算数と数学の接続における
2つの一般化に関する開発研究

平成17～18年度文部科学省科学研究費補助金・特定領域研究(2)
「新世紀型理数科系教育の展開研究」
A01「教育内容と学習の適時性に関する研究」
課題番号：17011049

研究成果報告書

平成19(2007)年3月

研究代表者 岩 崎 秀 樹
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は し が き

やや厚くなったが，この冊子は平成14年度より平成18年度にかけて実施された文部科学省科学研究費補助金「特定領域研究」新世紀型理数科系教育の展開研究（領域代表者：増本 健）の下でなされた，５年間の研究の成果報告である。本特定領域研究には総額21億円にも及ぶ巨費が投じられており，通常の科研にはない組織性と，全体を統括する一貫した研究テーマがある。

「新世紀型理数科系教育の展開研究」は，《児童生徒の「科学技術離れ」「理科離れ」》という今日的な教育課題から出発し，わが国の《「科学技術創造立国」を実現》するには，《将来の科学技術を担う児童生徒に科学技術や理科・数学に対する興味・関心を培い，自然についての知的好奇心，探求心を高め，論理的思考力や創造性を伸ばし，将来有意な人材を育成》しなければならないという立場から，証拠に基づく教育科学の展開を推進した。研究期間の５年は３期に分けられ，各期ごとに研究が公募されている。第１期には75件，第２期には79件，第３期には78件の研究課題が採択され，それぞれ研究班として特定領域に参集した。これらの研究班はA01からA05の５つの研究グループに分かれ，次に示す研究テーマの下で組織された。

A01「教育内容と学習の適時性に関する研究」
A02「論理的思考力や創造性，独創性を育むための教育内容や指導方法，教材等の研究」
A03「ITを活用した新たなカリキュラムの研究」
A04「ITを利用した先進的で実効性の高い，教授・学習システムの研究」
A05「情報化など，社会や学校の変化が児童生徒の心身の発達や理数科教育への学習意欲等に及ぼす影響及び対応に関する研究」

われわれの研究班はA01グループに属しているが，A01は３期の各期ごとに８件，８件，７件の研究班によって，編成されている。これらの研究班は次に示す３つの研究テーマの下で組織された。

１．学習指導要領に記載されている内容についての適時性に関する研究
２．新しい教育内容についての開発研究
３．カリキュラムの体系的な研究

以上われわれの親組織に当たる「新世紀型理数科系教育の展開研究」のフレームワークと直属の研究グループが担当する研究テーマを示してきたが，われわれのそれは，以下に示すように，上記１と２，つまり教育内容の適時性とその開発研究に該当する。すでに述べたように，この５年間は３期に分けられ，そのつど新たに研究計画調書を作成して応募することになったが，幸い３期とも採用され，親組織とともに最終年度を迎えることができた。３期それぞれの研究課題名をあげると次のようになる。

第１期　課題番号14022233
小学算数を中学数学に接続する分数による除法に関する学習指導の開発研究

第２期　課題番号15020241

算数を数学に接続する一般化に基づく教授単元の計画・実施・評価に関する開発研究

第３期　課題番号17011049

算数と数学の接続における２つの一般化に関する開発研究

上記３つの課題名に共通する鍵概念を括り出せば，研究対象としての「算数と数学の接続」であり，研究方法としての「一般化」であり，研究成果を示唆する「開発研究」である。われわれの課題意識は，一言でいえば数学的認識の質的変換を可能にする教材構成の原理にあった。そこに後で述べるように日本独自の制度的視点を加えて，「接続」という形で問題化した。すなわち「教育内容と学習の適時性」について，一般化の視座から教材の構成原理にアプローチしたといってよい。５年にわたる本研究の成果をまとめれば，次の２点に集約される。

１．小学算数と中学数学の接続を念頭に置きながら，移行の本質を記号論の視座から明らかにする枠組みを構築した。

４．移行を決定づける教材を開発した上で，その指導のあり方を提案し，その有効性および妥当性を実証的に検証した。

開発研究の成果をリストアップすれば，表１にあげる接続教材にまとめることができる。

表１　算数から数学への移行を促す教材

	移行前期	移行後期
算術から代数	分数による除法	正負の数の減法
図形から幾何	図形の相互関係	図形の作図

一つひとつの接続教材に関する考察は本書に収録された小論を読んでいただくとよいので，ここでは「接続」の背景にある日本独自の制度的視点と「一般化」の意義といったものについて簡単に触れておきたい。

グローバルな視野から見れば，「算数」という教科名よりは「数学」の方が一般的である。東南アジアの漢字文化圏ですら，今日では初等レベルの教科名に「算数」ではなく「数学」を用いている。したがって戦後のように学校システムが単線ともなれば，むしろ制度的な連続性からこれを「理科」と同じように，「数学」に統合すべきかもしれない。しかし教科にとってより本質的な課題は，そうした制度的整合性を意識することではなく，子どもの認識発達に寄与することであって，小学校と中学校における教科名の違いも，単なる戦前の歴史的・制度的な残滓ではなく，教科の内部的論理に基づく，学習段階の違いとして理解すべきであろう。このように理解するなら，「教育内容と学習の適時性」は「接続」の問題として捉えることができよう。

次に，われわれは「接続」の方法的視座として「一般化」を据えたが，それは数学の内部的な論理に基づくばかりでなく，数学の外にある現代社会の要請とみなせるからである。高度情報化社会に直面する今日，おびただしい情報の中で適切に判断するためには，多くの情報の核になる概念を形成することが，不可欠といってよい。すなわち限られた知識で様々な具体に適応できる知の形成であり，これを数学

では，一般化と呼んでいる。算数・数学の学習指導では教科の本質から，こうした知的態度の育成がめざされているものの，理論的に十分な考察がなされているとはいえない。これが一般化の視座から，学習過程を記述する理論的枠組みと教授過程をデザインする規範的枠組みの開発が，急がれなければならない理由である。

５年にわたって本研究を続けてこれたが，そのコツなりツボをいえといわれれば，企業秘密なのであまりいいたくないが，それはただ一つ，海外から著名な数学教育研究者を招聘し，毎年ワークショップを開催した点につきるであろう。この５年で次の方々を広島大学にお呼びすることができた。どの方をとっても，数学教育研究で世界に通用する業績をあげられた研究者である。

平成14年度：Prof. Dr. Willi Dörfler (University of Klagenfurt, Austria)
平成15年度：Prof. Dr. Norma Presmeg (Illinois State University, USA)
平成16年度：Prof. Dr. Erich. Ch. Wittmann (University of Dortmund, Germany)
平成17年度：Prof. Dr. Heinz Steinbring (Universität Duisburg-Essen, Germany)
平成18年度：Prof. Dr. Erich. Ch. Wittmann と Prof. Dr. Willi Dörfler の２名

こうした研究者をお呼びした成果は「第３部　招聘研究者論文編」として本書に収められているが，それは全体からみればほんの一端にすぎない。海外からお呼びするのであるから，かなり念入りな計画と面倒なペーパーワークや交渉を必要としたが，その苦労を遙かに凌駕する成果や効果や影響を，有形・無形に享受できたと考えている。

有形の部分は本書に収められた「第１部　和文編」と「第２部　英文編」の論文の数々に代表されるであろう。また無形の部分，むしろこちらの方が大であるが，これを３点に絞れば次のようになる。第１は，上述の研究者を招聘することによって，本研究課題に関する最新の研究成果を得た点である。第２は，ワークショップにおける発表や議論を通じて，招聘した研究者から，本プロジェクトの研究成果に関するレビューをはじめ，今後の研究の方向性に関するコメントを得ることができた点である。そして３番目に，多くの若手研究者にもワークショップでの発表や参加を呼びかけ，彼らに様々な角度からアカデミカルな刺激を与えることができた点である。３点目は国際ワークショップの波及効果といえるが，本プロジェクトのメンバーばかりでなく若手研究者にも，海外の著名な研究者との研究上の交流の機会をはかれたこと，また研究に対する彼らの真摯な姿勢や人間性に触れることができた点は，何事にも代えられない貴重な経験であった，と自負している。

最後に上記のワークショップを通じて得た，研究のあり方について付記して，やや長めの「はしがき」を閉じたい。

伝統的な学問が社会と時代の要請に応えきれなくなったとき，新たな学問が生まれる。無から有が生じるわけがなく，既存の学問の再編がそれに応えなければならない。したがって複合的な新領域の特徴は，伝統的な学問的基準からすれば「いかがわしさ」にある。しかしそうした批判は甘んじて受けるべきであり，その批判への真摯な対応が学として次の飛躍の発条になる。むしろ既存の学的規範に無批判に追従することこそ，警戒すべきであろう。

現代が抱える教育課題はその現実を反映して複雑である。しかし複雑さは安易に単純化されるべきでなく，それは複雑さによって正しく語られるべきであろう。そ

の意味で今後ますます数学教育研究は，数学と教育以外にも多様な学問的成分を含むことになる。「数学教育学」を研究の核にして複合的研究の内実を高めるには，対話型の研究は不可欠である。また教育研究である以上，程度の差こそあれ実践を視野に収めぬ理論研究は無意味であり，また理論的基盤を持たない実践研究は「学」の名に値しない。しかしそもそも理論と実践の調整が一人の研究者によってなされるとは思えないし，たとえなされたとしてもはなはだ覚束ない議論になりはしないか。理論と実践のあるべき関係は，複数の研究者によってこそ可能であり，その時はじめて独善に陥いることなく，両者の間に健全な関係が構築されると思う。

このように考えてくれば，ワークショップのような研究形態は，数学教育という複合的な研究領域にとって不可欠な研究手法といえると思う。それを自覚的に活用し，様々なアイディアをある課題の下で編集していく研究者たちとトポスは，数学教育学の新たな地平を切り開いていくにちがいない。そこに集う若い研究者の斬新な目差しと多彩な対話とコラボレーションによって，数学教育学の新たな展開が期待できると考えている。

最後になったが本書をこのような形に編集できたのも，研究分担者の山口武志さんの尽力に負うとことがほとんどであった。ここにその名を記して謝意を表したい。

平成19年３月 10 日

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【研　究　経　費】

平成１７年度	3,200 千円
平成１８年度	4,200 千円
合　　計	7,400 千円

【本特定領域研究に関する助成実績】

特定領域研究「新世紀型理数科系教育の展開研究」は，平成 14 年度から平成 18 年度までの５年間にわたる継続研究である。平成 14 年度から平成 16 年度の３年間についても，下記のような研究題目の下で，本特定領域研究に参画した。

○平成 14 年度　課題番号：１４０２２２３３
　小学算数を中学数学に接続する分数による除法に関する学習指導の開発研究
　　［研究経費：計 2,500 千円］
○平成 15 ～ 16 年度　課題番号：１５０２０２４１
　算数を数学に接続する一般化に基づく教授単元の計画・実施・評価に関する開発研究
　　［研究経費：計 4,800 千円（内訳　平成 15 年度：2,000 千円，平成 16 年度：2,800 千円）］

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【国際ワークショップの実績】

５年間にわたる研究においては，海外の著名な数学教育研究者を招聘した上で，国際ワークショップを計５回開催した。ワークショップで招聘した研究者とその講演題目は，下記のとおりである。

平成 14 年度：Prof. Dr. Willi Dörfler (University of Klagenfurt, Austria)
(1)The process of generalization: Theory and examples
(2)Protocol and diagrams: Means and product of generalization and abstraction

平成 15 年度：Prof. Dr. Norma Presmeg (Illinois State University, USA)
(1) Semiotic as a theoretical framework for linking mathematics in and out of school: Significance of semiotics for teachers of mathematics
(2)Lesson study characterized as a multi-tiered teaching experiment

平成 16 年度：Prof. Dr. Erich. Ch. Wittmann (University of Dortmund, Germany)
(1)From the multiplication table to the solution of quadratic equations
(2)Empirical research centered around substantial learning environments

平成 17 年度：Prof. Dr. Heinz Steinbring (Universität Duisburg-Essen, Germany)
(1)Children's construction of new mathematical knowledge: Analysis of classroom episodes from primary teaching
(2)What makes a sign a mathematical sign? : An epistemological perspective on mathematical interaction

平成 18 年度：Prof. Dr. Erich. Ch. Wittmann と Prof. Dr. Willi Dörfler の２名
Prof. Dr. Erich. Ch. Wittmann (University of Dortmund, Germany)
Mathematics as the science of patterns: A guideline for developing mathematical education from childhood to adulthood
Prof. Dr. Willi Dörfler (University of Klagenfurt, Austria)
Mathematical reasoning: Mental activity or practice with diagrams

【目　　次】

【第3部　招聘研究者論文編】(アルファベット順)

第　1　部

和　文　編

一般化分岐モデルに基づく分数除の教授・学習に関する研究

岩崎　秀樹（広島大学）・山口　武志（福岡教育大学）

要　約

同じ計算ではあっても，質的に異なる計算方法を想定することは可能である。「÷分数」はその好例であり，除法という形式的普遍性を維持しながら，「÷自然数」の発展としてそれを計算することも十分可能である。他方，いわゆる「×逆数」という計算アルゴリズムを新たに構成しながら計算を工夫することも可能である。この場合，「÷自然数」の形式的普遍性を逸脱しながら，除法を乗法の特殊とみなすことが理解されなければならない。筆者らは，両者の違いを一般化の質的相違ととらえ，前者を「内包的一般化」，後者を「外延的一般化」ととらえる。

本研究では，計算の意味や手続きの理解過程を一般化という視座から考察しながら，Dörfler に基づく「一般化分岐モデル」を提案する。その下で，「÷小数」と「÷分数」の達成度における乖離の要因を指摘する。さらに，二量の比例関係に基づく現行の「÷分数」指導に代わる対案を提案しながら，実験授業を通じて，その有効性を例証する。

1. 数学教育の今日的課題:算数と数学の接続

戦後の教育改革とともに，最初の6年の教科「算数」とその後の3年の教科「数学」とは，義務教育によって接続を余儀なくされた。1970 年代には現代化を背景に，数学から算数への乗り入れが図られたが，結果として「失敗」という総括が加えられたことはなお記憶に新しい。一方 1998 年の学習指導要領の改訂では「学習内容の先送り」という形で，実質的に算数から数学への乗り入れがなされたが，その理論的整備は十分とは思えない。

グローバルな視野からみれば，「算数」という教科名よりは「数学」の方が一般的であり，東南アジアの漢字文化圏ですら，今日では初等レベルで「数学」を用いている。したがって戦後のように学校システムが単線化すれば，むしろ制度的な連続性からこれを理科のように「数学」に統合すべきかもしれない。しかし教科にとってより本質的な課題は，そうした制度的整合性を意識することではなく，子どもの認識発達に寄与することであって，小学校と中学校における教科名の違いも，単なる戦前の歴史的残滓ではなく，教科の内部的論理に基づく，学習段階の違いとして理解すべきであろう。

実際 Piaget の明らかにした具体的操作と形式的操作の発達段階の相違(Piaget,1970, pp.42-51;メイヤ,1985,pp.151-172)は，学習内容の適時性や算数と数学の区分原理になりうるが，数学の教授・学習を念頭に設定されたわけでもなければ，算数と数学の接続について述べているわけでもない。その意味で，算数から数学への接続という視座から，比例的推論(proportional reasoning)の果たす役割に言及した Lesh らの次の指摘は示唆に富む。

《われわれは比例的推論をきわめて重要な概念と考える。つまりそれは算数(elementary school arithmetic)の頂点を飾る石(capstone)であると同時に，数学の礎 (cornerstone)とな

るからである。この章では移行(transition)のメカニズムと生徒の活動の2つの視点から，比例的推論の構成について議論する。》(Lesh, Post & Behr, 1988,pp.93-94)

Lesh らは，比例的推論が算数と数学を接続する鍵概念であることを主張しているが，彼らの関心はあくまで有理数概念の形成過程に果たす比例的推論の役割であって，算数を数学につなぐ教授原理にあったわけではない。彼らの用語を用いれば，「飾る石(capstone)」が「礎 (cornerstone)」に変成するプロセスに，我々の関心はある。

本研究では，次の2点を研究の目的とする。まず一般化の視座から，上記の課題に応えたいと考える。その理論的基盤は，抽象を一般化の前段に置き，両者の有機的連関を図る Dörfler の一般化理論に依拠する。しかし氏の一般化モデルを批判的に考察して「一般化分岐モデル」を新たに提唱し，認知的な視座から，内包的一般化と外延的一般化の役割を明確に区分する。第2の目的は，外延的一般化に基づく，教材の構成と教授・学習過程の分析にある。本研究では，教育課程実施状況調査において課題が指摘されている分数除（分数でわる除法）（以下，「÷分数」）をとりあげ，一般化分岐モデルの下で，「÷分数」の困難性の要因を明らかにする。

以上の分析をふまえ，「÷分数」指導に関する新たな枠組みを提案する。さらに，その枠組みに基づく教授実験を実施・評価しながら，枠組みの妥当性，有効性を検証する。これら一連の考察を通じて，接続に関する移行教材という視座から，「÷分数」教材の新たな価値を指摘し，指導への示唆を得たい。このことは一般化分岐モデルの記述性および規範性を明らかにすることと考えている。

2. Dörfler の一般化モデル

数学的な概念形成において決定的な役割を果たす抽象と一般化は，相補的な関係にある（中原,1995,pp.50-54）。見方を換えれば，互いに他を制約し，しかも容易に止揚を許すものではない。しかし，認識論的な視座からみれば，両者の統合を図らなければ，そもそも認識の更新や知の発明などありえない。いわば論理的な相補関係をエピステモロジカルなダイナミズムに組み替えていくシステムが Dörfler の一般化である，と考える。つまり，以下で詳述するように，Dörfler のいう一般化は，「記号の対象化」を転轍機としながら，個人的な「わたし」の認識を，共有された「われわれ」の認識に接続する過程とみることができる。言い換えれば，「記号の対象化」を前後して，主観的認識は客観的認識へと昇華することになる。Dörfler の一般化モデルを批判的に考察するに先立ち，ここでは，氏のとらえる数学的一般化について概観しておきたい。

Piaget が数学的認識を根拠に，経験的には成立しえない抽象すなわち反省的抽象を指摘したように，Dörfler は数学的概念を，経験的には成立しえない一般化すなわち理論的一般化の成果としている。両者に共通するのは，客体の経験ではなく，主体の活動に，抽象や一般化が始まるという点である。ただし Dörfler は反省的抽象ではカバーしきれない認識の社会的側面を，記号使用とそこに潜在する変数性に求め，反省的抽象を構成的抽象に再定式化した上で，それに一般化を接続させている。いわば活動の反省が抽象を誘導するように，記号の反省を可能にする認知機構を一般化とし，両者を統合する形で，数学的一般化を構想した（図1を参照）。

前段の構成的抽象について，認知論的な視座から説明を加えていけば次のようになる。数学の認知的起源は，活動と対象間の関係認識にある。したがって数学的概念は活動シェマと関係シェマからなる(Dörfler,1984a,ss.245-246)。両シェマは並行関係にあって，互いに他を制約し発展させ，相補的な関係をあくまで維持する。こうしたシェマは多数の経験から定着していくものではなく，教授学的に意図されたプロトタイプ（範例）の下で生起する（註１）。数学的対象と数学的操作は，プロトタイプにおける活動や関係意識の純粋な抽象ではなく，すなわち単なる捨象ではなく，記号的付加の結果と考えるべきであり，それ故この認知プロセスを，「構成的抽象」と名づけている(Dörfler,1986,p.149; 1987,p.4)。

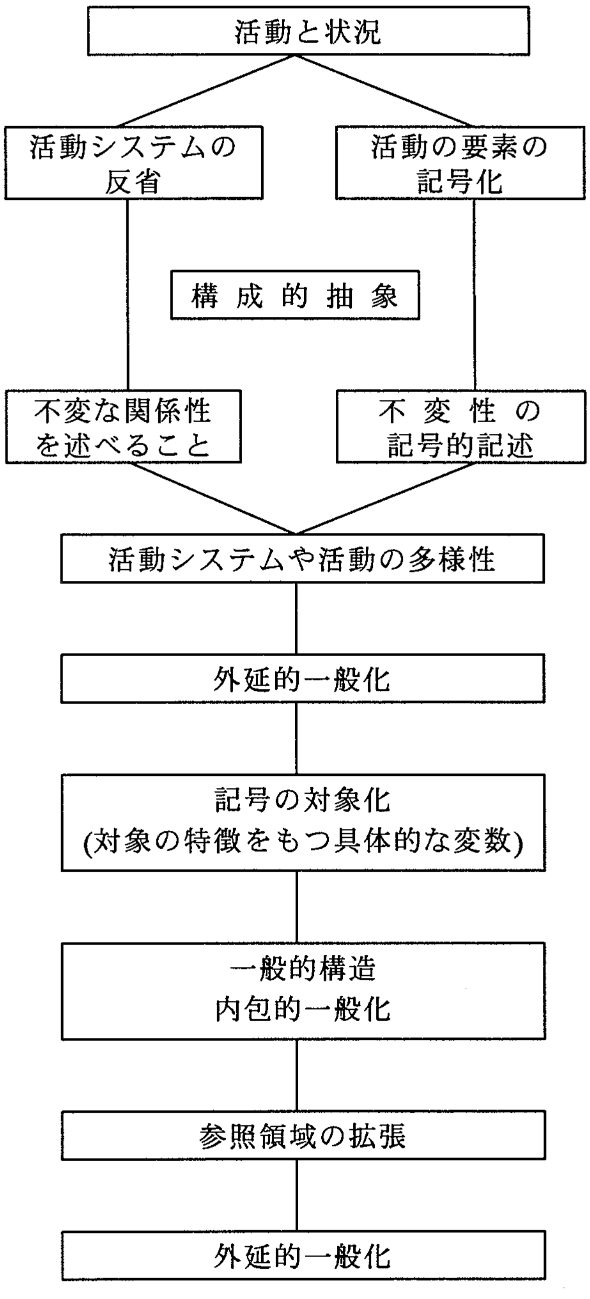


図１　Dörfler の一般化モデル(1991a,p.74)

さらに後段の一般化に認識論的な視座から説明を加えていけば次のようになる。構成的抽象は「わたし」の活動に裏打ちされているため，動詞的事態ないし認知心理学の水準といえるが，「わたし」の認識を「われわれ」の認識に変換するため，プロトコルが必要になる（註２）。すなわち構成的抽象の記号化あるいは名詞化がこれである。名詞化によって，自己は他者と交渉可能になるばかりでなく，認識は空間的に広範な具体性を確保でき，時間とは独立の不変性を手に入れる(Dörfler,1988,ss.77-78)。その確信を深めることが，Dörfler の印象的な言葉づかいを借用すれば，「抽象から具体への上昇(Aufsteigens vom Abstrackten zum Konkreten)」(Dörfler,1984a, s.260)ということであろう。

すなわち構成的抽象によってえた記号がある種の具体性を備えて，対象化を許す点にある。換言すれば，Freudenthal や Treffers のいう水平的数学化への起点となり，様々な具体へと適用領域を拡張する。このことは同時に記号の変数性を加速し，氏らのいう垂直的数学化への足がかりを与える。要するに記号的抽象を記号的な具体におきかえることで，数学的概念の内部的構造の拡充が始まる。いわば抽象は具体に変化し，さらなる抽象の基盤になる。そこには「わたし」の認識が，ポパー(1978)のいう世界２や世界３に開かれていくプロセス(註３)が示されており，Dörfler はこの開かれた知的プロセスに一般化を見いだしている。

以上が Dörfler の一般化に関する，多年にわたる堅密な論攷と込み入った議論(1983,1984a, 1984b,1985,1986,1987,1988,1991a,1991b,2000)を要約したものである。そして，こうした一連の研究の集大成として，Dörfler は，氏のとらえる一般化の過程を図１のようにモデル化することとなる(Dörfler,1991a,p.74; 岩崎・山口,2000)。

Dörfler の一般化モデルの特徴は，第１に活動から始まる点にある。第２にその記号過程を「構成的抽象」とし，これを基盤に一般化が展開される点にある。氏は構成的抽象も一般化に組み込むが，従来の一般化の議論では，論理学的に制約されるのが普通で，構成的抽象を除外した内包的一般化ないし外延的一般化について考察されてきただけといってよい。したがってその視野には，そもそも活動は収められてはいなかった。

Dörfler は出発点となる活動に，具体的なものばかりでなく，心的活動や記号的活動も含めている。まず活動の目的や手段や方針などが反省されて，一連の活動の構成要素の特徴や構成要素間の関係に注意が向けられる。氏はこの心理過程に注目し，「内包的抽象(intensionaler Abstraktion)」という術語すら準備している(1988,s.81)。内包的抽象の下に置かれる関係は，活動の繰り返しによって安定し不変なものとなる。この不変な状態を記述するために，言葉や図や記号が必要となる。したがって，活動の不変性を言葉や図や記号で記述する記号過程は経験的な諸属性の捨象のプロセスではなく，むしろ数学的特性の構成のプロセスといってよく，そのため「構成的抽象」と呼ばれる(1991a, p. 71)。

活動は構成的抽象によって操作に変換される。その結果，活動と操作は対照可能となり，表１のようにまとめられる。さらにいえば，両者の一つひとつの項目的対比すなわち特殊－一般，具体－抽象，実体－記号，内容－形式などが，数学の本性を浮き彫りにしているといえよう。それ故数学を内包的に捉えるなら Piaget のいうように，操作の構造は数学の構造に同型となり，数学は操作の体系となる(Beth & Piaget,1966,pp.163-190; Piaget,1970,pp.21-40;ピアジェ,1972, pp.26-49)。

表１　活動と操作(Dörfler ,1988, s.99)

活動(Handlung)	操作(Operation)
特殊(speziell)	一般(allgemein)
具体的(konkret)	抽象的(abstrackt)
外的(äußerlich)	内的(innerlich)
実体的(gegenständlich)	記号的(symbolisch)
内容的(inhaltlich)	形式的(formal)
経験的(empirisch)	理論的(theoretisch)
特殊的(spezifisch)	シェマ的(schematisch)

操作の諸属性は「記号的(symbolisch)」側面を発揮することで，数学的特性の端緒を開示する。とりわけ記号化された不変性が一般性をもつためには，記号が様々な活動の代替性をもつことが必要になる。いわば記号が，探求の画面(search-screen)として機能することが要請される。これを Dörfler は一種の「外延的一般化(extensional generalization)」と呼んでいる。

やがて不変性の記号的記述が反省されるようになる。そのため記号は参照領域から切り離されて，記号自体が対象化される。この段階は「記号の対象化(symbols as objects)」と呼ばれ，これをもって構成的抽象は完了するとしている。したがって「記号の対象化」は構成的抽象の終点であり，同時にその後の一般化への始点となる。

記号は対象の性質をもった「変数」であるため，内包は一般的構造をもつ。この構造に注目するとき，「内包的一般化(intensional generalization)」が成立する。ここで構成された一般性は，さらなる活動を導き，参照領域を拡張する。そして上位の外延的一般化へとつながる。これ以降必要に応じて，一般化は循環し，記号の対象化，内包的一般化，参照領域の拡張が繰り返される，と考える。それ故，一般化は開かれた認識となる。

図１は，こうした一般化のプロセスをまとめたものである。図１は一般化という認識作用の全体像と，注目すべき認知活動を適切に順序づけているが，認知的変容を促す心理過程は，直線で示されているに過ぎず，明示されてはいない。そのため，岩崎・山口(1998,2000)では，こうした直線部分の認知的変容をメタ認知的視座から補完する試みがなされている。ただし，メタ認知は，本来，問題解決の文脈で，問題を解決する以前の「わたし」の個人的営為を解明すべく開発された概念であって，問題解決以後の発展的文脈を視野に収めているわけではない。さらに学習集団を前提とする一般化の文脈を想定していたわけでもない。上述のように，「構成的抽象」に続く一般化のプロセスにおいては，「記号の対象化」を転轍機として，「わたし」の認識から「われわれ」の認識への昇華が図られる。したがって，「わたし」を「われわれ」にスイッチする認知過程を明確にするために，メタ認知概念の拡張を図る必然性が生じる。「拡張されたメタ認知」によって，図１を認知論的に補完することが可能となり，それによって，一般化を推進する認知過程を記述することが可能になる（註４）。

3. Dörfler の一般化モデルに関する批判的考察

(1)2つの一般化

無論初等段階であれば，計算指導は特殊・具体の相の下で図られる。計算結果の伴わない計算など，この段階ではありえない。だからといって，計算練習を繰り返せば，そのアルゴリズムの理解に至るものではない。しかるべき具体的場面をしつらえ，その計算結果ではなく，その処理過程つまり形式や方法に意識をシフトさせなければ，理解したことにはならない。その結果，特殊・具体を越えて，任意の数式の計算が可能になる。しかしこうした形式や方法を通した一般化にも，「÷分数」のように，２通りの場合がある。

［計算①］

$$\frac{5}{7}\div\frac{2}{3}=\frac{15}{21}\div\frac{14}{21}=1\cdots\frac{1}{21}$$

［計算②］

$$\frac{5}{7}\div\frac{2}{3}=\frac{5}{7}\times\frac{3}{2}=\frac{15}{14}$$

端的にいえば，①は既有の「÷自然数」の拡充であり，いわゆる除法の定義から「整数」という条件を解除することで可能になる。無論分数のこのケースは，算数の教材にはならないが，小数の場合は教材になっていることを確認しておきたい。こうした知的操作は，あえていえば自然数的な思考空間を拡張した，といえよう。一方②は除法が除法ではなく，乗法であることを主張している。いわば既有の知識の否定であって，同時に既有の知識にない知を創造することで，乗除の統合が果たされる。その結果，「分数」は「有理数」に変化し，既有の数概念は有理数の下で新たに再編される。このように，一般化の視座からみれば，両者の計算過程は質的に異なると考えることが可能である。

小数も分数も有理数であるが，その表記は大きく異なる。つまり，前者は十進位取り記数法の小数点以下への合理的展開であり，後者は自然数に基づく新たな有理数の構成である。それ故，小学校の算数において，小数の四則演算はそのまま保持されるが，分数では，除法を乗法の特殊とみなすことで，その演算は加法，減法，乗法へとまとめられる。

数学的思考の１つの本質は，その記号的特性にあるから，こうした記号の内的構造の差違が一般化の過程に影響を与えることは十分に予想される。前述の Dörfler (1991a)は，すでにこの点を強く意識しており，図１の「記号の対象化」にそれが反映されている。ただ，記号の内的構造にともなう一般化の質的な相違が，図１のモデルにおいて「記号の対象化」に続く一般化の過程に反映するという点は重要である。しかし，記号の内的構造の差違が「記号の対象化」に続く一般化の質的相違を左右するほど重要であるにもかかわらず，氏は内包的一般化と外延的一般化を一般化モデルにおいて区別しているものの，「記号の対象化」の後には，唯一，内包的一般化を置いているにすぎない。さらに，氏の論考では，内包的一般化と外延的一般化は明確に規定されておらず，両者の規定は論理学的規定を超えるものとは思えない。つまり，氏のいう内包的一般化は，性質（命題）の一般化といってよく，外延的一般化は対象（集合）の一般化と考えられる（註５）。そのため，記号の内的構造の差違による一般化の質的差違がモデルに十分反映されているとはいえない。見方を変えれば，記号の対象化に続く一般化が内包的である理由については，モデルに基づく説明であるため，「外延的でない」理由にはなっていない。要するに一般化のプロセスが内包的なものから始まる理由は，モデルから導けない。また，図１において，第１の外延的一般化と第２のそれとの違いは明らかでないし，さらにいえば「外延的一般化」と「参照領域の拡張」すらその違いは明らかとはいえない。

ただし，後者の問題は技術的な処理で，論理的に解決できよう。Dörfler のモデルの最大の特徴は，抽象に一般化を接続した点にある。そのため抽象の内部に一般化のあることはそもそも不自然であり，そのため第１の「外延的一般化」を「参照領域の拡張」に置き換えれば，この点は解消され，構成的抽象として一貫したものになる。さらにこの措置で，第１と第２の外延的一般化の区分も不要となる。しかし後半の問題は，外延的一般化と内包的一般化の規定なしには，解決しえない。

そこで一般化をやや広い概念である数学化に重ねて，内包的一般化と外延的一般化の相違や両者の規定を探るなら，２つの数学化すなわち垂直的な数学化と水平的な数学化の区別や規定で代替できるように思える。しかし数学化を数学教育で最初に強調した Freudenthal は，氏の最後の著書で２つの数学化について次のように述べている。

《Freudenthal は Treffers の２つの数学化の区別を採用して，それぞれの規定を次のように述べている。水平的に数学化するとは，日常の世界から記号の世界に進むことであり，垂直的に数学化するとは，記号の世界における移動をさす。》(Van den Heuvel-Panhuizen, M., 2003, p.12)

要するに水平的数学化は一般化モデルの構成的抽象に対応し，垂直的数学化は記号の対象化以降の認識をさしているにすぎず，内包的一般化と外延的一般化の区別を含むことにはならない。それでは一般化をそれより狭い概念に重ねたとしても，両者の比較や違いにまで至らないであろう。考えてみれば数学化という望遠鏡は一般化をみるにはマクロであり，一方顕微鏡を見つけたとしてもミクロすぎて役にはたたないであろう。したがって全く別の角度から，２つの一般化への近接を図らなければならない。

(2)一般化分岐モデル

こうした一般化の質的差違を明確にする上で，「創造工学(synectics)」の知見は示唆的である。創造工学は《人間の創造活動に存在する，前意識的な心理的メカニズムを，意識して使うための実践的理論》（ゴードン,1964,「序」,p.1）として提唱されたものである。創始者の Gordon によれば，創造的思考の基本的メカニズムには「異質馴化」と「馴質異化」の２つがあるという（ゴードン,1964,pp.31-59）。このうち，前者は「見慣れないものを見慣れたものにすること(making the strange familiar)」という心的機序であり，《見慣れないものを前から知っている資料と照合し，それによって見慣れないものを見慣れたものに変えてしまう》（ゴードン,1964,p.32）ことを意味する。一方，後者は「見慣れたものを見慣れないものにすること(making the familiar strange)」という心的機序であり，《前からある同じ世界，概念，感情，事物を意識的に新しい角度から見ようとする》（ゴードン,1964,p.33）ことを意味する。

勿論，両者は日常場面の差違に着目した区別であり，数学教育を念頭においてなされた区別ではない。しかし，日常場面の差違を記号の内的構造の差違に読みかえることで，「記号の対象化」に続く認知過程を区別する視点になり得るものと考える。

これらを２つの一般化に合わせるなら，異質馴化は次のような思考に相当しよう。つまり，未知の記号内部の構造的連関が，既有の知識構造に落とされて，既有知識の内部的充実を図る思考である。見慣れた記号世界の内包を切りつめ精度を上げることで，思考の空間的幅を拡げようとする操作と考えるなら，異質馴化を内包的一般化の賓辞にするのは妥

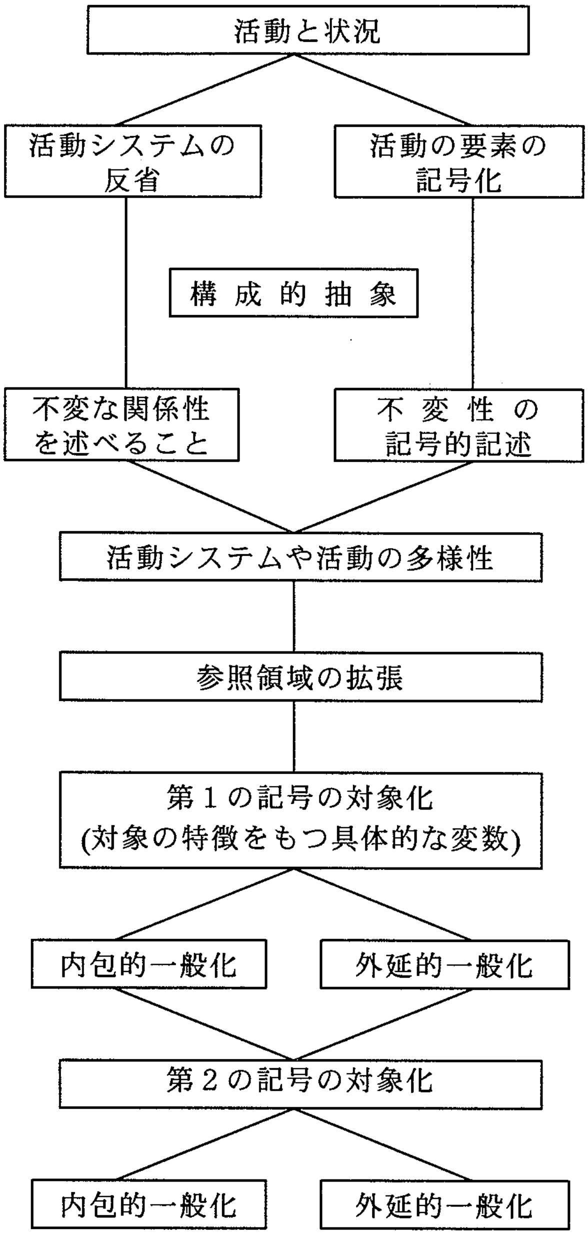


図2　Dörfler に基づく一般化分岐モデル

当といえよう。一方馴質異化は記号内部の構造的連関を，既有の知識構造に落とさずに，新たに基点を設けることで，様々な既有知識の総合と構造化を図るというものである。見慣れた記号的世界が新たな記号的連関を追加することによって，全体的な眺望の開ける様子は，馴質異化を外延的一般化の賓辞にするにふさわしいと考える。

