consider the following two cases:
(1) $G/K = P^n(\boldsymbol{C})$ $(n \geq 3)$ and $r = 2n-2$;
(2) $G/K = P^n(\boldsymbol{H})$ $(n \geq 3)$ and $r = 4n-3$.
If there is a solution Ψ of the Gauss equation in codimension r, then it is shown by Lemma 17 (1) and Proposition 18 that Ψ must satisfy the following condition:
Case (1) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m} \setminus \{0\} = \mathfrak{p}_{non}$;
Case (2) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_{non} = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m} \setminus \{0\} = \mathfrak{p}_2$.
We conjecture that in both cases (1) and (2) there are no such solutions Ψ. In other words:

$$\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-1 \quad (n \geq 3);$$
$$\mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-2 \quad (n \geq 3).$$

If this is true, then we obtain an improvement of Theorem 1:

$$\text{class}(P^n(\boldsymbol{C})) \geq 2n-1 \quad (n \geq 3);$$
$$\text{class}(P^n(\boldsymbol{H})) \geq 4n-2 \quad (n \geq 3).$$

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Obstructions in local isometric imbeddings of Riemannian manifolds

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概要

It is well known that any Riemannian manifold M can be isometrically imbedded into a sufficiently high dimensional Euclidean space. If M can be realized in a relatively low dimensional Euclidean space (for example, as a hypersurface), then the curvature of M must satisfy several conditions. These restrictions on curvature are the consequence of the integrability condition of a system of differential equations expressing local isometric imbeddings. And these conditions may be regarded as obstructions to the existence of (local) isometric imbeddings of M into a low dimensional Euclidean space. To find such conditions on the curvature in an explicit form is one important problem in considering isometric imbeddings of Riemannian manifolds. In this talk, I will review several obstructions obtained in collaboration with Professor E. Kaneda, and gave some explicit applications mainly when M is a Riemannian symmetric space.

§ 0. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. It is well known that M can be locally or globally isometrically imbedded into a sufficiently high dimensional Euclidean space $\mathbf{R}^N$.

For example, M^n can be locally isometrically imbedded into $\mathbf{R}^{\frac{1}{2}n(n+1)}$ in the real analytic category. This is a result of Janet-Cartan (1926, 1927). And in the global case, M^n can be realized in $\mathbf{R}^{\frac{1}{2}n(n+5)}$ if $n \geq 5$ (Günther, 1990). This result is the best (least dimensional) result known at present. Several mathematicians tried to improve the dimension of the ambient space $\mathbf{R}^N$. (Nash first proved the global existence theorem. And after Nash, Greene, Gromov, Günther successively improved the dimension of the ambient space $\mathbf{R}^N$.)

In the C^∞-category local existence theorem was also established. It is known that every n-dimensional Riemannian manifold of class C^∞ admits a C^∞-local isometric imbedding into $\mathbf{R}^{\frac{1}{2}n(n+1)+n}$. And it is one fundamental problem whether the dimension of the ambient space can be reduced to $\frac{1}{2}n(n+1)$. Especially 2 and 3 dimensional cases were studied deeply by several mathematicians, and perhaps these are the themes of Professor Maeda and Professor Han's talk.

But E. Kaneda and I continued to work on this problem from different viewpoint. In particular, we tried to find obstructions to the existence of local isometric imbeddings and its application to the explicit Riemannian manifolds from algebraic or representation theoretic viewpoint. In this talk, I will briefly review some results on this subject.

The CONTENT of this talk

§ 1. Differential equation of local isometric imbeddings

In terms of local coordinate $(x_1, \cdots, x_n)$ of M, the differential equation of isometric imbedding can be expressed as

$$(\#) \qquad g_{ij} = \sum_{a=1}^{N} \frac{\partial f^a}{\partial x_i} \frac{\partial f^a}{\partial x_j} \qquad i, j = 1, \cdots, n,$$

where $\boldsymbol{f} = (f^1, \cdots, f^N) : M \longrightarrow \mathbf{R}^N$ is a differential mapping from M to $\mathbf{R}^N$.

Problem. For given $\{g_{ij}\}$ and N, determine whether there is a family of functions $\boldsymbol{f} = (f^1, \cdots, f^N)$ such that $(\#)$ holds.

There are several equivalent formulation of isometric imbeddings. And for our purpose, it is convenient to express this differential equation in terms of the language "covariant derivative" ∇ associated with the Riemannian metric. (Covariant derivative is a quite useful language in describing the differential equation of isometric imbeddings of Riemannian manifolds.)

First we define the covariant derivative acting on the space of tensor field of type $(0, k)$ on M. We denote by $\mathfrak{X}(M)$ the set of vector fields on M and $C^\infty(M)$ be the set of C^∞ functions on M. Since there is given a Riemannian metric g on M, it is well known that there exists uniquely a differential operator $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ ($\nabla_X Y \in \mathfrak{X}(M)$ for X, $Y \in \mathfrak{X}(M)$) such that

$$(\#\,\#) \qquad \begin{aligned} &X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X,Y,Z \in \mathfrak{X}(M), \\ &\nabla_X Y - \nabla_Y X = [X,Y], \quad X,Y \in \mathfrak{X}(M), \end{aligned}$$

where $[X,Y]$ is the bracket of two vector fields. (Vector field is considered as a differential operator acting on $C^\infty(M)$. $[X,Y]$ is defined by $[X,Y]f = X(Yf) - Y(Xf)$ for $f \in C^\infty(M)$.)

Now let T be a tensor field on M of type $(0,k)$, i.e., $T : \overbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}^{k} \longrightarrow C^\infty(M)$, which is $C^\infty(M)$-multilinear. We define a new tensor field ∇T of type $(0,k+1)$ by

$$\nabla T(X, X_1, \cdots, X_k) = X(T(X_1, \cdots, X_k)) - \sum_{i=1}^{k} T(X_1, \cdots, \nabla_X X_i, \cdots, X_k).$$

In case $k=0$, i.e., T is a function f, we put $\nabla f = df$, i.e., $(\nabla f)(X) = Xf$. Note that the condition (##) can be expressed as $\nabla g = 0$. We define $\overbrace{\nabla \cdots \nabla}^{k} T$ inductively by $\nabla\nabla T = \nabla(\nabla T)$, $\nabla\nabla\nabla T = \nabla(\nabla\nabla T)$, $\cdots$.

Since $\nabla \cdots \nabla T$ is a tensor field, we can substitute tangent vector x, y, $\cdots \in T_p(M)$ instead of tangent vector fields X, Y, $\cdots$. And in the following we express $\nabla_x \nabla_y \nabla_z T = (\nabla\nabla\nabla T)(x,y,z) \in \mathbf{R}$, etc.

When the tensor field T is a function f, the covariant derivatives satisfy the following **integrability conditions**:

$$\begin{aligned} &\nabla_x \nabla_y f = \nabla_y \nabla_x f, \\ &\nabla_x \nabla_y \nabla_z f = \nabla_x \nabla_z \nabla_y f, \\ &\nabla_x \nabla_y \nabla_z f = \nabla_y \nabla_x \nabla_z f - \nabla_{R(x,y)z} f, \end{aligned}$$

where $x, y, z \in T_p(M)$. It should be remarked that the operator ∇ is not commutative. The difference can be expressed in terms of the curvature R. The last equality is usually called the **Ricci formula**. In terms of the Christoffel symbol Γ^k_{ij}, we have

$$\nabla_{X_i} \nabla_{X_j} f = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x_k},$$

where $X_i = \frac{\partial}{\partial x_i}$. Note that we have

$$
\begin{aligned}
(\nabla\nabla f)(X,Y) &= X(Yf) - (\nabla_X Y)f,\\
(\nabla\nabla\nabla f)(X,Y,Z) &= X(Y(Zf)) - X((\nabla_Y Z)f) - Y((\nabla_X Z)f)\\
&\quad - (\nabla_X Y)(Zf) + (\nabla_{\nabla_X Y} Z)f + (\nabla_Y(\nabla_X Z))f,
\end{aligned}
$$

where X, Y, $Z \in \mathfrak{X}(M)$.

The covariant derivative of the map $\boldsymbol{f} = (\cdots, f^a, \cdots) : M \longrightarrow \mathbf{R}^N$ is defined by

$$
\overbrace{\nabla_{x_1}\cdots\nabla_{x_k}}^{k}\boldsymbol{f} = (\cdots, \overbrace{\nabla_{x_1}\cdots\nabla_{x_k}}^{k} f^a, \cdots) \in \mathbf{R}^N, \quad p \in M,\ x_1, \cdots, x_n \in T_p(M).
$$

If $\boldsymbol{f}$ is an imbedding, then the vectors $\nabla_x \boldsymbol{f}$ span the vector space $(\boldsymbol{f}_*)_p T_p(M) \subset \mathbf{R}^N$, where $(\boldsymbol{f}_*)_p : T_p(M) \longrightarrow \mathbf{R}^N$ is the differential of $\boldsymbol{f}$ at $p \in M$. In terms of this language, the differential equation of isometric imbeddings (#) is equivalent to

$$
\langle \nabla_x \boldsymbol{f}, \nabla_y \boldsymbol{f} \rangle = g(x,y).
$$

Here $\langle\ ,\ \rangle$ implies the inner product of the ambient vector space $\mathbf{R}^N$. The sequence $(p, \boldsymbol{f}(p), (\nabla \boldsymbol{f})_p, (\nabla\nabla \boldsymbol{f})_p, \cdots)$ may be considered as the jet of $\boldsymbol{f}$. And this sequence must satisfy some conditions. By using the property $\nabla g = 0$ and the Ricci formula, we obtain successively the following formulas:

(1) $\langle \nabla_x \boldsymbol{f}, \nabla_y \boldsymbol{f} \rangle = g(x,y)$,

(2) $\langle \nabla_z\nabla_x \boldsymbol{f}, \nabla_y \boldsymbol{f} \rangle = 0 \longrightarrow \langle \nabla_y\nabla_x \boldsymbol{f}, \nabla_w \boldsymbol{f} \rangle = 0$

(3) $\langle \nabla_z\nabla_y\nabla_x \boldsymbol{f}, \nabla_w \boldsymbol{f} \rangle + \langle \nabla_y\nabla_x \boldsymbol{f}, \nabla_z\nabla_w \boldsymbol{f} \rangle = 0$,

(4) $\langle \nabla_y\nabla_x \boldsymbol{f}, \nabla_z\nabla_w \boldsymbol{f} \rangle - \langle \nabla_z\nabla_x \boldsymbol{f}, \nabla_y\nabla_w \boldsymbol{f} \rangle = -g(R(y,z)x, w)$,

(5) $\langle \nabla_v\nabla_y\nabla_x \boldsymbol{f}, \nabla_z\nabla_w \boldsymbol{f} \rangle + \langle \nabla_y\nabla_x \boldsymbol{f}, \nabla_v\nabla_z\nabla_w \boldsymbol{f} \rangle$
$\quad -\langle \nabla_v\nabla_z\nabla_x \boldsymbol{f}, \nabla_y\nabla_w \boldsymbol{f} \rangle - \langle \nabla_z\nabla_x \boldsymbol{f}, \nabla_v\nabla_y\nabla_w \boldsymbol{f} \rangle = -g(\nabla_v R(y,z)x, w)$.

$$\vdots$$

$$\vdots$$

If $\boldsymbol{f}$ is an isometric imbedding, $\nabla \boldsymbol{f}$, $\nabla\nabla \boldsymbol{f}$, $\cdots$ must satisfy these conditions. The condition (2) implies that the vector $\nabla_x\nabla_y \boldsymbol{f}$ belongs to the normal space $T_p(M)^{\perp}$. From the integrability condition, we know that this is symmetric with respect to x and y. We call $\nabla\nabla \boldsymbol{f}$ the **second fundamental form** of $\boldsymbol{f}$ at the point p. The second fundamental form satisfies the equality (4). We call this equation the **Gauss equation**. Thus the Gauss equation naturally appears as a part of the integrability condition of the original differential equation (#). (This formulation is due to N. Tanaka.)