上述の視座から，改めて内包的一般化と外延的一般化を規定すると，次のようになる。

<u>内包的一般化</u>：既知の対象を普遍化することによる一般化。対象となっている記号に含意された意味を，既有の知識に関連づけながら同化し，既有の知識を発展させる認知プロセスとしてとらえられる一般化。

外延的一般化：記号の内的構造に基づいて，未知の対象を構成するような一般化。記号に内在する意味を既有の知識に同化させることができないので，新たな知識を構成し，その知識の下で，既有の知識を統合する認知プロセスとしてとらえられる一般化。

このように内包的一般化と外延的一般化の規定を設ければ，いずれか一方が他方に先行する理由はなく，むしろ記号の対象化における学習状況に応じて，①のように内包的一般化に進む「÷分数」の理解もあれば，②のように外延的一般化に進む理解もあると考える方が自然である。他方，数学は記号に基づく科学であり，形式と内容にわたって絶えざる記号的更新を果たしていかねばならない。そのため幾度も「記号の対象化」は生起すると考える方が妥当である。以上のようなプランと概念の規定と発想を図１の一般化モデルに加えれば，新たに図２のような分岐型の一般化モデルが構想される。

4．一般化分岐モデルに基づく「分数除」の意味理解に関する考察

(1)教育課程実施状況調査

一般に，計算のアルゴリズムは，多くの場合，既習内容からの類推や，いくつかの具体からの帰納によって導出され，アルゴリズム化される。その意味で，計算の意味や手続きの理解の過程は，一般化の過程とみなすことができよう。

Vergnaud(1983)は，相互に関連する概念の獲得を別々に研究することは困難であるばかりか，不合理でもあるとしながら，概念間の有機的な関連に注目する全体論的な枠組みを「概念領域(conceptual field)」と命名した（岩合,1984）。例えば，乗法や除法といった演算は，小数や分数や比，比例，有理数などとともに「乗法構造」とよばれる１つの概念領域を構成する。そうした全体性の中で，それまで孤立した状態にあった除法と乗法は関連し合い，除法は乗法の特殊という論理的関係すら成立する。

しかし，「乗法構造」の理解は，いわば完成された概念領域であり，概念形成過程における子どもの理解が複雑に分岐していることは十分に予想される。例えば，平成５年度に実施された文部省（当時）の「教育課程実施状況調査」（以下，「実施状況調査」，註６）では，以下で詳述するように，小数と分数による乗除の立式に関する問題が出題され，その分析結果は，子どもたちの様々な乗法構造の有り様を報告している。

実施状況調査では，「小数をかける乗法」(以下，「×小数」) と「分数をかける乗法」(以下，「×分数」) の立式に関して，学年をかえて，次のような問題が設定されている。

(第5学年－B3)

1 l のねだんが 650 円のペンキがあります。このペンキを 0.7 l 買ったときの代金はいくらですか。答えを求める式を □ の中に書きましょう。（□ は省略）

(第6学年－A2)

じろうさんは，かべをペンキでぬろうとしています。1 l のペンキで 9 m^2 のかべをぬることができます。$\frac{3}{4}l$ のペンキでは，何 m^2 のかべをぬることができますか。答えを求める式を □ の中に書きましょう。（□ は省略）

実施状況調査に関する報告書（文部省初等中等教育局,1997,pp.81-82;pp.112-113）に基づいて，これら２題の調査結果をまとめれば，表２のようになる。なお，「小計」欄は，表に示した正答と誤答の割合の和を示している。

表２　「×小数」および「×分数」に関する調査結果（単位：％）

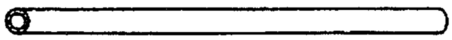
	正答	誤答（除法）	小計
×小数	66.0	19.7	85.7
×分数	64.8	17.3	82.1

両者とも通過率は65％前後となっており，おおむね良好との評価を得ている。また，典型的な誤答としては，両者とも，除法の立式を行っている子どもの割合が多く，両者の和はほぼ全体をカバーしているように思える。

他方，除法についても，次に示すような「小数でわる除法」（以下，「÷小数」）と「÷分数」の立式にかかわる達成度が調査されている。

（第５学年－A4）

次の図のような鉄のパイプがあります。長さは3.5mで，重さは4.2kgです。このパイプ１mの重さは何kgですか。答えを求める式を □ の中に書きましょう。（□ は省略）



（第６学年－B2）

水そうに水を入れています。$\frac{2}{3}$分間に$\frac{5}{6}l$ の水が入ります。同じ割合で水を入れていくと，１分間では何 l の水が入りますか。答えを求める式を □ の中に書きましょう。（□ は省略）

これら２題の調査結果は表３のようにまとめられる（文部省初等中等教育局,1997, pp.52-53; pp.141- 142）。「÷小数」の通過率は65.9％であり，おおむね良好と評価されている。それに対し，「÷分数」の通過率は27.2％と低く，「÷小数」のそれとの間には大きな差がある。

表３　「÷小数」および「÷分数」に関する調査結果（単位：％）

	正答	誤答		小計
		除数と被除数が逆	乗法	
÷小数	65.9	9.2	10.1	85.2
÷分数	27.2	20.1	18.1	65.4

誤答については，「÷小数」と「÷分数」の双方において，次の２種類が，ほぼ１対１の割合で同定されている。その１つは，除法の式ではあっても，除数と被除数が逆になっ

ている誤答であり，もう１つは，乗法の式になっている誤答である。特に，「÷分数」においては，これら２つの誤答が，いずれも約 20%となっている。通過率が 27.2%であったことを考えれば，これらは決して看過することのできない数値である。また，前者の誤答は，少なくとも除法の場面を意識しているわけであるから，後者とは質的に異なるともいえる。

(2) 乗法構造における「分数除」

Vergnaud(1983)は，乗法構造にかかわる問題を「量の同型(isomorphism of measures)」「量の積(product of measures)」「多重比例(multiple proportion)」の３つに分類している。このうち，「量の同型」とは，２つの測度空間 M_1，M_2 に属する４項間に，図３のような比例関係が成立する構造を意味しており，上記４題は，いずれも「量の同型」に該当する問題である。つまり，「×小数」および「×分数」は，a×p によって未知数 b を求める問題であり，「÷小数」および「÷分数」は，b÷p によって未知数 a を求める問題ということになる。

M_1	M_2
1	a
p	b

図３　「量の同型」の構造

実施状況調査の結果で興味深いことは，「÷分数」に関する子どもの意味理解の実態が，乗法や除法という演算の区別を超えて，きわめて特異であるという点である。表２や表３の「小計」欄が示すように，正誤はともかく，「÷分数」を除く３題では，少なくとも 80%以上の子どもが，問題場面に乗法構造を認めている。つまり，乗除の認識という点でみても，「÷分数」は，他の３題に比べて特異な位置を占めている。

ここで，「÷小数」と「÷分数」に注目するなら，これらの文章題は２量の比例関係に基づいて単位量を求める問題であって，図３の Vergnaud の枠組みに照らせば，構造的には同じであるといってよい。にもかかわらず，６年生の通過率や乗除認識は５年生のそれらより低く，なおかつこれほどまでに大きな差違が生じるのは何故であろうか。

小数・分数に限らず除法の意味理解を担うのは問題場面であり，通常それは，２文か３文からなる文章である。その問題場面は，マクロにみればどれも比例関係を含意しており，ミクロにみれば，様々な構成要素からなる。しかし文章による問題場面であれば，その構成要素も多岐にわたるものではなく，ある種の原型性を備えている。通過率の差違を構成要素の差違に求めるのは自然であるから，次の４つの構成要素について，その有無や差違あるいはその組み合わせを考えるだけで十分であろう。

①問題場面　　②除数の大きさ　　③図の併記　　④数値の形態

問題を構成する量が可視であったり不可視であったりすることで，通過率の変化は期待できようが，被験者が 16000 人にものぼり，それぞれが様々な問題場面で学習していることを考慮すれば，それが全体の通過率に与える影響は，「÷小数」と「÷分数」の通過率の差を埋めるほど，劇的であるとは思えない。②の除数の大きさについては，中学１年を対象に，小規模ながら追調査をおこなってみた（註７）。調査問題は，実施状況調査で出

された問題をそのままに，除数を1より大きい場合にあらためて，仮分数と帯分数の場合の2種類を出題した。調査の結果，通過率が高いのは除数が帯分数の場合であるが，それも約34％であって，①の場合と同様のことがいえよう。③の図を併記した場合，文字情報以外に視覚的な情報が増加することになり，問題解決にとって有利にはたらく。平成13年2月実施の同調査では，問題場面は異なるが数値を同一にして，図を併記したところ，その通過率は43.8％で，17ポイント近い伸びを示している（国立教育政策研究所教育課程研究センター,2003,pp.103-105）。この数値は図の問題解決に与える影響が重要であることを明らかにしているが，それでもなお，「÷小数」と「÷分数」の通過率の差を埋めるに十分な値とはいえない。④の数値の形態的違いについていえば，表2が示すように乗法の場合その差はほとんどない。一方，表3をみれば，両者の差は歴然としてある。数字の形態はアルゴリズムに直接響くため，何故に乗法と除法でこれほどの変化が生じるのか，教授学的にみて興味の尽きない課題といえよう。

(3)意味と形式の相互連関

こうした「÷分数」の立式の困難性や特異性を検討する上で，次の点を考慮することは重要であると考える。つまり，「÷小数」は除法であるが，「÷分数」は乗法になるという点である。つまり，「÷分数」においては，「ひっくり返してかける」（以下，「×逆数」）という「形式」が，「÷分数」の「意味」と結びつかなければ，「÷分数」の十分な理解には到底達し得ない。さらにいえば，ここで問題にすべきことは，どのように「形式」と「意味」が結びつくかという点であり，「÷小数」と「÷分数」において，その違いは何かということである。

算数の式はどれも数値形式であるため，計算に直結している。中学1年の生徒がフレーズ型の文字式を何とか計算しようとするのは，式と計算のつながりがいかに強く刷り込まれているかということを示唆している。すなわち計算において概念的理解と手続き的理解は不可分であり，意味の理解が形式の理解を促し，同時に形式の理解は意味の理解を確実なものにする。ただし加減乗除が加減乗除である限りこの状況は変わらないが，四則が学年進行とともに二則に統合されるとき，状況は異なった様相を呈する。すなわち手続きは単純化するが，その理解は込み入ったものになる。その結果，アルゴリズムの理解が伴わないのに計算はできる一方で，それ故に計算の意味理解が不確かになる，という事態が現れる。表3下段の数値はこれを雄弁に示しており，いかに多くの児童が除数と被除数の区分も定かでなく，乗除の区別も定まらないかということを伝えている。

いささか論理的に上記の事態をまとめれば，次のようになろう。分数と小数の除法に関し，形式の理解ができれば意味の理解は可能であるが，その裏は真とは限らない。すなわち「÷小数」の場合，その形式の理解は既有の乗法構造の中で，整数の除法として読み解くことができるが，「÷分数」の場合，その算法は「×逆数」になるため，既有の乗法構造を拡張しなければ，形式の理解に到達しない。前者では量の同型の下，形式の理解が意味理解の十分条件になるが，後者についていえば，形式の理解が十分でなければ，意味理解も確かではないであろう，としかいえない。裏が必ずしも真とはならない以上，この点をさらに明確にする必要が生じる。

算数教科書をみればわかることだが，整数における加減乗除の導入は，どれも具体的な

問題場面とセンテンス型の式から始まる。フレーズ型の式を立てさせてから，「計算しましょう」という順序ではない。このことは計算理解の初期段階では，意味に形式が加えられていくといってよく，その点で意味と形式は不可分にある。ところが小数や分数の加減乗除では事態はやや異なり，問題場面がフレーズ型の式で記され，その後で計算の仕方が指導される。しかも単にアルゴリズムの記銘だけが図られるのではなく，その理解が文脈の中で図られるため，形式に関する理解は意味の理解にフィードバックされるといってよく，したがって形式の理解がなければ，意味の理解も減衰することは，十分に予想される。言い換えれば，意味理解はアルゴリズムの形成と不可分であり，立式だけをその目安とすべきではない。このことから，実施状況調査で明らかにされた差違は，立式に不可分なアルゴリズムの理解によると考えることが可能であり，またそのように考える方が，数学教育的にはるかに生産的である，と考える。

(4)一般化分岐モデルに基づく「分数除」の分析

問題解決の過程は学習過程の短時間の再現と考えてよいであろう。大規模の調査であれば，問題解決過程に混在する偶発性は相殺されて，出力されたデータは学習過程に潜在する問題を示唆する重要な指標になる。実施状況調査の「÷小数」と「÷分数」の学習過程を「一般化分岐モデル」（図２）に基づいて分析し，それに通過率を重ねていけば，それぞれの問題点は，明確になると考える。

そこでまず図２の「状況」は文章で示される問題場面であるが，「活動」は学習という意識的活動の意識（註８）を明確にするプロセスと考えてよいであろう。したがって初発の段階では主客の区別はなく，状況と不可分な活動を全体として意識するか，あるいはその部分で意識するかで，左右の分岐が生じる。それぞれの意識的活動は状況と共振して，前者は「一方の量が変化すれば他方の量も変化する」という不変な関係性を述べることになり，後者は同種の量の比較つまり比によって，不変性が記される。全体と部分の分岐は比例関係によって統合され，比例関係は乗除の２通り，乗数・被乗数と除数・被除数の順序を入れれば４通りの式に意識される。その中の１つに意識を絞るには，状況の数値を整数・小数・分数にすることで，対象とすべき数式への方向性が決まる。

この段階でひとまず構成的抽象は完了するが，「÷小数」であろうと「÷分数」であろうと比例で学習する意識的活動に異同があるわけではない。そのため「÷小数」と「÷分数」の立式の通過率にあれほどの差がつくことは，他の要因を入れない限り，説明はつかない。その要因として問題場面や説明図の併記などが考えられたが，それらが決定的な要因になりえないことは，すでに述べたとおりである。かわって形式の理解が，場を共有することで，意味の理解に再起的に還元されるとすれば，通過率の本質的な差に「記号の対象化」に続くアルゴリズムの理解は決定的な意味をもつ。

「÷小数」の場合，そのアルゴリズムは「÷自然数」の場合に埋め込まれて，その形式的不変性を確認することで理解が成立する。子どもたちは未知のアルゴリズムを既知のアルゴリズムに同化させて，除法を拡張したことになる。その意味で「÷小数」のアルゴリズムは図２の内包的一般化に向かうといってよい。一方「÷分数」の場合，除法ではなく乗法であることが主張され，それが理解されなければならない。「主張する」とは，既に述べたように，除法として振る舞うことも可能だからである。しかし新たに有理数への展

開を図るなら，やはりそれは「×逆数」でなければならない。その意味で「÷分数」のアルゴリズムは，図２の外延的一般化に向かうといってよい。

以上のように考えれば，第５学年児は自然数の中で成功的に「÷小数」を読み解いているのであって，それが通過率 65.9％に現れているといえよう。一方第６学年児は有理数において成功的に「÷分数」を読み解けなかった，ということになる。結果からいえば当然のことであり，理解を支える数領域を欠いたまま論理だけを頼りに「×逆数」を導き，乗法になることを納得したとしても，理解の場がなければ，技能の習熟だけが強調され意味理解の決定的な阻害要因になるであろう。

5. 移行教材としての「分数除」

(1) 比例的推論に基づく「分数除」指導の課題

一般化分岐モデルに基づいて，「÷分数」と「÷小数」の困難性の違いは明確になったが，そのことによって，「÷分数」指導に関する具体的な改善が明らかになったわけではない。「÷分数」は，従来から難教材であるといわれており，カリキュラムというグローバルな視座からも，教科書に現れる具体的な指導法という視座からも，様々な改善が加えられてきたといえよう。

カリキュラム面での改善について，学習指導要領に基づきながら，従来の「÷分数」指導の取り扱いの変化を概観してみた場合，生活単元学習期の学習指導要領は別として，数学の系統性に基づくカリキュラム構成を原則とするようになった昭和 33 年，昭和 43 年，昭和 52 年告示の学習指導要領では，分数の除法に関する内容は第５学年と第６学年の２学年にわたって位置づけられている。具体的には，第５学年では，除数が整数の場合のみを扱い，第６学年において，除数が分数の場合を扱うこととなっている。その後，平成元年，平成 10 年告示の学習指導要領では，第６学年において両者を一括して扱うように改訂されている（小山,2000,pp.53-54）。ただし，学習指導要領をみる限りでは，そのような改訂の原理まで示されているわけではない。

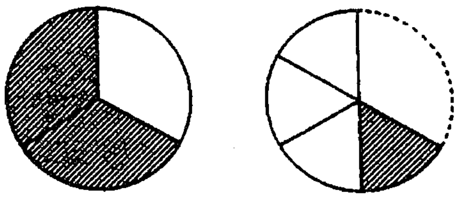
次に，指導法であるが，現行の教科書では，比の第３用法に基づく問題場面を提示した上で，「÷分数」の式を立式させ，さらに，面積図や線分図を駆使しながら，「×逆数」を導く方法がとられている。勿論，問題場面や数値の違いはあるが，立式にしろ，「×逆数」のアルゴリズムにしろ，２量の比例関係に基づく単位量を求める問題が基本的に採用されている。換言すれば，「等分のスキーマ」（等分除）に基づく学習指導が原則として採用されていることになる。

一方，従来の教科書の中には，ある量が別の量の何倍になっているかを求める問題場面を導入として採用しているものもみられる（加藤他,1971,p.16;山口寿穂,2000,p.22）。これは，比の第１用法に基づく導入といえ，いわゆる「測定のスキーマ（包含除）」に基づくものである。このように，数学的にみて，従来の指導には，等分のスキーマあるいは測定のスキーマの双方に基づく学習指導がみられるが，今日的には，比例的推論(proportional reasoning)に基づく「÷分数」指導がほぼ定着しているとみることができる。

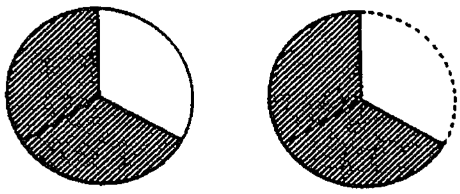
ただ，ここで注目したい点は，「÷分数」の場面であっても，等分のスキーマ（等分除）や測定のスキーマ（包含除）を調節しながら，除法としてそれを処理することも十分考えられるという点である。例えば，等分のスキーマの場合，リンゴを等分する問題場面にお

いて，以下のような図を示しながら，整数の除法との外挿的な類比によって，「$\div \frac{1}{4}$」の意味づけをある程度認めることも可能である。「$\frac{1}{4}$人」の解釈の問題はあるけれども，これも一種の等分のスキーマの調節といえよう。

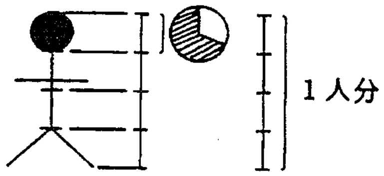
①$\frac{2}{3} \div 4$：「$\frac{2}{3}$個のリンゴを４人で分けたときの１人分」



②$\frac{2}{3} \div 1$：「$\frac{2}{3}$個のリンゴを１人で分けたときの１人分」



③$\frac{2}{3} \div \frac{1}{4}$：「$\frac{2}{3}$個のリンゴを$\frac{1}{4}$人で分けたときの１人分」



同様に，$\frac{6}{7}$の大きさのリンゴの中に$\frac{2}{5}$の大きさのリンゴが何個含まれているかという問いに対して，各々を$\frac{30}{35}$，$\frac{14}{35}$と通分した上で，測定のスキーマを調節しながら，「$\frac{6}{7} \div \frac{2}{5}$」を意味づけることも可能である。その場合，複雑な余りを考える煩わしさを伴うことになる。

このように，既存のスキーマにおける同化と調節によって，「÷分数」を意味づけし，答えをそのスキーマの中で構成することは可能であるかもしれないが，「÷分数」を「×逆数」に組み替えていく方向性や推進力はそこに期待できない。したがって，「÷分数」の計算方法を，「×逆数」にするには，等分と測定のスキーマとは別のスキーマで，「÷分数」の意味づけと「×逆数」の構成を図らなければならない。それが「比例」のスキーマであり，教科書で通常取りあげられている説明法がこれにあたる。

しかし，ここで問題になるのは，比例のスキーマに基づく学習の文脈が，「÷小数」の

意味理解すなわち立式にとって有効ではあっても，「÷分数」のそれには必ずしも機能しないという現実である。上述の実施状況調査の結果はそれを端的に示している。現行の比例に基づく指導では，2量の比例関係に帰着させながら，立式とアルゴリズムとしての「×逆数」の双方の理解を図っているが，上述のような「÷分数」の意味理解に関する問題に直面するとき，その解決策として現行の教授・学習とは異なる新たな学習展開を検討することが必要ではないであろうか。

(2)「分数除」指導の1つの対案

上述のように，「÷小数」ならびに「÷分数」に関する立式及びそのアルゴリズムの指導は，伝統的に2量の比例関係に基づいてなされるが，その意味で，比例的推論は両者の意味理解のいわば前提となっている。しかし，実施状況調査の結果は，比例の文脈において，「÷小数」にかかわる内包的一般化はある程度成功するが，「÷分数」にかかわる外延的一般化はうまくいかないことを示している。つまり，一般化分岐モデルによって，実施状況調査における「÷小数」と「÷分数」の達成度の違いを説明することが可能になっただけではなく，比例のスキーマに基づく「÷分数」の指導の困難さが明らかにされたといえよう。

このような問題を解消するためには，既有の除法スキーマのように，「÷分数」の立式が自然に出る場面と，「×逆数」が明瞭に導ける論理を準備することが必要不可欠である，と考える。その際，分岐モデルが次の点を示唆していたことは重要である。つまり，「÷分数」の教材としての１つの価値が，分数を有理数に展開する契機を提供する点にあるということである。Lesh 流にいうならば，「÷分数」を小学校算数の頂点(capstone)とみるだけではなく，数学学習への礎(cornerstone)とみるならば，そうした教育目標に見合った教授・学習の新たな枠組みが考えられてもよい，と考える。さらに，こうした枠組みの下で，「÷分数」は，算数を数学に接続する移行教材としての価値を有することになる，と考える。

本研究では，このような視座から，比例のスキーマに基づく現行の教授・学習にかわって，次のような指導の対案を提案したい。つまり，立式のためには「比較」のスキーマを前提にし，「×逆数」の説明には，既有の数学的知識を仮定する教授学的介入である。

表４　比例的推論に基づく「÷分数」指導と対案に基づく「÷分数」指導の比較

	比例的推論に基づく「÷分数」指導	対案に基づく「÷分数」指導
立式の指導	比例のスキーマ	比較のスキーマ
「×逆数」の指導	比例のスキーマ	分数や除法の性質に基づく演繹的推論

表４は，従来の比例的推論に基づく指導と対案に基づく指導とを比較したものである。比較のスキーマは，数学的には「差」と「比」によって表現される。比をとる場合，倍概念によって，除法は自然に導かれる。例えば，「$\frac{3}{5}$ mは$\frac{2}{7}$ mの何倍ですか」という問題場

面から，比較のスキーマによって，式「$\frac{3}{5}\div\frac{2}{7}$」を導くことは容易であろう。問題は比較のスキーマには，等分や測定や比例のスキーマのように，スキーマ自体に式変形ないし計算の牽引力がない点である。２量はスキーマによって併置される式になるが，除数が分数のため計算できない。そのため，ある種の数学的な知識をこのスキーマに付加して，「×逆数」を構成しなければならない。

たとえば，除法の性質や分数の性質といった，既有の数学的知識から，以下のように「×逆数」を導くことが想定される。

①$\frac{3}{5}\div\frac{2}{7}=(\frac{3}{5}\times 35)\div(\frac{2}{7}\times 35)$
$=(3\times 7)\div(2\times 5)=\frac{3\times 7}{2\times 5}=\frac{3}{5}\times\frac{7}{2}$

②$\frac{3}{5}\div\frac{2}{7}=\frac{3\times 14}{5\times 14}\div\frac{2}{7}$
$=\frac{3\times(14\div 2)}{5\times(14\div 7)}=\frac{3\times 7}{5\times 2}=\frac{3}{5}\times\frac{7}{2}$

①は，「わる数とわられる数に同じ数をかけても商が変わらない」という除法の性質に基づく説明である。また②は，「分子と分母に同じ数をかけても分数は変わらない」という分数の性質を用いつつ，さらに分数の乗法からの類推によって，「分母どうし，分子どうしをわる」ことを創意しなければならない。

こうしたアイデアは，数学的操作や概念に関する性質であるため，日常の経験や量に関係するわけではなく，そのため意味に煩わされず，論理的に，しかも比例の説明に比べればはるかに明瞭に「×逆数」が姿を現すことになる。

6. 対案に基づく教授実験の実際と考察

(1)「分数除」の立式に関する子どもの実態

「÷分数」の学習過程を「÷小数」のそれとの比較の下で，分岐モデルによって批判的に考察した結果，理解の基盤が脆弱であれば，意味理解は不確かであるという，常識的な結論を得たにすぎない。たとえていえば「÷分数」が「×逆数」に手品のように変わる様子を目の当たりにした子どもが，再度「なぜだろう」と思う間もなく，アルゴリズムの習熟に意識が向けられ，基盤より技能の強化に向かうのだから，計算の意味理解が半端になるのは，当然といえばいいすぎであろうか。

そこで「÷分数」の立式に関して，すでにこの単元の学習を終えた第６学年児を選んで，次のような調査を行った（註９）。児童の学習状況を正確に伝えるなら，１学期に「÷分数」を学習し，夏休みをはさんで２学期の初めに，28名の第６学年児に事前調査と実験的な授業をこころみた。事前調査の内容は数理構造の異なる問題場面から，「÷分数」の立式を求めるものである。問題１は従来の比例の文脈であり，問題２は比較の文脈である。

《事前調査の問題》

1．$\frac{3}{4}dl$ で板を $\frac{2}{5}$㎡ぬれるペンキがあります。このペンキ１dl で板を何㎡ぬれますか。

2．かおりさんのリボンの長さは $\frac{2}{3}$mです。しほさんのリボンの長さは $\frac{5}{7}$mです。しほさんのリボンの長さは，かおりさんのリボンの長さの何倍でしょうか。

問題1では$\frac{8}{15}$㎡を正答とし，問題2では$\frac{15}{14}$倍（帯分数を用いてもよい）を正答とする。また，解決途中のものも含め，それ以外の答を書いているものは誤答，白紙の状態のものを無回答として扱う。表5は正答・誤答・無回答の人数と，その解法について示したものである。立式後の計算処理は，二例を除いて正しく行われていた。比例の場面の正答者の解法を分析してみると，除法の式を正しく立式できた4人だけが正答である。誤答の中では$\frac{3}{4}\times\frac{2}{5}$という式が最も多く，比例の場面が乗法の式と結びついている。次に多い誤答は，除法の式を立式したものの，除数・被除数が逆になった式である。

次に，比例と比較ではどちらの場面が除法の式に結びつきやすいか，事前調査の答の正否に関わらず，除法の式を立式した子どもの人数を表5から抜き出しまとめてみた。表6から，倍概念を用いた比較の場面の方が除法の立式につながりやすいことは容易に推察できる。表6の誤答はすべて，除数と被除数の順序が入れ替わっているもので，これは問題の文中に出てくる数の順序が除法の式と逆になっているために起こった誤りであると考えられる。比例の場面から除法の式を立式した11人は「1あたりの量」を求めるための演算として除法を用いるということは理解している。しかし，誤答の7人についていえば，除数・被除数が逆になってしまっては無意味であり，除法の意味を理解しているとはいえない。一方，22人が比較の場面から除法の式を立式している。倍概念を用いた比較は2量の比の値を求めるもので，対象となる数が分数であっても，「片方がもう一方の量のいくつ分か」

表5 事前調査の結果　(N＝28)

	問題1（比例のスキーマ）		問題2（比較のスキーマ）	
	児童数	方　法	児童数	方　法
正　答	4	2/5÷3/4 (4)	9	5/7÷2/3 (9)
誤　答	20	3/4×2/5 (8) 3/4÷2/5 (7) 4/4×2/5 (2) その他　(3)	17	2/3÷5/7 (13) 5/7×2/3 (3) 5/7－2/3＝3/4 (1)
無　答	4		2	

（註：「方法」欄の括弧内の数は，人数を示している。）

表6　除法の立式を行った児童数

問　題	スキーマ	児　童　数		
1	比　例	11	正答	4
			誤答	7
2	比　較	22	正答	9
			誤答	13

という包含除の考え方がそのまま適用できるため，除法の式に結びつきやすいと考えられる。比例の場面と同様に比較の場面でも二数の順序が逆になる誤答が多く生じているが，この場合には二数の順序が入れ替わっても，計算の結果出てくる答はやはり「倍」である。除法の式の立式は分数で割ることの意味理解の問題に関わるものであるが，倍概念を用いる比較の方が比例の場面よりも立式は平易化されると考えられる。

(2)対案に基づく「分数除」の授業の実際と評価

大規模調査にしろ小規模調査にしろ，「÷分数」の意味理解は，いわゆる難教材であることに変わりなく，その理由は分岐モデルでも示したとおりである。しかし小規模調査から，それを比較の文脈に載せるなら比較的容易に立式に達することは示されており，分岐モデルでも次のように確認できる。問題２は問題１に較べれば数理構造が単純であるため，その状況は全体と部分に分岐することなく，いわば図２の右側だけの構成的抽象に沿って，記号の対象化に向かう。つまり差をとる比較と比をとる比較から，減法と除法の活動となって，いずれかの記号列がアルゴリズム理解の対象になる。減法の場合は量単位をそろえることで，整数の操作に同型になるが，除法についていえば，量の論理に基づいて「×逆数」を導くことは不可能であり，たかだか整数の比ないしその比の値をとるところで終わる。

そこで比較の文脈を設定した上で，以下のような教授実験を分岐モデルに従って，設計した。それは立式を導く構成的抽象と，そのアルゴリズムの理解をめざす一般化の２つの段階からなる。授業形態は小集団をベースに，適宜一斉指導を入れながら実施された。

はなちゃんはリボンを$\frac{3}{4}$m持っています。ゆきちゃんは$\frac{2}{5}$m持っています。はなちゃんのリボンの長さとゆきちゃんのリボンの長さを比べましょう。

まず，「長さを比べましょう」の問題から入り，各班の考えを式表示させ，その説明を促した。以下はその板書である。

①$\frac{3}{4}-\frac{2}{5}=\frac{3\times5}{4\times5}-\frac{2\times4}{5\times4}=\frac{15}{20}-\frac{8}{20}=\frac{7}{20}$　　答：はなちゃんのリボンが$\frac{7}{20}$m長い。

②$\frac{3}{4}\div\frac{2}{5}=\frac{3}{4}\times\frac{5}{2}=\frac{15}{8}=1\frac{7}{8}$　　答:はなちゃんのリボンは,ゆきちゃんのリボンの$1\frac{7}{8}$倍。

③$\frac{3}{4}:\frac{2}{5}=\frac{15}{20}:\frac{8}{20}=15:8$　　答：はなちゃんのリボン：ゆきちゃんのリボン＝15：8

どの計算も既習であるため，子どもたちはセンテンス型の式を立てている。事後のアンケートでも明らかにされたように，構成的抽象の段階はスムーズに展開した。そこで②に注目し，「なぜひっくり返してかけるのか」と，子どもたちに問いかけたところ，子どもたちはとまどうばかりであった。中には１学期に習ったことを思い出そうとした子どももいたかもしれないが，たとえ想起したところで，その考えは比較の文脈に当てはまるものではない。

比較の文脈自体に「×逆数」を導く論理がない以上，教師の積極的な介入が不可欠とな

る。通分や約分に現れる「分数では分母と分子に同じ数をかけても，同じ数でわっても，数の大きさは変わらない」という分数の性質や，

$$0.6 \div 0.2 = (0.6 \times 10) \div (0.2 \times 10) = 6 \div 2$$

といった小数の除法のように「わる数とわられる数に同じ数をかけても答は変わらない」という除法の性質を確認して，これら２つの性質を手がかりに，②の説明に取り組むことになった。

最初に分数を小数に直して計算する子どもが現れた。

$$\frac{3}{4} \div \frac{2}{5} = (3 \div 4) \div (2 \div 5) = 0.75 \div 0.4 = 75 \div 40 = \frac{75}{40}$$

子どもにとって既知の小数場面への翻訳は，内包的一般化といえるが，これでは「×逆数」には届かない。そこで教師から「この説明だと，わる数とわられる数にそれぞれ 100 をかけたことと同じですが，100 以外の数をかけてみてはどうでしょう」というヒントが与えられた。ここで授業はクライマックスを迎えることになった。ある児童が前に出て，次のように板書し口頭で説明を加えた。

$$\frac{3}{4} \div \frac{2}{5} = (\frac{3}{4} \times 20) \div (\frac{2}{5} \times 20) = (3 \times 5) \div (2 \times 4) = \frac{3 \times 5}{2 \times 4} = \frac{3 \times 5}{4 \times 2} = \frac{3}{4} \times \frac{5}{2}$$

この説明に知的に共振した子どもたちが，入れ替わり立ち替わり補足説明を加えていたとき，次のような質問が別の子どもから出され，議論はさらに深まることになった。

「かける数は，なぜ 20 なのですか。」

「20 は分母の 4 と 5 の最小公倍数です。」

「$(\frac{3}{4} \times 20) \div (\frac{2}{5} \times 20)$ がなぜ $(3 \times 5) \div (2 \times 4)$ になるのですか。」

「4 と 20，5 と 20 を約分するとこうなりました。」

「わかりません。」

「$\frac{3 \times 20}{4}$ になるから，4 と 20 が約分できて，（書きながら）こう約分して，3×5 になります。」

「$\frac{3 \times 5}{2 \times 4}$ は $\frac{3}{2} \times \frac{5}{4}$ になってしまうのではないですか。」

「分母の 2×4 を，先に 4×2 にしておいて，それから２つに分けたらどうですか。」

「いいと思います。」

上記の子どもたちのやりとりは，教室を取りまく知的な興奮に包まれ，しかも数学的な discourse にふさわしい内容に満たされていた。逆に数学が知的高揚のきっかけを与えていたのかもしれない。２つのルールだけを拠り所に，「÷分数」から「×逆数」を論理的に導くプロセスは，新たな知識について社会的な承認を取り付ける作業に似て，外延的一般化に十分に該当しよう。それは決して個人内部の認識変化に留まるものではなく，教室という社会の相互作用の下で，メンバー全員に分与される質を有していたと思う。

しかし以上の学習に成功したからといって，比例の文脈の立式にこれが転移するとは必

ずしもいえないであろう。状況の構造的特性は，調査問題(1)と(2)ではまったく異なる上に，比較の文脈におけるアルゴリズムの理解は，意味とは別の次元で，形式的になされるからである。しかし翻って，比例の文脈で立式できることは算数にとって重要なことであるかもしれないが，代数にとって重要であるとは思えない。それは算数の capstone になりえても，代数の cornerstone になるとは考えられない。代数にとってむしろ価値があるのは，そのアルゴリズムの理解であって，外延的一般化の質を考えるとき，比例の文脈の形式の理解が代数にとって適切なのか，それとも分数や除法のルールに基づく形式の理解が適切か，なお考慮の余地が残されているように思う。

7. 本研究のまとめと課題

本研究では，一般化の視座から，算数と数学の接続に関する理論化を試みた。本研究の成果と課題をまとめれば，以下のようになる。

(1)まず，一般化の考察にあたっては，Dörfler の一般化理論に依拠しながら，それを批判的に考察し，「一般化分岐モデル」（図２）を新たに開発した。一般化分岐モデルの開発にとって，内包的一般化と外延的一般化の規定は本質的であり，それを Gordon の創造工学に求め，前者を異質馴化すなわち新しい知識を既有の知識に同化する認知プロセスとし，後者を馴質異化すなわち新しい知識が既有の知識を調整し再編する認知プロセスとした。

(2) ２つの一般化の明確な区分に基づく分岐モデルによって，学習過程を記述する理論的枠組みばかりでなく教授過程をデザインする規範的枠組みの強化が図られたと考える。その上で，本稿では，教育課程実施状況調査において課題が指摘されている「÷分数」の学習過程に一般化分岐モデルを適用し，その困難性の要因や学習指導の改善について考察した。その結果，実施状況調査の結果を分析する記述レベルの精度が増し，「÷分数」の指導上の対案についても，その目標と行程は明確になり，評価は根拠を有する堅実なものになった，と考える。