§ 2. Gauss equation and curvature

Now we turn to the obstruction to the existence of local isometric imbeddings.

In the following, we express $\nabla\nabla \boldsymbol{f}$ simply as $\alpha \in S^2(T_p^*(M))\otimes\mathbf{R}^r$, where we put $N = n+r$ ($n = \dim M$). The number r is called the codimension of the imbedding $\boldsymbol{f}$. And we express the normal space $T_p(M)^\perp$ simply as $\mathbf{R}^r$. We recall the Gauss equation

$$(4) \qquad \langle\alpha(z,x),\alpha(v,y)\rangle - \langle\alpha(v,x),\alpha(z,y)\rangle = -g(R(z,v)x,y).$$

The right hand side of the equality is expressed by the curvature of M, which is an intrinsic quantity of the Riemannian manifold. This mean that the curvature R can be expressed in the above form for some $\mathbf{R}^r$-valued symmetric bilinear form α, if M can be (locally) isometrically imbedded into $\mathbf{R}^{n+r}$.

Of course, not all curvature type tensors R can be expressed in this form, if the value of codimension r is sufficiently small. And this is our start point of study.

We denote by $K_p(M)$ the set of curvature like tensors of $T_p(M)$:

$$K_p(M) = \{R \in \wedge^2 T_p^*(M)\otimes T_p^*(M)\otimes T_p(M) \mid g(R(x,y)z,w) + g(R(x,y)w,z) = 0, \quad \mathfrak{S}\, R(x,y)z = 0\}.$$

(The last condition is called the first Bianchi identity.) And we denote by $K_p^r(M)$ the set of curvature like tensors R which can be expressed as

$$-g(R(x,y)z,w) = \langle\alpha(x,z),\alpha(y,w)\rangle - \langle\alpha(x,w),\alpha(y,z)\rangle$$

for some $\mathbf{R}^r$-valued symmetric bilinear form α. We clearly have the following increasing sequence of the set $K_p^r(M)$.

$$\{0\} = K_p^0(M) \subset K_p^1(M) \subset K_p^2(M) \subset \cdots\cdots \subset K_p(M),$$

and we know that there is a positive integer r_0 depending on n such that the equality $K_p^{r_0}(M) = K_p(M)$ holds. (It is known that it suffices to put $r_0 = \frac{1}{2}(n-1)(n-2)+2$. But this value is not in general the least value.) This means that any curvature like tensor admits a solution of the Gauss equation in codimension r_0.

If we can show that $R \notin K_p^r(M)$ by some method, we know that any open neighborhood of $p \in M$ cannot isometrically imbedded into $\mathbf{R}^{n+r}$. And so, it is a fundamental problem whether a given curvature like tensor R belongs to $K_p^r(M)$ or not. Especially it is important to determined the least positive integer r such that $R \in K_p^r(M)$. (If we express the curvature R as a linear endomorphism $\wedge^2 T_p(M) \longrightarrow \wedge^2 T_p(M)$, then the Gauss equation is expressed in the form $R = \sum_{i=1}^r L_i \wedge L_i$, where the linear map $L_i : T_p(M) \longrightarrow T_p(M)$ is defined by $g(L_i(X),Y) = \langle\alpha(X,Y),\xi_i\rangle$. ($\xi_i$ is an orthonormal basis of $T_p(M)^\perp$.) By this formulation the least integer r_0 such that $R = \sum_{i=1}^{r_0} L_i \wedge L_i$ may be considered as a sort of "rank" of the curvature.) But in general this is a quite difficult algebraic problem, and unfortunately we only know a partial result on this question at present. Explicit characterization of the image of algebraic map is in general a quite difficult algebraic problem.

Example.

- $\mathbf{R}^n$: $R \in K_p^0(M)$,
- S^n : $R \notin K_p^0(M)$ and $R \in K_p^1(M)$,
- H^n : $R \notin K_p^{n-2}(M)$ and $R \in K_p^{n-1}(M)$.

The last example follows from the classical theorem of Ôtsuki (J. Math. Soc. Japan **6** (1954)).

Theorem 1 (Ôtsuki). *If M is an n-dimensional space of negative curvature, then $R \notin K_p^{n-2}(M)$.*

Note that the curvature of the space of constant curvature k is given by

$$R(x,y)z = k(g(y,z)x - g(x,z)y).$$

In the following, we mainly talk on this subject.

§ 3. Pseudo-nullity: An obstruction

We found several types of necessary conditions in order that $R \in K_p^r(M)$.

- Agaoka-Kaneda (Tôhoku Math. J. **36** (1984)),
- Agaoka (Hokkaido Math. J. **14** (1985), obstruction for the case $M^4 \subset \mathbf{R}^6$),
- Agaoka-Kaneda (Hiroshima Math. J. **24** (1994)),
- H.J.Rivertz (Thesis, (1999), obstruction for the case $M^3 \subset \mathbf{R}^4$, $M^5 \subset \mathbf{R}^7$).

We here explain one obstruction "pseudo-nullity", which is perhaps the strongest condition known at present. This obstruction was first appeared in Hiroshima Math. J. **24** (1994).

Let α be a solution of the Gauss equation of M in codimension r:

$$\alpha : T_p(M) \times T_p(M) \longrightarrow \mathbf{R}^r.$$

We express the normal space $T_p(M)^\perp$ simply as $\mathbf{R}^r$. For $x \in T_p(M)$ we define a linear map $\alpha_x : T_p(M) \longrightarrow \mathbf{R}^r$ by $\alpha_x(y) = \alpha(x,y)$. For convenience, we assume $n \geq r$ for some time. Then the space $\operatorname{Ker} \alpha_x \subset T_p(M)$ possesses the following property:

(*) If $y, z \in \operatorname{Ker} \alpha_x$, then $R(y,z)x = 0$.

This fact follows immediately from the Gauss equation

$$-g(R(y,z)x, w) = \langle \alpha(x,y), \alpha(z,w) \rangle - \langle \alpha(x,z), \alpha(y,w) \rangle.$$

Existence of such subspace imposes a strong condition on the curvature R. Holding this fact in mind, we introduce the following concept: Let $x \in T_p(M)$, and set

$$d(x) = \max_{W_x} \dim W_x,$$

where W_x moves in the set of subspaces of $T_p(M)$ satisfying the condition

(**) If $y, z \in W_x$, then $R(y,z)x = 0$.

If $R \in K$ admits a solution of the Gauss equation in codimension r, we have clearly the inequality $d(x) \geq n - r$ for any $x \in T_p(M)$, i.e., $r \geq n - d(x)$ (because we may put $W_x = \operatorname{Ker} \alpha_x$).

We define a function P_M on M by

$$P_M(p) = \min_{x \in T_p(M)} d(x).$$

Then we have the following theorem.

Theorem 2. *If M^n can be isometrically imbedded into $\mathbf{R}^{n+r}$, then the inequality $r \geq n - P_M(p)$ holds at any point $p \in M$. Hence any open Riemannian submanifold containing $p \in M$ cannot be locally isometrically imbedded into the Euclidean space with codimension $n - P_M(p) - 1$.*

The proof is obvious from the above arguments. We remark that the valued $P_M(p)$ is determined by the curvature of M at p, and hence it is an intrinsic quantity of (M, g).

Theorem 2 implies that the function P_M may be considered as one obstruction to the existence of local isometric imbedding of M which is useful for the case $r \leq n$. But to determine the value $P_M(p)$ is in general a difficult problem.

If M is a homogeneous Riemannian manifold, i.e., the group of isometries of M acts transitively on M, then the function $P_M(p)$ takes a constant value. In this case we simply express it as $P(M) \in \mathbf{Z}^+$.

Example. S^n $(n \geq 2)$: $P(S^n) = n - 1$.

The curvature of S^n is given by

$$R(y,z)x = g(x,z)y - g(x,y)z.$$

Hence the subspace $\langle x\rangle^\perp \subset T_p(S^n)$ satisfies the condition on W_x. If $x \neq 0$, the space W_x cannot be equal to the whole space $T_p(S^n)$. And hence $d(x) = n-1$. If $x=0$, we have clearly $d(x) = n$ because we may put $W_x = T_p(S^n)$. By these results, it follows that $P(S^n) = n-1$.

Thus by Theorem 2, the codimension of local isometric imbedding of S^n must satisfy the inequality $r \geq n - P(S^n) = 1$. But this is a trivial result because S^n is not flat in case $n \geq 2$.

§ 4. Example of isometric imbeddings

Among Riemannian manifolds there exist a special class of manifolds, called Riemannian symmetric spaces. This class contains the spaces of constant curvature, and they are locally characterized by the property $\nabla R = 0$, i.e.,

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W = 0$$

for X, Y, $Z \in \mathfrak{X}(M)$.

For these spaces Kobayashi (Tôhoku Math. J. **20** (1968)) constructed global isometric imbeddings of many Riemannian symmetric space. And these imbeddings give relatively low dimensional realization in Euclidean spaces even in the local standpoint. We call these imbeddings as **canonical isometric imbeddings** of Riemannian symmetric spaces. (Actually they should be called as symmetric R-spaces.) We give here some examples.

Example.
(1) $P^n(\mathbf{C}) \subset \mathbf{R}^{n(n+2)}$
Homogeneous coordinate : $[w_0, w_1, \cdots, w_n] \in P^n(\mathbf{C})$.
Inhomogeneous coordinate : $z_i = w_i/w_0 \in \mathbf{C}$ $(i = 1, \cdots, n)$.
The Fubini-Study metric:

$$g = \frac{1}{1+|z|^2}\sum dz_i\, d\overline{z_i} + \frac{1}{(1+|z|^2)^2}\left(\sum \overline{z_i}\, dz_i\right)\left(\sum z_j\, d\overline{z_j}\right).$$

Equivariant isometric imbedding $f : P^n(\mathbf{C}) \longrightarrow \mathbf{R}^{n(n+2)}$:

$$f([w_0, w_1, \cdots, w_n]) = \sqrt{-1}\left(\frac{\delta_{ij}}{n+1} - \frac{w_i\,\overline{w_j}}{|w|^2}\right)_{0\leq i,j\leq n} \in \mathfrak{su}(n+1) \cong \mathbf{R}^{n(n+2)},$$

or by inhomogeneous coordinate

$$f(z_1, \cdots, z_n) = \sqrt{-1}\left(\frac{\delta_{ij}}{n+1} - \frac{z_i\,\overline{z_j}}{1+|z|^2}\right)_{0\le i,j\le n},$$

where we put $z_0 = 1$. Actually $P^n(\mathbf{C})$ is expressed as $SU(n+1)/S(U(n)\times U(1))$, and g is $SU(n+1)$-homogeneous and the map f is $SU(n+1)$-equivariant.

(2) $SO(p+q)/SO(p)\times SO(q) \subset \mathbf{R}^{\frac{1}{2}(p+q)(p+q+1)-1}$ (real Grassmann manifold)

We put $G = SO(p+q)$ and $K = SO(p)\times SO(q)$. And we denote by $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ the canonical decomposition of the symmetric space G/K. The space $\mathfrak{m} \cong$ the space of (p,q)-matrices can be naturally identified with the tangent space at the origin of $SO(p+q)/SO(p)\times SO(q)$. The Riemannian metric on this space is given by

$$g(X,Y) = \text{trace}\,(X^tY), \quad X, Y \in \mathfrak{m}.$$

And g is extended to the whole space by the left action of $SO(p+q)$.

Elements of $SO(p+q)/SO(p)\times SO(q)$ is a p-dimensional subspace of $\mathbf{R}^{p+q}$. And let $\{{}^t(x_{1i},\cdots x_{p+q,i}) \in \mathbf{R}^{p+q}\}_{1\le i\le p}$ be an orthonormal basis of this subspace $V^p \subset \mathbf{R}^{p+q}$. Here we put

$$\begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{p+q,1} & \cdots & x_{p+q,p} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{p+q} \end{pmatrix}.$$

Then the isometric imbedding is given by

$$f(V^p) = \left(\frac{p}{p+q}\,\delta_{ij} - \langle \xi_i, \xi_j\rangle\right)_{1\le i,j\le p+q},$$

where $\langle\ ,\ \rangle$ is a positive definite inner product of the space of row vectors $\mathbf{R}^p$. Note that the matrix $f(V^p)$ is symmetric and traceless.

Similarly we have the global isometric imbedding for two remaining Grassmann manifolds:

$$SU(p+q)/S(U(p)\times U(q)) \subset \mathbf{R}^{(p+q)^2-1},$$
$$Sp(p+q)/Sp(p)\times Sp(q) \subset \mathbf{R}^{2(p+q)^2-(p+q)-1}.$$

Note that $\dim SU(p+q)/S(U(p)\times U(q)) = 2pq$ and $\dim Sp(p+q)/Sp(p)\times Sp(q) = 4pq$. The complex projective space $P^n(\mathbf{C})$ is a special case of the complex Grassmann manifold.

(3) $SO(n) \subset \mathbf{R}^{n^2}$, $U(n) \subset \mathbf{R}^{2n^2}$, $Sp(n) \subset \mathbf{R}^{4n^2}$

These are the natural inclusion of groups into the space of matrices.

$$SO(n) = \{A \in GL(n, \mathbf{R}) \mid {}^tAA = I_n\} \subset M(n, \mathbf{R}) \cong \mathbf{R}^{n^2},$$
$$U(n) = \{A \in GL(n, \mathbf{C}) \mid {}^t\overline{A}A = I_n\} \subset M(n, \mathbf{C}) \cong \mathbf{R}^{2n^2},$$
$$Sp(n) = \{A \in GL(n, \mathbf{H}) \mid {}^t\overline{A}A = I_n\} \subset M(n, \mathbf{H}) \cong \mathbf{R}^{4n^2},$$

where $\mathbf{H}$ is the set of quaternion numbers. It consists of numbers of type $a + bi + cj + dk$ $(a, b, c, d \in \mathbf{R})$ and $\overline{a + bi + cj + dk} = a - bi - cj - dk$.

The Riemannian metric is given as follows: In the case of $SO(n)$ the tangent space at the identity element is expressed as $\{X \in \mathfrak{g}(n, \mathbf{R}) \mid {}^tX + X = 0\}$. And the metric of this space is given by $g(X, Y) =$trace $(X^tY) = -$ trace(XY). This metric is extended to the whole $SO(n)$ by the left action of $SO(n)$.

For the cases $U(n)$ and $Sp(n)$ we put $g(X, Y) =$trace $(X^t\overline{Y})$.

Note that these compact Lie groups G are globally isometrically imbedded into the Euclidean space with dimension about $2 \dim G$. And these are relatively low codimensional global isometric imbeddings.

(4) $Sp(n)/U(n) \subset \mathbf{R}^{n(2n+1)}$, $P^2(\mathbf{Cay}) \cong F_4/Spin(9) \subset \mathbf{R}^{26}$

Among these spaces we have the following result.

Theorem 3. *The canonical isometric imbeddings of the followings spaces give the least dimensional isometric imbeddings into the Euclidean space even in the local standpoint :*

$$Sp(n) \subset \mathbf{R}^{4n^2},$$
$$Sp(n)/U(n) \subset \mathbf{R}^{n(2n+1)},$$
$$P^2(\mathbf{H}) \subset \mathbf{R}^{14},$$
$$P^2(\mathbf{Cay}) \subset \mathbf{R}^{26}.$$

Note that the quaternion projective plane $P^2(\mathbf{H}) = Sp(3)/Sp(2) \times Sp(1)$ is a 8-dimensional space and the Cayley projective plane $P^2(\mathbf{Cay}) = F_4/Spin(9)$ is a 16-dimensional space.

For the remaining Riemannian symmetric space (except spaces of constant curvature S^n and H^n) we cannot yet determine the least codimension of the Euclidean space into which $M = G/K$ can be locally isometrically imbedded.

But the order of the dimension of the ambient space is determined for most Riemannian symmetric space. We here show the list of known result:

TABLE : Local isometric imbedding of Riemannian symmetric spaces

	M	$\dim M$	$M \not\subset \mathbf{R}^N$	$M \subset \mathbf{R}^N$
AI	$SU(n)/SO(n)$ $(n \geq 3)$	$\frac{1}{2}(n-1)(n+2)$	n^2-2	$n(n+1)$
AII	$SU(2n)/Sp(n)$ $(n \geq 3)$	$(n-1)(2n+1)$	$3n^2-2n-2$	$2n(2n-1)$
$AIII$	$P^2(\mathbf{C})$	4	6	8
	$P^3(\mathbf{C})$	6	9	15
	$P^4(\mathbf{C})$	8	12	24
	$P^n(\mathbf{C})$ $(n \geq 5)$	$2n$	$[\frac{16}{5}n]-1$	$n(n+2)$
	$SU(p+2)/S(U(p) \times U(2))$ $(p \geq 2)$	$4p$	$\begin{cases} 12 \ (p=2) \\ 20 \ (p=3) \\ 27 \ (p=4) \\ 7p \ (p \geq 5) \end{cases}$	$(p+1)(p+3)$
	$SU(p+q)/S(U(p) \times U(q))$ $(p \geq q \geq 3)$	$2pq$	$\begin{cases} 4pq-2q-1 \\ \quad (p=q, q+1) \\ 4pq-p-q+1 \\ \quad (p \geq q+2) \end{cases}$	$(p+q)^2-1$
BDI	$*Q^3(\mathbf{C}) \simeq Sp(2)/U(2)$	6	9	10
	$Q^n(\mathbf{C})$ $(n \geq 4)$	$2n$	$[\frac{1}{5}(16n-3)]$	$\frac{1}{2}(n+1)(n+2)$
	$SO(p+q)/SO(p) \times SO(q)$ $(p \geq q \geq 3)$	pq	$\begin{cases} 2p^2-p-1 \\ \quad (p=q) \\ 2pq-p \\ \quad (p \geq q+1) \end{cases}$	$\frac{1}{2}(p+q)(p+q+1)-1$
$BDII$	$*S^n$ $(n \geq 2)$	n	n	$n+1$
	$*H^n$ $(n \geq 2)$	n	$2n-2$	$2n-1$
CI	$*Sp(n)/U(n)$ $(n \geq 1)$	$n(n+1)$	$2n^2+n-1$	$n(2n+1)$
CII	$*Sp(3)/Sp(2) \times Sp(1)$	8	13	14
	$Sp(p+1)/Sp(p) \times Sp(1)$ $(p \geq 3)$	$4p$	$\begin{cases} 8p-4 \\ \quad (3 \leq p \leq 4) \\ 7p \ (p \geq 5) \end{cases}$	$p(2p+3)$
	$Sp(p+2)/Sp(p) \times Sp(2)$ $(p \geq 2)$	$8p$	$\begin{cases} 16p-7 \\ \quad (2 \leq p \leq 5) \\ 15p-2 \\ \quad (p \geq 6) \end{cases}$	$(p+1)(2p+5)$
	$Sp(p+q)/Sp(p) \times Sp(q)$ $(p \geq q \geq 3)$	$4pq$	$\begin{cases} 8pq-4q-1 \\ \quad (q \leq p \leq q+3) \\ 8pq-p-3q+3 \\ \quad (p \geq q+4) \end{cases}$	$2(p+q)^2-(p+q)-1$
$DIII$	$SO(8)/U(4) \simeq Q^6(\mathbf{C})$	12	18	28
	$SO(2n)/U(n)$ $(n \geq 5)$	$n(n-1)$	$\frac{3}{2}n(n-1)-1$	$n(2n-1)$

TABLE (continued) : Local isometric imbedding of Riemannian symmetric spaces

	M	$\dim M$	$M \not\subset \mathbf{R}^N$	$M \subset \mathbf{R}^N$
EI	$E_6/Sp(4)$	42	77	702
EII	$E_6/SU(2)\cdot SU(6)$	40	59	650
$EIII$	$E_6/Spin(10)\cdot SO(2)$	32	47	78
EIV	E_6/F_4	26	37	54
EV	$E_7/SU(8)$	70	132	1463
EVI	$E_7/Spin(12)\cdot SU(2)$	64	95	1539
$EVII$	$E_7/E_6\cdot SO(2)$	54	80	133
$EVIII$	$E_8/Spin(16)$	128	247	?
EIX	$E_8/E_7\cdot SU(2)$	112	167	3875
FI	$F_4/Sp(3)\cdot SU(2)$	28	51	324
FII	$*F_4/Spin(9)$	16	25	26
G	$G_2/SO(4)$	8	13	27
$[A_{n-1}]$	$SU(n)$ $(n \geq 6)$	n^2-1	$2n^2-2n-2$	$2n^2$
$[B_n]$	$SO(2n+1)$ $(n \geq 5)$	$n(2n+1)$	$4n^2-2n-2$	$(2n+1)^2$
$[C_n]$	$*Sp(n)$ $(n \geq 1)$	$n(2n+1)$	$4n^2-1$	$4n^2$
$[D_n]$	$SO(2n)$ $(n \geq 5)$	$n(2n-1)$	$4n^2-6n$	$4n^2$
	$SU(3)$	8	12	18
	$SU(4) \simeq SO(6)$	15	24	32
	$SU(5)$	24	41	50
	$*SO(5) \simeq Sp(2)$	10	15	16
	$SO(5,\mathbf{C})/SO(5)$	10	16	?
	$SO(7)$	21	35	49
	$SO(8)$	28	47	64
	$SO(9)$	36	63	81
	E_6	78	139	1458
	E_7	133	238	3136
	E_8	248	459	?
	F_4	52	94	676
	G_2	14	23	49

$\simeq$ implies a local isomorphism of Riemannian symmetric spaces.

The symbol $*$ before M implies that the least dimension of the Euclidean space is determined.

Unfortunately there still remains between two columns even if the value $P(M)$ is determined.

For most cases the results in this table are obtained by determining the value of the invariant $P(M)$. But the determination of the value $P(M)$ is a difficult problem in representation theory, and there still remains some spaces whose value $P(M)$ is undetermined. We here give the value $P(M)$ for known spaces.

	M	$P(M)$
AI	$SU(n)/SO(n)$	$n-1$
$AIII$	$SU(p+1)/S(U(p)\times U(1))$	$\begin{cases} 1 & (p=1) \\ 2 & (p=2) \\ p-1 & (p\geq 3) \end{cases}$
	$SU(p+2)/S(U(p)\times U(2))$	$\begin{cases} 3 & (p=2,3) \\ 4 & (p=4) \\ p-1 & (p\geq 5) \end{cases}$
BDI, II	$SO(p+q)/SO(p)\times SO(q)$	$\begin{cases} p & (p=q) \\ p-1 & (p\geq q+1) \end{cases}$
CI	$Sp(n)/U(n)$	n
CII	$Sp(p+1)/Sp(p)\times Sp(1)$	$\begin{cases} 3 & (1\leq p\leq 4) \\ p-1 & (p\geq 5) \end{cases}$
	$Sp(p+2)/Sp(p)\times Sp(2)$	$\begin{cases} 6 & (2\leq p\leq 5) \\ p+1 & (p\geq 6) \end{cases}$
EI	$E_6/Sp(4)$	6
EV	$E_7/SU(8)$	7
$EVIII$	$E_8/Spin(16)$	8
FI	$F_4/Sp(3)\cdot SU(2)$	4
FII	$F_4/Spin(9)$	7
G	$G_2/SO(4)$	2
	$Sp(n)$	$2n$
	$SU(2)\simeq SO(3)\simeq Sp(1)$	2
	$SU(3)$	3
	$SU(4)\simeq SO(6)$	5
	$SU(5)$	6
	$SO(5)\simeq Sp(2)$	4
	$SO(7)$	6
	$SO(8)$	8
	$SO(9)$	8
	G_2	4

Example. In the case $Sp(n)$ we can prove that the value $P(Sp(n)) = 2n$. Hence the value $\dim Sp(n) - P(Sp(n)) - 1 = n(2n+1) - 2n - 1 = 2n^2 - n - 1$. From Theorem 2 it follows that the space $Sp(n)$ cannot be locally isometrically imbedded into the Euclidean space with codimension $2n^2 - n - 1$. On the other hand, the codimension of the canonical isometric imbedding of $Sp(n)$ is $2n^2 - n$. Hence this imbeddings gives the least dimensional isometric imbeddings even in the local standpoint.