(3) 比較に基づく対案の策定と評価によって，逆に比例に基づく小数・分数の乗除指導の一貫性が明らかになるばかりでなく，その限界が指摘された。つまりその指導は算術の頂点(capstone)になりうるが，代数への礎(cornerstone)にはなりがたいということである。アルゴリズムそのものの説明力に問題を抱えることは，本研究での考察で間接的に明らかにされた点である。そのため capstone と cornerstone をつなぐ教材の開発が急がれるが，それが比較による「÷分数」の意味指導とアルゴリズムの形式的説明になると考えた。また，それが有効に働くことが，教授実験によって示された。

本研究の今後の課題としては，一般化分岐モデルの下で，算数と数学の接続を促す様々な移行教材を開発することがあげられる。本研究では，移行教材として，「÷分数」を含め，表７に示すような題材を想定している（註 10）。

一般化分岐モデルの下で，こうした移行教材の設計，実施，評価を総合的に考察し，移行教材の完成を目指すことは今後の課題である。

表７　算数から数学への移行教材

	移行前期	移行後期
算術から代数	分数の除法	正負の数の減法
図形から幾何	図形の相互関係	図形の作図

［註］

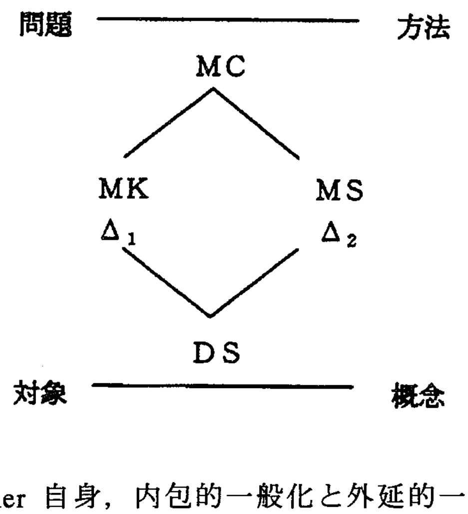
1．Dörfler のいう「プロトタイプ」は，単なる「典型例」という意味ではない。氏のいうプロトタイプは，数学的思考の対象を意味しており，それが何を表現し意味するかは，個々人の対象への働きかけに依存する。このことについて，氏自身は，次のように述べている(Dörfler,2000, p.103;岩崎・山口,2000,pp.3-5)。

《「プロトタイプ」という用語の私なりの用い方は，状況，活動，活動の体系，記号体系，そしてその他の実在といったものをプロトタイプの候補として認めることによって，より広範なものにすることである。さらに，プロトタイプの重要な側面は，あるカテゴリーに属する指示物の原型ということではなく，むしろ多くの様々な種類の活動を導く力にあり，そこには機能的な意味が込められている。数学にとって，そのような活動とは，計算すること，描くこと，証明すること，議論すること，読むこと，などである。》

2．プロトタイプと同様に，Dörfler のいう「プロトコル」は，単なる「発話」という意味ではない。氏のいうプロトコルは，活動に含意されている数学的属性の記述である(Dörfler,2000, p.120;岩崎・山口,2000,pp.3-5)。

3．ポパーによれば，世界１，世界２，世界３は，それぞれ物理的世界，思考過程のような主観的経験の世界，言明それ自体の世界として特徴づけられる(ポパー,1978)。

4．岩崎・山口(1998,2000)では，従来のメタ認知概念の拡張の必要性を論じるとともに，それを「拡張されたメタ認知」として理論化し，下記のようにモデル化している。また，Iwasaki & Yamaguchi(2000)では，「拡張されたメタ認知モデル」に基づいて，課題学習「カレンダーの数」における一般化の認知過程の分析を行っている。



MC：メタ認知
MK：メタ認知的知識
MS：メタ認知的技能
DS：ディレクターシステム

「拡張されたメタ認知モデル」
（岩崎・山口，1998,2000; Iwasaki & Yamaguchi,2000)

5．Dörfler 自身，内包的一般化と外延的一般化を明らかにしているわけではないが，氏の一般化モデルとの関連でそれらを規定すれば，次のようになろう。

内包的一般化：いくつかの対象や事象について，ある性質や命題が成り立っている

とき，それらを全称命題へと拡張すること。

外延的一般化：ある性質（命題）をそのまま保持して，その適用範囲を拡張すること。

6．平成6年（1994年）2月末にペーパーテスト調査が実施された。被験者は，第5学年児および第6学年児であり，各学年の被験者数は16000人であった。

7．次のような調査を実施した。

被 験 者:広島市内の公立中学校1年生

調査時期:平成13年4月上旬

調査方法:グループA（77名）には問題1を回答させ，グループB(75名)には問題2を回答させた。

（問題1）水そうに水をいれています。$\frac{4}{3}$分間に$\frac{5}{6}l$の水が入ります。同じ割合で水を入れていくと，1分間では何lの水が入りますか。答えを求める式を□の中に書きましょう。（□は省略）

（問題2）水そうに水をいれています。$1\frac{1}{3}$分間に$\frac{5}{6}l$の水が入ります。同じ割合で水を入れていくと，1分間では何lの水が入りますか。答えを求める式を□の中に書きましょう。(□は省略)

調査結果：正答率は次のとおりであった。

問題1：20.8%　　問題2：33.7%

8．図1の一般化モデルを提起したDörflerは，活動に具体的なものばかりでなく，心的活動や記号的活動を含めているが，活動の幅を念頭的な場合にまで拡げなければ，そもそも思考の方法たる数学は成立しない。Gattegnoの知見を借りるなら，心的活動や記号操作は意識性(awareness)の意識化に他ならず，《意識性だけが教育可能であり，この命題が，教育における様々な努力の基本的な解答を与える》(Gattegno, 1973, p.vii)と述べている。これを分数の乗除の学習指導に当てはめるなら，問題場面から立式を導くときには比例的推論が背景になければならず，アルゴリズムを理解するときには，それを前景に出して意識化しなければならない。要するに比例的推論という意識性と小数・分数の乗除の学習指導は表裏の関係にあり，乗除の理解のためには，その意識化に教育努力の大半が注がれることになる。

9．平成10年（1998年）9月，福岡県浮羽町立M小学校第6年児を対象として，田中志穂先生に教授実験をお願いした。

10．移行教材の開発にあたっては，Wittmannの「教授単元(Teaching Units)」の理論（例えば，Wittmann,1984,1995,2001）を一つの理論的基盤としているが，山口・岩崎・田頭(1997)やIwasaki & Yamaguchi(2003)では，一般化の視座から教授単元の補完を試みている。また，「図形の作図」に関する先行研究としては，例えば，岡崎・岩崎(2003)がある。

[謝辞]

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［付記］　本稿は，次の論文を加筆，修正したものである。
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一般化分岐モデルに基づく教授単元の計画・実施・評価に関する研究

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要　約

「数学化」とは，数学を活動的かつ創造的なものとして教え・学ぶという教授思想を敷衍する用語であるが，活動の源泉としての「状況」に特別な教育的意義を付与し，その設計に数学教育研究の核としての地位を与えたのが Wittmann の「Teaching Units」の思想であった。しかし，氏の Teaching Units には目的・題材・問題・背景という基本設計はあっても，教室内で継時的に展開される授業の規範となる下絵は含まれていない。他方，数学化を一種の記号過程とみなすことは可能であり，この視点にとって，活動に始まる一般化を組織的に構造化した Dörfler の一般化モデルは，極めて示唆的である。

本研究では，こうした視座から，Dörfler に基づく「一般化分岐モデル」に基づいて，Wittmann の Teaching Units を教授・学習の文脈で具体的に設計，実践，評価する。これが本稿の目的であり，同時に成果であると考える。方法論としては，上記の理論的枠組みに基づいて教授実験「Star Patterns」を構想し，その有効性を例証した。

1. 本研究の目的

算数・数学に限らず教科教育にとって，どのような目標の下で，どのような内容をどのような系列で扱うかということは，教育課程編成の本質であり，したがってその作成は教科教育学の最終的なシナリオといってよい。しかしわが国にあって，教育課程の編成と教科教育の研究はある意味で背馳している。すなわち前者は学習指導要領によって制度的に規定されており，その教育目標も学習内容の年次的な配当も決められているため，学の立場からあらためて議論することにはならない。あえていえば既成の事実に評論を加えることはあっても，その本質を問い，alternative な教育課程を編成するまでには至らないのは，自然といえよう。そのため教科教育学の守備範囲はいきおい「方法」に限定され，「学」とは名ばかりの議論が大勢を占めてきたことは，多くの研究者の同意するところであろう。

本研究は無論，最終的シナリオを目論むものではない。その基本的な準備段階として，教育課程を編成する単元の構成原理について考察する。教授単元の計画・実施・評価の根拠となる思想ならびに方法論を点検し，提案し，そして実践する。積極的な言葉づかいが許されるなら，健全な教授思想や教授・学習原理を下絵としながら，教授単元を適切に設計し，個々の学習段階と対照させながら，教材の適時性を考察していくことは，数学教育学の核心である，と考えている。

本研究の具体的な目的は次の２点である。１つは，Wittmann の教授単元の理論と方法を明らかにするとともに，そこに含まれる問題点を指摘することである。２つには，その問題に回答を与えることである。すなわち，後述する Dörfler の一般化理論に基づく「一般化分岐モデル」によって，実践レベルで教授単元の設計と評価を図る点にある。以上，本

研究の2つの目的を一言でくくるなら，Wittmann の展開する共時的な理論構成に通時的な構造をあたえるべく，一般化分岐モデルで補完するところにある。

2. Wittmann の教授単元

(1)教授単元の哲学

数学教育に数学は不可分である。しかしその数学の捉え方は一様ではない。例えば数学教育現代化の時代は，構造主義的な数学観が一世を風靡していたが，その後に世界的に展開した「問題解決」は，構成主義の数学観に支えられた教育動向といえよう。両者はいわば数学教育を挟んで対極する2つのポールといえ，数学教育を研究する者は意識するしないにかかわらず，両極の間のどこかに自らの数学観の碇を降ろすことになる。Wittmann の場合それは，次のように述べられている。

《（数学教育学の）核心における研究は，人間的認知の根源的でかつ自然な要素としての，数学的活動から始まらなければならない。さらにこの研究では，「数学」は広汎な社会現象と考えられねばならない。その多様な使用法と表現様式は，大学の数学科に典型的にみられるような専門数学には，部分的にしか反映していないと考えるべきである。私はこのもっとも広義の数学的取り組みを大文字で MATHEMATICS と書くことを提案したい。これは，科学，工学，経済学，コンピュータ科学，統計学，工場生産，商業，工芸，美術，日常生活等々で開発され使用されている数学を含んでおり，これらの文脈に特有の習慣や要求に従っている。確かに専門的数学は MATHEMATICS の本質的要素であり，これらの専門家によってなされた仕事がなくては，より広い解釈も用をなさない。しかしながらこの逆も真である。すなわち専門的数学も，より広い科学的・社会的活動にその多くの発想の源を負うているのであって，決して「数学(mathematics)」の専売だと考えるわけにはいかない。》(Wittmann,1995,pp.358-359，括弧内筆者)

ここに現れる Wittmann の数学観を一言に要約すれば，MATHEMATICS である。教育にとって数学的認識の起源を，数学の普遍性の影に隠れる文化や社会に求めることは，健全な思想といえよう。しかし文化的要因や社会的成分を抱える MATHEMATICS が，学校数学に組み込まれるとき，氏はさらに構造的な発展性を求めている。「mathe 2000」プロジェクトの数学教育の綱領[(1)]に従えば，数学は《「構造指向(Strukturorientierung)」と「応用指向(Anwendungsorientierung)の2つの相補的な側面をもつ「範例的な科学(Wissenshaft der Muster)」》(Müller, Steinbring und Wittmann,2002,p.14)でなければならない。よって Wittmann の数学観をまとめるなら，構成的な MATHEMATICS の上で，構造的な「範例的科学(Wissenshaft der Muster)」に向かう数学化こそ，氏の求める学校数学の本質といえよう。

しかしここで注意すべきことは，Wittmann の数学観と数学教育学の8つの核心[(2)]との関係である。上記 MATHEMATICS の引用にみられるように，その核心は数学的活動に起源をもたねばならないし，この数学的活動は MATHEMATICS において成立するため，核心と MATHEMATICS は数学的活動を媒介としながら，数学教育という一枚の布地の経緯のような関係にある。別の角度からみれば，2つの理念に支えられた数学的活動を教室で具現化する作業こそ氏の教授単元であったといえようが，その作業にはなお次の保留が付けられている。

《数学教育者は，「学問知の教育知への教授学的変換(transposition didactique du savoir savant au savoir enseigné)」(3)によって，専門数学から学校数学が導かれると考えるべきではない。学校数学は数学以前にある人間的能力の延長にあると理解すべきであって，この能力は MATHEMATICS によって与えられる，より広い社会的文脈の下で育成されるべきである。》(Wittmann,1995,pp.359-360)

この注意書きには，学校数学を外在する客観的な知的体系とするのではなく，人間の自律的な活動によって構成されるとする，氏の内在的数学観が反映している。と同時に内在する数学が，社会的な相互作用という他律的な枠組みを通して立ち現れることも，示唆されている。そのため，自律と他律のバランスを，何に基づきどのようにとるのかが，Wittmann の教授学的課題であった。この課題への具体的方策が，MATHEMATICS のインターフェイスとして，子どもたちの数学的活動を組織する教授単元の開発であり，それらの集合体を生命論的な視座(systemischer Sicht)からカリキュラムに編集していく原理の，自覚的適用であった。このことは数学教育学の６番目の核心「本質的教授単元や教授単元集やカリキュラムの開発と評価」に深く関わることになるが，氏が数学教育学を「設計科学」と特徴づけるのは，この点にあると思う。

「教授単元」と「設計」をキー・ワードに，両者の関係を氏の論説から検索していくと次のようになる。

《教授単元，ひとそろいの一貫した教授単元，さらにはカリキュラムの設計は，MATHEMATICS に起源をもたねばならない》(Wittmann,1995,p.359)

見方をかえれば，設計という思想なしに，MATHEMATICS は教授単元にはなりえない。さらに計画だけではなく，実施と評価という実践的吟味を経て初めて数学教育学の成果になることが，次のように述べられている。

《数学教育の研究における最も重要な成果は，基礎的な理論的原理に基づきながら，慎重に設計され，実験的にも検討された，教授単元の集まりである。》(Wittmann, 1995,p.369)

数学教育研究の全成果を教授単元に集約するという見識は，一面で過度の実践重視という印象につながるが，教授単元を支える数学教育研究の学的基盤には，次のような留保もつけられている。

《教科教育学(Fachdidaktik)は次の条件の下で，その役割を最大限に果たす。すなわち学習環境やカリキュラムを設計し，実験的な企画の下でその効果性を実証し，授業の建設的な改善を研究の中心に据えるときである。要するに教科教育学が「設計科学」として組織されるときである。》(Müller, Steinbring und Wittmann, 2002,pp.27-28)

しかしながら次節「２．教授単元の設計原理」で具体的に取り上げるように，数学的活動を組織し設計する教授単元には，教材を目的・題材・問題・背景に分節する項目はあっても，学習指導過程を分節する項目は用意されていない。Wittmann は，教室集団を背景として，教授単元・子ども・教師の間に展開される力動的な相互作用を，氏の生命論的な視座から「教育的生態学」と比喩するが，教育的生態学の構造特性なり生態系が，授業のことばで語られているわけではない。つまり子どもに内在する数学が，状況の MATHEMATICS とどのように共振し，知識としてあるいは見方・考え方として，どのようにクラスの中で共有されるかについて，そのプロセスを設定し分析するシステムが教授単

元には想定されていない。おそらくそれは数学教師の専任事項として，教授単元に織り込まれているのであろう。

(2)教授単元の設計原理

Wittmannは教授単元を教師が子どもと共に「数学する」文脈として位置づけ，そのような状況の下で初めて，子どもは数学の具体に触れるとしている(1984,p.29)。したがって教授単元の設計は教師の重要な仕事であり，同時にその成果は蓄積可能な教育資産を形成することになる。ただし教授単元にはそれが充たさなければならない４つの条件があり，それらは次のように記されている(2001,p.2)。

《1) それは数学における学習指導の中心的な目的・内容・原理をある水準において明示している。

2) それは上記の水準を越える意味ある数学的な内容やプロセスや方法と結びついており，豊かな数学的活動の源である。

3) それは柔軟な取り扱いが可能であり，個々のクラスの特殊な条件に対応できる。

4) それは数学教授の数学的・心理学的・教授学的側面を統合し，実証のための豊かなフィールドを提供する。》

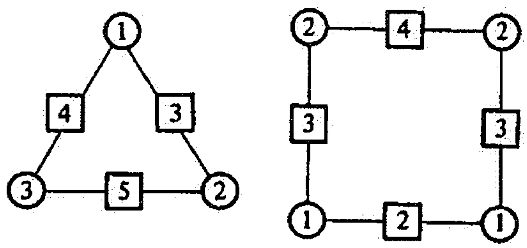
Wittmannはいくつもの論文の中で教授単元について言及しているが，それに明確な規定を与えているわけではないし，この用語を一貫して使用しているわけでもない。「教授単元」に「本質的(substantial)」がつく場合もあれば，用語自身「本質的学習場(substantial learning environment)」と様変わりする場合もある。しかし上記の４条件については不変であり(1984,pp.31-32; 1995,pp.365-366; 2001,p.2)，それぞれ「目的」「題材」「問題」「背景」として，教授単元を構成する要素になる(Wittmann,1984)。すなわち「目的」は条件1)を基盤に持ち，「題材」「問題」とはそれぞれ条件2)と3)の要請に応えなければならない。そして「背景」は条件4)を的確に明示しなければならない。したがって教授単元はこれら４項目からなる教材事例あるいはその集合体と考えてよい。

Wittmannはこれまで述べたように高い理念と単純な枠組みの下で，具体的に教授単元の開発に取り組んでいる。次に示す教授単元「Arithmogons」(Wittmann,1984,p.31)はその一例であり，前記の４つ条件を充たしている。

教授単元　Arithmogons

目的：加法・減法。さらにこれらの混合計算に関する演算の研究。探究と発見。

題材：三角 Arithmogon・四角 Arithmogon（ワークシートの使用。次図参照）



問題：いくつかの頂点や辺上の数が与えられている。その他の数を求めよ。
背景：頂点や辺上の数の一次独立性。方程式の体系に基づく組織的解法。計算の原理。

教授単元「Arithmogons」が「本質的教授単元」とよばれるのは，４つの構成要素で記述されるばかりでなく，前記４つの条件を満たすからに他ならない。すなわち子どもたちの興味・関心あるいは数学的能力に応じて，多様な数学的展開の可能性が潜在しているからである。たとえば三角 Arithmogon の学習展開を，低学年児童の「おはじきを置いたり，移したりする」活動と絡めれば，図１のようになる(ミューラー他,2004,p.131)。そこには実質的に方程式を解く活動が込められている（２から４）し，作ろうと思えば不定の場合も不能の場合も設定できる。

また図１の１のケースをプロトタイプとし，３つの領域のおはじきに変化を加えていけば，その和はどのように変化するのか（５から８），子どもたちの意識はその変化に向かうであろう。さらに学年を上げれば，その本質を変えることなく，おはじきや整数を，分数や文字にまで発展させられることはいうまでもない（９から 12)。

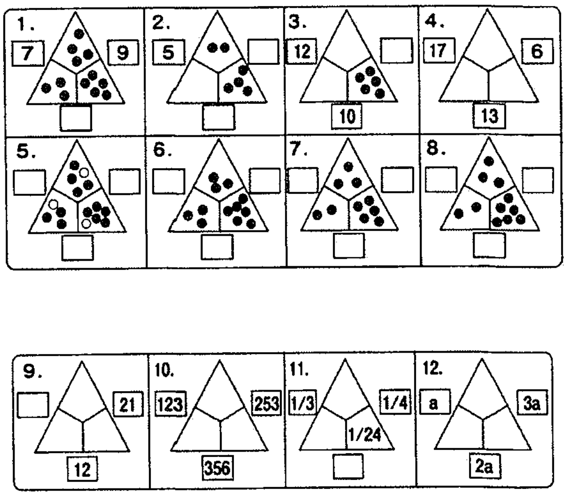


図１　三角 Arithmogon

やや抽象度を上げてこの種の数学的活動をみていくと，そこに次のような数学が現れてくる。

《三角 Arithmogon の背後にある数学はきわめて高度である。内側の３つの数は１つのベクトルを作り，外側の３つの数も同様にみられる。このようにみれば上記の操作は，実数上の３次元ベクトル空間からそれ自身への一次変換を定義する。対応する行列は正規である。この構造をｎ-gon に一般化できることは，McIntosh と Quadling(1975)によって示されている。》(Wittmann,1995,p.367)

重要なことは，その展開の数学的展望ばかりでなく，そうした数学的発展の文脈に子どもを置くことの教授学的意義である。上記のプロセスはいわば数学化の典型であり，そのプロセスを経験することによって，未知と既知の区別，演繹的な推論，一般化の考え，記号化の考えなどが育成される。そこには測り知れない数学教育的価値が込められていると

いえよう。

3. 一般化分岐モデルによる教授単元の補完

理念として，MATHEMATICS と数学教育学の核心をバックボーンに持つ教授単元であっても，その実践型である設計図の骨格は単純である。学習指導の出発点と方向性は明快に語られるが，その行程は教師に委ねられているといってよい。そのため学習指導のプロセスを行程表として設計するには，別の理論的原理を必要とする。この点が，教授単元の重要な課題である，と考える。

数学学習の自律性と他律性を媒介するのは記号である。活動に内在する諸関係を記号化するプロセスと，それを社会化する認知プロセスを前提にしなければ，数学的知識の客観化はありえない。この点について，オーストリアの数学教育研究者 Dörfler は，数学的な概念形成の本質を一般化に求め，活動に始まる一般化の過程を，記号化の視座からモデルに表している（例えば，Dörfler,1991）。氏の一般化理論の特徴は，抽象と一般化を接続させている点にある。その接点こそ，「記号の対象化」である。教室を前提に，抽象を個人的な思考の営為，一般化を集合的な思考の営為とすれば，教室における認識の更新は，記号あるいは表現のレベルで把握されることになる。

「記号の対象化」における記号は，対象の性質をもった変数であるため，一般化のプロセスで質的な変異を起こす場合が考えられる。例えば，小数による乗除は既習の自然数の乗除に帰着するため，記号の質的変位は退行的ともいえるが，分数の除法では，除法を乗法に切り替えるため，その変位はいわば未習の有理数に向かっているといえる。Dörfler の一般化モデルでは，こうした記号の質的変位を予想した上で，２つの用語「内包的一般化」と「外延的一般化」が準備されているわけではなく，両者は論理的な区分に基づいて設定されているにすぎない(4)。一般化が認知的なプロセスである点を再確認すれば，やはり下記のように，認知的な成分を含む形で，両者の再定義は必要になり，したがって，一般化モデルそのものの再編は不可欠となる（図２・Dörfler に基づく一般化分岐モデル,山口・岩崎,2005,pp.8-9）。

内包的一般化：既知の対象を普遍化することによる一般化。対象となっている記号に含意された意味を，既有の知識に関連づけながら同化し，既有の知識を発展させる認知プロセスとしてとらえられる一般化。

外延的一般化：記号の内的構造に基づいて，未知の対象を構成するような一般化。記号に内在する意味を既有の知識に同化させることができないので，新たな知識を構成し，その知識の下で，既有の知識を統合する認知プロセスとしてとらえられる一般化。

既にみてきたように，Wittmann の教授単元には，数学的活動の動因となる初期値と方向を与えるベクトルが４つの構成要素によって示されているが，その学習指導過程について，何も述べられていない。学習指導にはいうまでもなく始まりと終わりがある。従ってそこには時間的な構造がなければならず，時間的な構造を与えることが学習指導になる。それを通時的な設計図とすれば，それが準拠すべき枠組みがなければならない。数学化をめざす教授単元であれば，「導入」・「展開」・「まとめ」といった通常の枠組みで間に合うわけもなく，ねらいにふさわしい準拠枠を設定すべきであろう。われわれはそれを Dörfler

の一般化モデルに求めたが，すべに述べた理由から，それを図2の「Dörfler に基づく一般化分岐モデル」にシフトした。

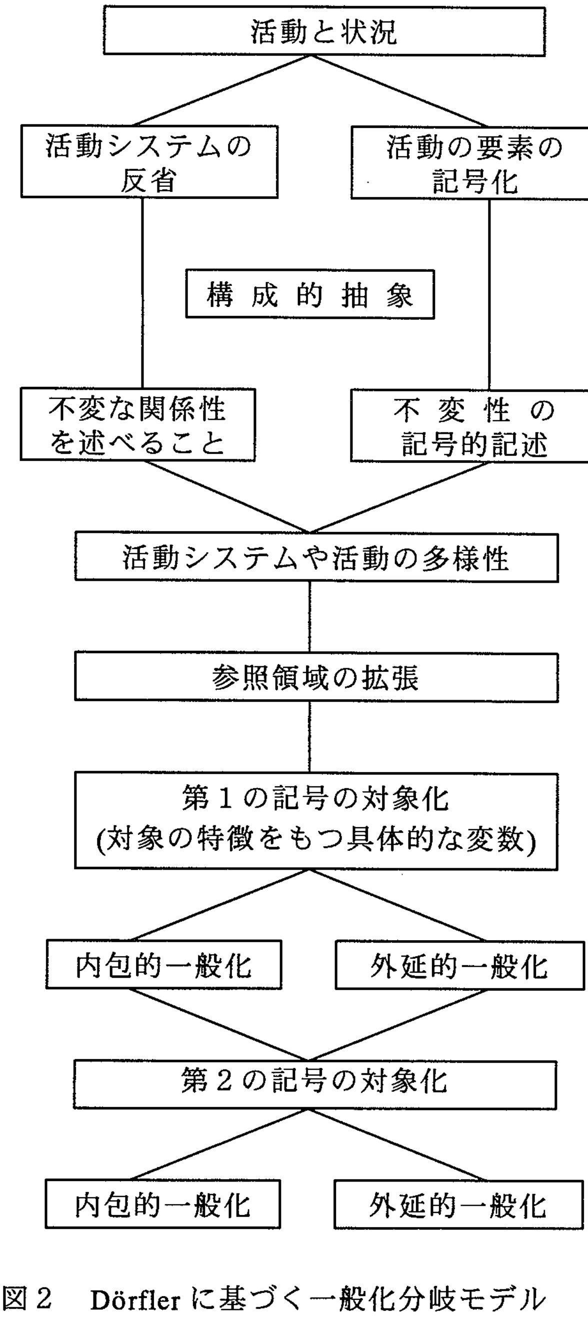


図2　Dörfler に基づく一般化分岐モデル

次節では，一般化分岐モデルに基づいて教授単元「Star Patterns」を具体的に設計するが，そのための準備として，Dörfler による一般化モデルと一般化分岐モデルに共通する特徴を概観しておきたい。両者ともに活動に始まる「構成的抽象」を一般化の前段に置いて，一般化と抽象との統合をはかっている。そして「記号の対象化」に始まる記号過程を一般化とする点で共通する。

「記号の対象化」以前の「構成的抽象」では，まず，活動の目的や手段などの反省がな

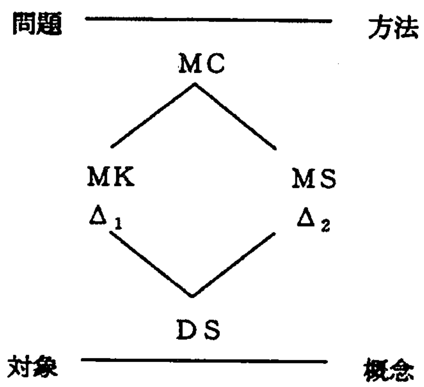
され，活動の要素の性質や要素間の関係が抽出される。不変な状態を記述するためには，言葉や図や記号が必要となる。構成的抽象の終点である「記号の対象化」を境にして，記号は参照領域から切り離されるため，それはある性質ないし条件を担った変数としてふるまう。このことで個人の思考は，程度の差こそあれ，社会的相互作用のきっかけを得て，学習集団の思考へと変容する。つまり，「記号の対象化」を境として，個人の主観的認識は，具体的な表現をえて，教室集団に共有され，社会的相互作用を経てある種の客観性を獲得する。その意味で，「記号の対象化」をめぐる児童・生徒の発話なり記述は，教授単元がねらう数学化を考察する上で，重要な情報になる。

しかし変数と一般化の関係は次のような形而上的言及に止まり，変数が「認知モデル」となる契機や条件について，何も述べられていない。

《一般化することは変数を構成することである。この文脈での「変数」とは（主題を認識するための）代替性を備えた認知モデルとみなしうるし，また客観的知識の一部とも考えられる》（Dörfler,1991,p.84）

そこで形而下的な議論を導入すれば，「代替性を備えた認知モデル」に何らかの管理装置がつかなければ，「変数」は「客観的知識の一部(a part of objective knowledge)」にはなり得ない。その管理装置として注目されるのが「メタ認知」である。しかし，メタ認知は，本来，問題解決の文脈で，問題を解決する以前の「わたし」の個人的営為を解明すべく開発された概念であって，問題解決以後の発展的文脈を視野に収めているわけではない。さらに学習集団を前提とする「われわれ」の活動が想定されていたわけでもない。そのため，一般化という知的発展の文脈を明確にするためには，メタ認知概念そのものの拡張を図る必然性が生じる。

このような課題意識から，筆者ら(岩崎・山口,1998; Iwasaki & Yamaguchi,2000)は，一般化の過程を支える認知プロセスを記述し得るよう，従来のメタ認知概念を拡張し，その結果を「拡張されたメタ認知」として，図３のようにモデル化した。「拡張されたメタ認知モデル」の本質は，van Hiele のいう「方法の対象化」を認知レベルで記述した点にある。つまり，図３は，当面する問題の解決によって得られた「方法」を，新たな学習の「対象」へと変換させ，その概念化を図る際の認知プロセスを示している。



MC：メタ認知　　　MK：メタ認知的知識
MS：メタ認知的技能　DS：ディレクターシステム

図３　拡張されたメタ認知モデル

教授単元に基づく教授-学習は，一般化を背景とした，「数学化」のプロセスである。したがって，それを支える記号化のプロセスは，一般化分岐モデルによって記述可能であり，他方，その記号化を支える認知プロセスは「拡張されたメタ認知」によって説明される，と考える。そのためどのような教授単元であれ，そこに数学的活動と社会的相互作用が組み込まれるなら，その学習指導の設計は一般化分岐モデルによって適切に補完されうる。あえていえば，教授単元という共時的な規範システムは，一般化分岐モデルによって，通時的な規範性と記述性を確保し，そのことによって学習指導の過程を形成したり分析したりすることが可能になる。

4. 一般化分岐モデルに基づく教授単元の計画・実施・評価

(1)教授単元「Star Patterns」の設計

銀林浩によれば，Dieudonné がはなばなしく「教室から幾何よ出て行きなさい(Euclid Must Go!)」と論じた(OEEC,1961,pp.31-49) New Math 開始の時代，それへの静かな異議申し立てをしているのが，コクセターの「幾何学入門」ということである(1965,p. i)。その第2章「正多角形」の第8節を「星形多角形」が占めている。このトピックに寄せてきたヨーロッパの数学者達の関心が，次のように記されている。

《星形多角形を最初に数学的に論じたのは，トマス・ブラッドウォーディン(Thomas Bradwardine,1290～1349)で，この人は生涯の最後の月にカンタベリーの大司教になった。これを研究した人に，またドイツの大科学者ケプラー(J.Kepler,1571～1630)がいる。数学的記号として $\{p/d\}$ を最初にあてたのは，スイスの数学者シュレーフリ(L.Schläfli,1814～1895)である。$\{n\}$ に対して成り立つ公式で，n が整数であっても分数であっても通用するものが多いことからもこの記法は支持される。》(コクセター, 1965,p.39)

教会建築の装飾に用いられた星形への関心は，ヨーロッパに固有の民族数学的なものかもしれないが，それ故 Wittmann の MATHEMATICS として，十分な資格を備えていると考える。

教授単元「Star Patterns」の目的は，円周上の等分点を線分で結んだときにできる図形と点の結び方との間に成り立つ関係を探求し，法則を見出して一般化することにある。具体的に授業実践の分析に入る前に，教授単元「Star Patterns」の本研究における位置づけを簡単にしておきたい。

星形多角形は数学的な考え方を育成するための格好の教材としてしばしば取りあげられ，いくつかの教材化とその実践例なども内外において報告されている(Bennet,1978; Hirsch,1980)。しかしながらそれらは，生徒の興味・関心を引き起こし，意欲的な探求を惹起する教材例として紹介されるのが常であって，生徒の認知的展開や数学的発展に関する教授学的可能性にまで，言及されているわけではない。本稿で「計画・実施・評価」として問題にしたいのは，この点である。

一般に優れた教授・学習モデルの条件として，「規範性」と「記述性」の両面が指摘される。本教授実験の主たる目的は，教材「Star Patterns」の実践に「一般化分岐モデル」を適用し，その規範性と記述性を示す点にある。このことは同時に「Star Patterns」の学習指導に一般化を保証することにもなり，本教材が教授単元にふさわしいことを示す。こうした視座より本研究では，一般化分岐モデルの「規範性」に基づいて教授単元の展開を具体

化することを「計画」ととらえるとともに，その「記述性」に基づく生徒の活動の分析を「評価」ととらえることにする。

教授単元「Star Patterns」

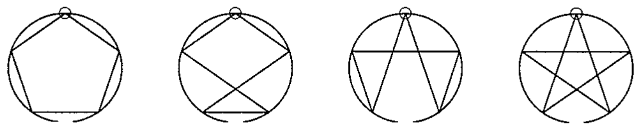
目的：幾何的諸性質の代数記号による考察。具体的には，幾何図形「星形」に内在する幾何的性質を代数的な記号列に置き換え，記号列の考察によって，幾何的性質を明確にする。

題材：Star Patterns

問題：

(1) 円周上の２等分点，３等分点，４等分点を一筆がきの要領で線分で結ぶと，どんな図形ができるだろう。

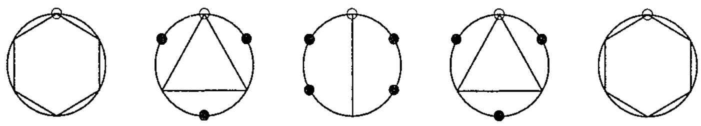
(2) 円周上の５等分点を結んでできる図形をかき，それぞれに名前を付けてみよう。



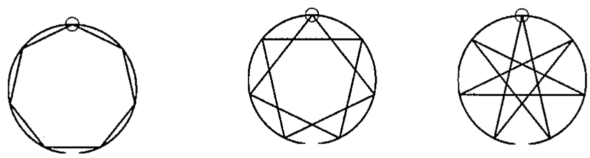
(3) ((2)でかいた図形のうち）星形に注目してみよう。どのようなかき方をしたときにその図形ができましたか。

(4) 他にも星形をかいてみよう。円周上の６等分点，７等分点，…を線分で結び，星形をかいてみよう。

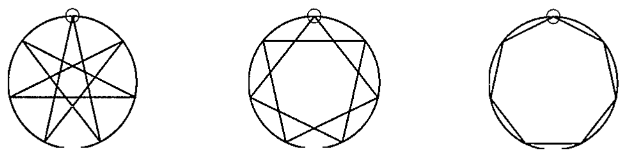
6・1　6・2　6・3　6・4　6・5



7・1　7・2　7・3



7・4　7・5　7・6



(星形のかき方の約束として，一筆がき，線分で結ぶ，○の点からスタートして （　） 個目ごとの点を結び，スタートの点にもどるまで続けることを確認。また描画された星形の名付け方を決める。)

(5) 星形はどんなときにかける（または，かけない）のだろう。またなぜそうなるのか，理由もあわせて考えてみよう。

背景：円周上の n 等分点を一筆がきの要領で d 個めごとに線分で結ぶとき，次の関係が成り立つ。

① n と d が互いに素のとき，一筆ですべての点を通るような星形多角形がかける。

② n が d の倍数のときは，正 $\frac{n}{d}$ 角形がかける。

③ $g=\mathrm{GCD}(n,\ d)$ とするとき，n と d が既約でない場合にできる図形は，円周上の $\frac{n}{g}$ 等分点を $\frac{d}{g}$ 個目ごとに結んだ星形多角形と同じになる。

一般化分岐モデルを規範にすると，教授単元「Star Patterns」による学習指導のプロセスは次のように計画される。問題(1)と(2)は，図２の一般化分岐モデルの「活動と状況」にあたり，そこでは円周上の等分点が一筆がきで結ばれる。(3)は「不変性の記号的記述」に対応し，特に円周上の５等分点を結んで星形ができた場合に焦点化し，「いつも２つ目ごとの点を結ぶ」という不変な活動を抜き出し記述することになる。続く(4)では「活動システムや活動の多様性」を設定し，(3)で見つけた性質が他の星形でも成り立つかどうかを探求する。この設定は「参照領域の拡張」と表裏の関係にあり，活動に潜在する数学的システムが具体的な場の拡張によって，顕在化する。それが次の段階で「対象の特徴を持つ具体的な変数」として，6・k(k=1,2,･･･5)や 7･k(k=1,2,･･･6)のように，現れる。つまり(5)の作業では「記号の対象化」が設定されており，この課題の目標である数学的な関係を確立し，教室集団での共有化を図ることになる。

一方，一般化分岐モデルの記述性に注目すれば次のようになる。問題(4)において，例えば６等分点を２つ目ごとに結んでできる図形に対して，生徒は 6･2 などの記号をつけるが，この種の幾何的事態に対する数値による記号表現こそ，その後の考察の対象になる。さらにそれらが，n 等分や d 個目などの文字を用いて，変数的な側面があらわになれば，論証を代数的な状況でおこなう外延的一般化に向かう環境が整ったといえよう。つまり，ここでいう外延的一般化とは，星形という図形の性質を幾何学的に考察することにとどまるのではなく，$n\cdot d$ といった表記を導入することによって，代数的な視座から星形を新たに考察し，その性質を理解することを意味する。以上の段階が一般化分岐モデルの「記号の対象化」にあたる。したがって，問題(4)と(5)における数値から文字への記号表現の移行が「記号の対象化」の判断の要になる。このように，生徒の発話や記述から，一般化の進捗状況を分析することができる。

(2)一般化分岐モデルに基づく授業の分析と評価

前節(1)の計画に従って，広島市内の公立中学校第２学年の生徒を対象に３時間の授業を行った。３時間にわたる授業では，生徒たちを６～７人から成る６つのグループに分け，議論しやすい授業形態をとっている。授業を担当した教師のクラスでは，「生活班」と称するグループを設定しており，授業では，生活班に基づくグループ学習が日常的に展開されていた。本授業のグループも，この生活班に基づくものである。グループを編成することによって，自分の考えをグループ内で確認し合う場が保証されていた。また，そのこと

によって，一斉指導の場においても，各自の考えやグループの考えが積極的に発表され，活発な授業が展開された。

本授業の分析と評価にあたり，まず，３時間にわたる授業の実際を概観しておきたい。第１時では，まず，上述の問題(1)から(5)がプリントとして配布された。生徒たちは，問題(1)や(2)に取り組む過程において，当初，円周上の点を自由に結んでいたが，一定の規則で点を結んでできる図形に自然な形で関心を寄せていった。このことをふまえて，第１時の後半では，「星形」という用語を教師が導入するとともに，問題（3）に取り組むように指示した。問題(3)に取り組む過程では，正五角形の場合も含めて，円周上の５等分点を何個目かごとに結んだときに星形ができるということが，生徒たちの意見に基づいて実際に確認された。第１時はこの段階で終了することになるが，授業の最後において，生徒たちからは，問題(4)を先取りする形で，次時の課題が設定された。つまり，円周上の点が６つや７つなどになった場合には，どのような結び方をしたときにどのような星形になるか，という課題である。

第２時は，上述の問題意識に基づいて，問題(4)に取り組むことから始まった。星形になる場合の結び方について，各グループで考察することが，第２時の活動の中心である。問題(4)に取り組む過程では，第１時において曖昧であった星形のかき方に関するルールが確認され，グループや教室全体における議論は，できあがった様々な星形の分析に焦点化されていった。第２時のポイントは，できあがった星形をどのように表現するかということである。実際，生徒たちは，円周上の等分点を何個目ごとに結んでできた星形であるかについて，当初，様々な表現を工夫したが，各自の表現が異なるため，議論が噛み合わない場面がしばしば見受けられた。教師は，こうしたコミュニケーションの不便さを十分意識させた上で，以下で詳述する「6・2」や「9・3」といった表記を導入している。一般化分岐モデルにおける「記号の対象化」を意識しながら，「$n \cdot d$」という表記の導入の適時性を慎重に検討することは，本教授実験における教師の重要な役割といえる。それ故変数性を意識しながら，生徒たちが「$n \cdot d$」という表記をどのように内化，専有し，星形の性質をどのように定式化していくかということが，本節における分析ならびに評価の主たる対象である。実際，第２時の後半から第３時にかけて，活動の水準は様々ではあるものの，生徒たちは「$n \cdot d$」という表記を駆使しながら，星形多角形の性質を探究していった。

３時間にわたる授業展開を確認した上で，以下ではまず，生徒の発言や記述に関する記号論的な分析とその評価を行いたい。円周上の等分点を規則的に結ぶ作業を続けていると，同じ図がいくつかできるため，生徒たちは，「同じであること」に着目して関係を見出していくことになる。生徒が発見した関係は，まず単に「同じ」からはじまり，図形が同じになる場合の記号，すなわち数の組がもつ性質の分析へと向かう。発見される関係は「線分ができるのは，n が d の２倍のとき」「正多角形ができるのは，n が d で割りきれるとき」「n と d が公約数をもつときは，最大公約数で割った数の組の図形と同じ」「n と d が既約なとき，星形ができる」などである。個々の生徒の発見が，単純な図形から複雑な図形へ，公約数をもつ２数から公約数をもたない２数へ順に発展していることは，授業の後で回収したワークシートから明らかであった。

次に，生徒がワークシートに記述した関係を記号的な視点から分類すると，それらは次のｉ)からｖ)に分けられる。

ⅰ) 6・2と6・4と9・3は同じ形である。
ⅱ) 6・2と9・3は同じ形である。6÷2も9÷3も3で，正三角形ができる。
ⅲ) 6・2や9・3のように，わり算の答えが3になるときは正三角形ができる。
ⅳ) □・1は正□角形になる。
ⅴ) $x \cdot y$ で $x \div y$ が割り切れないときに星形ができる。

ⅰ)は単に合同な図形の記述である。ここでの6・2や6・4や9・3という記号は，かかれた図につけられた名前に過ぎない。したがってこうした書き方に限定されるのではなく，2数の対であれば，どのような組み合わせも形態も可能である。ⅱ)以降は，かかれた図形に一組の数対を対応させるとき，数対の側で見いだされる関係である。そのため幾何図形は一組の数対によって考察される。しかし，ⅱ)は特定のかかれた図形についてのみ述べられたものであるのに対して，ⅲ)では6・2や9・3は1つの典型であって，それ以外にも，同様にこの関係が成り立つことが示唆されている。そのことは，生徒の発話や記述表現に現れる「～のように」「～など」「例えば～」の語からとらえることができる。ⅲ)のように，一組の数対と商と幾何図形との関係が「～のように」で結ばれるとき，他の図形への広がりを予想させるが，しかし学習者の意識は記号（数対）と指示対象（図形）との意味論的関係に留まっている。ⅳ)は，具体的な数値から，変数としての文字へと移行する過渡的な段階にあるといえよう。ⅴ)では，等分点の数が x，そしていくつ目ごとの点を結ぶかが y として表されている。こうした明確な変数化によって，幾何的な対象は認識の背景に退き，代数的な関係性がその前景にくる。しかし x や y は幾何的な性質や条件を担いながら変数としてふるまう。一般化分岐モデルとの対応でいえば，このように変数性を認識しながら，星形の性質を根拠づけることによって，「記号の対象化」の段階を越えることが可能となる。

これまでⅰ)からⅴ)の記号的側面に注目してきたが，認知的な側面に注目すると次のようになる。ⅰ)は図形の視覚的類似性にのみ着目する思考によって得られる表現であるのに対し，ⅱ)以降は，数の性質とかかれた図形との関係を抽象しようとする思考によってはじめて得られる記述である。また，そうした見方・考え方が意識されない限り，ⅰ)からこれらの水準への移行はあり得ない。他方，ⅱ)とⅲ)にみられる数学的認識の差は，思考が「参照領域の拡張」に向かっているかどうかによる。認知的な推進役を果たすのは，星形の描画をコントロールする意識性に反省を加える「拡張されたメタ認知」である。生徒の意識が数学的な活動の普遍性に拘るかどうかで，ⅱ)とⅲ)の記述の差が決まる。しかしⅱ)やⅲ)を支える思考は，全体として具体的な図形に基づく推論の域を出ておらず，その意味で「わたし」のシェマがとりあえず定式化されたにすぎない。

そのためⅱ)やⅲ)の記述は具体的な図形に張り付いたメモであって，他者には開かれていない。しかし他の生徒や教師による「なぜ，どうして」という問いかけで，言明としての限界に直面する。納得を説得に切り替えるためには，演繹的推論が不可欠であり，それは関係や事態の記号化や変数化と不可分になされる。ⅳ)やⅴ)の記述は関係の記号化に至る過渡的段階あるいはそれが達成された段階とみなすことができるが，それ以前とそれ以後では数学的活動に内在する認識は一変する。つまり経験的知識が理論的知識に変容する，演繹への端緒がそこに指摘できる。換言すれば，指示対象と記号との意味論的な関係がいったん解かれて，記号列の中に関係性を見いだし，結語論的なアクセスをかけようとする

「拡張されたメタ認知」が働かなければ，ⅳ)やⅴ)にはならない。

Star Patternsといった幾何的事態が，代数的な記号列を媒介にするだけで，理論的な認識に変容するとは思えない。無論個人のレベルでそれが達成できるとも思えない。演繹の前提に，教室という学習集団による社会的相互作用をおかなければ，子どもたちの認識は閉じたままであり，一般化への回路は開かれない。MATHEMATICSを背景とする教授単元に一般化分岐モデルを組み込むとき，子どもたちの認識を数学に押し広げていく的確なタイミングを，はかることができる。つまり，教師は，一般化分岐モデルの各分節に対応させながら，教授単元にかかわる学習過程を具体的に構築することが可能になる。特に，一般化にとって，記号の変数性を意識することは不可欠となるが，「記号の対象化」に対応する学習段階を特定した上で，そこで適切な表記を導入しながら変数性を意識させ，一般化を効果的に促すことが可能になるといえる。

(3)教授単元「Star Patterns」の発展的展開

2．の(2)で述べているように，Wittmannの構想する教授単元は，児童・生徒の実態に即応する柔軟性に優れるばかりでなく，深い数学性を備えていなければならない。その意味で「Star Patterns」における「外延的一般化」へのさらなる展開を検討する余地は残されている。

教室での取り扱いでは，円周上のn等分点を一筆がきの要領でd個めごとに結び，スタートの点に戻るとそれで作業は完了する。例えば円周上の6等分点を2個目ごとに結んでいく6・2であれば図4のようになり，3つの点を残す。そこでこの条件をはずして，すべての点を結んでできる図形について考察すると，そこに新たな展開が生じる。例えば円周上の9等分点を3個目ごとに結んでいく9・3であれば，図5のようになる。

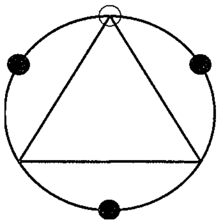


図4　星6・2

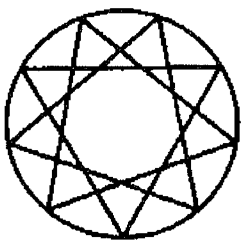


図5　星9・3

図5の星の個数3は，9と3の最大公約数である。このことは，次のように一般化できる。すなわち，すべての点を通るために必要な星の数をm，GCD(n，d)＝gとすると，

$m=g$ である。これより，星形 $n \cdot d$ は g 個の星 $\frac{n}{g} \cdot \frac{d}{g}$ からできる，と言い換えることもできる。

さらに，すべての点を通って星形をかくとき，スタートしてからかき終えるまでに，円周上を何周するかを考察する。この回転数こそ，星形のとがり具合を決める要因となっている。例えば，前述の星形 9・3 では，1つの正三角形をかくのに，円周上を1回転する。したがって，9等分点をすべて通って3つの正三角形をかくには，3回転を要することになる。正三角形は，9・3 の各数を最大公約数3で割った 3・1 に相当し，3回転の3は，「9等分点を3個目ごとに結ぶ」の3と一致する。このように回転数については，次の関係が成り立つ。すなわち，m 個の星形のうち，一つの星をかくときの回転数を f とすると，$f=\dfrac{d}{\mathrm{GCD}(n,d)}$ あるいは $f=\dfrac{\mathrm{LCM}(n,d)}{n}$ である。したがって，$mf=\dfrac{\mathrm{GCD}(n,d)\times d}{\mathrm{GCD}(n,d)}=d$ となるから，d がすべての星形をかくために必要な回転数である。

上述のように，ここでは，円周上の n 等分点を d 個目ごとに結ぶことによってできる星形図形の性質を，n と d に着目し，数理的な観点からの考察を行ったが，星形多角形は，他にも多様な性質をもっている。実際，辺や角に着目して，図形的な観点から考察を行うこともできる。n と d が既約であるときにできる星 $n \cdot d$ は，正 n 角形の概念を拡張することによって，正 $\frac{n}{d}$ 角形という新たな正多角形につながる。また，シンメトリーの視点からは，回転と鏡映からなる対称変換群を導くこともできる。さらに，3次元の星形多面体への発展的展開も可能である。

5. 本研究のまとめと今後の課題

本論文では一般化の視座から Wittmann の教授単元を批判的に考察した。教授単元は数学化の学習指導をめざすものであり，数学化はそれが垂直・水平いずれにせよ，一般化の認知過程と見なしてよい。それ故，教授単元の通時的なパートは，「拡張されたメタ認知」によって補完された「一般化分岐モデル」によって，設計されそして分析されうる。本研究の結論をまとめると次のようになる。

(1) Wittmann にとって，教授単元は《数学教育研究の最も重要な成果》(Wittmann,1995, p.369) である。本研究では，Wittmann の論考に基づきながら，教授単元の理論的な背景について明確するとともに，次のような課題を指摘した。つまり，教授単元は，「目的」・「題材」・「問題」・「背景」の4つの構成要素によって明確にされるが，その学習指導の設計について何も述べていない，という点に課題をみいだした。これを解決するには，4つの構成要素によって特徴づけられた教授単元の学習軌道を予測するような設計原理は不可欠であり，本研究ではそれを一般化分岐モデルに求め，教授単元の通時的な設計と評価に適用した。

(2) 具体的に議論を構築するため，教授単元として「Star Patterns」を提案し，学習指導過程を一般化分岐モデルによって計画し，実践に移した。その上で，「記号の対象化」の段階に焦点を当てながら，生徒の記述を記号論的に分析し，評価した。一方，生徒の認知過程については，「拡張されたメタ認知モデル」を用いて分析し，「記号の対象化」

が数学化にとって鍵になることを明確にした。

Wittmannは，群論の進歩が数多くの有限群の研究成果の上に築かれるように，数学教育学の発展も教授単元の実証的研究の蓄積に依存するとしている(1995,p.369)。今後の課題としては，Wittmannの指摘をふまえながら，一般化分岐モデルに基づいて，様々な教授単元の通時的設計をおこない，実践に移し，評価し，その結果を蓄積していかねばならないと考えている。その際，一般化分岐モデルで示された「内包的一般化」と「外延的一般化」の質的な差違に注目しながら，教授単元の学習指導を分析することが重要になる。こうした作業を通じて，逆に，一般化分岐モデルの規範性と記述性の精度を上げることも可能になる。

［註］

1．数学教育の6つ綱領は次のように記されている(Müller, Steinbring und Wittmann, 2002,pp.13-14)。

① 学校段階全体に行きわたるように，授業を学習能力(Lernfähigkeit)の発達に向きづけること（スローガン「活動的－発見的学習」）

② 数学的認識過程や学習過程を「数学化する・探求する・推論する・表現する」のような学習目標として記し，それらを学校教育全体に浸透させること

③ 「構造指向(Strukturorientierung)」と「応用指向(Anwendungsorientierung)の2つの相補的な側面をもつ「範例的な科学(Wissenshaft der Muster)」として数学を捉えること

④ 内容領域：整数論／離散数学・代数・初等幾何・解析そして統計の基本的アイディアを明確に関連づけること。これらを基礎学校から卒業試験まで連続的に展開すること

⑤ 自然科学に関する教科との間に明示的な連関をつけること

⑥ 学習の補助手段や道具として，新しいメディアを取り入れること（ただし，特別な教科「情報技術の基礎教育」のように，それ自体を目標とするわけではない。）

2．数学教育学の核心は次の8つの項目から構成されている(Wittmann, 1995,pp.356-357)。

①数学的活動と数学的考え方の分析

②局所的理論の展開（例えば，数学化，問題解決，証明と実践技能）

③学習者にとって近づきやすい数学内容の探求

④数学教授の一般目標からみて，内容を批判的に検討したり，是認したりすること

⑤学習のための準備，教授，学習過程の前提条件の探求

⑥本質的教授単元や教授単元集やカリキュラムの開発と評価

⑦授業計画，教育実践，授業観察そして授業分析に関する方法の開発

⑧数学教育の歴史的把握

3．FreudenthalはChevallardの「学問知の教育知への教授学的変換」の書評の中で，大学数学を学校数学に安易におろす姿勢に対し，強い調子で批判を加えている。

《学問知を教育知に教授学的に変換するという理論は，最初から間違っている。なぜならこの場合，上から下へということが考えられており，逆のことは考えられていないからである。将来市民となる多くの人が学ぶ数学は，専門家の数学に対応しない。以前の学校数学は（教育的であろうとなかろうと）学問知の翻案であった。数世紀前までそれは教養ある人たちの数学であった。しかしながら今日の青年の大多数は，数学者になる

ために教育されるのではなく，（様々な水準で）知識の習得方法を身につけるために教育されるのである。今日の科学技術社会において，大学数学は四半世紀以前に考えられていたほどの役割を有していない。もはや大学の数学は学校数学の主人というより召使いである。》（Freudenthal,1986,pp.326-327）

4．Dörfler 自身，内包的一般化と外延的一般化を明らかにしているわけではないが，氏の一般化モデルとの関連でそれらを規定すれば，次のようになろう。

内包的一般化：いくつかの対象や事象について，ある性質や命題が成り立っているとき，それらを全称命題へと拡張すること。

外延的一般化：ある性質（命題）をそのまま保持して，その適用範囲を拡張すること。

[謝辞]

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［付記］　本稿は，次の論文を加筆，修正したものである。

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記号論的連鎖2項モデルによる内省的記述表現の分析

二宮　裕之（埼玉大学）

要　約

本稿は、Presmeg(2001)において述べられている「記号論的連鎖(Semiotic Chaining)」について検討した上で、二宮(2002(a))、二宮(2002(b))において述べられた「内省的記述活用学習」及び「キャラクター学習」を記号論的連鎖の観点から分析・考察したものである。よりよい記述表現には記号的連鎖が必ず見いだされ、さらにその記号論的連鎖は最後に「内省的記述表現」により総括される。具体的な「記述表現」が存在することで記号論的連鎖はより活性化され、さまざまな「つながり」を学習者に認識させることを可能とする。このような記述表現活動は、単なる学習方法としてのみならず、それ自体を算数・数学学習の「目標・目的」として位置づけられるべきであることが、改めて認められた。

1. はじめに

数学教育における「表現」は、学習の目標ともなり、学習の内容ともなり、学習の方法ともなる(中原,1995,p.193)。そして、近年における「表現力を育てる指導」の強調(文部省,1993,p.104)を見るまでもなく、表現の検討は従来から数学教育学の大きな課題である(中原,1995,p.194)。その中でも特に、数学的活動の一つの特質はそれが最終的にはなんらかの形で「表記」に結ばれる（平林,1987,p.374）という指摘から、記述による表現が数学教育の活動において重要な役割を担っていることが伺える。二宮(2001)は、「内省的記述表現」及び「数学的記述表現」を規定した上で、「内省的記述」の具体例を示すことで「内省的記述表現」を内包的及び外延的に規定した。続いて二宮(2002(a))において、算数・数学学習における『他者』の役割を明確にした上で、内省的記述表現活動を活用する学習モデル（内省的記述活用学習モデル）の構築を行った。更に二宮(2002(b))では更に精緻な理論化を行い、「対象表記的記述表現とともに狭義の内省的記述表現を併せて記述する表現」を『総体としての内省的記述表現』と規定した。また記述表現活動のもつ「相互構成性」に注目し、内省的記述活用学習の一つの形態である「キャラクター学習法」による小学校４年生の学習事例について、記述表現の観点からの分析を試みている。

2001 年 7 月にオランダ・ユトレヒトで開催された PME25(The 25th Annual Conference of the International Group for the Psychology of Mathematics Education)の Discussion Group、DG03 において、Saenz-Ludlow、Presmeg をコーディネーターとして「Semiotics in mathematics education」と題されたセッションが行われた。このセッションは PME25 において初めて設定されたものであり、翌年イギリスの Norwich で行われた PME26 では更なる講演者を交え活発な活動が展開されている。ここでは数学教育学を学際的領域にあるものとして捉え、その背景の一つとして「記号論(Semiotics)」を取り上げることを提案している。そして主たる目的は以下の３点とされる。

(a) 授業データの分析に記号論を用いようとする研究者間の交流を深めること
(b) 記号論における異なる視座について共有し議論すること
(c) 研究成果を共有すること

2001 年のこのセッションにおいて、コーディネーターの一人でもあるアメリカ・Illinois State University の Norma Presmeg 女史が「Progressive mathematizing using semiotic chaining」と題した講演を行った。本稿は、Presmeg 女史の言う「Semiotic Chaining(記号論的連鎖)」という考え方に注目し、これを算数・数学学習における記述表現活動の分析に援用しようとするものである。具体的には Presmeg(2001)を手がかりとして、まず始めにパースの記号論、ソシュールの記号学といった先行研究の成果をもとに、数学教育における記号論的連鎖の事例を検討しその枠組みを明らかにする。そして、二宮(2002(a))で構築された「内省的記述活用学習モデル」、二宮(2002(b))において検討された「キャラクター学習法」による事例について、記号論的連鎖の観点からの分析・検討を

試みる。その結果として、よりよい記述表現の特質を記号論的連鎖の観点から明らかにするとともに、内省的記述表現活動それ自体を、算数・数学学習の「目標・目的」として位置づけるべきであることを記号論的連鎖の観点から指摘しようとするものである。

2. 記号論的連鎖の2項モデル

Presmeg(1998)で提唱された記号論的連鎖は、Presmeg(2001)においてその理論的背景を、パース(Charles S. Peirce)が提唱した記号論（Semiotics）における記号論モデルから検討を始めている。パースのモデルでは、以下にあげる３つの基本的構成要素をあげており、「３項モデル」として捉えられる。(Presmeg,2001,p.2)

①『対象物(object)』: 他の何ものからも独立した対象物の存在
②『記号(representamen)』: 対象物とそれを指示する記号との間の関係
③『解釈項(interpretant)』: 対象物、記号、及び解釈項と呼ばれる第三の要素との関係を考慮した、記号の解釈

そして、先行研究において述べられている具体例を参考に、以下のような例をあげてパースのモデルを解釈した。

表１　パースの３項モデルの事例

対象物 (object)	記号 (representamen)	解釈項 (interpretant)
ミシン（それ自体）	ミシンの写真	ミシンの機能が分かるような写真を理解すること
「多数は常に正しいのか？」という疑問	［複数のサイン］ 書籍の中での言語的表現 少数民族の宗教に対する視点が差別的傾向を持つ人	「多数は常に正しいとは限らない」という解釈
雨の降る可能性	雨が降る兆候	傘を持参することの決断

パースの３項モデルに対して、それとはまた異なる解釈を行っているのが、ソシュール(Ferdinand de Saussure)である。パースの枠組みが記号論（Semiotics）と呼ばれているのに対し、ソシュールの枠組みは記号学（Semiologie）と呼ばれる。氏は記号(signe)を「意味を担うもの(記号表現)」と「担われる意味(記号内容)」とに類別し、前者をシニフィアン(signifiant)、後者をシニフィエ(signifié)と名付けた。Presmeg(2001)は２項モデル（記号内容(シニフィエ)と記号表現(シニフィアン)）に関連して、「これはそれぞれ３項モデルの構成要素となっている。しかし第三の要素(解釈項:interpretant)は、記号の解釈において暗黙的である(p.2)」と述べ、それらを統合した新しいモデルを以下のように提案している。

「記号内容が記号表現よりも優先される」とするソシュールのモデルを逆転して捉えたラカンは、記号内容よりも上位に位置する記号表現が強調されるモデルを考えた。そしてそれは、「記号表現が何らかの形で記号内容の支配下に置かれることが暗黙のうちに了解された場合には容易に認識されえない、記号表現による動的で継続的な生産作用における広遠な自律性(far ranging autonomy)」を意味するものとなった（Whitson, 1994, p.40）。このような見識に対してPresmeg(2001)は、「前段階の記号の組み合わせにおける記号表現が、新たな記号の組み合わせの記号内容となり、さらにそれは繰り返される」という連鎖の過程を想定した。Presmeg(2001)はこのような連鎖を『記号論的連鎖(Semiotic Chaining)』と命名し、その具体的事例として、Walkerdine(1988)において取り上げられた「幼い娘が飲み物を注ぐ際の母親とのやりとり(pp.129-138)」を次のように分析し示している。

母親が、飲み物を注ぐべきお客さん(５名)の名前を順番に述べる。それを聞いた娘は、それぞれの名前に対して手の指を１本ずつ曲げる動作で対応させた。

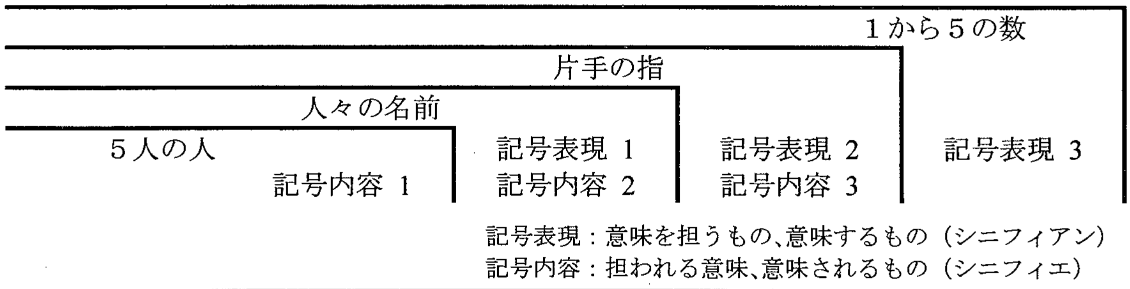


図１　記号論的連鎖の２項モデル

3. 内省的記述活用学習における記号論的連鎖

二宮(2002)は、算数・数学学習における内省的記述表現活動の必要性・重要性に鑑み、「学習活動」の一環として「他者」の視点からの記述表現活動を取り入れる学習モデルを、図２に示す『内省的記述活用学習モデル』のように提起した。

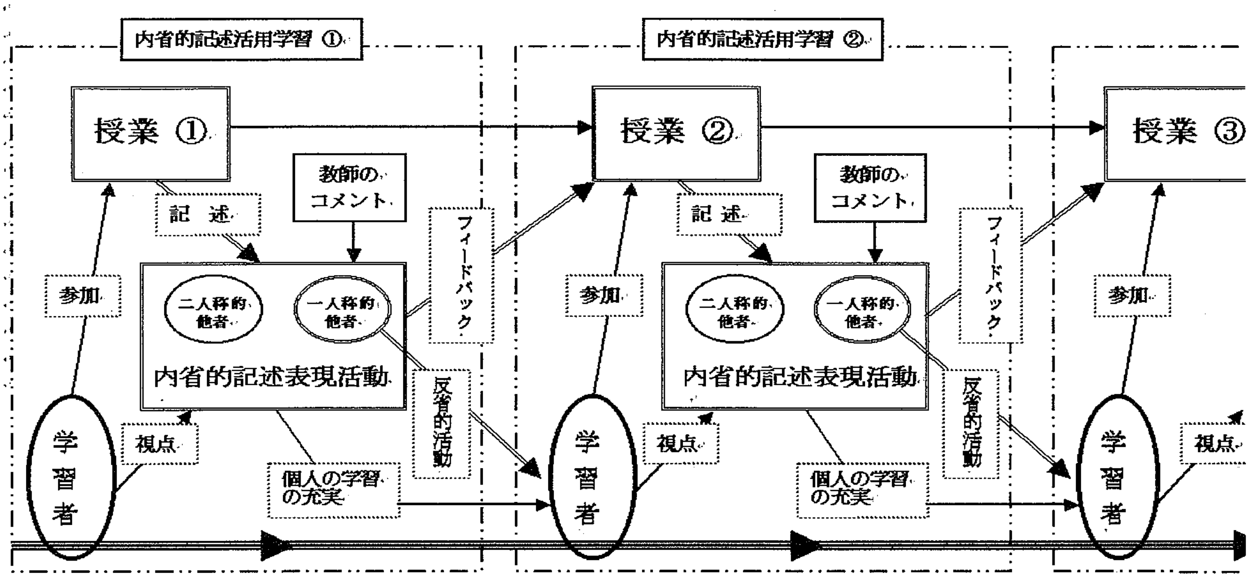


図２　内省的記述活用学習モデル

この学習モデルでは、学習者が授業へ参加するとともに、その授業の様子や学習内容について児童・生徒それぞれが内省的記述表現活動を行う。この際、学習者個人の視点だけではなく、「二人称的他者(互いに学び合う関係の級友)」や「一人称的他者(学習者の内部にいるもう一人の自分)」を登場させる形で記述することで、内省的記述活動そのものが「『文化的実践』としての算数・数学」を記述の中で再現させることとなる。ここで、一人称的他者のもつ機能から、内省的記述表現により反省的な活動を促すことができる。また、教師からのコメント（児童へのフィードバック）により更なる内省的活動がなされることで、児童の学習はより充実したものとなる。一方、内省的記述表現を次の授業へフィードバックすることで、前時とのつながりを児童に認識させるとともに、児童による記述をそのまま活用することで主体的な学習をも促そうとするものである。このように

① 学習者→ 授業→ 内省的記述活動→ 学習者
② 授業→ 内省的記述活動→ 次の授業へのフィードバック

という二つの大きなサイクルで数学的記述表現活動を算数･数学の授業の中に捉える学習を「内省的記述活用学習」と規定した。

ここで述べられた二つの大きなサイクルは、ある種の『連鎖』として捉えることもできる。つまり、記号論的連鎖におけるシニフィアン(意味を担うもの､意味するもの､記号表現)とシニフィエ(担われる意味､意味されるもの､記号内容)とを、内省的記述活用学習モデルの中に同定することができるのである。例えば上述の①のサイクルについては図３に示すような、②のサイクルについては図４に示すような記号論的連鎖が、それぞれ内在していると捉えることができる。

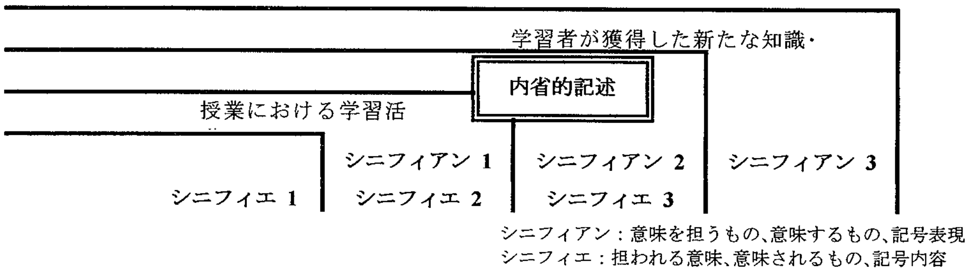


図３　内省的記述活用学習における記号論的連鎖 ①

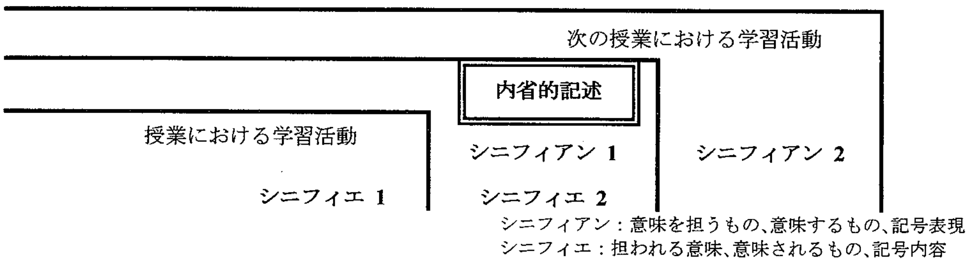


図４　内省的記述活用学習における記号論的連鎖 ②

4. キャラクター学習に見られる記号論的連鎖

4.1 事例の概要

二宮(2002(b))は、内省的記述表現活動を活用する学習方法の一つとして「キャラクター学習法」について検討を行っている。キャラクター学習法とは、二人称的他者や一人称的他者の存在を児童･生徒に強く意識させることを目的として､人物や動物などのキャラクターを意図的に用いるものである。それぞれのキャラクターは、児童･生徒自身にもなり、二人称的他者ともなり、また一人称的他者(もう一人の自分)ともなる。児童･生徒は、記述表現の中においてそれぞれのコメントを述べる主体を目の当たりにすることができ、また誰がどのコメントを述べているかを区別することができるので、自らの学習環境において「他者」の存在を容易に介在させることができる。

導入 (5 分)　課題を提示する。

教師は１枚の紙を取り出し、それを折り始める。

T: １回折ったら、折り目は何本になるかな？
S: １本。
T: ２回折ったら？
S: ３本。
T: では３回では？
S: ７本。
T: ４回、５回、６回、７回------, そうすると、７回折ったときには線が何本になりますか？

内省的記述表現活動 (2 分)

課題解決への見通し、今の気持ちを書く。

各児童による問題解決 (10 分)

各児童は、紙を折り問題解決に臨む。

課題の確認 (6 分)

T: 何か困ったことにならない？
S: だんだん折れなくなってきた。7 回目は折れない。
T: それじゃ、どうしたらいいかなぁ？

Yoji の考えの紹介 (6 分)

おった数	1	2	3	4	5	6	7
おり目の数	1	3	7	15	31	63	127
	2	4	8	16	32	64	

Yoji: 折り目の数の次に出てくる数を足すと、次の折り目の数になっている。だんだん、足す数が２倍になっている。

Yuri の考えの紹介 (4 分)

T: この数字はどこから出てきたの？
Yuri: 折った時に長方形が何個できたかということ。

両者の考えの比較 (5 分)

T: どう、Yoji くん、これでいい？
Yoji: ぼくの考え方は違う。１回から順番に折っていって、16 までやってみて、１回折ると増える数が前の２倍になると気づきました。

Ayu の考えの紹介 (2 分)

Ayu: 折り目の数は部屋の数より１少ない。

児童の考えのまとめ (5 分)

T: 何を２倍にしているのかな？？
S: 部屋の数。前に折ってできた部屋の数。
T: 「前に折ってできた部屋の数」×２－１ ＝「折り目の数」

授業のまとめ (3 分)

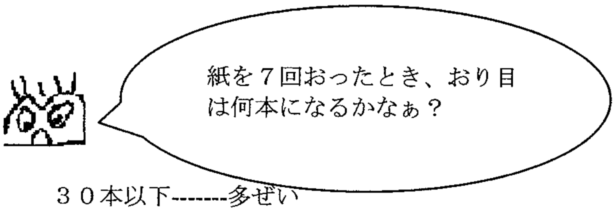
本時の内容を教師が簡潔にまとめる。

内省的記述表現活動(4 分)

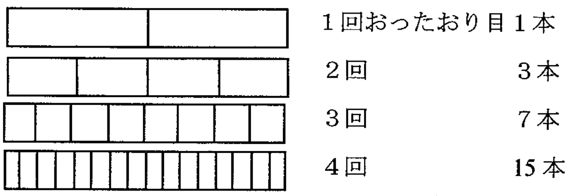
数名の児童のコメント発表 (1 分)

２名の児童がまとめを発表する。

図５　授業の概要



３０本以下-------多ぜい
３０本以上----------　７人



Yoji:

おった数	1	2	3	4	5	6	7
おり目の数	1	3	7	15	31	63	127
	2	4	8	16	32	64	

答え　127 本

Yuri:　+2　+4　+8　+16　部屋の数
Yoji:　前の２倍になっていそう
+16　(２倍)→ +32

Ayu:

	へやの数	おり目の数
１回	1*2=2	1
２回	2*2=4	3
３回	4*2=8	7
４回	8*2=16	15
５回	16*2=32	31
６回	32*2=64	63
７回	64*2=128	127