We can obtain the same best result for the space $Sp(n)/U(n)$.

The exact value of $P(M)$ for two groups $SU(n)$ and $SO(n)$ is still undetermined for large n. At present we have the following estimates:

$$\begin{aligned} &SU(n) : [3n/2]-1 \leq P(SU(n)) \leq 2n-1, \\ &SO(2n+1) : 2n \leq P(SO(2n+1)) \leq 4n+1, \\ &SO(2n) : n+2[n/2] \leq P(SO(2n)) \leq 4n-1. \end{aligned}$$

We conjecture that the left values give the exact value of $P(G)$. But in Theorem 2 we may substitute a larger value in $P(M)$, and from this we obtain the following theorem.

Theorem 4. *The Lie groups $SU(n)$ and $SO(n)$ cannot be isometrically imbedded into $\mathbf{R}^{2n^2-2n-2}$, $\mathbf{R}^{n^2-3n}$ even in the local standpoint.*

Unfortunately there still remains some gap between the imbedding dimension of Kobayashi $2n^2$ and n^2, respectively.

For general compact simple Lie group G we conjecture that the following equality holds:

$$P(G) = \begin{cases} 2\,\mathrm{rank}\, G & G \neq SU(n), SO(2n), E_6, \\ n-1+[n/2] & SU(n), \\ n+2[n/2] & SO(2n), \\ 10 & E_6. \end{cases}$$

For two projective plane $P^2(\mathbf{H})$ and $P^2(\mathbf{Cay})$ we have the following data:

$$\begin{aligned} &\dim P^2(\mathbf{H}) = 8, \quad P(P^2(\mathbf{H})) = 3, \\ &\dim P^2(\mathbf{Cay}) = 16, \quad P(P^2(\mathbf{Cay})) = 7. \end{aligned}$$

Hence by Theorem 2 we know that these spaces cannot isometrically imbedded into $\mathbf{R}^{12}$ and $\mathbf{R}^{24}$ even locally. On the other hand the canonical isometric imbedding is

$$P^2(\mathbf{H}) \subset \mathbf{R}^{14}, \quad P^2(\mathbf{Cay}) \subset \mathbf{R}^{26}.$$

For both spaces there remains 1-dimensional gap. But by analyzing the space Ker α_x : $T_p(M) \longrightarrow \mathbf{R}^r$ for both spaces, we can prove that the canonical isometric imbeddings give the least dimensional isometric imbeddings in the local standpoint for both spaces. (For details see our preprints available from: http://www.mis.hiroshima-u.ac.jp).

§ 5. Complex projective plane: Higher order obstruction

Perhaps one interesting and difficult problem is to determine whether there exists a local isometric imbeddings of the complex projective plane $P^2(\mathbf{C})$ into the 3-codimensional Euclidean space $\mathbf{R}^7$.

This is still an open problem. (For remaining three projective planes $P^2(\mathbf{R}) \simeq S^2$, $P^2(\mathbf{H})$ and $P^2(\mathbf{Cay})$ the problem is completely solved, including "rigidity".)

We explain the present situation concerning this problem. Our present knowledge is summarized in the following form:

Theorem 5. (1) $P^2(\mathbf{C})$ *cannot be locally isometrically imbedded into* $\mathbf{R}^6$.

(2) $P^2(\mathbf{C})$ *can be globally isometrically imbedded into* $\mathbf{R}^8$ (*Canonical isometric imbedding*).

(3) *The Gauss equation of* $P^2(\mathbf{C})$ *in codimension* 3 *admits a solution. The set of solutions forms a* 10*-dimensional algebraic subset of* $S^2(T_p^*(P^2(\mathbf{C}))) \otimes \mathbf{R}^3 \cong \mathbf{R}^{30}$.

(4) *"Generic" solutions of the Gauss equation in codimension* 3 *cannot be the second fundamental form of an actual isometric imbedding of* $P^2(\mathbf{C})$ *into* $\mathbf{R}^7$.

(1) and (3) are results of Agaoka (Hokkaido Math. J. **14** (1985)). (2) is a result of Kobayashi (1968) which we explained before. (4) is a result of Kaneda (Hokkaido Math. J. **19** (1990)).

In the following we explain the results (1), (3) and (4).

Obstruction for the case $M^4 \subset \mathbf{R}^6$.

We express the curvature of 4-dimensional Riemannian manifold as

$$R : \wedge^2 T_p(M) \longrightarrow \wedge^2 T_p(M).$$

Actually this map is defined by $(R(x \wedge y), z \wedge w) = -g(R(x,y)z, w)$ in terms of the Riemannian metric g. $(\ ,\ )$ is the metric of $\wedge^2 T_p(M)$ naturally induced from g. We once fix an orientation of M (precisely an orientation of the tangent space $T_p(M)$). Then in the 4-dimensional case there exists a $*$-operator

$$* : \wedge^2 T_p(M) \longrightarrow \wedge^2 T_p(M).$$

In terms of an orthonormal basis $\{e_1, \cdots, e_4\}$ $*$ is given by

$$*\,(e_1 \wedge e_2) = e_3 \wedge e_4, \quad *(e_1 \wedge e_3) = -e_2 \wedge e_4,$$
$$*\,(e_1 \wedge e_4) = e_2 \wedge e_3, \quad *(e_2 \wedge e_3) = e_1 \wedge e_4,$$
$$*\,(e_2 \wedge e_4) = -e_1 \wedge e_3, \quad *(e_3 \wedge e_4) = e_1 \wedge e_2.$$

Then we have

Theorem 6. *If M^4 can be locally isometrically imbedded into $\mathbf{R}^6$, then the eigenvalues of the map $R \circ * : \wedge^2 T_p(M) \longrightarrow \wedge^2 T_p(M)$ are expressed as $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$ at each point of M^4.*

If we change the orientation of $T_p(M)$, then the map $*$ is changed to $-*$. And hence the result of this theorem 6 is unaltered.

In the case of the complex projective plane $P^2(\mathbf{C})$ the curvature of $P^2(\mathbf{C})$ is given by

$$\begin{aligned} R(x_1 \wedge y_1) &= 4x_1 \wedge y_1 + 2x_2 \wedge y_2, \\ R(x_2 \wedge y_2) &= 2x_1 \wedge y_1 + 4x_2 \wedge y_2, \\ R(x_1 \wedge x_2) &= x_1 \wedge x_2 + y_1 \wedge y_2, \\ R(y_1 \wedge y_2) &= x_1 \wedge x_2 + y_1 \wedge y_2, \\ R(x_1 \wedge y_2) &= x_1 \wedge y_2 + x_2 \wedge y_1, \\ R(x_2 \wedge y_1) &= x_1 \wedge y_2 + x_2 \wedge y_1, \end{aligned}$$

where $\{x_1, y_1, x_2, y_2\}$ is an oriented orthonormal basis of $T_p(P^2(\mathbf{C}))$. Then we can easily check that the eigenvalues of $R \circ *$ is given by $\{6, -2, -2, -2, 0, 0\}$. And hence we know that $P^2(\mathbf{C})$ cannot be locally isometrically imbedded into $\mathbf{R}^6$.

If we change the value 4 in the above curvature to 0 (in two places), then the eigenvalues become $\{\pm 2, \pm 2, 0, 0,\}$ and the condition in Theorem 6 is satisfied. And we can show that in this case the Gauss equation admits a solution in codimension 2 if we complexify all variables. In the real category we can show that the Gauss equation does not admit a real solution.

Next we explain the result (4). We put

$$\begin{aligned} J^0(M, \mathbf{R}^N) &= \{j_p^0(f) = (p, f(p))\}, \\ J^1(M, \mathbf{R}^N) &= \{j_p^1(f) = (p, f(p), (\nabla f)_p)\}, \\ J^2(M, \mathbf{R}^N) &= \{j_p^2(f) = (p, f(p), (\nabla f)_p, (\nabla\nabla f)_p)\}, \\ &\cdots\cdots\cdots, \end{aligned}$$

and we denote by Q the set of elements of $J^2(M, \mathbf{R}^N)$ which satisfy the conditions (1), (2) and (4) in § 1. Similarly we denote by $Q^{(1)}$ the set of elements of $J^3(M, \mathbf{R}^N)$ which satisfy the conditions (1) $\sim$ (5). The space $Q^{(2)} \subset J^4(M, \mathbf{R}^N)$, $Q^{(3)} \subset J^5(M, \mathbf{R}^N)$, $\cdots$ can be similarly defined by using the integrability conditions and the differential of the Gauss equation. (Precisely, we must restrict them to an open dense subset because Q, $Q^{(1)}$, etc.

are not in general manifolds.) Symbolically we express the jet ∇f as ω and $\nabla\nabla\nabla f$ as β etc.

Then there is a sequence of natural projections induced from $\pi : J^{k+1}(M, \mathbf{R}^N) \longrightarrow J^k(M, \mathbf{R}^N)$:

$$\cdots\cdots \longrightarrow Q^{(4)} \longrightarrow Q^{(3)} \longrightarrow Q^{(2)} \longrightarrow Q^{(1)} \longrightarrow Q.$$

If α is a second fundamental form of some local isometric imbedding of $f : P^2(\mathbf{C}) \longrightarrow \mathbf{R}^7$, then there exists a sequence of elements of $Q^{(k)}$ which projects to the given $\alpha = \nabla\nabla f$. These elements are of course obtained by differentiating f several times.

In this situation Kaneda showed the following result: Let α be a "generic" solution of the Gauss equation of $P^n(\mathbf{C})$ at the point p in codimension 3. Then any element $(p, \omega, \alpha, \beta) \in Q^{(1)}$ is not contained in the image of the projection $Q^{(2)} \longrightarrow Q^{(1)}$. The assertion (4) follows immediately from this fact.

This fact implies that if a local isometric imbedding from $P^2(\mathbf{C})$ into $\mathbf{R}^7$ exists, its second fundamental form must be "singular", i.e., α must satisfies some additional conditions in addition to the Gauss equation. This condition may be considered as a higher order obstruction to the existence of local isometric imbeddings. And thus the problem of local isometric imbedding $P^2(\mathbf{C}) \subset \mathbf{R}^7$ is a delicate and difficult problem because we must consider higher obstruction in addition to the Gauss equation. (Compared with the case of $Sp(n)$ or $Sp(n)/U(n)$, where the least dimension of the ambient Euclidean space is determined by only considering the Gauss equation.)

§ 6. Remaining problems

- Find new obstruction which is useful for higher codimensional case.

At present we only know the obstruction which is useful only for the range $r \leq n$. But this is a quite unsatisfactory situation, because generic n-dimensional Riemannian manifolds cannot be locally isometrically imbedded into $\mathbf{R}^{\frac{1}{2}n(n+1)-1}$ and hence there must be some obstruction for this case $r = \frac{1}{2}n(n-1) - 1$.

For the complex projective space $P^n(\mathbf{C})$ we know that

$$P^n(\mathbf{C}) \not\subset \mathbf{R}^{[16n/5]-1}, \qquad P^n(\mathbf{C}) \subset \mathbf{R}^{n(n+2)}.$$

And the gap of these two dimensions is quite large. If we find new obstruction which is useful for codimension $r \sim \frac{1}{2}n^2$, we can apply it to $P^n(\mathbf{C})$, and it may be possible to show that the canonical imbedding gives almost the least dimensional isometric imbedding.

Concerning this problem, Rivertz (1999) found the obstruction for the case $M^3 \subset \mathbf{R}^4$ and $M^5 \subset \mathbf{R}^7$, which involves the curvature R and its covariant derivative ∇R. It is an interesting problem to find a similar obstruction for the case $M^3 \subset \mathbf{R}^5$, because in this case the Gauss equation always admits a solution and hence the obstruction must necessary involves the covariant derivative ∇R or its higher order covariant derivative. (Find higher order obstructions involving R, ∇R, $\nabla\nabla R$, $\cdots$.)