へやの数 –1 = おり目の数

「前におってできた部屋の数」×２－１ ＝「おり目の数」

図６　板書の様相

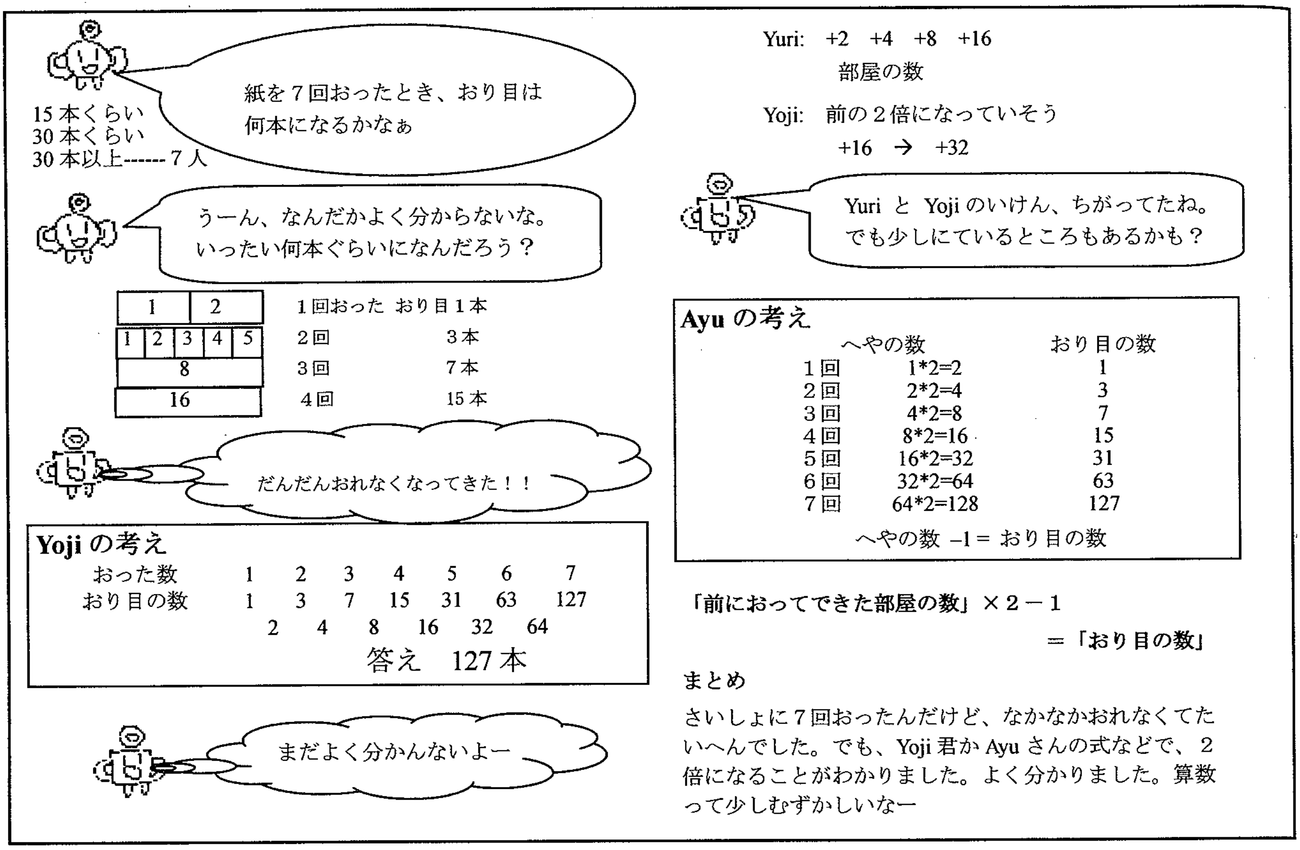


図７　Miku の記述

図３、図４において、内省的記述活用学習における典型的な活動サイクルでの記号論的連鎖を示した。学習活動におけるこのように大きな枠の中で、それぞれの活動どうしの連鎖として内省的記述活用学習を捉えることは、学習活動の大きな流れを見るための一つの方法となろう。また一方で、内省的記述表現の一つ一つを精緻に分析していくことで、子どもが行った１時間の学習活動の場面場面における活動を、連鎖として捉えることが可能となる。本稿では二宮(2002(b))で言及した小学４年生の事例を取り上げ、記号論的連鎖の観点から児童 Miku の学習の様相を分析する。

本稿で取り上げる事例は、これまでキャラクター学習法を用いて算数の学習を進めてきた公立小学校４年生１クラス（男子 18 名、女子 19 名）を対象とした、ある１時間の授業である。授業内容は「１枚の紙を折って」というトピック教材で、その授業の概要を図５に、板書を図６に示す。更に、児童 Miku の書いた内省的記述表現を図７に示す。

4.2 事例の分析

本小節では、図７に示した Miku の記述表現を拠り所に、Miku の学習軌跡の分析を試みることとする。Miku が行ったこの記述表現は授業終了後にまとめたものではなく、授業の進展に伴って漸次書き進められたものであった。そのため、図７に示した記述表現を完成させていくことそれ自体が、彼女にとって本時の学習活動であったと捉えることもできる。図５に示した「授業の概要」、図６に示した「板書の様相」とを比較検討することで、Miku の学習軌跡には数多くの連鎖が存在していることが伺える。１時間の授業をその学習の様相に従い、導入、Yoji の考えの検討、Yoji の考えと Yuri の考えとの比較、まとめ、の４つに区分した上で、Miku の学習活動における記号論的連鎖の様子を表２から表５にまとめる。尚、表の下にある「W①」「I②」はそれぞれ「学級全体の活動①」「Miku の学習活動②」のようにそれぞれの活動を表し、間の矢印はそこに記号論的連鎖が存在すると見なせることを意味する。また表２から表５に示す「学級全体の活動(W)」は、Miku の学習活動と関わりをもつと思われるものを中心にまとめている。ここには記載されて

いない他の児童の活動が少なからず存在している点には留意されたい。尚、Miku の学習活動(総体としての内省的記述表現)に示される「内省的記述表現」とは『狭義の内省的記述表現』を意味し、それは以降の考察においても同様である。

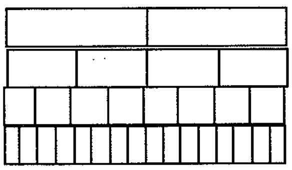
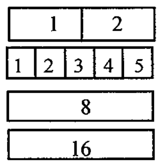
表２　記号論的連鎖の様子①（導入）

	学級全体の活動　(W)	Miku の学習活動(総体としての内省的記述表現)　(I)
①	教師による導入 「７回折ったときには線が何本になりますか」	
②	板書（課題の導入） 「紙を７回おったとき、おり目は何本になるかなぁ？」	**内省的記述表現** 「紙を７回おったとき、おり目は何本になるかなぁ」
③	板書（課題解決への見通し） ３０本以下-------多ぜい ３０本以上---------- ７人	**対象表記的記述表現** 15 本くらい 30 本くらい 30 本以上------７人
④	全員が内省的記述表現活動を行う	**内省的記述表現** 「うーん、なんだかよく分からないな。いったい何本ぐらいになんだろう？」

W①→W②→I②

W②→W③→I③→I④

表３　記号論的連鎖の様子②（Yoji の考えの検討）

	学級全体の活動　(W)	Miku の学習活動(総体としての内省的記述表現)　(I)
①	教師による「課題の確認」 T: 何か困ったことにならない？ S: だんだん折れなくなってきた。７回目は折れない。 板書  １回おったおり目１本 ２回　３本 ３回　７本 ４回　15 本 「だんだんおれなくなってきた！！」 T: それじゃ、どうしたらいいかなぁ？	**対象表記的記述表現**  １回おった おり目１本 ２回　３本 ３回　７本 ４回　15 本 **内省的記述表現** 『だんだんおれなくなってきた！！』
②	Yoji の考えの紹介 板書 おった数　1　2　3　4　5　6　7 おり目の数　1　3　7　15　31　63　127 2　4　8　16　32　64 答え　127 本 Yoji による説明 「折り目の数の次に出てくる数を足すと、次の折り目の数になっている。だんだん、足す数が２倍になっている。」	**対象表記的記述表現** **Yoji の考え** おった数　1　2　3　4　5　6　7 おり目の数　1　3　7　15　31　63　127 2　4　8　16　32　64 答え　127 本
③		**内省的記述表現** 『まだよく分かんないよー』

W①→I①

W①→W②→I②→I③

表４　記号論的連鎖の様子③（Yojiの考えとYuriの考えとの比較）

	学級全体の活動　(W)	Mikuの学習活動(総体としての内省的記述表現)（I）
①	Yuriの考えの紹介 板書：Yuri:　+2　+4　+8　+16　部屋の数 T: この数字はどこから出てきたの？ Yuri: 折った時に長方形が何個できたかということ。	**対象表記的記述表現** Yuri:　+2　+4　+8　+16 部屋の数
②	両者の考えの比較 板書：Yoji:　前の２倍になっていそう +16　(２倍)→ +32 T: どう、Yojiくん、これでいい？ Yoji: ぼくの考え方は違う。１回から順番に折っていって、16までやってみて、１回折ると増える数が前の２倍になると気づきました。	**対象表記的記述表現** Yoji:　前の２倍になっていそう +16　→　+32
③		**内省的記述表現** 『Yuri と Yojiのいけん、ちがってたね。でも少しにているところもあるかも？』

W①→I①
W①→W②→I②→I③

表５　記号論的連鎖の様子④（まとめ）

	学級全体の活動　(W)	Mikuの学習活動(総体としての内省的記述表現)（I）
①	Ayuの考えの紹介 板書： **Ayu:** へやの数　おり目の数 １回　1*2=2　1 ２回　2*2=4　3 ３回　4*2=8　7 ４回　8*2=16　15 ５回　16*2=32　31 ６回　32*2=64　63 ７回　64*2=128　127 へやの数 −1＝ おり目の数 Ayu: 折り目の数は部屋の数より１少ない。	**対象表記的記述表現** **Ayuの考え** へやの数　おり目の数 １回　1*2=2　1 ２回　2*2=4　3 ３回　4*2=8　7 ４回　8*2=16　15 ５回　16*2=32　31 ６回　32*2=64　63 ７回　64*2=128　127 へやの数 −1＝ おり目の数
②	児童の考えのまとめ T: 何を２倍にしているのかな？？ S: 部屋の数。前に折ってできた部屋の数。 T: 「前に折ってできた部屋の数」×２−１ ＝「折り目の数」 板書 **「前におってできた部屋の数」×２−１** **＝「おり目の数」**	**対象表記的記述表現** **「前におってできた部屋の数」×２−１** **＝「おり目の数」**
③	授業のまとめ 本時の内容を教師が簡潔にまとめる。	
④	全員が内省的記述表現活動を行う	**内省的記述表現**（まとめ） さいしょに７回おったんだけど、なかなかおれなくてたいへんでした。でも、Yoji君かAyuさんの式などで、２倍になることがわかりました。よく分かりました。算数って少しむずかしいなー

W①→I①
W①→W②→I②(→I④)

4.3 考察

第一に気づく点として、記号論的連鎖は授業の様相が異なる場面に渡っては存在していない。つまり、導入、Yoji の考えの検討、Yoji の考えと Yuri の考えとの比較、まとめ、という授業における４つに区分に対して、その区分を越えるような記号論的連鎖は見いだされなかった。このことは逆に、それぞれの学習活動間に存在する記号論的連鎖を個別に見ていくことで、授業の大きな流れを把握することや、授業内容や様相の大きな区切りを見いだすことが可能であることを示すものとも言える。

具体的な記号論的連鎖について見た場合、学習活動の一区切りの中で複数の記号論的連鎖が存在していることが分かる。例えば表２に示す「導入」では、「W①→W②→I②」という連鎖と「W②→W③→I③→I④」という連鎖の２つが見いだされた。前者は、教師による導入→板書(課題の導入)→内省的記述表現、という一連の記述表現であり、後者は、板書(課題の導入)→板書(課題解決への見通し)→対象表記的記述表現→内省的記述表現、という記述表現の連鎖である。同様に「Yoji の考えの検討」では、教師による課題の確認→対象表記的記述表現→内省的記述表現、教師による課題の確認→Yoji の考えの紹介→対象表記的記述表現→内省的記述表現、といった連鎖が認められる。それぞれの授業場面における記号論的連鎖を、その活動の様相に即してまとめると図８のようになる。

導入

教師による導入→板書(課題の導入) →**内省的記述表現**

板書(課題の導入)→板書(課題解決への見通し) →対象表記的記述表現→**内省的記述表現**

Yoji の考えの検討

教師による課題の確認→対象表記的記述表現 →**内省的記述表現**

教師による課題の確認→Yoji の考えの紹介 →対象表記的記述表現 →**内省的記述表現**、

Yoji の考えと Yuri の考えとの比較

Yuri の考えの紹介→対象表記的記述表現

Yuri の考えの紹介→両者の考えの比較 →対象表記的記述表現→**内省的記述表現**

まとめ

Ayu の考えの紹介→対象表記的記述表現

Ayu の考えの紹介→児童の考えのまとめ →対象表記的記述表現 (→**内省的記述表現**)

図８　記号論的連鎖の様相

図８において特徴的なことは、Miku の学習活動において見いだされた記号論的連鎖は「内省的記述表現」を行うことで総括されている点である。二宮(2002(b))では、Miku ともう一人の典型的な児童 Yoji の２名が行ったそれぞれの内省的記述表現について分析・考察を行った結果、以下の諸点が見いだされている。

・Yoji と比較して、Miku は内省的方法で表現する能力がとても高い。
・この違いは単なる表現力の違いだけに留まらず、学習能力の違いであると見なすべきである。
・内省的記述表現活動は一つの学習方法ではあるものの、同時に「内省的記述表現ができること」それ自体を算数・数学学習の「目標・目的」として位置づけるべきである。

二宮(2002(b))におけるこのような帰結は、記号論的連鎖の観点から解釈するとその意味がより明確となる。Miku の記述表現と Yoji の記述表現の一番大きな違いは、記述表現の中に記号論的連鎖を見いだすことができるか否かである。更に Miku の記述表現に特徴的なことは、記号論的連鎖が内省的記述表現をもって総括される点にある。数学とは本来、連鎖的な本性をもつものである点に考慮すると、算数・数学学習においてもまた連鎖的思考がとても重要であろう。ここで、その連鎖的思考をきちんと記述表現にとどめた上で、一連の記号論的連鎖の最後に「反省的活動、メタ認知、メタ知識、情意、などの心的活動」の記述表現である『内省的記述表現』を行うことにより、一連の連鎖的思考を総括することが可能となる。そしてその総括をもとに、「獲得した新たな知

識・情報」を深く再認識するとともに、更なる学習活動へとつながっていくのである。このような記述表現を行うことのできる能力はもはや単なる「表現力」の問題ではなく、本質的な「学習能力」に関わる問題として捉えられるべきである。つまり、学習内容に内在するつながり(連鎖)を認識すること、そしてその一連の連鎖をきちんと総括できることこそが「学習能力」なのである。
更に、(総体としての)内省的記述表現活動はそのような「学習能力」を伸ばすためにとても有効な学習方法でもある。このような記述表現こそが、算数・数学の学習内容に内在する「つながり」をきちんと認識・把握し、『内省的記述表現』によりその一連の連鎖的思考を総括したものになるのである。このような点に鑑み、『内省的記述表現ができること』それ自体を算数・数学学習の「目標・目的」の一つとして位置づけていくことも、今後の課題となろう。

5. おわりに

本稿は、Presmeg(2001)において述べられている「記号論的連鎖(Semiotic Chaining)」について検討した上で、二宮(2002(a))、二宮(2002(b))において述べられた「内省的記述活用学習」及び「キャラクター学習」を記号論的連鎖の観点から分析・考察したものである。よりよい記述表現には記号的連鎖が必ず見いだされ、さらにその記号論的連鎖は最後に「内省的記述表現」により総括される。もちろんそのような記述表現がなされていなくても、学習内容をきちと総括することは不可能なことではない。しかし、具体的な「記述表現」が存在することで記号論的連鎖はより活性化され、さまざまな「つながり」を学習者に認識させることを可能とするであろう。つまりこのような記述表現活動は、単なる学習方法としてのみならず、それ自体を算数・数学学習の「目標・目的」として位置づけられるべきであることが、改めて認められるのである。
本稿における事例分析では、記号論的連鎖という観点から記述表現の分析を試みた。つまり、記号論的連鎖という「道具」を用いて分析・検討を行ったことになる。そして、記号論的連鎖の存在をもとに児童の学習活動や思考を推定し考察を進めた。このような点において、記号論的連鎖とは分析・研究のための枠組みと捉えることができる。しかし一方で記号論的連鎖そのものは、算数・数学学習において非常に重要な役割を果たすものでもある。つまり、学習における有機的な記号論的連鎖がなされることそれ自体が算数・数学学習において極めて重要なのである。このような点に鑑み、記号論的連鎖を明示的に表現する手だてである「内省的記述表現活動」の有用性もまた、より深く認識されるべきであると言えよう。

付記

本稿は、本プロジェクトの国際セミナー『The 2nd International Seminar on the Transition from Elementary Mathematics to Secondary Mathematics（平成 15 年 9 月 24 日～28 日、於:広島大学）』において、Norma Presmeg 先生より直接ご指導をいただいた諸点をもとに、二宮裕之(2003)「数学教育における内省的記述表現の分析－記号論的連鎖（Semiotic Chaining）を手がかりとして－」『全国数学教育学会誌数学教育学研究』第 9 巻, pp.117-126 に加除修正を行ったものである。

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記号論的連鎖3項モデルによる内省的記述表現の分析

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要　約

本稿は、Presmeg(1998)で提唱された「記号論的連鎖」の枠組みと、平林(1987)において示されている「メタ表記」の枠組みについての考察を通して、ノート記述のあり方、特に「メタ認知」の記述に関する知見を導き出そうとするものである。最初に、パースの記号論(semiotics)における３項モデルをもとに、「対象物」「記号」「解釈項」の３つの要素から成り立つ「記号論的連鎖の入れ子型モデル」を同定する。また平林(1987)における「メタ表記」についての指摘と「記号論的連鎖の３項モデル」との関連について考察を行った上で、二宮(2005)において示された「内省的記述表現活動」の事例を「記号論的連鎖の３項モデル」により分析する。その結果として、３種類のノート記述のあり方を同定するとともに、より望ましいノート記述のあり方を記号論的連鎖の観点から指摘する。

1．記号論的連鎖の2項モデルと3項モデル

Presmeg(1998)で提唱された記号論的連鎖は、Presmeg(2001)においてその理論的背景を、パース(Charles S. Peirce)が提唱した記号論（Semiotics）における記号論モデルから検討を始めている。パースのモデルでは、以下にあげる３つの基本的構成要素をあげており、「３項モデル」として捉えられる。(Presmeg,2001,p.2)

①『対象物(object)』: 他の何ものからも独立した対象物の存在
②『記号(representamen)』: 対象物とそれを指示する記号との間の関係
③『解釈項(interpretant)』: 対象物、記号、及び解釈項と呼ばれる第三の要素との関係を考慮した、記号の解釈

そして、先行研究において述べられている具体例を参考に、以下のような例をあげてパースのモデルを解釈した。

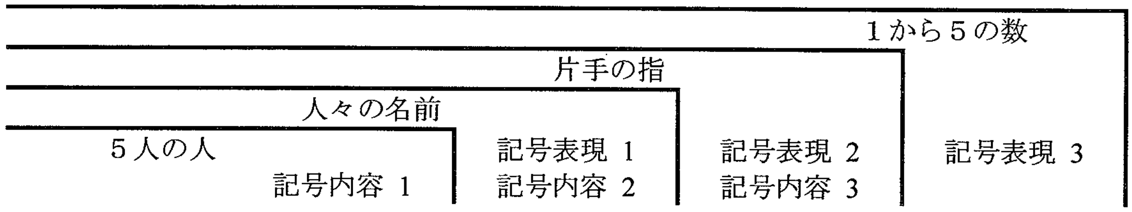
表１　パースの３項モデルの事例

対象物 (object)	記号 (representamen)	解釈項 (interpretant)
ミシン（それ自体）	ミシンの写真	ミシンの機能が分かるような写真を理解すること
「多数は常に正しいのか？」 という疑問	［複数のサイン］ 書籍の中での言語的表現 少数民族の宗教に対する視点が差別的傾向を持つ人	「多数は常に正しいとは限らない」という解釈
雨の降る可能性	雨が降る兆候	傘を持参することの決断

パースの３項モデルに対して、それとはまた異なる解釈を行っているのが、ソシュール(Ferdinand de Saussure)である。パースの枠組みが記号論（Semiotics）と呼ばれているのに対し、ソシュールの枠組みは記号学（Semiologie）と呼ばれる。氏は記号(signe)を「意味を担うもの(記号表現)」と「担われる意味(記号内容)」とに類別し、前者をシニフィアン(signifiant)、後者をシニフィエ(signifié)と名付けた。Presmeg(2001)は２項モデル（記号内容(シニフィエ)と記号表現(シニフィアン)）に関連して、「これはそれぞれ３項モデルの構成要素となっている。しかし第三の要素(解釈項:interpretant)は、記号の解釈において暗黙的である(p.2)」と述べ、それらを統合した新しいモデルを以下のように提案している。

「記号内容が記号表現よりも優先される」とするソシュールのモデルを逆転して捉えたラカンは、記号内容よりも上位に位置する記号表現が強調されるモデルを考えた。そしてそれは、「記号表現が何らかの形で記号内容の支配下に置かれることが暗黙のうちに了解された場合には容易に認識されえない、記号表現による動的で継続的な生産作用における広遠な自律性(far ranging autonomy)」を意味するものとなった（Whitson, 1994, p.40）。このような見識に対してPresmeg(2001)は、「前段階の記号の組み合わせにおける記号表現が、新たな記号の組み合わせの記号内容となり、さらにそれは繰り返される」という連鎖の過程を想定した。Presmeg(2001)はこのような連鎖を『記号論的連鎖(Semiotic Chaining)』と命名し、その具体的事例として、Walkerdine(1988)において取り上げられた「幼い娘が飲み物を注ぐ際の母親とのやりとり(pp.129-138)」を次のように分析し示している。

母親が、飲み物を注ぐべきお客さん(５名)の名前を順番に述べる。それを聞いた娘は、それぞれの名前に対して手の指を１本ずつ曲げる動作で対応させた。



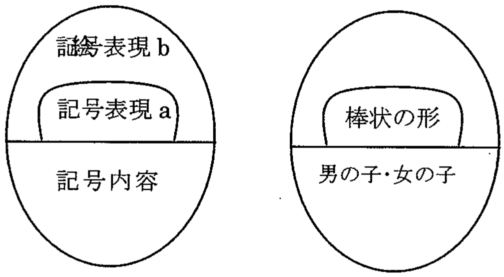
記号表現：意味を担うもの、意味するもの（シニフィアン）
記号内容：担われる意味、意味されるもの（シニフィエ）

図１　記号論的連鎖の２項モデル

図１において、５人の人（記号内容１）に対して最初に『名前』が「記号表現１」として登場する。しかしそれはすぐに、新しい記号表現(指)のための記号内容へと変容する。その後、「指」は再び記号内容となり、１から５までの数が新たな記号表現となる。

このような記号論的連鎖の枠組みに対し、Presmeg(2001)では更に新たな知見が導き出されている。Hall(2000)は上述の記号論的連鎖の枠組みを用いて子どもたちの学習活動を分析したが、このような二項モデルを基盤とする枠組みでは示しきれない、より複雑な学習過程が観察された。それは、一つの記号内容に対して複数の記号表現が同定されるものであった。その一例としてPresmeg(2003)は、Hall(2000)に述べられた次のような事例をあげている。

この活動は、子どもたちに男の子と女の子の人数を数えさせるものである。子どもたちは問題場面を絵に描いて考えるとともに、棒状の形を描いた。この場合、棒状に描かれたもの、絵に描かれたもの、の両者とも、同じ記号内容(男の子・女の子)に対する記号表現であると考えることができる。それは以下のような図で示されるものである。



このような事実に対しては、記号論的連鎖の枠組みを単純に採用するのではなく、意味の構成について考慮する必要がある。つまり、記号論的連鎖の２項モデルを拡張し、記号表現への連鎖のみならず、それぞれの連鎖における意味の生成に関わる構成要素を考えるべきなのである。記号論的連鎖における記号表現は、その連鎖における前段階の記号内容を表現している。この記号表現はその意味すること全てを含めて、新たな記号内容となる。このように、新たに構成された記

号表現や記号内容は、連鎖のその時点に至るまでの全てを内包するのである。記号表現が数学教育において重要であるのは、このような「内包関係」があるからである。

(Presmeg、2003, pp.6-7)

このような知見に対し Presmeg(2003)では、記号論的連鎖における記号の再構成という観点から、図2のようなモデルを示した。

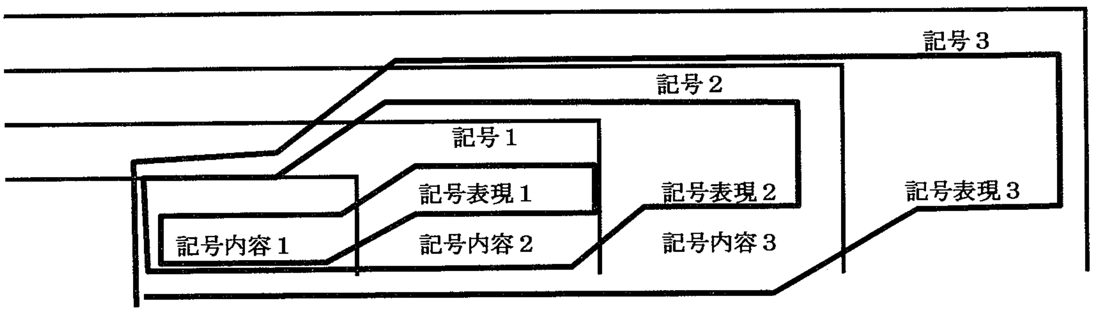


図2　記号論的連鎖における記号の再構成

そして、数珠繋ぎの連鎖(A chain)はこのような入れ子型の様相をきちんと表現しきれていない点において、必ずしも適切なモデルとは言い難いことを指摘するとともに、パースの3項モデルを手がかりに新たに図3に示すモデルを提示した(1)。

O = Object (signified)　: 対象物(記号内容)
R = Representamen (signifier)　: 記号(記号表現)
I = Interpretant : 解釈項

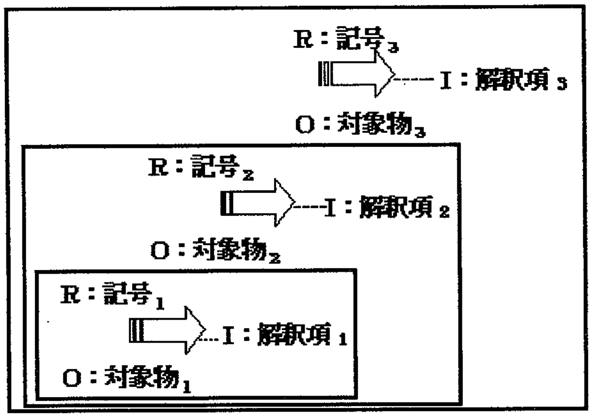


図3　記号論的連鎖の入れ子型モデル

図1に示した事例を図3に当てはめてみると、次のようになる。

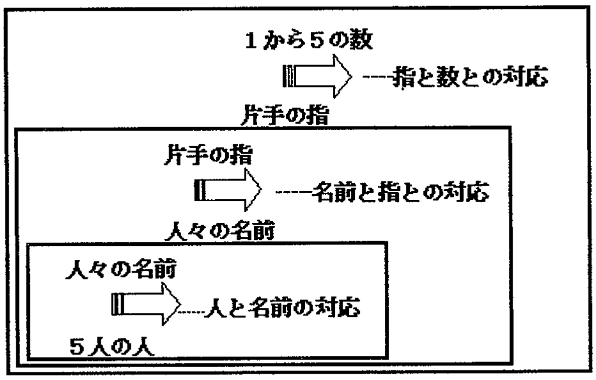


図4　入れ子型モデルの具体的事例

2. 対象表記とメタ表記

平林(1987)は一般言語学における「対象言語」「メタ言語」という知見に言及している。対象言語は数学の学習内容であり、それについての教育論は数学教育内容論であるといえる。また、メタ言語は、学習・指導に用いられる言語であり、それに関する議論は教育学的には数学学習指導論というべきであろう(平林,1987,p.390)。更に平林(1987)は数学教育における2種類の表記について言及し、対象言語に対応する「研究の対象とされる表記」を「対象表記」として、メタ言語に対応する「研究の方法として使用される表記」を「メタ表記」としてそれぞれ規定した(平林,1987,p.388)。このような対象表記とメタ表記との本性上の区別は、その構成規則の形式性の有無にある。つまり、対象表記は明確な構成規則に従って構成されるのに対して、メタ表記については、かような規約を明記することは要請されない(平林,1987,p.389)。

例えば一冊の教科書の中にも、これら二種の言語・表記が混在しているものと考えられる。しかし、それらの表記を対象表記とメタ表記とに区分しようとすると、それはなかなか難しい。その理由として平林(1987)は、対象表記に対する構成規則の複雑さと、学習活動の有機的・生命的な本性をあげている。特に、「一旦学習された対象言語は忽ちメタ言語として他の対象言語の学習に利用されることもあれば、また逆に、メタ言語が意識的に洗練され、形式化されて、対象言語化される(平林,1987,pp.390-391)」という算数・数学学習の本性に対する指摘には大いに注目したい。平林(1987)による「対象表記」「メタ表記」の特質をまとめると図5のようになろう。

ここで、平林(1987)による知見を表1に示したパースの3項モデルとの関連で考えていくと次のように捉えていくことができる。

対象表記
- 算数・数学学習の対象になっている表記
- 算数・数学の学習内容
- 一旦学習された対象言語は、忽ちメタ言語として他の対象言語の学習に利用される

メタ表記
- 算数・数学の学習のために使用される表記
- 算数・数学の学習方法
- メタ言語が意識的に洗練され、形式化されて、対象言語化される

図5　対象表記とメタ表記の特質　(平林,1987,pp.388-391)

対象物(object)と記号(representamen)は、それぞれ算数・数学学習における「内容」や「対象」である点に鑑み「対象表記」として捉えることができる。一方,解釈項(interpretant)は、記号の解釈であること、並びにその解釈が数学学習の方法として用いられる点に鑑み「メタ表記」として捉えられよう。以上をまとめると図6のようになる。

『対象物 (object)』	⇔	対象表記
『記号 (representamen)』	⇔	対象表記
『解釈項 (interpretant)』	⇔	メタ表記

図6　パースの3項モデルと表記との関係

3. 内省的記述表現活動

重松(1990)では、メタ認知を擬人化して「内なる教師」という概念を導入している。算数・数学の学習におけるメタ認知は、児童・生徒にとって教師となる者の影響が内面化することによって形成されていく(重松, 1990,p.85)。そして「内なる教師」とは、メタ認知があたかも学習者の内面に存在する「教師」のような役割を果たす内的操作であることに由来している。

二宮(2005)は、学習者と学級全体でなされる学習活動との間の相互作用を記述する「内省的記述表現活動(2)」を提起した。学習者は、①学習者自身、②他者、③学習者に内在するもう一人の自己、という3つの観点から内省的記述を行う。内省的記述は基本的には学習者自身による学習活動の一環ではあるが、学習者による記述は単に「答え」や「解法」を記すだけにはとどまらない。学習者が自分の解法をふり返ることにより、何らかの内的操作(メタ認知)が生起し、学習

者はその内的操作を記述するように促される(3)。ところが学習者にとって、自分自身の認知的活動と、メタ認知などの内的操作とを峻別すること、或いはそれらの違いを意識的に認識することが難しい場合も十分想定できる。そこで学習者は、「自分の中にいるもう一人の自分（一人称的他者(4)）」という視点から自らの学習をふり返ることを促される。そしてそれはあたかも教師（内なる教師）からの言葉がけのように、一人称的他者からのコメントとして学習者自身へ何らかのフィードバックをなすのである。

このように内省的記述は、①学習者自身、②他者、③学習者に内在するもう一人の自己、という３つの観点からなされるものであるが、学習者の記述は単に一つのコメントで終わらない点に留意したい。例えば、児童・生徒が自分自身の考えを記述した後に、続けて友だちの考えを記述し、更にそれを自分の考えと比較するかもしれない。その比較を通して、児童・生徒は何らかのメタ認知、或いは内省的活動を行い、今度は一人称的他者の観点から何らかのコメントを記述するであろう。そのようなコメントを受け、児童・生徒は自分自身の考えをさらに深め、また別の考えを得るかもしれない。このようにして、記述表現活動と児童・生徒の学習活動とは相互作用を通して互いに深化していく(二宮,2002,p.145)。

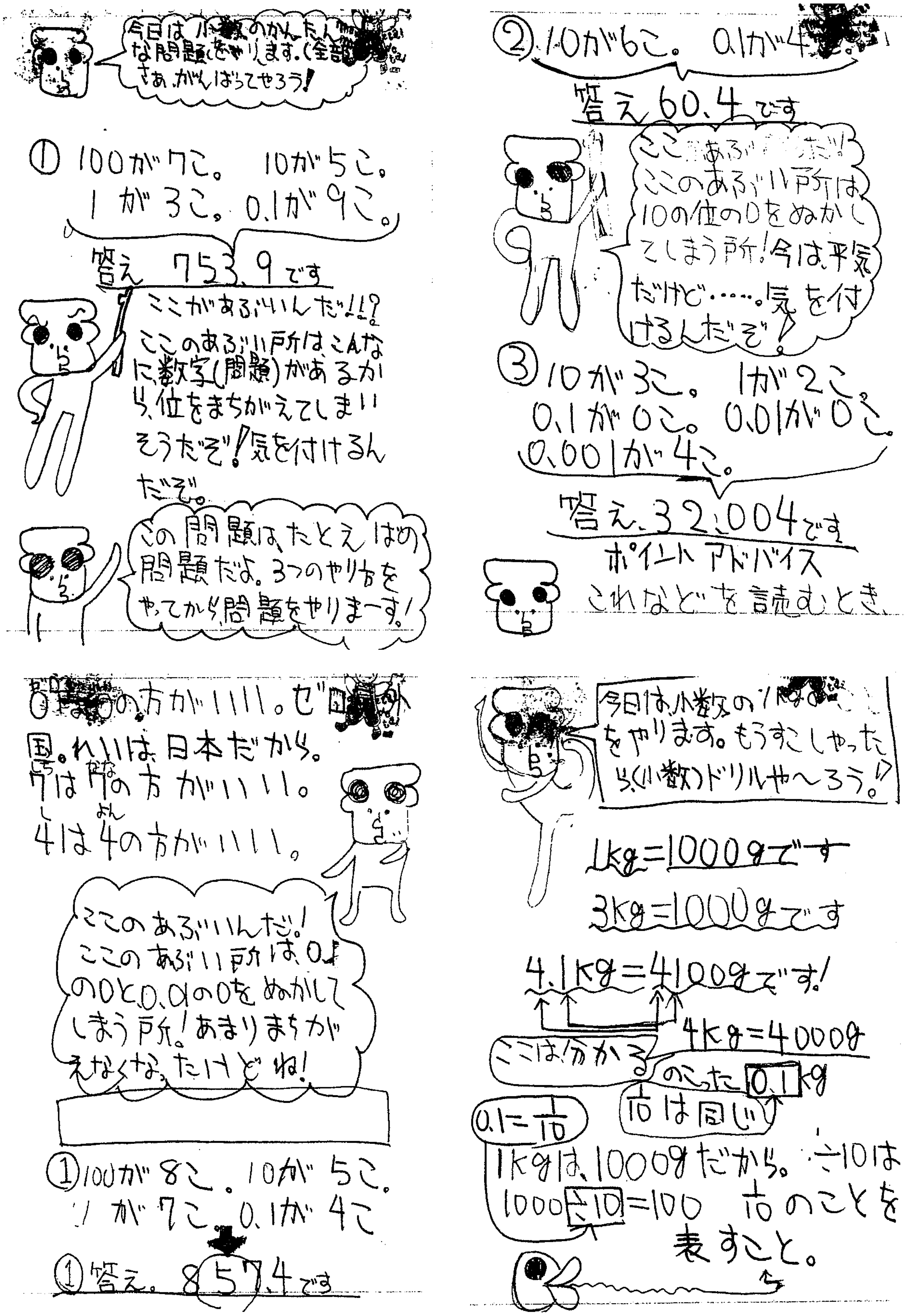
このような学習活動における学習者のノート記述からは、「学習者自身による答えや解法」と「一人称的他者からのコメント」という大きく２つの部分を見いだすことができる。ここで前者は学習者の算数・数学学習における「学習の対象」となるものであり、後者は「学習の方法」或いは「学習の対象についての説明」である。言い換えれば、前者は対象表記、後者はメタ表記であるとともに、パースの３項モデルとの関連で言えば、前者は「対象物」及び「記号」、後者は「解釈項」であると捉えることができよう。