- Inequality for the case $M^n \subset \mathbf{R}^{n+1}$

The curvature of a hypersurface $M^n \subset \mathbf{R}^{n+1}$ satisfies the following inequality at each point of $p \in M$:

$$\begin{vmatrix} R_{ijij} & R_{ijik} & R_{ijjk} \\ R_{ikij} & R_{ikik} & R_{ikjk} \\ R_{jkij} & R_{jkik} & R_{jkjk} \end{vmatrix} \geq 0.$$

Our next problem is to find a similar inequality for higher codimensional case. These inequalities also serve as an obstruction to the existence of local isometric imbeddings. For example in the case $M^4 \subset \mathbf{R}^6$ there exists an example such that the Gauss equation does not admit a solution, but the complexified Gauss equation admits a complex solution. In this situation by only polynomial relation of R, ∇R, $\nabla\nabla R$, etc. we cannot show the non-existence of local isometric imbeddings.

- $M^5 = SU(3)/SO(3)$

For the 5-dimensional Riemannian symmetric space $SU(3)/SO(3)$ (type AI) the following results are known:

$$R \in K_p^5(M), \quad R \in \overline{K_p^4(M)}, \qquad R \notin K_p^2(M),$$

where $\overline{K_p^4(M)}$ is the Zariski closure of $K_p^4(M)$ and R is the curvature of $SU(3)/SO(3)$. But the least dimension where the Gauss equation admits a solution is not yet determined. The space $SU(3)/SO(3)$ is globally isometrically imbedded into the Euclidean space with codimension 7.

- $P^2(\mathbf{C})$, $P^n(\mathbf{C})$ and $P^n(\mathbf{H})$

Determine the least dimensional Euclidean space into which the complex projective plane $P^2(\mathbf{C})$ can be locally isometrically imbedded. It is $\mathbf{R}^7$ or $\mathbf{R}^8$.

For general n $P^n(\mathbf{C})$ admits a solution of the Gauss equation in codimension $= n^2 - 1$. Hence it is a problem whether $P^n(\mathbf{C})$ can be locally isometrically imbedded into the Euclidean space with codimension $n^2 - 1$ or not, i.e., we can decrease the dimension of the ambient space of the canonical isometric imbedding to $\mathbf{R}^{n(n+2)-1}$.

As for the quaternion projective space $P^3(\mathbf{H})$ it is known that

$$P^3(\mathbf{H}) \not\subset \mathbf{R}^{20}, \qquad P^3(\mathbf{H}) \subset \mathbf{R}^{27}.$$

We must fill this gap by using a new obstruction. We conjecture that the canonical isometric imbedding $P^n(\mathbf{H}) \subset \mathbf{R}^{n(2n+3)}$ gives the least dimensional isometric imbedding even in the local stand point.

- $P(G/K)$

Determine the value $P(M)$ for all Riemannian symmetric space $M = G/K$.

- Non-compact Riemannian symmetric space

Find or construct an example of local (or global) isometric imbedding of Riemannian symmetric space of non-compact type. Such example is known only for the case $H^n \subset \mathbf{R}^{2n-1}$, which is the space of constant negative curvature. This is almost unbelievable situation compared with the long history of isometric imbeddings of Riemannian manifolds in differential geometry. And perhaps this means that to find such an example is a difficult problem.

リーマン多様体の等長埋め込み論小史，あるいは外史

A Historical View on the Theory of Isometric Imbeddings of Riemannian Manifolds — unofficial version

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概 要

We present principal events appeared in the theory of isometric imbeddings of Riemannian manifolds in chronological order, mainly concerning with the existence or non-existence of isometric imbeddings into low dimensional Euclidean spaces.

研究集会では“対称空間の等長埋め込み”という題目で講演しました．しかし対称空間に関する主な結果は，最近共同研究者の兼田英二さんと共著で‘数学’の論説に纏めたばかりなので，ここではタイトルを変え，初日にお話ししたリーマン多様体の局所的あるいは大域的な等長埋め込みの存在・非存在に関する主要な結果を時間の順に並べてみることにしました．この問題に関し，現在何がわかって何がわかっていないのか，これによりその輪郭を理解していただけたらと願っています．

ただ特に古い出来事については，原典が入手不可能であったり，あるいは入手できても私にはそれが読めないということが多く，その場合は自分で内容を確認せずに，安易な方法で申し訳ないのですが他の文献からの孫引きですませてしまいました．文献により記述に矛盾の生ずることも当然 (!?) あって，調べる程に何を信用すればよいのか判断の付きかねることも多々ありました．また最近の結果であっても，私の力不足で内容の正当性を確認せず (できず)，論文の記述をそのまま写しただけの項目が多くあります．私の好みにより，各論文のコメントにかなりの濃淡が生じていますし，中には著者の意図とは別の文脈で等長埋め込みの歴史に強引に含めてしまったものもあります．これらが題目を‘小史，あるいは外史’とした所以です．私の間違い・勘違い，あるいは記述漏れ等について何か気付かれた場合，ご指摘いただけたら誠に幸いに存じます．

リーマン多様体の等長埋め込み問題全般に関する歴史については次の著作に詳しく記されています．本文を書くにあたって，これらの著作を参考にさせていただきました．

[1] B. Y. Chen, Riemannian submanifolds, in "Handbook of Differential Geometry Vol.I" (ed. F. J. E. Dillen, L. C. A. Verstraelen), 187–418, Elsevier Science, Amsterdam, 2000.

[2] M. L. Gromov and V. A. Rokhlin, Embeddings and immersions in Riemannian geometry, Russian Math. Surveys, **25**(5) (1970), 1–57.

[3] 松本誠, Riemann 空間の局所的 imbedding について I, II, 数学, **5** (1953), 210–219, **6** (1954), 6–16.

[4] É. G. Poznyak and D. D. Sokolov, Isometric immersions of Riemannian spaces in Euclidean spaces, J. Soviet Math., **14** (1980), 1407–1428.

[5] M. Spivak, A Comprehensive Introduction to Differential Geometry Vol.V, Publish or Perish, Boston, 1975.

本文中敬称はすべて省略しました．ご了解お願い致します．微分可能性の程度 (C^1-級, C^2-級, $\cdots$) については解析学的に大切な問題ですが，ここでは簡単のため C^∞-級という記述ですませてしまったところがあります．正確な主張については原論文にあたっていただくようお願いします．何も記されていない場合は C^∞-級と理解して下さい．

では，年代順に出来事を並べます．どこから等長埋め込み問題の歴史が始まったと見做すかが既に問題ですが，ここではガウスに敬意を表し，ガウスから始めることにしましょう．

- C. F. Gauss　1827　曲面に関する一般的考察

  [いわゆるガウスの "曲面論". 寺阪英孝・静間良次, 19 世紀の数学　幾何学 II, 共立出版 (1982) に翻訳があります.]

- G. F. B. Riemann　1854　幾何学の基礎をなす仮説について

  [教授資格取得講演．邦訳はいろいろあります．Nash の論文 (1956) の参考文献には，このガウス，リーマンの仕事が最初に並べて挙げられ，更に一人おいてヒルベルトの 1901 年の仕事が置かれています.]

- E. Beltrami　1868　Teoria fondamentale degli spazzi di curvatura costante, Ann. di Mat. Pura Appl. II **2** 232–255.

  [2 次元の負定曲率空間を $\mathbf{R}^3$ 内の回転面として実現．ただし E. R. Rozendorn の Surfaces of negative curvature (in Encyclopaedia of Math. Sci. **48**, Geometry III, Springer (1992)) の p.90 には 1839 年に F. Minding がこのような例を見つけていたとの記述があります.]

- L. Schlaefli　1873　Nota alla Memoria del sig. Beltrami, ((Sugli spazzi di curvatura costante)), Ann. Mat. Pura Appl. (2) **5** 178–193.

[問題提起：n 次元リーマン多様体は $\frac{1}{2}n(n+1)$ 次元のユークリッド空間へ等長に埋め込めるか．この次元のとき等長埋め込みを表す偏微分方程式 $g_{ij} = \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{\partial f^k}{\partial x_i} \frac{\partial f^k}{\partial x_j}$ は，未知関数の個数と方程式の個数とが一致する決定系となります.]

- G. Ricci 1884 Principii di una teoria delle forme differenziali quadratiche, Ann. di Mat. Pura Appl. (2) **12** 135–167.

  [クラス数 = ‘ユークリッド空間に等長に埋め込める M の最小余次元’ を定義．既にこの頃から等長埋め込み可能な最小次元を決定せよ，という問題意識があったわけです．多様体という概念がまだなかった頃の話です.]

- F. Schur 1886 Ueber die Deformation der Räume constanten Riemann'schen Krümmungsmaasses, Math. Ann. **27** 163–176.

  [負定曲率空間 H^n の $\mathbf{R}^{2n-1}$ への局所等長埋め込みを構成．具体的な式が与えられています．実解析的なカテゴリーにおいて，H^n の $\mathbf{R}^{2n-1}$ への局所等長埋め込みは 1 変数関数 $n(n-1)$ 個分の自由度があり (Cartan)，剛性からかけ離れた状況にあります.]

- D. Hilbert 1901 Ueber Flächen von Constanter Gaussscher Krümmung, (On surfaces of constant negative curvature), Trans. Amer. Math. Soc. **2** 87–99.

  [完備な 2 次元負定曲率空間を $\mathbf{R}^3$ へ C^4-級等長にはめ込むことはできない．幾何学の基礎 (第 7 版, 1930) の付録 V に同じ結果が紹介されています (初版は 1899 年)．邦訳：ヒルベルト 幾何学の基礎 現代数学の系譜 7 共立出版 (1970) 194–202. ヒルベルトの結果は N. V. Efimov (1964) により微分可能性を弱めた形で次のように拡張されました：ガウス曲率が $K \leq -c$ (c は正の定数) を満たす完備な 2 次元リーマン多様体は $\mathbf{R}^3$ へ C^2-級等長にはめ込めない．M. Berger の A Panoramic View of Riemannian Geometry (Springer, 2003) p.52 には Hilbert's theorem is of fundamental historical importance. It explains why hyperbolic geometry has to be defined abstractly, and can never be obtained as the inner geometry of a surface in $\mathbb{E}^3$. とあります.]

- G. Fubini 1903 Sulle metriche definite da una forma Hermitiana, Atti Ist. Veneto **6** 501–513.

  E. Study 1905 Kürzeste Wege im komplexen Gebiet, Math. Ann. **60** 321–377.

  [いわゆる Fubini-Study 計量を複素射影空間に導入.]

- J. A. Schouten, D. J. Struik 1921 On some properties of general manifolds relating to Einstein's theory of gravitation, Amer. J. Math. **43** 213–216.

  [Ricci 平坦だが平坦でない n 次元リーマン多様体は $\mathbf{R}^{n+1}$ へ等長に埋め込めないことを証明.]

- L. Bianchi 1924 Lezioni di Geometria Differenziale, VII, PII, Bologna, 554–555.

  [H^n $(n \geq 3)$ は $\mathbf{R}^{n+1}$ へ等長に埋め込めないことを証明.]

- M. Janet 1926 Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien, Ann. Soc. Polon. Math., **5** 38–43.

E. Cartan　1927　Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien, Ann. Soc. Polon. Math., **6** 1–7.

[n 次元実解析的なリーマン多様体は $\mathbf{R}^{\frac{1}{2}n(n+1)}$ へ実解析的に局所等長埋め込み可能であることを証明. Janet の証明には初期条件の設定に関して不十分な点があり, 1931 年に C. Burstin が証明を補っています. Janet の 1 年後に同じ雑誌に同じタイトルで Cartan の論文が載るというのは一体何があったのでしょうか. M. A. Akivis, B. A. Rosenfeld, *Élie Cartan* (1869–1951), Transl. Math. Monographs **123** AMS (1993) の p.185 には自分の方法に対する Janet のこだわりが書かれていて興味深く感じました.]

- E. Cartan　1926–27　Sur une classe remarquable d'espaces de Riemann, Bull. Soc. Math. France **54** 214–264, **55** 114–134 等.

[対称空間の導入・分類. ちなみに, カルタン全集ではこの論文は Partie I — Groupes de Lie に収められています. 話はそれますが, Spivak は [5] の文献欄でカルタン全集について, Few have read his works, many pretend to have read them, and every one agrees that every one should read them. と記しています.]

- S. Cohn-Vossen　1927　Zwei Sätze über die Starrheit der Eiflächen, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 125–134.

[コーン-フォッセンの定理 : 卵形面 (正のガウス曲率をもつ $\mathbf{R}^3$ 内のコンパクトな曲面) は剛性をもつ. Cohn-Vossen は実解析的な場合にこの定理を示しましたが, 後 O. K. Zhitomirsky (1939), G. Herglotz (1943) 達が微分可能性の仮定を弱めた簡単な証明を発表しています.]

- K. H. Weise　1934　Beiträge zum Klassenproblem der quadratischen Differentialformen, Math. Ann., **110** 522–570.

T. Y. Thomas　1936　Riemann spaces of class one and their characterization, Acta Math., **67** 169–211.

N. A. Rozenson　1940–43　On Riemann spaces of class one, Izv. Akad. Nauk SSSR Ser. Math. **4** 181–192, **5** 325–351, **7** 253–284.

[余次元 = 1 の場合の研究. Rozenson の論文は露語です.]

- A. E. Liber　1938　On a class of Riemannian spaces of constant negative curvature, Uchen. Zap. Saratov Gos. Univ. Ser. Fiz.-Mat. **1** (14) 105–122.

[H^n は $\mathbf{R}^{2n-2}$ へ局所的に等長に埋め込めないことを証明. 文献を遡れば, E. Cartan, Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien, Bull. Soc. Math. France **47** (1919) 125–160, **48** (1920) 132–208 にこの事実は既に述べられているようです.]

- C. B. Allendoerfer　1939　Rigidity for spaces of class greater than one, Amer. J. Math., **61** 633–644.

[剛性の研究. タイプ数を導入. ユークリッド空間の余次元 $\leq \frac{1}{3}\dim M$ の generic な部分多様体は剛性をもつ.]

- C. Tompkins　1939　Isometric embedding of flat manifolds in Euclidian space, Duke Math J., **5** 58–61.

[コンパクトで平坦な n 次元リーマン多様体は $\mathbf{R}^{2n-1}$ へ等長にはめ込めないことを証明．後の Chern–Kuiper, 大槻富之助の仕事の原型となる結果です．標準的な平坦トーラス $\mathbf{R}^n/\mathbf{Z}^n$ は $\mathbf{R}^{2n}$ へ等長に埋め込めるので，この結果はある意味で best possible なものです．しかし数多く存在している他の平坦トーラスの等長埋め込み可能な次元については何も知られていないのが現状のようです．Tompkins は更に平坦なクラインの壷は $\mathbf{R}^4$ へ等長はめ込み可能であることを示しました (1941)．これについては $\mathbf{R}^4$ へ等長に埋め込めるか否かが未解決問題です．$\mathbf{R}^5$ へ埋め込めることはわかっていますので.]

- D. Blanuša　1947　Le plongement isométrique des espaces elliptiques dans des espaces Euclidiens, Glasnik Mat. Fiz. I Astrn. **2** 248–249.

  [正の定曲率空間である実射影空間 $P^n(\mathbf{R})$ の $\mathbf{R}^{\frac{1}{2}n(n+3)}$ への大域的な等長埋め込みを構成．この埋め込みは，後に小林昭七により一般の対称 R 空間に拡張されました．$P^n(\mathbf{R})$ の大域的な等長埋め込みについては, 未だにこれが知られているものの中で次元最小です.]

- 松本誠　1950　Riemann spaces of class two and their algebraic characterization, J. Math. Soc. Japan **2** 67–76, 77–86, 87–92.

  [余次元 = 2 の場合の研究．Finsler 幾何学を研究される前の話です.]

- A. Borel　1950　Le plan projectif des octaves et les sphères comme espaces homogènes, C.R. Acad. Sci. Paris Ser.A **230** 1378–1380.

  [Cayley 射影平面と対称空間 $F_4/Spin(9)$ とを初めて同一視．カルタンによる対称空間の分類からはかなり時間がたっています．Cayley 射影平面の自己同型群に関しては，他に C. Chevalley–R. D. Schafer (1950), H. Freudenthal (1951) 達の代数方面からの貢献があります.]

- S. S. Chern and N. H. Kuiper　1952　Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space, Ann. of Math. **56** 422–430.

  [index of nullity と index of relative nullity を導入．すべての点において index of nullity $\geq r$ となるコンパクト n 次元リーマン多様体は $\mathbf{R}^{n+r-1}$ へ等長にはめ込めない．また，コンパクト非正曲率 n 次元リーマン多様体は $\mathbf{R}^{2n-1}$ へ等長にはめ込めない (後に大槻が証明を完成させる).]

- L. Nirenberg　1953　The Weyl and Minkowski problems in differential geometry in the large, Comm. Pure Appl. Math. **6** 337–394.

  [ガウス曲率が正のコンパクト向き付け可能な曲面は $\mathbf{R}^3$ 内の凸曲面として一意的に等長埋め込み可能．H. Weyl が 1916 年に提出した問題で，H. Lewy (1938), A. D. Aleksandrov (1942), E. Heinz (1962) 達による貢献があります．N. V. Efimov, Qualitative problems of the theory of deformation of surfaces, Amer. Math. Soc. Transl. Ser.I **6** (1962) 274–423, R. S. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. **7** (1982) 65–222 を参照して下さい．この定理の中の‘一意的’の部分がコーン–フォッセンの定理です.]

- 立花俊一　1953　On the imbedding problem of spaces of constant curvature in one another, Natur. Sci. Rep. Ochanomizu Univ. **4** 44–50.

[定曲率空間の間の等長埋め込みの研究は Cartan (1919) に始まり，多くの結果があります.]

- 大槻富之助　1954　Isometric imbedding of Riemann manifolds in a Riemann manifold, J. Math. Soc. Japan, **6** 221–234.

  [n 次元負曲率空間は $\mathbf{R}^{2n-2}$ へ等長に埋め込めない.]

- D. Blanuša　1954　Le plan elliptique plongé isométriquement dans un espace à quatre dimensions ayant une courbure constante, Glasnik Mat. Fiz. I Astrn. **9** 41–58.

  [実射影平面 $P^2(\mathbf{R})$ の $\mathbf{R}^4$ への大域的な "C^1-級"等長はめ込みを具体的に構成. $K = 1/r^2$ の場合それは次式で与えられます：

$$x_1 = \frac{r}{2}\sin^2 u \sin\left(2v - 2\sqrt{3}\cos u - 2\sqrt{3}\log\left|\tan\frac{u}{2}\right|\right),$$
$$x_2 = \frac{r}{2}\sin^2 u \cos\left(2v - 2\sqrt{3}\cos u - 2\sqrt{3}\log\left|\tan\frac{u}{2}\right|\right),$$
$$x_3 = \frac{r}{2}\sin 2u \sin\left(v - \sqrt{3}\cos u\right), \quad x_4 = \frac{r}{2}\sin 2u \cos\left(v - \sqrt{3}\cos u\right).$$

  ただし $-\pi/2 \leq u,\ v \leq \pi/2$ で u は北極からの緯度，v は経度を表します.]

- J. Nash　1954　C^1 isometric imbeddings, Ann. of Math. **60** 383–396.

  N. H. Kuiper　1955　On C^1-isometric imbeddings I, II, Indag. Math. **17** 545–556, 683–689.

  [C^1-級等長埋め込みの研究. 後年の S. Smale–M. W. Hirsch の研究成果と合わせると，n 次元 C^1-級多様体 M 上の C^0-級リーマン計量について

  (1) M は局所的に $\mathbf{R}^{n+1}$ へ等長埋め込み可能,

  (2) M は大域的に $\mathbf{R}^{2n}$ へ等長埋め込み可能,

  (3) M は大域的に $\mathbf{R}^{2n-1}$ へ等長はめ込み可能

  を示したことになります. これらは'接続・曲率'という概念のない世界での話です.]

- D. Blanuša　1955　Über die Einbettung hyperbolischer Räume in euklidische Räume, Monatsch Math. **59** 217–229.

  [大域的な等長埋め込み $H^2 \longrightarrow \mathbf{R}^6$, $H^n \longrightarrow \mathbf{R}^{6n-5}$ を構成. H^n 全体を有限次元ユークリッド空間に実現した最初の例です. ユーゴスラビアの数学者 Blanuša はこの他にも実に様々な等長埋め込み・はめ込みを構成しています. 例えば無限に延びた平坦なメビウスの帯の $\mathbf{R}^4$, S^4, H^4 への等長はめ込み等. 詳しくは [4] の p.1413 を参照して下さい.]

- J. Nash　1956　The imbedding problem for Riemannian manifolds, Ann. of Math. **63** 20–63.

  [Nash の埋め込み定理. コンパクト C^r-級 $(3 \leq r \leq \infty)$ リーマン多様体は $\mathbf{R}^{\frac{1}{2}n(3n+11)}$ へ C^r-級に等長埋め込み可能. 非コンパクトの場合は $\mathbf{R}^{\frac{1}{2}n(n+1)(3n+11)}$ への埋め込みが存在する. ここで $r = 2$ は除外されていることに注意.

The Essential John Nash (eds. H. W. Kuhn and S. Nasar, Princeton Univ. Press, 2002) p.209 の Author's Note に

In June 1998 I was notified by an e-mail from Professor R. M. Solovay of a fault in the arguments of the last part (part D) of my paper …… It seems, surprisingly, that before then no reader had actually detected the error! With regard to the question of repair or repairs of the error, I feel that the whole issue of what to do for non-compact manifolds has been changed by the contributions of Mikhail Gromov.

とあります．その後 ambient space $\mathbf{R}^N$ の次元の評価は R. E. Greene, C. J. S. Clarke, M. L. Gromov–V. A. Rokhlin, M. Günther 達により大幅に改良されました.]

- A. Lichnérowicz 1958 Géométrie des Groupes de Transformations, Dunod, Paris.

  [コンパクトなエルミート対称空間 $M = G/K$ の $\mathfrak{g}$ ($= G$ のリー環) への大域的等長埋め込みを構成．小林の埋め込みはこれを対称 R 空間に拡張したものにあたります.]

- E. R. Rozendorn 1960 Realization of the metric $ds^2 = du^2 + f^2(u)dv^2$ in five dimensional Euclidean space, Dokl. Akad. Nauk Armjan SSR **30** No.4 197–199.

  [H^2 の $\mathbf{R}^5$ への大域的な等長はめ込みを構成．露語．H^n の等長埋め込みに関しては，ロシアの人達の膨大な研究があります．ヒルベルトの結果 (1901) とあわせて，'H^2 は大域的に $\mathbf{R}^4$ へ等長はめ込み可能かどうか' が未解決の問題となります.]

- 高橋恒郎 1966 Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan **18** 380–385.

  [M の球面への極小な等長はめ込みと M のラプラシアンの固有関数との関係．この結果により等質空間 G/K の球面 (従ってユークリッド空間) への大域的な等長はめ込みが多く構成されます．N. R. Wallach, Minimal immersions of symmetric spaces into spheres, In: Symmetric Spaces (eds. W. M. Boothby, G. L. Weiss), Pure Appl. Math. **8**, Marcel Dekker (1972) 1–40 も参照して下さい．小林の埋め込みは，このようにして得られる対称 R 空間の埋め込みの中で最も次元の低いもの，ということもできます.]

- 小林昭七 1968 Isometric imbeddings of compact symmetric spaces, Tôhoku Math. J. **20** 21–25.