４．事例分析

５年生女子による内省的記述表現活動の事例を以下に示す。図７に示す記述活動は「小数の位取り」について児童がまとめたもので、学校の授業が終わった後の自主学習課題として家庭学習の一環として行われたものである。

ここで、図７に示した内省的記述表現の事例に、図３に示す記号論的連鎖の入れ子型モデルを適用すると次のようになる。最初に問題①「100 が７こ、10 が５こ、1 が３こ、0.1 が９こ」が示される。それに対して児童は「753.9」という解答を示すとともに、キャラクターを用いてその解答に対するコメントを示している。ここで最初の問題①は、学習の内容であるとともに、「意味されるもの(記号内容)」として「記号表現(＝解答)」によりその意味を担われていることから、『対象物(記号内容)』であると捉えることができる。（図１「記号論的連鎖の２項モデル」より） そしてその解答である「753.9」は、問題①の意味を担っていることから『記号(記号表現)』として解釈されよう。更にキャラクターによる解答に対するコメントは、一人称的他者によるもの(メタ認知の記述)であり、学習のための表記であるという点において「メタ表記」であると考えることができる。そしてそれは、記号を解釈するものでもあることから『解釈項』として捉えられよう。

さらに、Presmeg(2001)において「記号論的連鎖における記号表現(記号)は、その連鎖における前段階の記号内容(対象物)を表現している。この記号表現(記号)はその意味すること全てを含めて、新たな記号内容(対象物)となる。」と示されているように、ここでは引き続いて第二の『対象物(記号内容)』が見いだされる。つまり、続いて出された問題②「10 が６こ、0.1 が４こ」は、前段階における問題①とその解答並びに一人称的他者によるコメントの全てを内包する形で、それらを踏まえた新たな問いとして示されていると捉えることができるのである。このような解釈のもと、図７に示された内省的記述表現の事例には、記号論的連鎖の様相を見いだすことができる。それをまとめると図８のようになろう。 尚、ここで示された記号論的連鎖の中で解釈項 $_5$（I_5）は空欄になっているが、それはここでの対象物 $_5$（O_5）ならびに記号 $_5$（R_5）の示すところについて、児童のノートにはその解釈がきちんと記述されているとは見なされないからである。



今日は小数のかんたんな問題をやります。
さあ、がんばってやろう!
① 100が7こ。10が5こ。1が3こ。0.1が9こ。
答え 753.9です
ここがあぶないんだ!!?
ここのあぶない所はこんなに数字(問題)があるから、位をまちがえてしまいそうだぞ!気を付けるんだぞ。
この問題は、たとえはの問題だよ。3つのやり方をやってから問題をやりまーす!
② 10が6こ。0.1が4
答え 60.4です
ここがあぶないんだ!
ここのあぶない所は10の位の0をぬかしてしまう所!今は平気だけど……。気を付けるんだぞ!
③ 10が3こ。1が2こ。0.1が0こ。0.01が0こ。0.001が4こ。
答え 32.004です
ポイントアドバイス
これなどを読むとき、
の方がいい。
れいは日本だから。
7はなの方がいい。
4はよんの方がいい。
ここがあぶないんだ!
ここのあぶない所は0.1の0と0.01の0をぬかしてしまう所!あまりまちがえなくなったけどね!
①100が8こ。10が5こ。1が7こ。0.1が4こ
①答え。857.4です
今日は小数の
をやります。もうすこしやったら(小数)ドリルやへろう!?
1kg=1000gです
3kg=1000gです
4.1kg=4100gです!
4kg=4000g
ここは分かる
のこった0.1kg
1/10は同じ
0.1=1/10
1kgは、1000gだから。÷10は
1000÷10=100 1/10のことを表すこと。

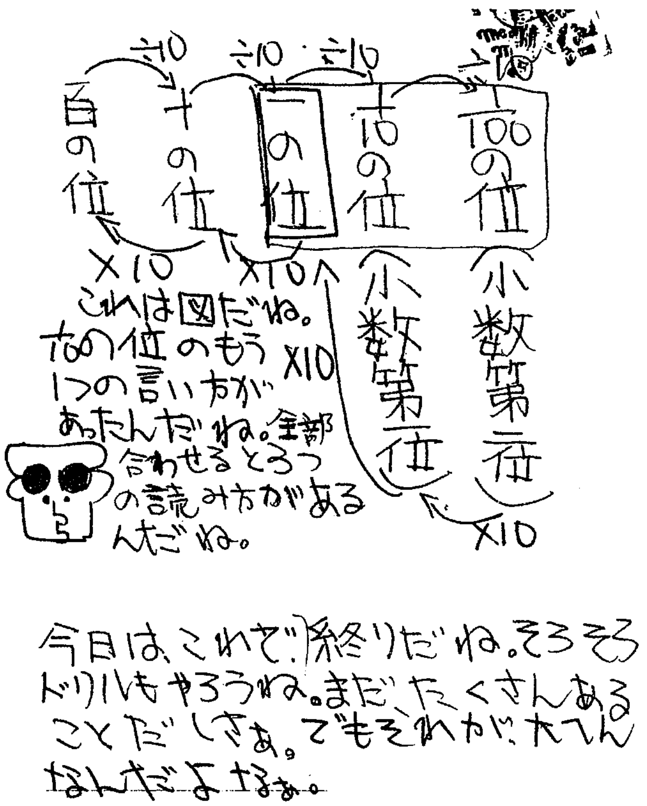


図７　内省的記述表現の事例（５年生「小数の位取り」）

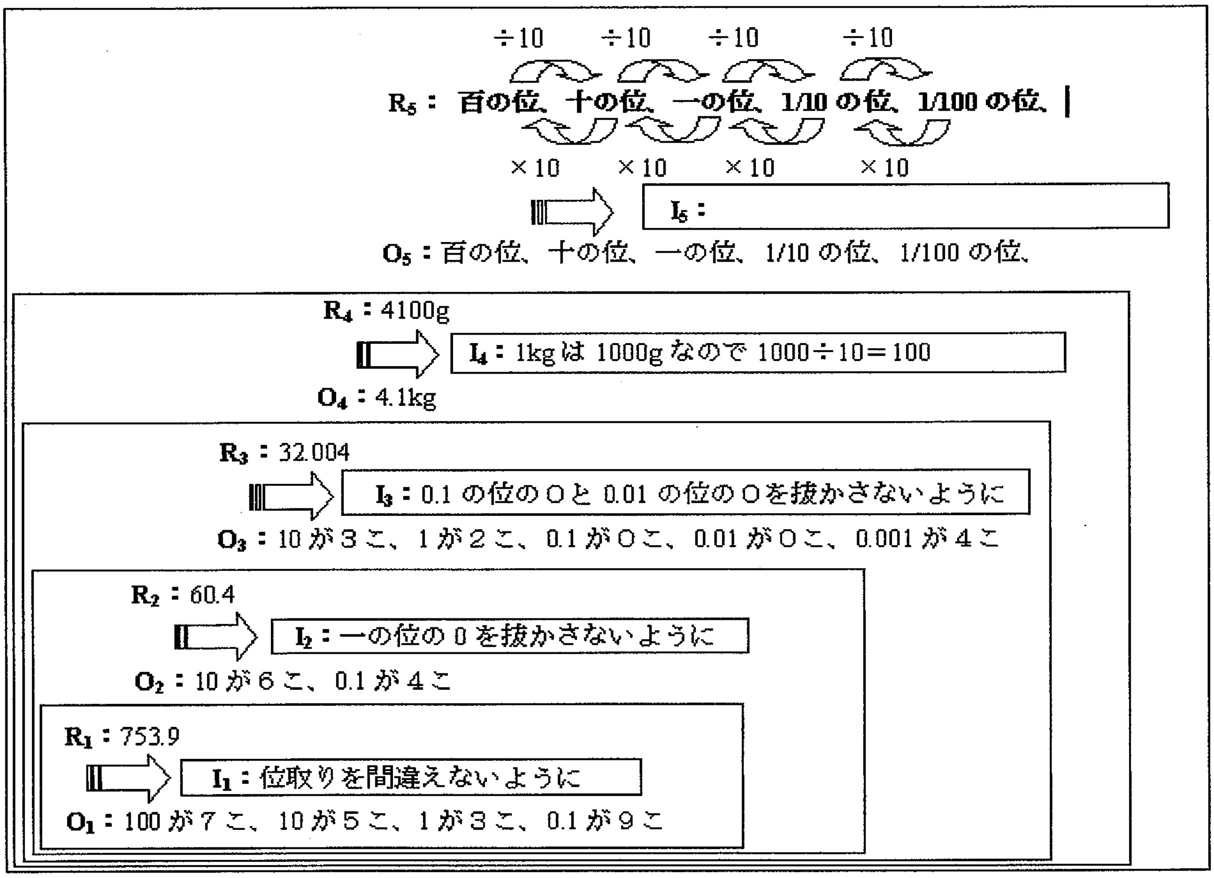


図８　記号論的連鎖の入れ子型モデルによる内省的記述表現の分析

4. 考察

内省的記述表現活動は、対象表記とメタ表記の両者を表現させる学習･指導方法の一つである。例えば図７の事例にあるように、「問題」と「解答」という形で対象表記が記述され、「コメント」という形でメタ表記が記述されている。これをパースの３項モデルにあてはめて考えてみると、「問題」は『対象物(記号内容)』として、「解答」は『記号(記号表現)』として、「コメント」は『解釈項』として、それぞれ位置づいていることが見いだされた。一人称的他者によるコメントがメタ表記として機能する「内省的記述表現活動」の効果については、二宮(2002, 2003, 2005)などにおいて既に議論されている。

ここで、記号論的連鎖の入れ子型モデルによる内省的記述表現の分析（図８）から、次に述べるような３通りのノート記述を想定することができる。それぞれ、① 解答（＝記号(記号表現)）のみが列挙されているもの、② 問題（＝対象物(記号内容)）と解答（＝記号(記号表現)）が記述されているもの、そして③ 問題（＝対象物(記号内容)）と解答（＝記号(記号表現)）に加えてその説明（＝解釈項）がなされているものである。本稿ではそれぞれ、① 記号型、② 対象物－記号型、③ 解釈項型、と名前をつけることにする。

①記号型

$R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_4 \rightarrow$

②対象物－記号型

$O_1 \rightarrow R_1 \rightarrow O_2 \rightarrow R_2 \rightarrow O_3 \rightarrow R_3 \rightarrow$

③解釈項型

$O_1 \rightarrow R_1 \rightarrow I_1 \rightarrow O_2 \rightarrow R_2 \rightarrow I_2 \rightarrow \rightarrow O_3 \rightarrow R_3 \rightarrow I_3 \rightarrow$

O = Object：対象物(記号内容)
R = Representamen：記号(記号表現)
I = Interpretant：解釈項

図９　ノート記述の典型的な３つのタイプ

図７に述べた事例に則して考え、仮にこれが「記号型」のノート記述により表現されていたとすると、その様相は概ね次のようになるであろう。

① 753.9　　② 60.4　　③ 32.004　　④ 4100g

⑤　÷10　÷10　÷10　÷10
百の位、十の位、一の位、1/10 の位、1/100 の位、
×10　×10　×10　×10

図 10　記号型ノート記述の例

このタイプのノート記述は、記号(記号表現)（＝解答）のみが列挙されている。問題集にある問題を次々と解いていくような形の学習において、このような記述を行うケースは多々見られるであろう。仮に、問題の中に示されている規則や結果をとにかく丸暗記すればことが足りるのであれば、このような記述活動を介した学習もある意味効果的であるのかもしれない。例えば英単語を暗記しようとすることと同じように、図形の名称や計算の公式を次々に列挙し暗記するような学習において大いに活用される記述活動である。もちろん、このような記述を行う算数･数学の学習は児童･生徒に対して、認知的にも情意的にも望ましくない結果をもたらすことは言うまでもない。

同様に、「対象物－記号型」のノート記述の事例を想定するなら、その様相は次のようになろう。

① 100 が 7 こ、10 が 5 こ、1 が 3 こ、0.1 が 9 こ

答え 753.9

② 10 が 6 こ、0.1 が 4 こ

答え 60.4

③ 10 が 3 こ、1 が 2 こ、0.1 が 0 こ、0.01 が 0 こ、
0.001 が 4 こ

答え 32.004

④ 4.1kg をグラムに直すと　答え 4100g

⑤ 小数の位取りのまとめ

÷10　÷10　÷10　÷10

百の位、十の位、一の位、1/10 の位、1/100 の位、

×10　×10　×10　×10

図 11　対象物－記号型ノート記述の例

このタイプのノートでは、対象物(記号内容)（＝問題）と、記号(記号表現)（＝解答）とが交互に記述されている。算数･数学の授業でよくなされる「ノート指導」には、このようなタイプのものを意図する場合が多いかもしれない。記号内容(対象物)と記号表現(記号)とが交互に記述されている点では、図 1 に示した「記号論的連鎖の 2 項モデル」に沿った形でのノート記述とも言える。しかし、図 6 に示した「パースの 3 項モデルと表記との関係」即して考えてみると、「対象物(記号内容)」も「記号(記号表現)」もともに『対象表記』として類別されるものである。つまり対象物－記号型のノート記述では、学習の対象である『算数･数学の内容』についてはきちんと記されているものの、「考え方」「説明」「学習のやり方」といった『学習の方法』についての記述が欠如している。

更に、図 7 に示した内省的記述表現の事例は「解釈項型」ノート記述の一例であると言える。これも同様にまとめ直してみると図 12 のようになる。

① 100 が 7 こ、10 が 5 こ、1 が 3 こ、0.1 が 9 こ

答え 753.9

⇒位取りを間違えないようにしよう

② 10 が 6 こ、0.1 が 4 こ

答え 60.4

⇒ 一の位の 0 を抜かさないように気をつけよう

③ 10 が 3 こ、1 が 2 こ、0.1 が 0 こ、0.01 が 0 こ、
0.001 が 4 こ　答え 32.004

⇒ 0.1 の位の 0 と 0.01 の位の 0 を抜かさないように

④ 4.1kg をグラムに直すと　答え 4100g

⇒ 1kg は 1000g なので 1000÷10＝100

⑤ 小数の位取りのまとめ

÷10　÷10　÷10　÷10

百の位、十の位、一の位、1/10 の位、1/100 の位、

×10　×10　×10　×10

図 12　解釈項型ノート記述の例

図 7 に示した事例ではキャラクターを用いて一人称的他者からのコメントを記述している。キャラクターや吹き出しを使用することの是非はとにかくとして、このような形でメタ表記（＝解釈項）を記述することで、メタ認知のような「学習者の内面に存在する『教師』のような役割を果たす内的操作」を学習者により強く意識させることができる。

ノート記述を「記号論的連鎖の入れ子型モデル」により分析することで、内省的記述表現活動

の本性が見いだされるとともに、３通りのノート記述のタイプを同定することができた。また、内省的記述表現がノート記述として望ましいとされる従来からの指摘（二宮 2002,2005,など）に対し、それが「対象物(記号内容)」「記号(記号表現)」「解釈項」の全てを書き表すノート記述（解釈型ノート記述）であるという点において、算数･数学学習においてより望ましい方法であることが明らかとなった。そして、本稿における分析を通して、「記号論的連鎖の入れ子型モデル」が記述表現を分析するための枠組みとして十分に機能するものであることが明らかとなった。

5. おわりに

本稿では、記号論的連鎖の枠組みを援用することで、算数･数学学習におけるノート記述の分析を試みた。Presmeg(1998)で提唱された「記号論的連鎖」の枠組みと、平林(1987)において示されている「メタ表記」の枠組みについて考察することで、算数･数学教育における「記述表現」と記号論的連鎖との間の有機的なつながりを見いだすことができた。また「内省的記述表現活動」の事例を「記号論的連鎖の３項モデル」により分析することで、３種類のノート記述のあり方が同定されるとともに、より望ましいノート記述のあり方が記号論的連鎖の観点から指摘された。そして、記述分析を行うための枠組みとして「記号論的連鎖の３項モデル」が有効であることが示された。

今後の課題として、記号論的連鎖に関する更なる理論的検討があげられる。特に、連鎖を促す原動力になっていると考えられる「比喩」の観点から検討を進めていく必要があろう。

付記

本稿は、本プロジェクトの国際セミナー『The 2nd International Seminar on the Transition from Elementary Mathematics to Secondary Mathematics（平成 15 年 9 月 24 日～28 日、於:広島大学）』において、Norma Presmeg 先生より直接ご指導をいただいた諸点をもとに作成された論文、二宮裕之(2005)「数学学習におけるノート記述とメタ認知－記号論的連鎖とメタ表記の観点からの考察－」『全国数学教育学会誌数学教育学研究』第 11 巻, pp.67-75 に加除修正を行ったものである。

注

(1) Presmeg 教授は、記号論的連鎖の２項モデルと入れ子型モデルについて、それぞれのモデルによさがあり、用途に応じて使い分けるべきであるとの立場をとっておられる。２項モデルは連鎖の様相、つまり記号のつながりの様子が分かりやすい。一方、入れ子型モデルは意味の構成の様相を捉えるのに適している。
(2) 「内省的記述表現活動」とは、算数･数学の学習における思考や活動を振り返って考えたことや、情意面に関わることなどを記述する活動である。
(3) 二宮(2005)では、有効な内省的記述表現の要件を、「『核となる記述』を中心に、豊富な『具体例』を伴い、『メタ知識的記述』によりそれらを関係づけた記述 (p.230)」と規定している。
(4) 一人称的他者とは、二宮(2002)において提起された概念で、自分自身の学習活動を見つめ、それに関わる心的操作を行う主体としての「もう一人の自己」である。

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記号論的連鎖3項モデルによる教授単元の分析

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要約

本稿は、Presmeg(1998)で提唱された「記号論的連鎖」の枠組みを「数学化」と「比喩」の観点から精緻化するとともに、Witmann(1984)の提起する「教授単元」へと援用することで、教授単元における「数学化」に関する示唆を得ようとするものである。最初に、パースの記号論(semiotics)における３項モデルをもとに、「対象物」「記号」「解釈項」の３つの要素から成り立つ「記号論的連鎖の入れ子型モデル」を同定する。続いて、Freudenthal の示した「数学化」に関する Treffers(1987)の見解を取り上げ、「水平的な数学化(Horizontal Mathematization)」と「垂直的な数学化(Vertical Mathematization)」を同定する。更にこれら２通りの「数学化」の本性へと迫ることで「数学化」の過程に内在する『比喩』の存在に言及し、水平的な／垂直的な数学化のそれぞれがなされるメカニズムと、「比喩」「具象化」により記号論的連鎖が促進されるモデルにおける『比喩』の役割について明らかにする。そして具体的な事例の検討として Witmann の教授単元を取り上げ、その概要について言及した後、教授単元の事例に対する記号論的連鎖の枠組みを用いた分析を試みる。

1. 記号論的連鎖における「数学化」と比喩の役割

Presmeg(2003)ではさらにいくつかの事例をあげ、記号論的連鎖における「水平的な数学化(Horizontal Mathematization)」と「垂直的な数学化(Vertical Mathematization)」について説明している。ここで、『数学化』とは Freudenthal の示した「数学が有用であるためにはどのように教えるべきか(How to teach mathematics so as to be useful)」という設問に対する解答である。氏は以下のように述べている。

> 人間が学ばなければならないことは、閉じたシステムとしての数学ではなく、活動としての数学であり、現実を数学化するプロセスであり、可能なら数学すら数学化するプロセスである。(Freudenthal,1968,p.7)

Freudenthal の示した「数学化」を更に精緻に捉えたのが Treffers である。Treffers(1987)は数学化を「未知の規則や関連･構造について、知識･技能を用いて構造化･組織化する活動 (p.247)」と規定した上で、それを更に「水平的な数学化(Horizontal Mathematization)」と「垂直的な数学化(Vertical Mathematization)」の２つに区分している。そして前者を「問題場面を数学の文脈へと転換すること」、後者を「数学的体系内での手続き」とし、以下のように特徴づけた。

> 水平的な成分において、数学への道筋はモデルの形成･図式化･記号化を通して開かれる。一方で垂直的な成分は、数学の処理過程や考察中の問題領域構造におけるレベルの向上と関わりを持つ。しかし、水平的な成分と垂直的な成分との間のこのような区分は些か理論的過ぎ、それらは相互に関係のあるものである点を我々は認めざるを得ない。(Treffers,1987,p.247)

このような異なる二種類の数学化プロセスに関連して、岩崎(1996)は数学的理解の水平的な成分と垂直的な成分について言及した。そして前者を「物理的世界を基盤とする数学化」として、後者を「活動に基づく数学化」としてそれぞれ特長づけている(p.12)。このような岩崎(1996)による特徴づけと Treffers(1987)によるそれとを比較すると表１のようになろう。

表１　水平的／垂直的な数学化に関する Treffers(1987)と岩崎(1996)の見解の比較

	Treffers(1987)	岩崎(1996)
水平的な数学化	・**問題場面を数学の文脈へと転換すること** ・数学への道筋はモデルの形成･図式化･記号化を通して開かれる。	・物理的世界を基盤とする数学化
垂直的な数学化	・**数学的体系内での手続き** ・数学の処理過程や考察中の問題領域構造におけるレベルの向上と関わりを持つ。	・活動に基づく数学化

両者の見解はほぼ一致していることが分かる。即ち、水平的な数学化とは「**問題場面を数学の文脈へと転換すること**」であり、それは物理的世界を基盤とし、物理的世界における異なる対象を結びつけることでもある。そしてそのための手続きとして「モデルの形成･図式化･記号化」がなされ、水平的な数学化は推移していく。

一方、垂直的な数学化とは「**数学的体系内での手続き**」であり、数学の活動(数学的処理や考察)による変化を伴う対象間の関係に基づく。その結果、「問題領域構造におけるレベル」が向上することで、垂直的な数学化は推移していく。

ここで、岩崎(1996)はさらに次のように指摘を続けている。

> こうした二分法は、Piaget の経験的抽象と反省的抽象という抽象の区分にも、そして比喩に関連させれば、Saussure の連合関係と統辞関係という言語使用の基本構造にも、深層で通じていると考えられる。(岩崎,1996,p.12)

そして楠見(1991)などを拠り所にして、『類似性に基づき、知識構造における異なるカテゴリーの対象を結び付け、カテゴリーを組み替えることによって成立する比喩』として隠喩（metaphor）を、『隣接性に基づき、カテゴリーを組み替えずに、カテゴリー内の上位－下位関係や、場面内の時間的･空間的隣接関係に依拠する比喩』として換喩（metonymy）を、それぞれ規定している(p.13)。

水平的な数学化は「問題場面を数学の文脈へと転換すること」である。物理的世界における異なる対象を結びつけるものであり、カテゴリーを組み替える（異なる文脈を結びつける）ことによって成立する。そして対象（文脈）間の『類似性』がその結びつきの拠り所となる。従って、水平的な数学化は「対象間の類似性」に基づく転義がなされるものである点から**『隠喩(metaphor)』的なつながり**を持つものと捉えることができる。

一方、**垂直的な数学化**とは「数学的体系内での手続き」である。数学的処理や考察といった「同一の文脈内における」数学の活動により生じる変化を伴う対象間の関係に基づき、問題領域構造におけるレベルの向上を伴うものである。「カテゴリーを組み替えずに、カテゴリー内の上位－下位関係に依拠する」点において、対象間の『隣接性』がその結びつきの拠り所となる。従って、垂直的な数学化は「対象間の隣接性」に基づく転義がなされるものである点から**『換喩(metonymy)』的なつながり**を持つものと捉えることができよう。

別の言い方をするなら、「類似性に基づく転義(隠喩)」では、対象どうしの間に類似性が存在すること、つまりその対象どうしは本質的に同質のものであることが想定されるのに対し、「隣接性に基づく転義(換喩)」では、対象どうしが質的に異なるものであることに留意したい。これを数学の文脈に即して言い直すのであれば、内包される数学に注目した時に、水平的な数学化ではそれぞれに内包される数学の質は基本的に同じものであるのに対して、垂直的な数学化においては内包される数学が変容するものであると言える。

以上をまとまると表２のようになる。

表２　水平的／垂直的な数学化における比喩

水平的な数学化 問題場面を数学の文脈へと転換すること	対象間の類似性に基づく転義	隠喩的(metaphor)
	対象どうしの本質は同じ	
	内包する数学の質は同じ	
垂直的な数学化 数学的体系内での手続き	対象間の隣接性に基づく転義	換喩的(metonymy)
	対象どうしが質的に異なる	
	内包する数学の質が変容	

ところで、Presmeg(2001)は記号論的連鎖のプロセスを次のように説明している。

> このような定式化により、前段階の記号における記号表現は新たな段階における記号の記号内容となり、それが繰り返しなされることとなる。ここで記号表現は記号内容に対して換喩的関係にある。また新たな記号内容は、前段階の記号表現の具象化により構築される。(p.3)

この指摘は、記号論的連鎖において、それぞれの記号間に「比喩的関係」並びに「具象化」のプロセスが存在することを示すものと解釈できる。もっとも、記号内容から記号表現への比喩的関係を、Presmeg(2001)では「換喩」と限定されている点は問題である。ここでは、なされる数学化が水平的なものであるのか垂直的なものであるのかに因って、その比喩が「隠喩」となるか「換喩」となるかが異なると解釈すべきである(2)。 Presmeg(2001)による記号論的連鎖のプロセスにおける記号間の関係についての記述は、図１および図２のように表すことができよう。

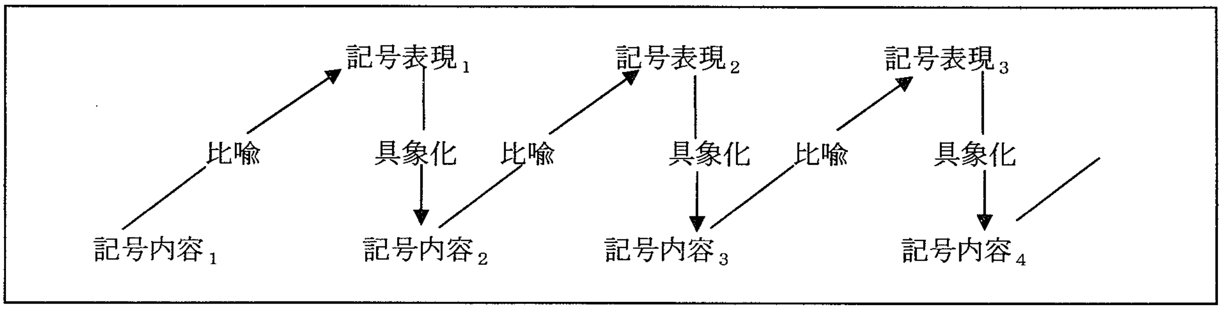


図１　記号論的連鎖における記号間の関係（２項モデル）

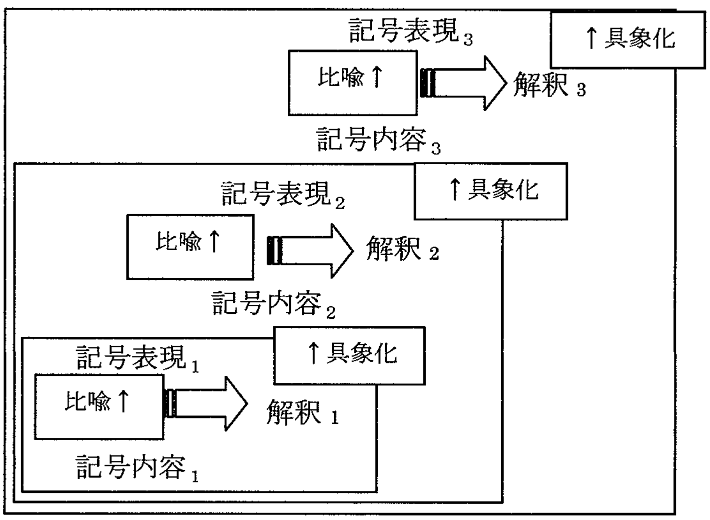


図２　記号論的連鎖における記号間の関係（入れ子型モデル）

例えば、Walkerdine(1988)において取り上げられた「幼い娘が飲み物を注ぐ際の母親とのやりとり(pp.129- 138)」の事例を図２に基づいて解釈すると以下のようになる。

最初に、記号内容$_1$(５人の人)が「人と名前の対応」という解釈を介して、記号表現$_1$(人々の名前)へと結びついている。ここでの連鎖は、「人」と「名前」という「**異なるカテゴリー**」間を結びつけることでなされている。カテゴリーを組み替える（異なる文脈を結びつける）ことによる連鎖は、対象(文脈)間の類似性がその結びつきの拠り所となる。従ってここでの連鎖は、「対象間の類似性に基づく転義」がなされたと見なすことができ、『隠喩的なつながり』をもつ数学化、すなわち『水平的な数学化』がなされたものと判断することができる。

記号表現$_1$(人々の名前)は、解釈$_1$(人と名前の対応)を介して具象化され、新たな記号内容（記号内容$_2$：人々の名前）として更なる連鎖を続ける。

次の段階での連鎖では、記号内容$_2$(人々の名前)が「名前と指との対応」という解釈を介して、記号表現$_2$(片手の指)へと結びついている。これも同様に異なるカテゴリー間を結びつけ、カテゴリーを組み替える（異なる文脈を結びつける）ことによってなされる連鎖である。従ってここでの連鎖も、「対象間の類似性に基づく転義」がなされたと見なすことができ、『水平的な数学化』がなされたものと判断される。

再び、記号表現は解釈を介して具象化され、新たな記号内容（記号内容$_3$：片手の指）へと連鎖を続ける。

更なる連鎖は、記号内容$_3$(片手の指)が「指と数との対応」という解釈を介して、記号表現$_3$(１から５の数)へと結びつく。ここでの連鎖は、これまでのものとは若干様相を異にする。「人」「名前」「指」と異なるカテゴリーの対象を結びつける「水平的な数学化」により展開されてきた連鎖が、今度は『同じ指』を「単なる指」として見るか、それとも「数」としてみるか、という異なる連鎖により展開されている。つまり同一文脈内における、数学の活動による変化を伴う対象間の関係に基づき、問題領域構造におけるレベルの向上を伴う連鎖がなされているものと判断できる。そしてここでの連鎖は、対象間の類似性ではなく『隣接性』に基づく転義がなされたものと見なすことができる。 つまりここでの連鎖は、『換喩的なつながり』をもつ数学化、すなわち『垂直的な数学化』がなされたものと判断することができるのである。

これら記号間の関係を図６に倣い図示すると次のようになる。

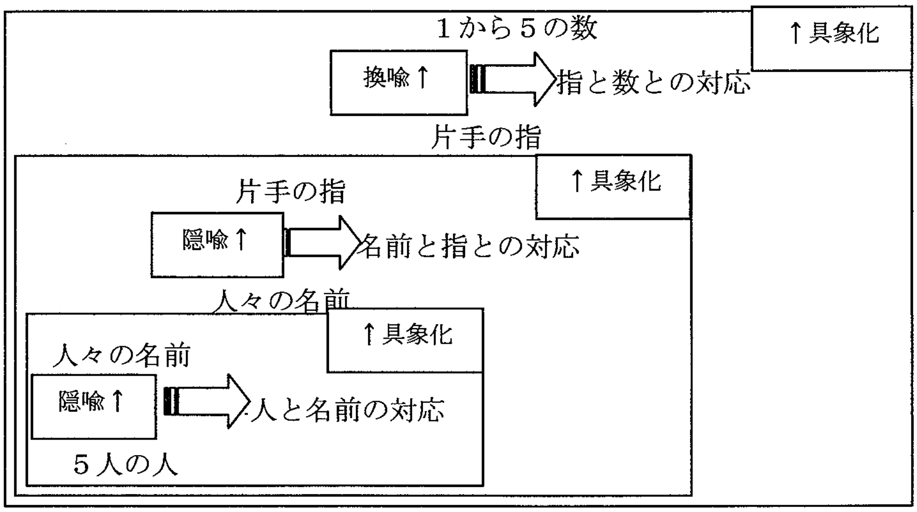


図３　記号論的連鎖における記号間の関係（Walkerdine(1988)の事例）

２. Wittmann の教授単元

ドイツで 1987 年に立ち上げられた数学教育改革プロジェクト「mathe200」を主催する Wittmann は、最近の著書で「mathe 2000」プロジェクトの 4 人の始祖(vier Erzväter)として、John Dewey, Johannes Kühnel, Jean Piaget, Hans Freudenthal をあげている(Müller, Steinbring und Wittmann, 2002, s.3)。数学に関する始祖は Freudenthal であり、Wittmann もまた Freudenthal と同様に『数学化』を

一つの手がかりとして教授学的実践を進めている。そして、水平的な数学化と垂直的な数学化の統合を図るべく設計科学を導入し、教授単元（Teaching Units, Unterrichtsbeispiel）という枠を与えた。

氏の数学観は次のように示されている。

> （数学教育学の）核心における作業は、人間的認知の根源的でかつ自然な要素としての、数学的活動から始まらなければならない。さらにこの作業では、「数学」は広汎な社会現象と考えられねばならない。私はこのもっとも広義の数学的作業を大文字でMATHEMATICSと書くことを提案したい。（中略）専門的数学も、より広い科学的・社会的活動にその多くの発想の源を負うているのであって、決して「数学(mathematics)」の専売だと考えるわけにはいかない。(Wittmann,1995, pp.358-359)

そして氏は、数学教育を「設計科学(Design Science)」と呼び、学習環境デザインの原理を次のように述べている。

(1) 数学は、相互に研究され、形作られ、(再)発明されることのできる(応用可能な)パターンの科学とみなされている。
(2) 学習は、能動的で、社会的に調停された知識構成の過程だと理解される。
(3) 教師の主要な役割は、教授学習過程を組織し、子どもに寄り添い、適切なフィードバックを与える点に見られる。 (Wittmann,2004,pp.4-5)

さらに「教授単元」について以下のように述べている。

・教授単元、ひとそろいの一貫した教授単元、さらにはカリキュラムのデザインは、MATHEMATICSに起源をもたねばならない(Wittmann,1995,p.359)。
・数学教育の研究における最も重要な成果は、基礎的な理論的原理に基づきながら、慎重にデザインされ、実験的にも検討された、教授単元の集まりである (Wittmann,1995,p.369)。

そして教授単元の設計原理を次のようにまとめている。

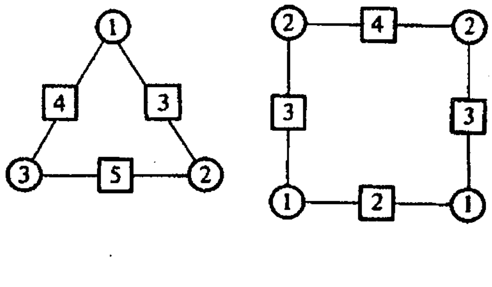
1) それは数学における学習指導の中心的な目的・内容・原理をある水準において明示している。
2) それは上記の水準を越える意味ある数学的な内容やプロセスや方法と結びついており、豊かな数学的活動の源である。
3) それは柔軟な取り扱いが可能であり、個々のクラスの特殊な条件に対応できる。
4) それは数学教授の数学的・心理学的・教授学的側面を統合し、実証のための豊かなフィールドを提供する。 (Wittmann,2001,p.2)

教授単元の具体例の一つに「アリスモゴン(Arithmogons)」がある。(Wittmann,1984,p.31)

教授単元 アリスモゴン

目的：加法・減法。さらにこれらの混合計算に関する演算の研究。探究と発見。

題材：三角アリスモゴン・四角アリスモゴン（ワークシートの使用。下図参照）



問題：いくつかの頂点や辺上の数が与えられている。その他の数を求めよ。
背景：頂点や辺上の数の一次独立性。方程式の体系に基づく組織的解法。計算の原理。

教授単元「アリスモゴン」は、単にその設計原理を満たすのみならず、子どもたちの興味･関心あるいは数学的能力に応じて、多様な数学的活動の発展への道を兼ね備えている。例えば、三角アリスモゴンの展開を、低学年児童の「おはじきを置いたり、移したりする」活動と絡めれば、図８のようになる(ビットマン et al., 2004, p.131)。そこには実質的に方程式を解く活動が込められているし（２から４)、作ろうと思えば不定の場合も不能の場合も設定できる。また１のケースをプロトタイプとし、３つの領域のおはじきに変化を加えていけば、その和はどのように変化するのか（５から８)、子どもたちの意識はその変化に向かうであろう。さらに学年を上げれば、その本質を変えることなく、おはじきや整数を、分数や文字にまで発展させられることはいうまでもない（９から12)。