  [対称 R 空間のユークリッド空間への大域的な等長埋め込みを構成．これは群論的にみて様々な美しい性質を持った埋め込みです．これにより，多くの対称 R 空間はおおよそ 2×dim$M$ 次元のユークリッド空間へ等長に埋め込めることになります．カルタンの次元 '$\frac{1}{2}n(n+1)$' と比べ，対称 R 空間は極度に低い次元に実現されていることがわかります．特に $S^n = SO(n+1)/SO(n)$, $Sp(n)/U(n)$, $P^2(\mathbf{H}) = Sp(3)/Sp(2) \times Sp(1)$, $P^2(\mathbf{Cay}) = F_4/Spin(9)$, $Sp(n)$ については局所的にみても小林の標準埋め込みが最小次元の等長埋め込みを与えていることがわかります．またこの中で S^n ($n \geq 3$), $P^2(\mathbf{H})$, $P^2(\mathbf{Cay})$, $Sp(2)$ については局所的な剛性が成り立ちます．残りの対称 R 空間についても，小林の埋め込みが 'ほぼ' 最小次元の局所等長埋め込みを与えているのであろう，というのが現在の私達 (A+兼田) の感触です.

  コンパクト単純リー群の中で対称 R 空間となるのは古典型のものだけです．コンパクトな例外群についても球表現を利用して等長埋め込みを構成することができますが，古典群の場合に比べいずれも異様に高い次元への埋め込みとなります．例えば 248 次元の E_8 につい

てはカルタンの次元 $(=\frac{1}{2}n(n+1)=30876)$ より低い次元の埋め込みは知られていません. $SO(n)\subset\mathbf{R}^{n^2}$ 等と比べてみるに, 何かの仕組みがあって対称 R 空間とそうでない空間との違いが生じているのでしょうが, それが何であるのか私にはわかりません.]

- R. E. Greene　1970　Isometric embeddings of Riemannian and pseudo-Riemannian manifolds, Mem. Amer. Math. Soc. **97**.

  [擬リーマン多様体の不定値計量を持ったユークリッド空間への C^∞-級等長埋め込み論を展開.]

- M. L. Gromov, V. A. Rokhlin　1970　Embeddings and immersions in Riemannian geometry, Russian Math. Surveys **25** No.5 1–57.

  [この時点での等長埋め込み論を総括した大論文. Nash の埋め込み次元を (非コンパクトの場合も含め) $\frac{1}{2}(n+2)(n+5)$ まで下げています. これ以外にも実に様々な結果が記されています. その中からいくつかピックアップしますと,

  ・n 次元 C^∞-級リーマン多様体は局所的に $\mathbf{R}^{\frac{1}{2}n(n+1)+n}$ へ C^∞-級等長埋め込み可能. (上記 Greene の論文 (1970) では, 擬リーマン多様体の場合に同じ結果を独立に示しています.)

  ・n 次元リーマン多様体上の C^∞-級リーマン計量で局所的に $\mathbf{R}^{\frac{1}{2}n(n+1)-1}$ へ等長に埋め込めるもの全体は C^∞-級リーマン計量全体の成す空間の中で nowhere dense な集合となる. (話を簡単にするためここでは C^∞-級としましたが, 正確な主張と証明はこの論文の Appendix 1 を見て下さい.) この結果, 低次元のユークリッド空間に埋め込めるリーマン多様体は何らかの意味で特殊なリーマン多様体と言うことができます. しかしどのような意味で'特殊'なのか, それはおそらく曲率を使って表現されるはずの条件ですが, 残念ながらそれを具体的に表現する言葉を私達はまだ掌握していません. p.8 にも Without doubt, theorems on the local isometric non-embeddability of specific n-dimensional manifolds in $\mathbf{R}^{\frac{1}{2}n(n+1)-1}$ would be of great interest. Unfortunately, for $n>2$ such results are not known; and the existing results concern instead of $\mathbf{R}^{\frac{1}{2}n(n+1)-1}$ euclidean spaces of comparatively low dimension. とあります.

  ・任意の n 次元実解析的なリーマン多様体は $\mathbf{R}^{\frac{1}{2}(n+2)(n+5)}$ へ実解析的に大域等長埋め込み可能.

  ・$P^2(\mathbf{R})$ は $\mathbf{R}^3$ へ等長はめ込み不可能. n が 2 のべき乗のとき, $P^n(\mathbf{R})$ は $\mathbf{R}^{2n}$ へ等長埋め込み不可能. (従って Blanuša (1947) の結果と合わせて, $P^2(\mathbf{R})$ の大域等長埋め込み可能な最小次元は 5 であることが確定したことになります. しかしはめ込みについては $\mathbf{R}^4$ に C^∞-級等長はめ込み可能か否かが未解決問題です. また最近出版された論文 Z. Tang, Some existence and nonexistence results of isometric immersions of Riemannian manifolds, Comm. Contemp. Math. **6** (2004) 867–879 では, n が 2 のべき乗のとき正のスカラー曲率をもつ $P^n(\mathbf{R})$ は $\mathbf{R}^{2n-1}$ へ等長はめ込み不可能であることが示されています.)

  ・コンパクト 2 次元リーマン多様体は $\mathbf{R}^{10}$ へ大域的に C^∞-級等長埋め込み可能. 特に S^2 の場合, 任意のリーマン計量に対して $\mathbf{R}^7$ への大域的等長埋め込みが存在. 実解析的の場合にも同じ結果が成立する.

  また初めにも記しましたように, 歴史的な事柄についても詳しい記述があります.]

- R. E. Greene, H. Jacobowitz　1971　Analytic isometric embeddings, Ann. of Math. **93** 189–204.

[n 次元コンパクト実解析的なリーマン多様体は $\mathbf{R}^{\frac{1}{2}n(3n+11)}$ へ実解析的に大域等長埋め込み可能．]

- 田中昇　1973　Rigidity for elliptic isometric imbeddings, Nagoya Math. J. **51** 137–160.

  [コンパクトなリーマン多様体の楕円型等長埋め込みの剛性定理．無限小的剛性をもてば局所的な剛性が成り立つことを示しています．'局所的' と書きましたが，これは写像空間の中での局所という意味であって，M 自身はコンパクトであることに注意して下さい．用語がまぎらわしいので，正確な主張については原論文を参照して下さい．この剛性定理はコンパクトエルミート対称空間の標準埋め込みに適用できます．この定理の (多様体としての)'局所版' が兼田英二–田中昇 (Rigidity for isometric imbeddings, J. Math. Kyoto Univ. **18** (1978) 1–70) で示されています．この局所的な剛性定理も多くの対称 R 空間 G/K の標準埋め込みに適用できますが，G/K の群論的な性質により埋め込みの型に種々の違いが生じます.]

- S. S. Chern, J. Simons　1974　Characteristic forms and geometric invariants, Ann. of Math. **99** 48–69.

  [コンパクトで向き付け可能な 3 次元リーマン多様体の $\mathbf{R}^4$ への共形はめ込みの 2 次障害類を構成．例えば $P^3(\mathbf{R})$ は大域的に $\mathbf{R}^4$ へ共形はめ込み不可能．この不変量は J. L. Heitsch–H. B. Lawson,Jr, H. Donnelly, J. J. Millson, 坪井堅二達により様々な空間について計算され，種々の共形はめ込み不可能定理が得られています．Chern–Simons 不変量自身については，最近では別の文脈で語られることが多くなったようです.]

- J. Gasqui　1975　Sur l'existence d'immersions isométriques locales pour les variétés riemanniennes, J. Diff. Geom. **10** 61–84.

  [Janet–Cartan の埋め込み定理を H. Goldschmidt 流偏微分方程式論の formulation の下で再証明．兼田英二–田中昇 (1978) の Appendix にも別証明があります.]

- J. D. Moore　1977　Submanifolds of constant positive curvature I, Duke Math. J. **44** 449–484.

  [Chern–Simons 不変量を使って $P^3(\mathbf{R})$ は大域的に $\mathbf{R}^5$ へ等長に埋め込めないことを証明．Moore にはこの他にも等長埋め込み問題への様々な貢献があります．また Gromov–Rokhlin の所で引用した Tang の論文 (2004) によれば，$P^n(\mathbf{R})$ は $\mathbf{R}^{2n-1}$ へ等長にはめ込めないことを示したプレプリントがあるそうです.]

- J. Vilms　1977　Local isometric imbedding of Riemannian n-manifolds into Euclidean $(n+1)$-space, J. Diff. Geom. **12** 197–202.

  [n 次元リーマン多様体 ($n \geq 5$) が $\mathbf{R}^{n+1}$ へ局所等長的に埋め込めるための必要かつ十分な条件を曲率作用素が非退化という仮定の下で求めている.]

- J. Vargas　1981　A symmetric space of noncompact type has no equivariant isometric immersions into the Euclidean space, Proc. Amer. Math. Soc. **81** 149–150.

  [タイトル通りの論文．群論的手法では非コンパクト型対称空間の等長埋め込みは構成できないということで，ある意味で深刻な結果です．私の知る限り，非コンパクト型対称空間の中

で具体的な等長埋め込みが得られているのは負定曲率空間の場合だけのようです．一般の非コンパクト型対称空間に対して，ある程度の対称性をもった等長埋め込みの実例が何とかして作れないものでしょうか．]

- 江尻典雄　1981　Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc. **83** 759–763.

  [$P^3(\mathbf{R})$ は大域的に $\mathbf{R}^7$ へ等長にはめ込める．実際には 6 次の調和多項式を使った $S^3(1/16)$ から $S^6(1)$ への全実極小等長はめ込みの存在を示しています．間下克哉 (1985) はこのはめ込みの群論的な意味付けを明確にし，また F. Dillen–L. Verstraelen–L. Vrancken (1990) はこの 6 次式を具体的に書き下しました．Blanuša–小林により $P^3(\mathbf{R})$ は大域的に $\mathbf{R}^9$ へ等長に埋め込めることがわかっていますが，Moore の結果 (1977) と合わせても，大域的な等長埋め込み，あるいははめ込みが可能となるユークリッド空間の最小次元は $P^3(\mathbf{R})$ についてはまだ確定していません.]

- W. Henke　1981　Isometrische Immersionen des n-dim. Hyperblolischen Räumes H^n in E^{4n-3}, Manuscripta Math. **34** 265–278.

  [H^n の $\mathbf{R}^{4n-3}$ への大域的な等長はめ込みを構成．これが現在知られている H^n の最小次元の大域的等長はめこみです．有名な未解決問題として，‘H^n $(n \geq 3)$ の $\mathbf{R}^{2n-1}$ への大域的等長はめ込みは存在しないことを示せ’ というものがありますが，現時点では次元の差はまだまだ大きいと言わざるをえません．この問題に関しては Y. A. Aminov, J. D. Moore, F. Xavier, その他の人達の様々な貢献があります．Y. A. Nikolayevsky, Non-immersion theorem for a class of hyperbolic manifolds, Diff. Geom. its Appl. **9** (1998) 239–242 には，非単連結完備な n 次元負定曲率空間は $\mathbf{R}^{2n-1}$ へ等長にはめ込めないことが示されています.]

- E. J. Berger　1981　The Gauss map and isometric embedding, Ph. D. Thesis, Harvard Univ.

  [n 次元リーマン多様体のガウス方程式は余次元が $\frac{1}{2}(n-1)(n-2)+2$ 以上であれば必ず解をもつ．つまりガウス方程式だけならば，‘カルタンの次元’$-n = \frac{1}{2}n(n-1)$ より小さな次元で方程式が解けてしまうということです．逆の言い方をすれば generic なリーマン多様体をユークリッド空間に等長に埋め込むためには $\frac{1}{2}n(n-1)$ 以上の余次元が必要でしたから，ガウス方程式以外にも局所等長埋め込みの obstruction が存在しているということを意味します．なお，Berger の出したガウス方程式が解をもつ余次元の評価式は最良のものではないことに注意しておきます．例えば $n = 2, 3$ のとき解をもつ最小の余次元はそれぞれ 1, 2 となります．一般の n の場合の最小余次元はどのような式で表されるのでしょうか.]

- H. Jacobowitz　1982　Local isometric embeddings, Ann. of Math. Studies **102** 381–393.

  [2 次元 C^∞-級リーマン多様体において，一点 p におけるガウス曲率が 0 でなければ点 p の十分小さな開近傍は $\mathbf{R}^3$ へ C^∞-級等長に埋め込める．一般に C^∞-級のカテゴリーにおいて，n 次元多様体上の generic なリーマン計量は $\mathbf{R}^{\frac{1}{2}n(n+1)}$ へ局所等長埋め込み可能か，という偏微分方程式の問題があります．上記の結果は $n = 2$ のとき答は yes であることを示していますが，誰が最初にこれを示したのか確認できませんでした．Gromov–Rokhlin (1970) の p.7 には $K > 0$ のときは Bianchi が，$K < 0$ のときは Pogorelov が示した，と記されているのですが.]

- E. Berger, R. Bryant, P. Griffiths 1983 The Gauss equations and rigidity of isometric embeddings, Duke Math. J. **50** 803–892.

  [$\dim M \geq 8$ のとき，ユークリッド空間内の余次元 $\leq \dim M$ の generic なリーマン部分多様体 M は剛性をもつ．またかなり高い余次元のところまで，generic な等長埋め込みの変形は有限個のパラメータのみに依存するという主張が示されているようです．ただしここでいう generic とは，第二基本形式及び曲率の高階 jet がある代数多様体に含まれないという形で定義されるもので，具体的な埋め込みに対してそれが generic であるか否かを判定できるような代物ではなさそうです.]

- R. L. Bryant, P. A. Griffiths, D. Yang 1983 Characteristics and existence of isometric embeddings, Duke Math. J. **50** 893–994.

  [generic な 3 次元 C^∞-級リーマン多様体は $\mathbf{R}^6$ へ C^∞-級に局所等長埋め込み可能．$\mathbf{R}^6$ の 6 は今の場合丁度カルタンの次元 $\frac{1}{2}n(n+1)$ に一致します．またここでいう generic とは Ricci テンソルを用いて記述できる条件です.]

- C. S. Lin 1985 The local isometric embedding in $\mathbf{R}^3$ of 2-dimensional Riemannian manifolds with nonnegative curvature, J. Diff. Geom. **21** 213–230.

  [ガウス曲率が非負の 2 次元リーマン多様体は局所的に $\mathbf{R}^3$ へ等長に埋め込める．微分可能性の程度については原論文を参照して下さい．Lin はこの他にも同様の埋め込み定理を得ています (1986): 一点において $K = 0$, $\nabla K \neq 0$ ならばその点の開近傍は $\mathbf{R}^3$ へ等長に埋め込める．中村玄 (1987) は更に $K = \nabla K = 0$, $\mathrm{Hess}\, K < 0$ のときでも同じ結果の得られることを示しました．一方，局所的にすら $\mathbf{R}^3$ に等長に埋め込めない 2 次元リーマン多様体を構成したというプレプリントがあります (N. Nadirashvili–Y. Yuan, math.DG/0208127).]

- M. L. Gromov 1986 Partial Differential Relations, Springer.

  [大域的な等長埋め込みの次元の評価を改良．現在知られている最良の評価は，M. Günther, Isometric embeddings of Riemannian manifolds, Proc. Intern. Congr. Math. Kyoto Vol.II, Math. Soc. Japan (1991) 1137–1143 にある結果のようです：n (≥ 5) 次元 C^∞-級リーマン多様体は $\mathbf{R}^{\frac{1}{2}n(n+5)}$ へ大域的に C^∞-級等長埋め込み可能．局所的に等長埋め込み可能な次元との差がほとんど無いところまで評価が改良されました．Günther の得た次元は一般論としては最良の値といってよいのでしょうか.]

- 中村玄, 前田吉昭 1989 Local smooth isometric embeddings of low dimensional Riemannian manifolds into Euclidean spaces, Trans. Amer. Math. Soc. **313** 1–51.

  [3 次元 C^∞-級リーマン多様体において一点 p における曲率が 0 でなければ p の開近傍は $\mathbf{R}^6$ へ C^∞-級に局所等長埋め込み可能．R. L. Bryant–P. A. Griffiths–D. Yang (1983) の仮定を弱めた形の埋め込み定理といえます．このような結果の高次元化は難しいということだそうです.]

- H. J. Rivertz 1999 On isometric and Conformal Immersions into Riemannian Manifolds, Ph. D. Thesis, Univ. Oslo.

  [$M^3 \subset \mathbf{R}^4$, $M^5 \subset \mathbf{R}^7$ の場合について，曲率及びその共変微分の満たすべき恒等式を計算機を用いて具体的に求めた．与えられた M^n に対して $\mathbf{R}^{n+r}$ への局所等長埋め込みの存在・非

存在を示すには，同様の関係式あるいは不等式を次元・余次元の高い場合にも求める必要があります．しかし，これは多変数版の‘終結式’を具体的に求めよ，という古典的不変式論において難問とされた問題にあたり，現実には(計算機を用いたとしても)実行不可能であるかもしれません．全く別の方向からのアプローチが必要かもしれません．一見出鱈目に出現しているかのように見える不変式にも実はある構造が隠されていて，それが等長埋め込みの問題にも生かせるのではないか，と私は信じているのですが.]

以上です．講演２日目に紹介した対称空間に関する私達(A+兼田)の結果については，最近出版された‘数学’の論説を見ていただけたら幸いです．解かねばならない問題が多くある，というのが私達の実感するところです．

小林昭七–野水克己, Foundations of Differential Geometry Vol.II, John Wiley & Sons (1969) の p.355 に

Once we know that every Riemannian manifold can be isometrically imbedded in a Euclidean space of sufficiently large dimension, we naturally seek for a Euclidean space of smallest possible dimension in which a Riemannian manifold can be isometrically imbedded.

と記されています．最後にこの問題に関する幾何学者の発言を二つ引用して，この小史(外史)を閉じることにしましょう．

However, if we pose the question of the greatest possible lowering of the codimension $N-p$, then even in the local formulation the problem is far from completely solved, and in the global formulation presented above we are only at the first steps of the development. "The problem of immersing a Riemannian metric in Euclidean space", said A. D. Aleksandrov in one of his public lectures (Moscow State University, May 1970), "is a tangle of non-linear problems". (From p.101 of E. R. Rozendorn, Surfaces of negative curvature, Geometry III, Encyclopaedia of Math. Sci. Vol.**48**, (eds. Y. D. Burago, V. A. Zalgaller), Springer, 1992.)

In the past we have had some very special results about the non-existence of isometric imbeddings of certain Riemannian manifolds in other Riemannian manifolds. For example, a compact surface of everywhere negative curvature cannot be isometrically imbedded, or even immersed in $\mathbb{R}^3$, nor can a complete surface of constant negative curvature be isometrically immersed in $\mathbb{R}^3$. Ideally, differential geometry should be replete with such results, so that we could have a reasonable chance of finding the smallest dimensional Euclidean space into which a given Riemannian manifold can be isometrically imbedded. But at present only quite isolated facts are known, and a general theory can hardly be said to exist. (From p.192 of M. Spivak, A Comprehensive Introduction to Differential Geometry Vol.V, Publish or Perish, 1975.)

対称リーマン空間の局所等長埋め込み

	G/K	$\dim G/K$	class (G/K)
AI	$SU(n)/SO(n)$ $(n \geq 3)$	$\frac{1}{2}(n-1)(n+2)$	$\frac{1}{2}n(n-1) \sim \frac{1}{2}n(n+1)+1$
AII	$SU(2n)/Sp(n)$ $(n \geq 3)$	$(n-1)(2n+1)$	$n(n-1) \sim n(2n-1)+1$
$AIII$	$P^2(\mathbf{C})$	4	$3 \sim 4$
	$P^n(\mathbf{C})$ $(n \geq 3)$	$2n$	$2n-2 \sim n^2$
	$SU(p+2)/S(U(p)\times U(2))$ $(p \geq 2)$	$4p$	$\min\{[\frac{1}{2}(7p-3)], 3p+1\} \sim p^2+3$
	$SU(p+q)/S(U(p)\times U(q))$ $(p \geq q \geq 3)$	$2pq$	$2pq-p-q+\min\{p-q,2\}$ $\sim p^2+q^2-1$
BDI	$*Q^3(\mathbf{C}) \simeq Sp(2)/U(2)$	6	4
	$Q^n(\mathbf{C})$ $(n \geq 4)$	$2n$	$[\frac{1}{5}(6n+2)] \sim \frac{1}{2}n(n-1)+1$
	$SO(p+q)/SO(p)\times SO(q)$ $(p \geq q \geq 3)$	pq	$pq-p+\min\{p-q,1\}$ $\sim \frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)-1$
$BDII$	$*S^n$ $(n \geq 2)$	n	1
	$*H^n$ $(n \geq 2)$	n	$n-1$
CI	$*Sp(n)/U(n)$ $(n \geq 1)$	$n(n+1)$	n^2
CII	$*P^2(\mathbf{H})$	8	6
	$P^n(\mathbf{H})$ $(n \geq 3)$	$4n$	$4n-3 \sim n(2n-1)$
	$Sp(p+2)/Sp(p)\times Sp(2)$ $(p \geq 2)$	$8p$	$\min\{8p-6, 7p-1\} \sim p(2p-1)+5$
	$Sp(p+q)/Sp(p)\times Sp(q)$ $(p \geq q \geq 3)$	$4pq$	$4pq-p-3q+\min\{p-q,4\}$ $\sim p(2p-1)+q(2q-1)-1$
$DIII$	$SO(8)/U(4) \simeq Q^6(\mathbf{C})$	12	$7 \sim 16$
	$SO(2n)/U(n)$ $(n \geq 5)$	$n(n-1)$	$\frac{1}{2}n(n-1) \sim n^2$

対称リーマン空間の局所等長埋め込み (続)

	G/K	$\dim G/K$	class (G/K)
EI	$E_6/Sp(4)$	42	$36 \sim 660$
EII	$E_6/SU(2)\cdot SU(6)$	40	$20 \sim 610$
$EIII$	$E_6/Spin(10)\cdot SO(2)$	32	$16 \sim 46$
EIV	E_6/F_4	26	$12 \sim 28$
EV	$E_7/SU(8)$	70	$63 \sim 1393$
EVI	$E_7/Spin(12)\cdot SU(2)$	64	$32 \sim 1475$
$EVII$	$E_7/E_6\cdot SO(2)$	54	$27 \sim 79$
$EVIII$	$E_8/Spin(16)$	128	120 以上
EIX	$E_8/E_7\cdot SU(2)$	112	$56 \sim 3763$
FI	$F_4/Sp(3)\cdot SU(2)$	28	$24 \sim 296$
FII	$*F_4/Spin(9) = P^2(\mathbf{Cay})$	16	10
G	$G_2/SO(4)$	8	$6 \sim 19$
	$SO(n)$	$\frac{1}{2}n(n-1)$	$\begin{cases} \frac{1}{2}n(n-1)-2\left[\frac{n}{2}\right] \sim \frac{1}{2}n(n+1) & (n=7,8,9) \\ \frac{1}{2}n(n-5)+1 \sim \frac{1}{2}n(n+1) & (n \geq 10) \end{cases}$
	$U(n)$	n^2	$\begin{cases} n^2-\left[\frac{3}{2}n\right] \sim n^2 & (n=3,4,5) \\ n^2-2n \sim n^2 & (n \geq 6) \end{cases}$
	$*Sp(n)\quad (n \geq 1)$	$n(2n+1)$	$n(2n-1)$
	E_6	78	$62 \sim 1380$
	E_7	133	$106 \sim 3003$
	E_8	248	212 以上
	F_4	52	$43 \sim 624$
	G_2	14	$10 \sim 35$

• クラス数 class (G/K) は，対称リーマン空間 G/K が $\mathbf{R}^N$ に局所等長に埋め込み可能となる最小余次元を示す．

• G/K の前の $*$ は，クラス数 class (G/K) が確定していることを示す．これらの空間のうち，$S^2 = Sp(1)/U(1)$, H^n $(n \geq 2)$ 以外については，最小次元の局所等長埋め込みは剛性をもつことが示せた．