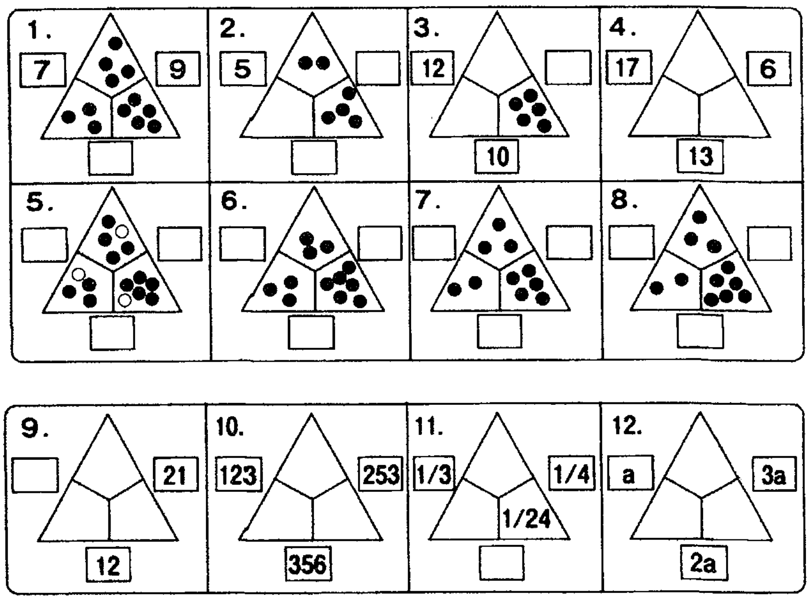


図４．三角アリスモゴン

やや抽象度を上げてこの種の数学的活動をみていくと、そこに次のような数学が現れてくる。

> 三角 arithmogon の背後にある数学はきわめて高度である。内側の３つの数は１つのベクトルを作り、外側の３つの数も同様にみられる。このようにみれば上記の操作は、実数上の３次元ベクトル空間からそれ自身への一次変換を定義する。対応する行列は正規である。この構造をｎ-gon に一般化できることは、McIntosh と Quadling(1975)によって示されている。 (Wittmann, 1995, p.367)

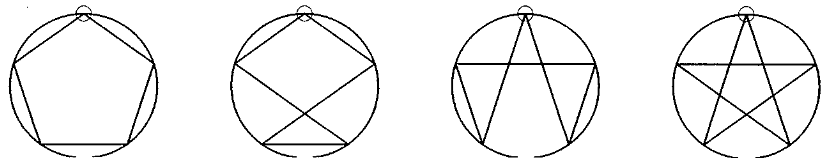
また別の事例として、教授単元「星形多角形(Star Patterns)」がある。この教授単元のねらいは、円周上の等分点を線分で結んだときにできる図形と点の結び方との間に成り立つ関係を、特に星形に注目して探求し、法則を見出して一般化することにある。

教授単元　星形多角形
目的：幾何的諸性質の代数記号による考察。具体的には、幾何図形「星形」に内在する幾何的性質を代数的な記号列に置き換え、記号列の考察によって、幾何的性質を明確にする。
題材：星形多角形

問題：

(1) 円周上の２等分点、３等分点、４等分点を一筆がきの要領で線分で結ぶと、どんな図形ができるだろう。

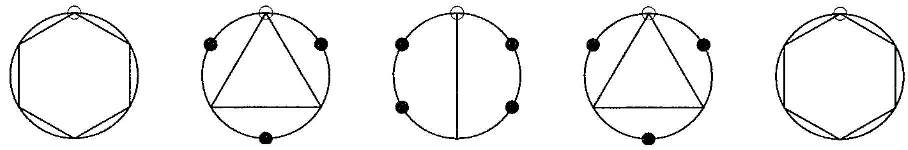
(2) 円周上の５等分点を結んでできる図形をかき、それぞれに名前を付けてみよう。



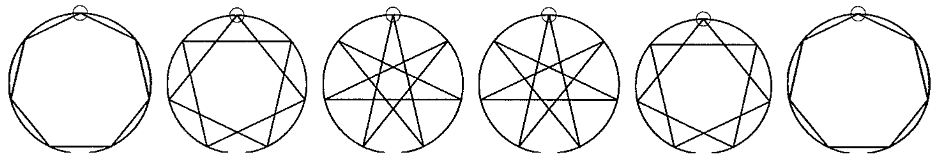
(3) ((2)でかいた図形のうち）星形に注目してみよう。どのようなかき方をしたときにその図形ができましたか。

(4) 他にも星形をかいてみよう。円周上の６等分点、７等分点、...を線分で結び、星形をかいてみよう。

6・1　6・2　6・3　6・4　6・5



7・1　7・2　7・3　7・4　7・5　7・6



（星形のかき方の約束として、一筆がき、線分で結ぶ、oの点からスタートして　(　　)個目ごとの点を結び、スタートの点にもどるまで続けることを確認。また描画された星形の名付け方を決める。）

(5) 星形はどんなときにかける（または、かけない）のだろう。またなぜそうなるのか、理由もあわせて考えてみよう。

背景： 円周上のｎ等分点を一筆がきの要領でｄ個めごとに線分で結ぶとき、次の関係が成り立つ。

① ｎとｄが互いに素のとき、一筆ですべての点を通るような星形多角形がかける。

② ｎがｄの倍数のときは、正 n/d 角形がかける。

③ g=GCD(n,d)とするとき、ｎとｄが既約でない場合にできる図形は、円周上の n/g 等分点を d/g 個目ごとに結んだ星形多角形と同じになる。

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lが既約であるときにできる星ｎ・ｄは、正角形の概念を拡張して、正 n/d 角形という新たな正多角形が生まれる。また、シンメトリーの視点からは、回転と鏡映からなる対称変換群を導くこともできる。さらに、３次元の星形多面体への発展的展開も可能である。

3. 教授単元の批判的考察

前節において、Wittmann の教授単元の具体例について検討を進めてきた。教授単元は、数学的活動を組織し設計する卓越した手法であり、その有効性は mathe2000 の実践を通して実証されてきた。しかし教授単元には、教材を目的・題材・問題・背景に分節する項目はあっても、学習指導過程を分節する項目は用意されていない。子どもに内在する数学が、状況の MATHEMATICS とど

のように共振し、知識としてあるいは見方・考え方として、どのようにクラスの中で共有されるかについて、予見し分析するシステムが教授単元に具備されていないのである。

そこで、本節では、Presmeg(2003)において示された記号論的連鎖の入れ子型モデルを援用し、教授単元を記号論的連鎖の視点から捉えることで、「学習指導過程」の分節を試みることとする。

アリスモゴンの学習指導過程を記号論的連鎖の枠組みを用いて表すと図５のようになる。

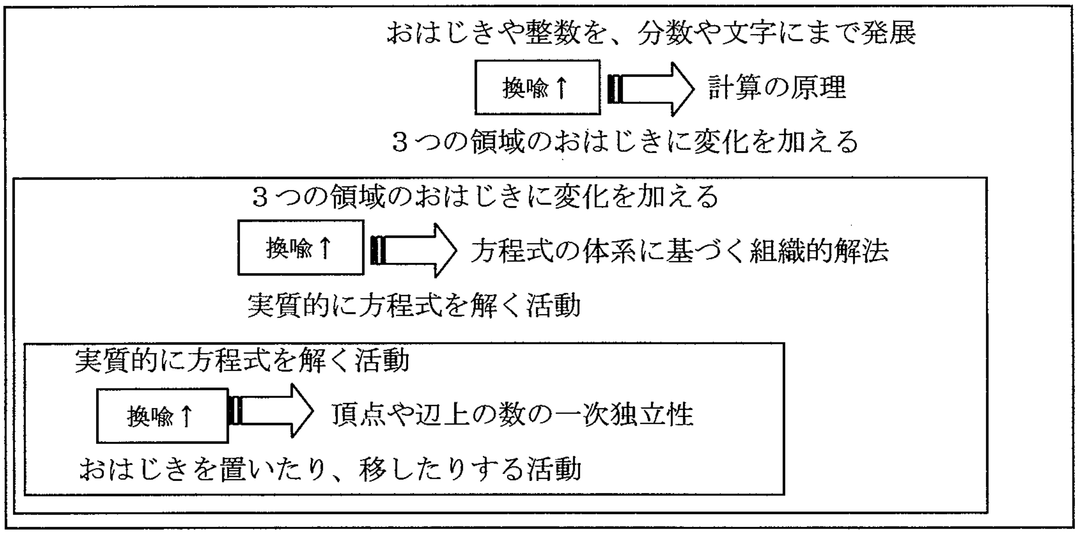

図５　教授単元「アリスモゴン」の記号論的連鎖

一方、星形多角形の学習指導過程を記号論的連鎖の枠組みを用いて表すと図 10 のようになる。

以上のことから、以下の諸点を見いだすことができる。第一に、記号論的連鎖の枠組みを用いて示すことで、教授単元の学習指導過程を分節することができる点である。特に、記号内容から記号表現へ至る比喩を同定することで、そこでの「数学化」が水平的なものであるか、垂直的なものであるかを同定することができる。第二として、学習指導過程において存在する連鎖が換喩的なつながりであることが、教授単元を本質的な学習場として機能させる点である。換喩的なつながりによる連鎖、即ち「垂直的な数学化」がなされているかどうかを記号論的連鎖の枠組みを用いて分析することで、教授単元の各学習指導過程における「数学化」の本性を見いだすことができる。更に第三として、垂直的な数学化がなされる場合、即ち換喩的なつながりによる連鎖が生じている場合には、その「解釈項」は教授単元設計原理の『背景』が相当する点である。

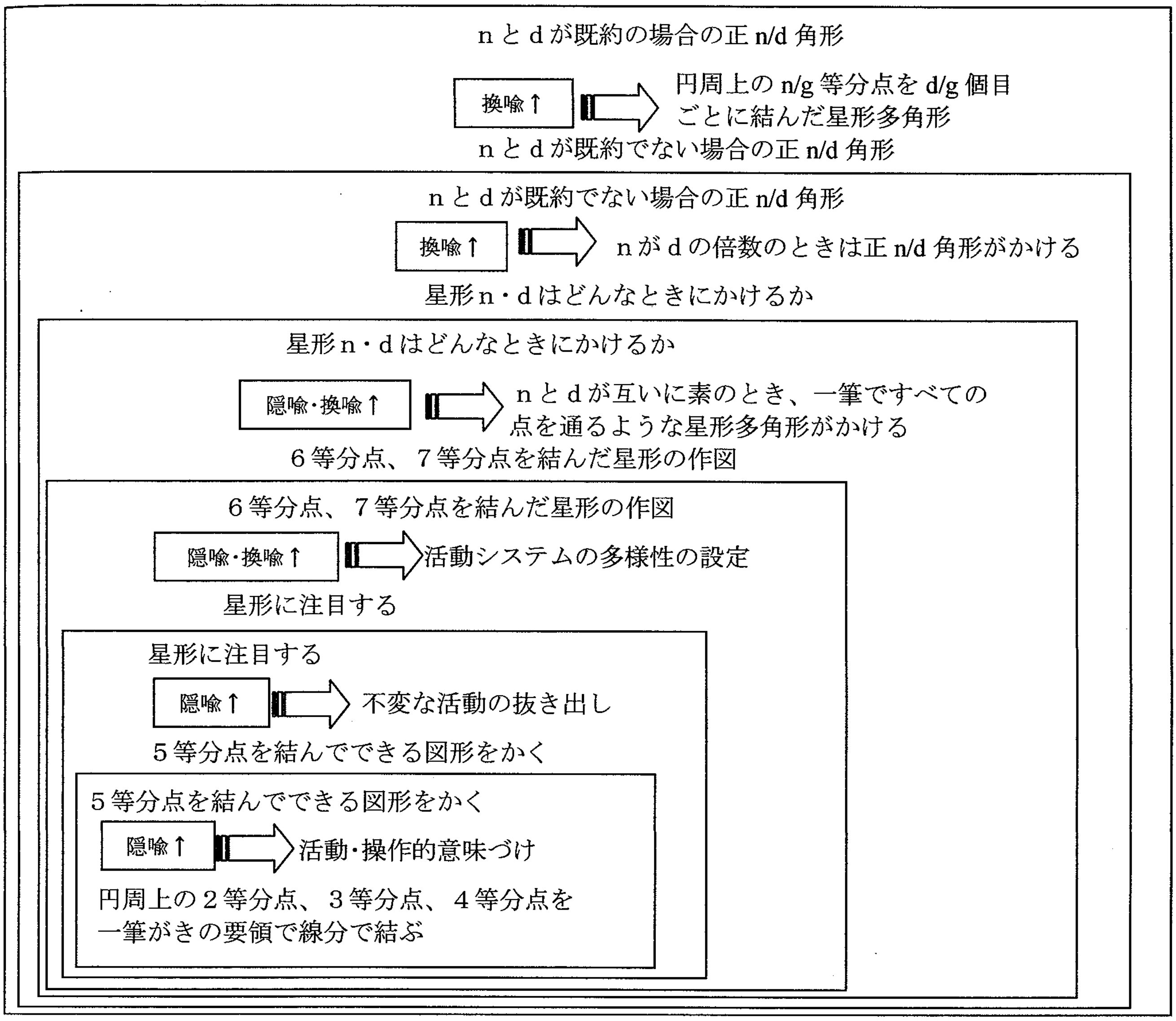

図6　教授単元「星形多角形」の記号論的連鎖

4. おわりに

本稿では、Presmeg(1998)で提唱された「記号論的連鎖」の枠組みを「数学化」と「比喩」の観点から精緻化するとともに、Witmann(1984)の提起する「教授単元」へと援用することで、教授単元における「数学化」に関する示唆を得た。その結果、以下の諸点が明らかとなった。

- 教授単元における「学習指導過程」を分節するシステムとして、記号論的連鎖を用いることの有効性が示された。
- 換喩的なつながり(垂直的な数学化)が促される連鎖により、教授単元は本質的学習場としての機能を果たすことができる。
- 記号論的連鎖の「解釈項」は、その連鎖が換喩的なつながり(垂直的な数学化)においてなされる場合には、教授単元設計原理の「背景」がそれに相当する。

今後の課題として、更なる事例の分析と、分析結果をフィードバックすることによる枠組みの精緻化があげられる。

付記

本稿は、二宮他(2005)「数学教育における記号論的連鎖に関する研究－Wittmann の教授単元の分析を通して－」『愛媛大学教育学部紀要』第 52 巻第 1 号, 139-152 に加除修正を行ったものである。

注

(1) Presmeg 教授は、記号論的連鎖の２項モデルと入れ子型モデルについて、それぞれのモデルによさがあり、用途に応じて使い分けるべきであるとの立場をとっておられる。２項モデルは連鎖の様相、つまり記号のつながりの様子が分かりやすい。一方、入れ子型モデルは意味の構成の様相を捉えるのに適している。

(2) 図６に示した「星形多角形」の記号論的連鎖では、「換喩的な隠喩」「隠喩的な換喩」などが混在している。

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第　2　部

英　文　編

RESEARCH PROBLEM AND PERSPECTIVE ON MATHEMATICAL LITERACY IN JAPAN: TOWARDS CURRICULUM CONSTRUCTION

Yoshitaka ABE (Graduate Schools of Education, Hiroshima University)

ABSTRACT

In this paper, I attempt to clarify research problems on mathematical literacy in Japan and to get suggestions for construction of a curriculum and developing practice by considering of references. For that purpose, the past mathematical literacy researches in Japan are surveyed first. And, in order to explore the trend of today's mathematical literacy research, prominent mathematical literacy researches are considered. Finally, suggestion towards curriculum construction and practice is described.

1. INTRODUCTION

Education is the working depending deeply on society and a time. Moreover, if that is not right, change of a time and a demand of society will not be recognized and it will become difficult to ask society and an individual for response corresponding to the time and scale. Literacy appeared in the school education of the modern West as "reading, writing, calculation", and supported and spread through the concept of the basic skills required for social independence (Sato, 2003). However, in the case (like today's Japan), which progresses quickly high advancement in information technology and internationalization, concept of literacy required for should be changed (Kikuti, 2003). Indeed, for instance PISA (Programme for International Student Assessment) instituted by OECD (Organisation for Economic Co-Operation and Development) and Project2061 by AAAS (The American Association for the Advancement of Science), educational reform based on new literacy concept as key-concept is advancing. Also in Japan, I think that the concept of literacy and the role of the mathematics education to foster it should be considered. Therefore, the purpose of this research is reconsidering of purpose of mathematics education, and curriculum for this, with setting the concept of mathematical literacy to a centre.

2. THE TREND OF THE PAST MATHEMATICAL LITERACY RESEARCH IN JAPAN

Since 2006, the project to define science, mathematics and technology literacy was inaugurated In Japan (cf. Kitahara (Ed), 2006). In this project, science, mathematics and technology literacy is "knowledge, skill and thinking related to the science, mathematics and technology which wanted all adult people to learn".

The references on literacy were investigated in the project last year (cf. Nagasaki (Ed.), 2006). The papers of the academic journal and the journal in connection with the technology, science education, mathematics education, technical education, and museum education, et al, (since 1970) was taken up, in order to analyse the trend of literacy research. And they are analysed about the present condition of literacy research. Here, in order to survey the past mathematical literacy research in Japan, the analysis (Abe, et al., 2006) is described.

2.1. Purpose of Analysis

Various mathematical literacy researches have been made in Japan until now. It is the purpose of this analysis to clarify the research situation. This clarifies changes of a mathematical literacy theory and an outline.

2.2. Method of Analysis

In this analysis, the papers about literacy are collected and analysed from the paper carried in 1970 and afterwards. The viewpoint of the analysis is as follows.

1. What is mathematical literacy?
2. Why is mathematical literacy necessity?
3. What kind of influence does mathematical literacy receive in the peculiar culture of each country?
4. What is the relation between mathematical literacy and a curriculum?
5. What is the standard, which specifies the components of mathematical literacy?
6. What are the contents suitable for fostering mathematical literacy?
7. How are mathematical literacy evaluated?

Also, the objects of analysis are 15 kinds of reference (since 1970) relevant to mathematics education and mathematics

2.3. Statement and Discussion of Result

The papers about literacy in mathematics education are 197 in a total. The findings show that the arguments have been developing into the argument of the literacy in mathematics education[1] gradually from the argument on the computer literacy of the beginning of the 1980s as an overall trend of this relation reference in Japan (see figure 1).

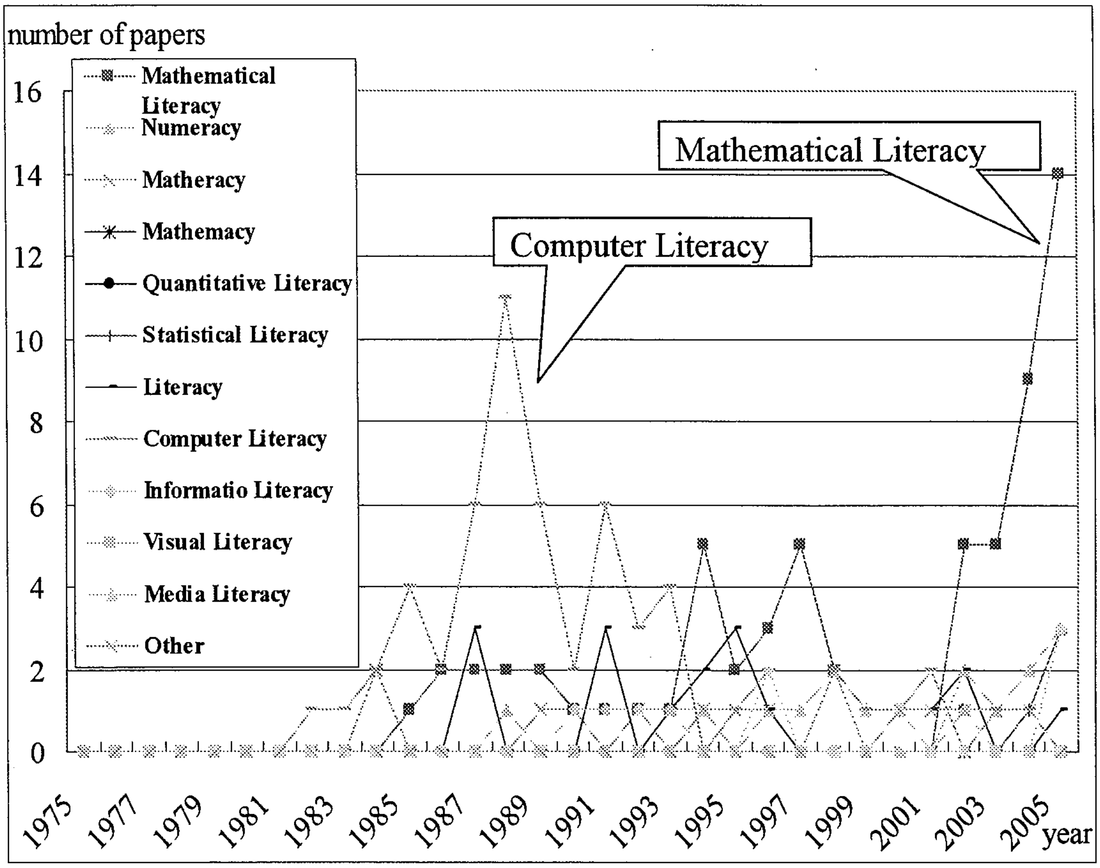

Figure 1: The overall tendency of an argument of literacy in a mathematics education

The papers about mathematical literacy (that is limited to a mathematically) are 97 in a total. In the process, although the literacy theory that depends on each author's theoretical background is developed, in recent years, the reference that depends on the mathematical literacy in OECD/PISA has a large number (see figure 2). About mathematical literacy theory in Japan, following three big positions of the peculiar to Japan was identified (cf. Nagasaki et al., 2006): First, Matheracy as a

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[1] In this analysis, "literacy in mathematics education" means all literacy (for example, computer literacy, visual literacy, and media literacy). As opposed to this, "mathematical literacy" is limited to a mathematically (except not mathematically).

principle of curriculum construction for fostering a student with mathematical literacy (Kawaguchi's theory); Secondly, Mathematical literacy for a majority of high school student (Fujita and Mogi's theory); Thirdly, computer literacy corresponding to high advancement in information technological society (Japan Society of Mathematical Education and Uetake's theory) .

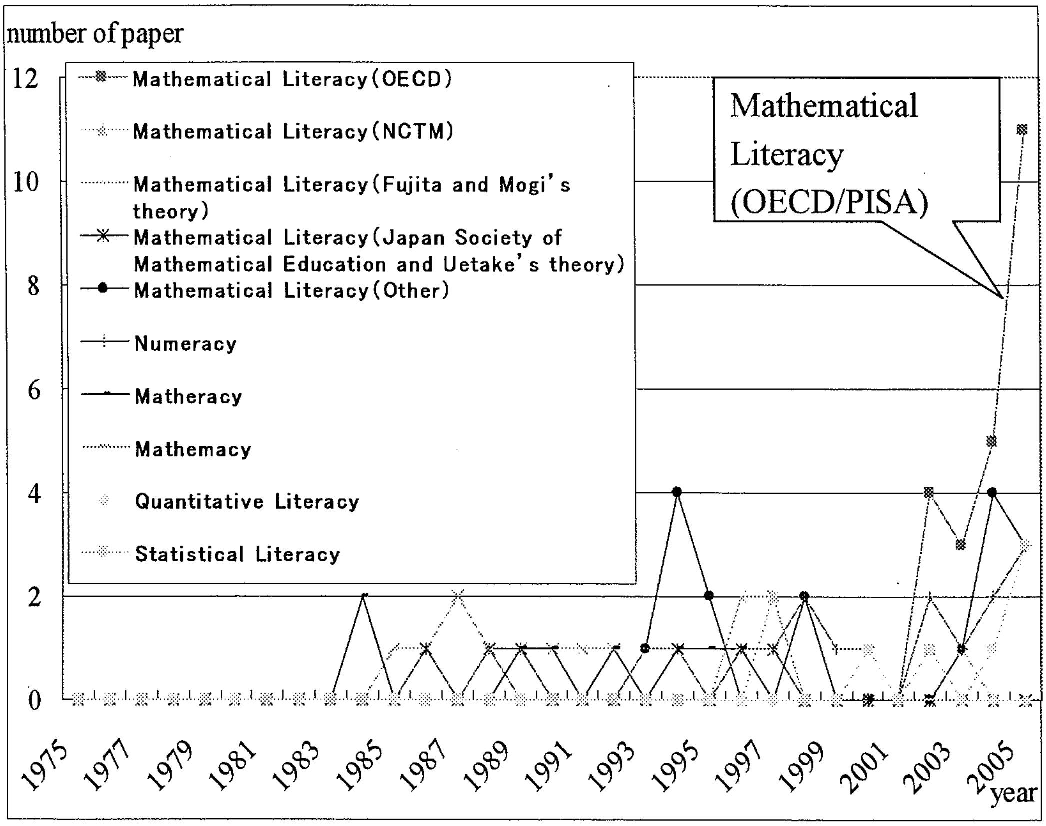

Figure 2: The overall tendency of an argument of mathematical literacy

From the analysis result, the feature of the literacy research in the mathematics education of Japan is as follows:

In the literacy research in the mathematics education of Japan, research by the individual is almost. On the other hand, they had not become a big flow even if Japan Society of Mathematics and Japan Society of Mathematical Education had respectively proposed mathematical literacy respectively. It seems that it receives social cognition as an educational purpose theory just because a literacy theory has both of the argument by the individual research, and research by societies recognized socially.

In the literacy research in the mathematics education of Japan, the almost motive for research reflect on trend in foreign countries. For instance, they are a computer or OECD/PISA. Although mathematical literacy for a majority of high school student (Fujita and Mogi's theory) was the opportunity, which solves a problem peculiar to our country, it could not be referred to as having succeeded. Now, the porous of the mathematics education of a high school is biased to the entrance examination, therefore the essence tends to be missed. Furthermore, the extent of mathematical literacy in an adult phase is also doubtful. Thus, it is thought that the necessity of fostering mathematical literacy is increasing more.

In the literacy research in the mathematics education of Japan, the various definitions of literacy have been made. However, there was no the argument on the denotation about the concept of the mathematical literacy, and furthermore, the argument was not made from the following viewpoints: "what kind of influence mathematical literacy receives in the peculiar culture of each country?" and "what kind of standard specifies the contents of mathematical literacy?" In short, in the literacy research in the mathematics education of Japan, there were no arguments about the structure and denotation of mathematical literacy, and relation between mathematical literacy and a curriculum.

In the literacy research in the mathematics education of Japan, An educational tendency is too strong. Their most researches described the definition of literacy abstractly after describing the necessity for mathematical literacy. The argument on the contents and evaluation to which they were suitable for fostering about literacy by describing generally is made, moreover, the argument on the contents suitable for the fostering and evaluation is made. It seems that there are few arguments about mathematical literacy itself and how education is considered from this.

3. CONSIDERING PRESENT PROMINENT MATHEMATICAL LITERACY RESEARCH

3.1. Overviews of mathematical literacy in two researches

OECD/PISA (cf. OECD, 2003) and AAAS/Project2061 (cf. AAAS, 1989) were considered as prominent mathematical literacy research.

OECD has been attempting PISA, as part of an educational indicator enterprise, since 1997. This aims at the following: To clarify various indices for comparing the educational system and policy of each country; namely, to develop framework which measures internationally the result (output) to the educational measure (input) as a country with the indices which can be compared, and to perform the measurement. In PISA, it measures reading literacy, mathematical literacy, and scientific literacy.

On the other hand, in AAAS (American Association for Advancement of Science), in the trend of the science reform of the educational system in the United States of the 1980s, Project2061 was launched in 1985 and the pacesetting roles have been played. This aims at “fostering of all citizens' scientific literacy” and made "scientific literacy" which integrate science, mathematics, and technology. And it has been published the report “science for all Americans” (American Association for Advancement of Science, 1989).

OECD/PISA mathematics frameworks have following three components:

- “The situation or contexts”: Four situation-types are defined: personal, educational/occupational, public, and scientific.
- “The mathematical content”: OECD considers mathematics as a pattern. Mathematical contents are redefined by a phenomenological approach to describing the mathematical concepts, structures, or ideas. And the domain is called “the overarching ideas”: quantity, space and shape, change and relationships, and uncertainty.
- “The competencies”: In OECD/PISA, a fundamental process that students use to solve real-life problems is referred to as “mathematisation”. It requires eight competencies: (1) Thinking and reasoning, (2) Argumentation, (3) Communication, (4) Modeling, (5) Problem posing and solving, (6) Representation, (7) Using symbolic, formal and technical language and operations, (8) Use of aids and tools.

On the other hand, the mathematical literacy in AAAS/Project2061 (AAAS, 1989) consists of an item of “the nature of mathematics”, “the mathematical world” (contents), “habits of mind”.

- “The nature of mathematics”: Mathematics is first considered to be "science of a patterns and relationships", it has two sides of theoretical principal and applied science, and mathematics is characterized from the both viewpoint. Moreover, using mathematics to express ideas or to solve problems is considered as “mathematical processes” which have three stages; (1) representing some aspects of things abstractly, (2) manipulating the abstractions by rules of logic to find new relationships between them, and (3) seeing whether the new relationships say something useful about the original things. I think that it can be considered that this is “mathematisation”
- “The mathematical world”: Mathematical contents focus on ‘basic mathematical ideas, especially those with practical application, that together play a key role in almost all human endeavours. ... the focus is on seven examples of the kinds of mathematical patterns that are available for modeling: the nature and use of

numbers, symbolic relationships, shapes, uncertainty, summarizing data, sampling date, and reasoning.'(AAAS, 1989, p. 101)

- "Habits of mind": This is "recommendations about values, attitudes, and skills in the context of science education".

3.2. Comparison between OECD/PISA and AAAS/Project2061

Comparison between OECD/PIAS and AAAS/Project2061 is described from four viewpoints.

1. "Functional mathematics" and "Theoretical mathematics"

If both mathematical literacy theories are compared from the viewpoints of "mathematics for the purpose on reality" (functional mathematics) and "mathematics for mathematics" (theoretical mathematics) which are two sides of mathematics, OECD/PISA's one focus in the former, and AAAS's one is described from the side of both viewpoints. However, in the stage, which constitutes mathematical literacy from AAAS, educational perspective is not contained.

2. Mathematical contents

Mathematics is considered as "the science of patterns" in both researches. In OECD/PISA, mathematical contents are constricted by a phenomenological approach. On the other hand, in AAAS, mathematical contents are constructed by a viewpoint from science. It seems that constructing mathematical contents depend on a view of mathematics. I think that the past mathematical literacy research in Japan (cf. Nagasaki et al., 2006), had been described from the only viewpoint of "theoretical mathematics"

3. Mathematisation

In OECD/PISA, mathematisation is key concepts. Also, in AAAS, mathematisation is the essential mathematical activity. Therefore, mathematisation is important concept for mathematical literacy.

4. Attitude and value

In OECD/PISA, although attitude and value is not elements of mathematical literacy, the necessity is described. Also, in AAAS, they are included as elements of mathematical literacy.

4. TOWARDS CURRICULUM CONSTRUCTION AND PRACTICE TO FOSTER MATHEMATICAL LITERACY

The suggestion for curriculum construction and developing practice from the viewpoint of fostering of mathematical literacy is described below.

The problem "how to define mathematical literacy" can be put in another way with "how mathematics is reconstructed from the perspectives of mathematical literacy". If it says in a previous example, mathematical literacy of OECD/PISA is reconstructing mathematics from phenomenology approach, and AAAS/Project2061 is reconstructing mathematics from the viewpoint of the mathematics as a part of scientific. Moreover, activity there is mathematisation and many capabilities for it is needed. However, the mathematical literacy in this stage is in the mathematical context, and "what kind of education is performed for the purpose of fostering for constructed that" is another new problem. That is, it is asked how a curriculum should be constituted in the context of mathematics education. For example, although AAAS constitutes mathematical literacy from a viewpoint of science, it has not described about the fostering [2] (in this stage). Moreover, although OECD has stated mathematical literacy in the context of evaluation, it has not described about the fostering too.

In that stage in the context of mathematics education, we have to think "what kind of viewpoints a curriculum and practice should consist of?" It seems that If the viewpoint "mathematics for the purpose on reality" (functional mathematics) and "mathematics for mathematics" (theoretical mathematics), past Japanese research consists of latter viewpoints. Namely it seems that the mathematical literacy was considered from academic mathematics. Also, the trend to OECD/PISA in recent years is considered that it is shown that concept of mathematical literacy has shifted to "functional mathematics".

Wittmann critiqued the trends to the literacy of OECD/PISA as too pragmatic from the mathematical position, and argued importance the complementarity of the application-oriented and the structure-oriented (cf. Wittmann et al., 2002). It suggests to importance that to consider that "From what kind of viewpoint is fostering of mathematical literacy considered?" and "How are both viewpoints "functional mathematics" and "theoretical mathematics" balanced?"

From the past research trend in Japan, mathematical literacy has been developed in more "theoretical mathematics". However, it is thought that it cannot respond only with the mathematical literacy, which consisted of this viewpoint in the lifelong learning type education in today's Japan. Also that can be seen from expression of various educational problems, and the trend to present PISA-type literacy. Namely, it means the necessity of considering the mathematical literacy

[2] This is restricted to the stage of AAAS (1989). By the next argument, it has proposed about curriculum construction.

which can respond to lifelong learning type education, which is the complementarity mathematical literacy not only "theoretical mathematics", but also "functional mathematics".

In this research, I have been describing introducing the perspective of environmental education as one approach to lifelong learning type education based on such awareness of above subject. To mathematics education having positioned "functional mathematics" as an application scene of "theoretical mathematics", this begins from "functional mathematics" and is connected to "theoretical mathematics". However, we should notice the balance of "functional mathematics" and "theoretical mathematics". And I think that concepts of "mathematisation" and "mathematics as the science of patterns" (cf. Devlin, K., 2003) are key concept as a bridge.

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The Design of Mathematical Learning Environment Based on the Theory of Situated Learning

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ABSTRACT

The purpose of this paper is to establish principles for the design of mathematical learning environment based on the theory of situated learning and to design the environment based on these principles. First, the theory of situated learning is examined. Secondly, mathematical learning is defined based on the theory and the principles for the design of mathematical learning environment are established. These principles are as follows: DMLE 0. - Decide mathematical-collaborative activities as aims and design the mathematical learning environment that includes activities, artifacts, and others (DMLE 1/2/3); DMLE 4. - Provide children with opportunities to actualize the mathematical-collaborative activities. Finally, the mathematical learning environment - 'let's divide confectionery' – that was designed based on the principles is presented.

1. INTRODUCTION

According to Third International Mathematics and Science Study (TIMSS) by The International Association for the Evaluation of Educational Achievement (IEA), Japanese children's mathematics record is ranked among the tops. However, the mean value in the ratio of the children who think 'Studying mathematics is interesting', 'In the future, I want to work with mathematics' and 'Mathematics is important in daily life' is each less than the international average (National Institute for Educational Research of Japan, 1996, 1997). Therefore, the ratio of the children who think well of mathematics is very small[1].

The Course of Study revised in 1998, 1999, which is based on the above situation, aims at 'mathematics education which regard children's independent learning to be important', and related to real life' (Nakahara, 2001), and 'children's awareness of values of mathematics and significances of learning it' (Ministry of Education, 1999, p.18).

However, the aims would be difficult to achieve by usual mathematics lessons based on the view of learning as 'acquisition of mathematical knowledge' (Imai, 2004).

Inagaki & Hatano (1998) point out some problems of the lessons. One is that in these lessons, teachers choose learning contents and methods and control learner's activities, so learner's independent learning would be impossible. The other is that general ways of dealing with different problem situations are regarded as important and activities in abstract/non-practical situations apart from concrete/real contexts are designed, so products of the lessons only have indirect usefulness and result in children's low motivation to learn.

In response to the limitations of the current mathematics lessons in realizing the aims of the course of study, the aim of my research is to design mathematical learning environment based on 'the theory of situated learning' (Lave & Wenger, 1991) and to realize that children are aware of the values of mathematics and significance of learning it in relation to real life and learn independently. Consequently, the purpose of this paper is to establish principles for the design of mathematical learning environment based on the theory of situated learning and to design the environment based on these principles. First, the theory of situated learning is examined. Secondly, mathematical learning is defined based on the theory; and the principles for the design of mathematical learning environment are constructed. Finally, mathematical learning environment - 'let's divide confectionery' – is designed based on the principles.

2. Theoretical examination of the theory of situated learning

'Practice theory'

'Practice theory' (Lave, 1988) is the basis of the theory of situated learning and refers to the nature of human's activity. It is to "focus on everyday activity, and its constitution in relation between social system and individual experience" (Lave 1988, p.14) and contents of the theory can be summarized as follows (Imai, 2005):

Individual activities (practices) in formal environments arise depending on how he/she grasps the environments - which include activities, artifacts and others, through experiences - memories, expectations and activities/practices in other environments, and it develop with time. Furthermore, the development is considered to be caused by changes of the environments including the individual.

It is important to argue that the theory claims individual's activities (practices) in environments cannot be judged without considering the environments. For example, consider the situation that a child isn't active in mathematics lessons but is active in social studies. Then, is the child originally active or not? The judgment must be difficult. That is, the claim of the Practice theory is that human activity should not be attributed to his/her 'ability' or 'character', but to the 'relation' between him/her and the environment in which he/she belongs.

Figure 1 represents the contents.

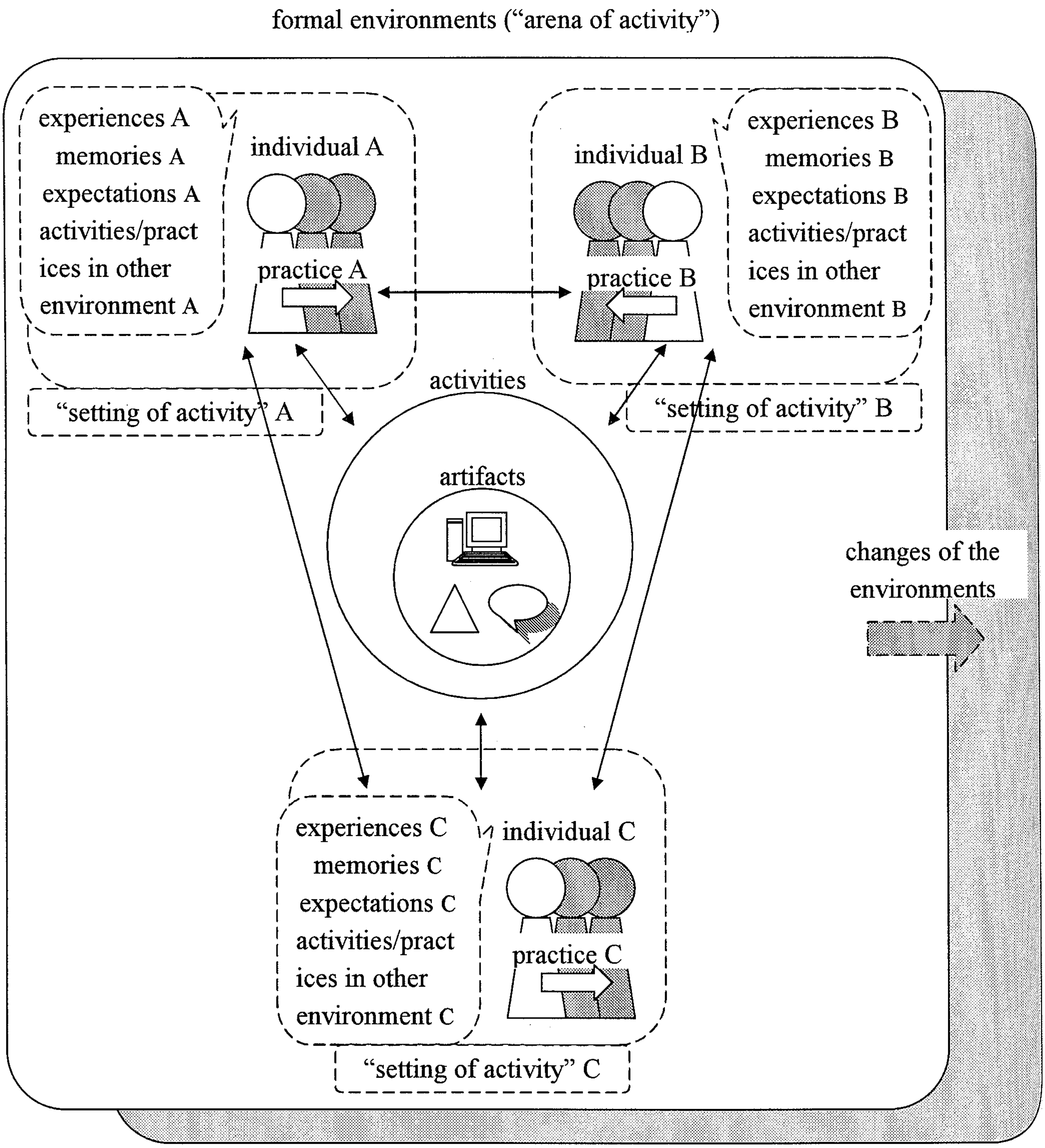


Figure 1. Practice Theory[2]

The theory of situated learning

In Lave & Wenger (1991), 'learning' is explained as "situated activity"[3] (p.29), "as increasing participation in communities of practice" (p.49). The 'activity', 'participation' corresponds to the 'individual activities (practices)' and 'increasing participation' means the change of it in 'Practice theory' which explains individual activities (practices) in formal environments. That is, 'learning' is taken as 'changes of individual activities in environments' in the theory of situated learning. Furthermore, judged from Practice theory, it depends on how he/she grasps the environments which includes activities, artifacts and others.

'How learning is caused' is explained as "learning occurs through centripetal participation in the learning curriculum[4] of the ambient community" (p.100). The 'participation' represents 'individual activities (practices)' in Practice theory. That is to say, learning is regarded to be caused by individual's actual activities (practices).

3. The definition of mathematical learning based on the theory of situated learning

Based on the interpretation of Practice theory - how individuals grasp formal environments - and the contents of the theory of situated learning, mathematical learning is defined as below (Imai, 2005):

The nature of mathematical learning

0. Mathematical learning is changes of individual mathematical-collaborative activities in mathematical learning environments - It depends on the following changes: 1/2/3

1. Changes of how an individual grasps the mathematical activities
   - 1-1 Changes of individual senses of purpose to the mathematical activities
   - 1-2 Changes of individual understandings of the mathematical activities
2. Changes of how an individual grasps artifacts used in the mathematical activities
   - 2-1 Changes of artifacts used in the mathematical activities
   - 2-2 Changes of individual understandings of artifacts used in the mathematical activities
3. Changes of how an individual grasps others he/she interacts with in the mathematical activities
   - 3-1 Changes of others an individual interacts with
   - 3-2 Changes of individual understandings of others he/she interacts with in the mathematical activities – changes of relations between him/her and others

The way mathematical learning is caused

Mathematics learning is caused by individual's actual mathematical-collaborative activities.

Table 1. The definition of mathematical learning based on the theory of situated learning[5]

4. Principles for the design of mathematical learning environment based on the theory of situated learning

The theory of situated learning doesn't refer to 'norms' such as what kind of learning environments are desirable, as can be understood from Table 1: "it is an analytical viewpoint on learning, a way of understanding learning" (Lave & Wenger, 1991, p.40). Therefore, to construct 'principles for the design of mathematical learning environment (DMLE)', norms of design needs to be added to 'the definition of mathematical learning based on the theory of situated learning'[6]. Table 2 shows the principles constructed by adding norms to the definition. The principles 'are based on the theory of situated learning' in the meaning that it is constructed based on the definition.

DMLE 0. Decide mathematical-collaborative activities as aims and design the mathematical learning environment. (In the design, DMLE 1/2/3 are included.)
DMLE 1. Design the environment concerned with the mathematical activities
1-1 Make it possible for children to have senses of purpose to the mathematical activities
1-2 Enable children to understand the mathematical activities
DMLE 2. Design the environment concerned with artifacts used in the mathematical activities
2-1 Prepare the artifacts used in the mathematical activities
2-2 Enable children to understand the artifacts used in the mathematical activities
DMLE 3. Design the environment concerned with others he/she interacts with in the mathematical activities
3-1 Make it possible for children to collaborate with other children in the mathematical activities, and for teachers to support the children's activities.
3-2 Make it possible for children to have good relation with other children and teachers.
DMLE 4. Provide children with opportunities to actualize the mathematical-collaborative activities

Table 2. Principles for the design of mathematical learning environment (DMLE)

5. The design of mathematical learning environment: 'Let's divide confectionery'

The aim in the mathematical learning environment: let's divide confectionery is to be able to do mathematical activities such as 'count', 'compare', and 'divide' using various artifacts in cooperation with others in real life. And in the environment...

1. the second grade is the target.
2. ‘the number up to one hundred’ (first grade) and ‘addition and subtraction up to two place values’ (second grade) are supposed to have been learned, while ‘multiplication’ and ‘division’ are not.- in the bracket is the teaching grade which is decided in Japanese current course of study (Ministry of Education, 1999, p.32).
3. children play active parts in groups.

First, teachers actually present two boxes each of the following four kinds of confectioneries to children. At the time, children can know the kind they don’t know the number of sticks in the boxes.

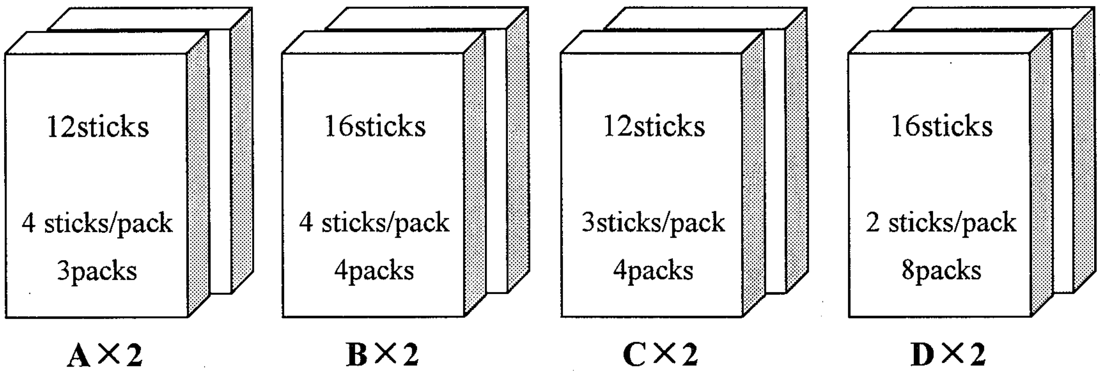
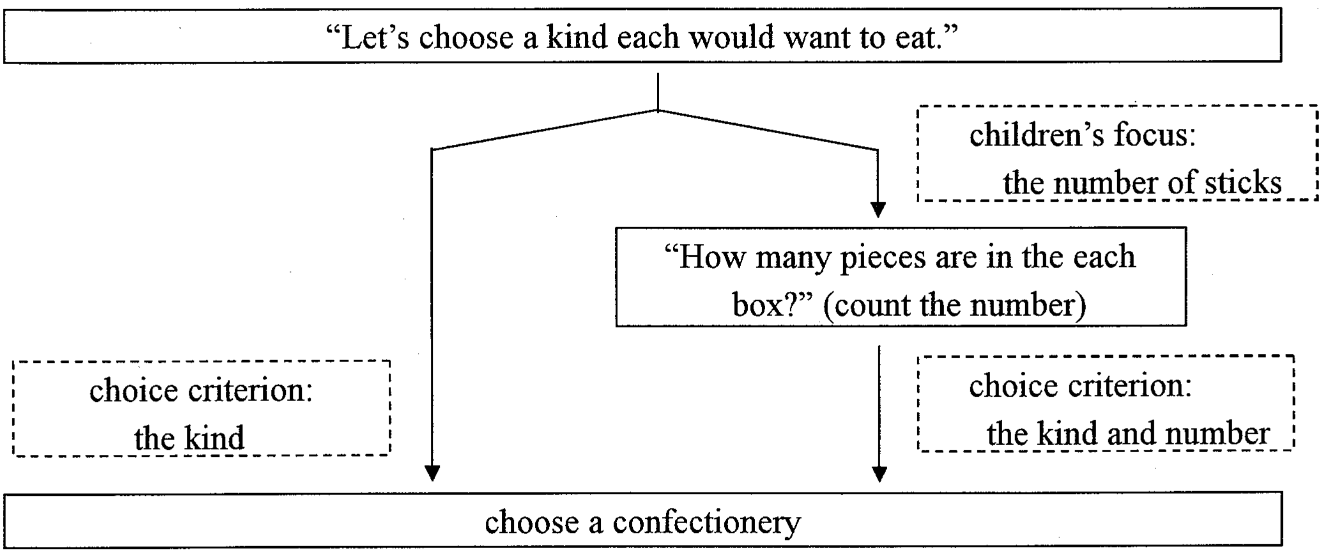


Figure 2. Confectioneries

After that, teachers say “let’s decide the confectioneries that each would like to eat”. At this point, children decide the way of division, not teacher.

Later developments of children’s activities are supposed as shown in Figure 3.



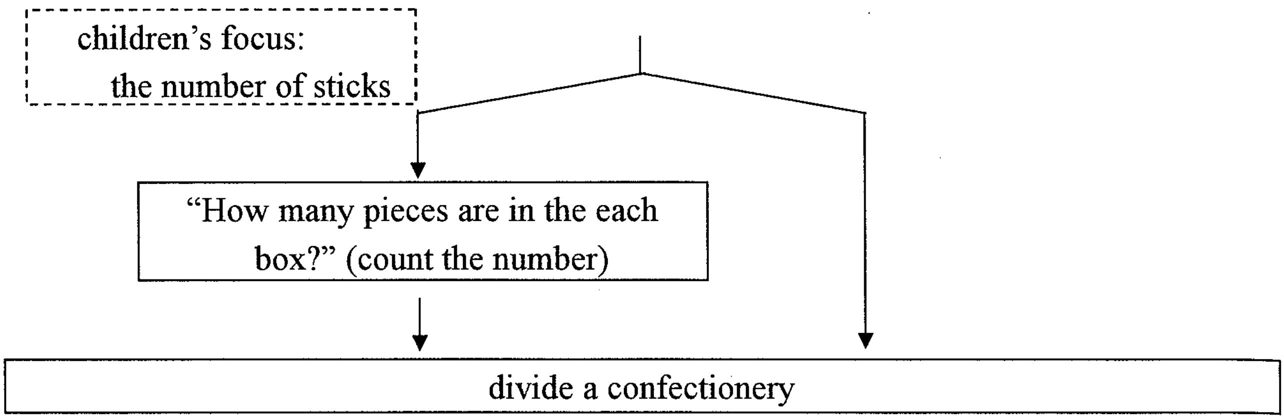


Figure 3. Hypothetical developments of children's activities in the mathematical learning environment of 'let's divide confectionery

Group1	Group2	Group3	Group4
A×2	B×2	C×2	D×2

Figure 4. An example of assignment of confectionaries

Group A	Group C
A×2	C×2
Group B	**Group D**
B×2	D×2

Figure 5. Groups after the choice of a confectionary

In the case that children answer "Let's choose a kind each would want to eat", it is supposed that children are going to choose a confectionery by the 'kind' only, or by the 'kind' and 'number' by asking "How many pieces are in the each box?"

In the latter case, for example, teachers assign the four kinds of confectioneries to four groups of children as in Figure 4, and children count the number in collaboration with each other. After the counting, they are asked to confirmed the
number of pieces in each box, how are the pieces packed, and how they counted the number. The teachers then summarize the result and process of counting activities in each group on blackboard. After the number of pieces of confectioneries is revealed, children choose it by the 'kind' and 'number'. This leads to the formation of four new groups of children as shown in Figure 5 and children divide it in collaboration with each other. After the dividing, the number of pieces for each child and how it is divided is confirmed. The teachers summarize the result and process of dividing in each group on blackboard. (In the case that children chose it by the kind only, it is supposed they now count the number.)

In these activities, if necessary, teaching aids can be used.

After these activities, a series of children's activities which was summarized by the teachers are reflected on. First is the looking back on 'counting' activities. The four kinds of confectionery the teachers present are packed as shown in Figure 2, therefore, following ways of counting are assumed:

"In each box there are three packs each containing four sticks, giving 4+4+4=12 sticks. Therefore, there are 24 (12+12) sticks." (A)
"In each box there are eight packs each containing two sticks. Therefore, in total there are 2+2+2... +2 = 32" (D)

In the latter case, it is inconvenient to use addition. After the children feel the inconvenience, the teachers present the notation of multiplication and the children check its meaning.

Second is the looking back on 'dividing' activities. The four kinds of confectionery the teachers present are packed as shown in Figure 2, therefore, following ways of dividing are supposed:

"24 sticks divided among 8 children with each child getting three sticks" (A)
"24 sticks divided by three sticks is given to 8 children" (C)

In this point, teachers present the notation of division and children check the meaning.

The application of the principles to the mathematical learning environment

DMLE 0.

To be able to 'count', 'compare', and 'divide' using various artifacts in cooperation with others in real life are aimed at and the mathematical learning environment of dividing confectioneries is designed.

DMLE 1-1.

The mathematical learning environment would be familiar with children. And to divide confectioneries, children would be interested in the number and want to count them. These would enable children to have senses of purpose to the mathematical activities such as 'count', 'compare', and 'divide'.

DMLE 1-2.

Dividing confectioneries would be familiar with children, so it would be easy to understand the activities. More detailed children's understandings of the mathematical activities are to be considered based on the actual conditions of the class and development of activities.

DMLE 2-1.

If necessary, teaching aids can be used.

DMLE 2-2.

Children's understandings of the teaching aids are considered. More detailed children's understandings of the artifacts used in the mathematical activities are to be considered based on the actual conditions of the class development of activities.

DMLE 3-1.

In counting and dividing confectionaries, playing an active part in the groups enable children to collaborate with others.

Teachers play roles of supporting children's activity, such as answering children's question, summarizing the process and result of children's activities, and helping children to look back on a series of their activities.

DMLE 3-2.

The relation with other children and teachers are considered based on the actual conditions of the class development of activities.

DMLE 4.

The opportunities for children to actualize the mathematical-collaborative activities using various artifacts in cooperation with others are provided in dividing the confectionaries.

6. Summary

In this paper, the mathematical learning environment is designed based on the principles for the design of mathematical learning environment based on the theory of situated learning. In the environment, it can be expected that children would be aware of values of mathematics and the significances of learning it by carrying out mathematical activities independently in real life situation and knowing mathematics as products of the activities.

Notes

1. In TIMSS 2003, the Japanese students are ranked fifth in average score of eighth grade in each country/region. In the question whether mathematics is interesting or not, the ratio of the students who answer 'very interesting' (9%) is less than the

international average (29%).

(http://www.mext.go.jp/b_menu/houdou/16/12/04121301/002.htm)

2. In this research, “arena of activity”, “setting of activity” and “artifacts” are each defined as follows:
   “Arena of activity”: “an identifiable, durable framework for activity, with properties that transcend the experience of individuals, exist prior to them, and are entirely beyond their control” (Lave, 1988, p.151)
   “Setting of activity”: “personally ordered and edited version of the arena” and “experienced differently by different individuals” (Lave, 1988, p.151)
   “Artifacts”: “things made artificially and used by people, for instance, tools, characters, and symbols” (Saeki, 1993, p.112)
3. ‘Practice theory’ claims that individual activities (practices) arise depending on how he/she grasps formal environments. Thus, in the case that activities are ‘situated’, the meaning of ‘situation’ is considered to be ‘formal environments and individual way of grasping it’: formal environments (“arena of activity”) and “setting of activity”
4. ‘Learning curriculum’ consists of situated opportunities for the improvisational development of new practice, that is, it is “a field of learning resources in everyday practice viewed from the perspective of learners” (Lave & Wenger, 1991, p.97).
5. In this research, ‘mathematical-collaborative activities’ and ‘mathematical learning environment’ are each defined as follows:
   ‘Mathematical-collaborative activities’: various activities concerned with mathematics where children have senses of purpose and do in collaboration with others
   ‘Mathematical learning environment’: an environment where mathematical-collaborative activities are done and artifacts and others are included

   Moreover, ‘machines and tools’, and ‘general language’ is included in ‘artifacts used in the mathematical activities’.
6. Suzuki, et al. (2002) point it out that when school reformation is tried from situated perspective, the reformers from the perspective need to realize that they have to refer to the necessity and direction of the reformation because the perspective doesn’t refer to it. In the same way, when principles for the design of learning environment are constructed, one needs to refer the ‘norms’ of the design.

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GEOMETRIC CONSTRUCTION AS A THRESHOLD OF PROOF: THE FIGURE AS A COGNITIVE TOOL FOR JUSTIFICATION[*1]

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Masakazu OKAZAKI (Joetsu University of Education)

ABSTRACT

Geometric construction is usually considered as the procedure of drawing a figure bounded by a straightedge ruler and a compass. It, however, could be didactical intermediate between a cause-effect and an assumption-conclusion explanation because it is not only a series of systematic activities using only two tools but also mathematical proof of existence in itself. We, therefore, consider it as a educational threshold of proof in system although most mathematics teachers emphasize its procedural aspect.

We analyzed the justification of the geometric constructions made by seventh grade students in the classroom lessons in terms of image schema. Image schema was developed for a meaning-making function by Dörfler (1991). It was made available for us to clarify the logical or cognitive base of the students' justification and change of thought. This research finally showed some factors for the transition from construction to proof in geometry.

1. INTRODUCTION

Mathematics in the secondary level must try to switch knowledge from one track of experience to another track of theory. It, therefore, is a worldwide problem in mathematics education to come to know how the theoretical component would be dealt with in the mathematics for the secondary level.

In Japan, geometric construction is taught in the seventh grade. This serves as a gateway to the logical proof in the geometry for the compulsory eighth grade. However, most teachers give more emphasis on the procedural aspect rather than the theoretical aspect. Hence, they fail to see or realize that geometric construction could

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play the role of an intermediate towards proof in logical system.

In this paper students of junior high school were asked to give justification on the procedures of the construction. This activity serves as a steppingstone for a logical proof. We investigated the process of change of thought as shown in the geometric figures that the students performed. The concept of image schema suggested by Dörfler (1991,1996) and Johnson (1987) worked well for this investigation. According to Johnson, "···image schemata are not rich and concrete images or mental pictures, either. They are structures that organize our mental representations at a level more general and abstract than that at which we form particular mental images."(pp.23-24) and "In order for us to have meaningful, connected experiences that we can comprehend and reason about, there must be pattern and order to our actions, perceptions, and conceptions. A schema is a recurrent pattern, shape, and regularity in, or of, these ongoing ordering activities. These patterns emerge as meaningful structures for us chiefly at the level of our bodily movements through space, our manipulation, and our perceptual interaction."(pp.99-100)

In the recent PME-Proceedings, papers on this topic show that students begin their experiences with the exploration of geometric construction (Mariotti *et al.*, 1997). The computer software is made workable to provide them with such experiences and to help them understand the reasons why they have arrived at such constructions. Clements and Battista , however, point out the pitfall of the use of computer for constructions as follows(1992, p.438): "···even with computer constructions, ... we might need to worry about the pitfalls of promoting an empiricist approach." With the Cabri software, in other words, students still focus on the procedural aspects just like what they do when they are using the compass and ruler.

As mentioned by Marriotti and Bussi (1998), it is important to go beyond the procedural process which is referred to as the product of construction for the development of justification. At this point compass and ruler construction is little less than computer construction. We observed several classroom lessons on the geometric construction by means of compass and ruler in the seventh grade. The discourse of students was analyzed to specify some factors of the success and failure in the justification on the construction.

2. THEORETICAL FRAMEWORK

Let us begin with the property of parallelogram for example which is presented with $P \rightarrow Q$. Needless to say, the logical symbol "$\rightarrow$" or the conjunction "if-then" is not written in the figure of parallelogram. The figure, however, could be conceptualized if this symbol or conjunction is used in it, and furthermore its universality should be proved under the logical system. On the other hand, the symbol "$\rightarrow$" or the conjunction "if-then" itself does not have any meaning just like any other conjunctions that never exist in the real world. Therefore, the figure of parallelogram is nothing less than something which gives meaning to the logical symbol of "$\rightarrow$".

Thus a geometric concept is composed of both figure and mathematical symbol which are complementary to each other. It is the human mind that provides a geometric concept with complementarity between them. We call it image schema following the idea of Dörfler (1991). Image schema, therefore, plays an important role not only to analyze, interpret, and transform shape to a mathematical symbol, but also to simplify and integrate a set of formalized relations into a new shape. Dörfler (1991) made four categories under the image schema; namely figurative, operative, relational, and symbolic. In geometric construction and its justification, operative and relational image schemata help to organize temporal order of the ruler and compass operations to spatial location when making a geometric figure.

Relational image schema develops the relations or the meta-relations from spatial location and arranges them to one copula i.e. "be". On the other hand, operative image schema develops the relations or the meta-relations from temporal order and arranges them to another copula i.e. "become".

Justification in geometric construction, therefore, means to translate spatial location into temporal order. For instance of an angle bisector, the reason why a segment OA becomes a bisector of an angle POQ is that a quadrangle POQA became a kite shape which is defined in Japan as the shape whose adjoining two sides are each equal. A relationship between assumption and conclusion in mathematics never has a temporal component although a causal relationship in science usually has it. In geometric construction, there is time for mathematical operations as well as the existence of mathematical proof. Temporal order is immanent in activities of students. They must reflect their activities and be aware of the series of "become" in the geometric construction and its justification. On the contrary, they should trace and erase the temporal component for their explanation.

3. STUDENTS' GEOMETRIC CONSTRUCTIONS AND THEIR JUSTIFICATION

The discourse mentioned below is an excerpt from the record of classroom lessons on geometric construction in Japan. The main topics for these lessons were to draw an angle bisector and to justify the procedure of construction. The teacher permitted the students to use the corners of any square or a protractor. He emphasized that the students should not only complete the construction but should also show the procedure and explain its validity.

3.1. The characteristics of students' geometric construction

First, the teacher drew an angle on the blackboard and posed a problem "draw the bisector of this angle". The students were told to make use of all the instruments for drawing, but they were negative to use a protractor.

Student: Even if the one side of the angle is on the graduations of a protractor, we can draw a bisector approximately, however we can never draw it accurately.

The students in this class did not admit even a subtle deviation. Similar discussion happened when Taro was explaining the following way of drawing. Taro put points A, B that are 20cm distant from the vertex, and said "*the length of the segment AB is divided by 2 and I put the point at a half of the length of the segment*", then drew a line connecting the vertex with this point. Just then, some other students questioned this method saying: "*What if the length is not divisible by 2? If so, then the ruler is inappropriate.*"

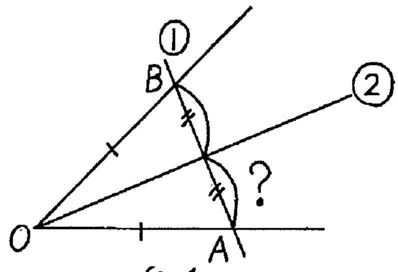

fig.1

This shows that Taro had the relational image schema of an isosceles triangle. His image schema developed the relation in it. That is, in an isosceles triangle, the vertex angle is divided into halves by the segment from the vertex to the midpoint of the base. But some other students claimed that this had a limitation because the midpoint of the base is not always determined by the graduation. One of them proposed another method. A bisector was practically built up by folding the angle written on his notebook. Many students said that this method was more accurate than Taro's method. Most students had the figurative image schema of an angle and seemed to be at the transitional stage from empirical to theoretical.

After some active introduction of this topic, many of the students developed the method shown in fig.2 when they were restricted to the use of compass and ruler. Gen presented it on the board and explained as follows:

Gen: I used a compass, so this length (OA) is same to this (OB). ... To be honest, I made two same isosceles triangles here and here by placing this (AB) as the base. Since I could draw a perpendicular here, as what was done with the rhombus yesterday, the angle was just bisected. When we draw a perpendicular through the midpoint of the base, it meets with these vertexes of this triangle and that triangle. ... Because there exists a rhombus.

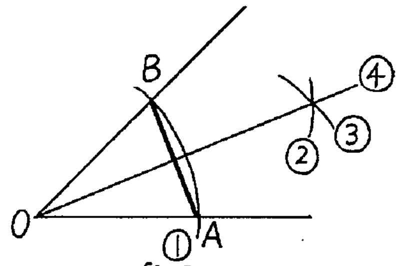

fig.2

Gen had a relational image schema of an isosceles triangle or a rhombus. He, therefore, could develop his ideas as the method of construction. He indeed read the relation embedded or written in the shapes of them.

3.2. Justification (1): from visual to theoretical correctness

Taka planned another way of construction to overcome the limitation of Taro's method. He copied out fig. 3 on the blackboard from his notebook and explained his way like this.

Taka: I put points A, B 20 cm distant from O. And I drew the perpendiculars ① and ② using the right corner of a set square, and then drew the bisector ③ connecting the intersection of ① and ② to the vertex O.

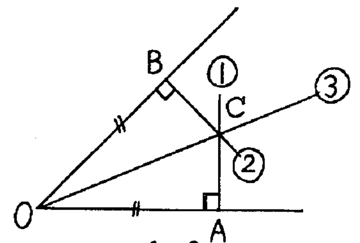

fig. 3

During the interaction process in the classroom, Taka derived his criterion of justification from visual to theoretical correctness. The following is a scene when the teacher asked him to mark the parts he made equal in his drawing.

Taka: OA and OB are equal. Angle A and angle B are right angles. ... And AC becomes equal to BC.

Teacher: Does AC become equal to BC? Did you make those lengths equal to each other?

Taka: We didn't make them equal. But when you put those two triangles side by side and right angles are constructed, ... then the heights... AC is equal to BC because they are same triangles.

The teacher tried to help Taka differentiate visual equality from derived one in construction. He suggested that the case of AC=BC was quite different from that of OA=OB because the latter was the condition of this construction. Taka had once said, "*I see*", but later he said again "*Anyway triangle OAC is congruent to OBC when we look at the shapes. So, it is divided into halves necessarily. ... If we measure lengths of OA, AC, OB and BC, we can ascertain the equalities*".

Taka tried to show that OC became the bisector of angle O by the congruence of two triangles. He read the relation of the two triangles in the kite shape CBOA through the relational image schema. But AC=BC was the result of his visual perception. This failed him to convince the teacher and other students of his justification. His failure made his justification more correct.

Taka: ... If that AC equals BC is ascertained, the angle could be determined definitely. If AC is equal to BC,....

Teacher: Are you saying that two triangles are congruent to each other if the two sides have the same length ?

Taka: Maybe. ... because both angles A and B are really 90 degree. If we can ascertain one angle and two sides, those are congruent.

Taka's statements gradually came to include "if-then" form and he eagerly needed the logical correctness of the premise at last. The relational image schema he derived gave him a logical base to reflect his construction and caused him to finally change his visual equality of length to the theoretical one.

3.3. Justification (2): lack of externalizing the image of shape

Syun put a triangle at the vertex and confirmed 10 cm length with the vertex as the midpoint. Next, he slid the triangle along the other triangle placed perpendicular to it. He stopped the translation where the angle cut the segment of 10 cm, and then drew a ray from the midpoint through the vertex. Some students tried to refute this way of drawing.

S1: I think Syun's drawing is doubtful.

S2: When he first put a triangle, accuracy is needed. But, that way cannot work well.

S3: I think so.

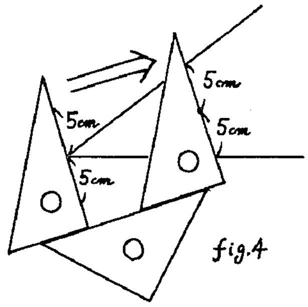

fig.4

The students questioned Syun's approach. They thought that Syun just put a triangle by his eyes without making any basis. Consequently, one student implemented same way of drawing by slightly moving the triangle. He found out that the drawing deviated from Syun's. Syun also refuted, saying "*You don't understand what I want to say. ... If you did so (moved a little), the segment of 5 cm wouldn't fit to the angle anymore*", but no one understood his idea.

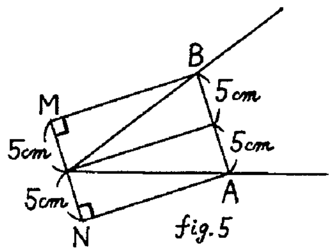

fig.5

His method is right if the angle really cut the segment of which he has been conscious at the start. We think it is "rectangle ABMN" that Syun has imagined (fig.5). He actually traced the segment MN with his finger when making the drawing. But he just did not verbally express his image of rectangle. That image enabled him to construct an angle bisector, but he failed to convince his classmates of his procedure. His failure seems to result mainly from the thought that he couldn't or doesn't externalize his image of shape.

3.4. Justification (3): based on an operative image schema

Yoshi also aimed to overcome the limitation on putting a midpoint of AB using graduations of a ruler and devised the following drawing: Put a ruler along AB, move the other triangle along it and stop at the vertex O. With this way, most of the students are convinced. Yoshi justified his procedure as follows.

Yoshi: I revised Taro's method. This side OA is equal to this side OB.

Teacher: Certainly those are what we did.

Yoshi: Therefore, when we connect A to B, the angle OAC is equal to the angle OBC.

Teacher: Well, everybody, what do you think about it?

Students : I see.

Yoshi: Because.... When we consider both cases when this triangle is put in this way (Yoshi put his triangle at the triangle OAC on the blackboard) and when it is put in that way (he flipped it to the other side and found out that it coincides with the triangle OBC), these are two same triangles. Therefore, these angles are equal to each other since those sides (OA and OB) have an equal length.

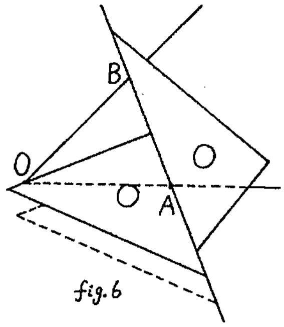

fig.6

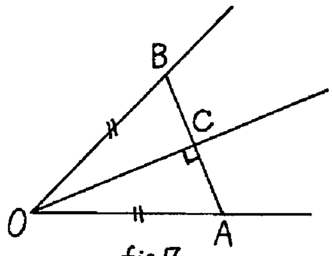

fig.7

Yoshi's justification started from the theorem of the base angles of an isosceles triangle: if OA=OB then angle OAB is equal to angle OBA. His reasoning was based on the physical manipulation of flipping the triangle, and the students in the class were convinced and agreed to his explanation. Since his explanation has been accepted, his justification is almost completed at this moment. Because these triangles are similar, the remaining angles are also equal. But in this discussion, there was a context about point C such that AC=BC must be constructed in order to overcome the limitation of Taro's work. Therefore, it is AC=BC that Yoshi tried to show next.

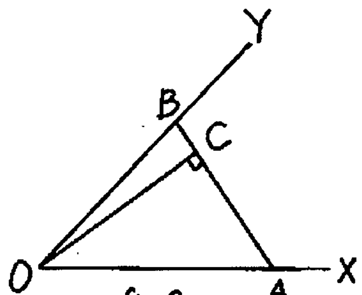

fig.8

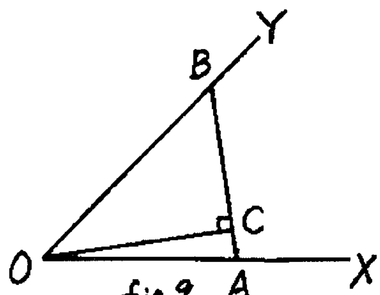

fig.9

Yoshi: For example, if we put point B arbitrarily on ray OY, connect A to B, and draw a line perpendicular to BA through point O, the line does not necessarily touch the midpoint. It is inclined to a direction where the distance from point O is shorter. ... The line makes angle OBA greater than angle OAB. Conversely, in the case where OA is shorter than OB, the perpendicular is inclined to the other direction and therefore makes angle OBA smaller than angle OAB. Therefore, when angle OBA becomes equal *to angle OAB, the line is not inclined to any directions but at the midpoint of AB.*

Yoshi's justification includes a distinction of cases about the lengths of OA and OB or the measures of the angles OAB and OBA. We think it was resulted from his conception of the angle bisector, which was a figure constituted as the perpendicular from the vertex to the base of a triangle in the continuous transformation. His explanation about AC=BC convinced his classmates and he finally justified that angle AOC is equal to angle BOC based on the congruence of the triangles OAC and OBC. The reasoning that contains a kind of transformations is discussed in Simon (1996) or Harel and Sowder (1998) as an excellent justification and as a necessary precedent to the axiomatic proof scheme. Yoshi's image schema was operative as well as relational, and his case suggests that the operative image schema can be used effectively to develop one's reasoning.

4. DISCUSSION AND DIDACTICAL IMPLICATIONS

4.1. Cognitive tool of construction

We consider construction as a didactical intermediate between concrete and formal operation rather than a mathematical proof of existence. In this paper we examined the cognitive tool in justification of construction. The image schema activates cognitive tools to make students possible to see another figure in the figure, to read the theoretical relations in and between figures, to give them a logical form, and to estimate failure or success in their justification of construction.

Most students have discovered a rhombus and a kite in the construction of an angle bisector just like as van Hiele-Geldof had shown (1984). As a matter of fact, they explained their procedure of construction as follows: "This is the matter of a rhombus", "This is the same way as drawing a kite", "If we fold a rhombus or a kite along the diagonal, they always become axial symmetry", "There is the same principle as drawing a rhombus and a kite when you draw an angle bisector", and so forth.

Besides a rhombus and a kite, students made good use of a right triangle, an isosceles triangle, and a rectangle etc. to justify the construction. They had already learned all of them with their properties in the elementary level. We do not need to restrict the students' logical basis to a triangle and its congruence as described in the Euclidian geometry. What a student should learn in the classroom lesson is to locally organize several pieces of knowledge against the question beginning with "how" and "why" (cf. Freudenthal, 1973). Mathematically speaking, he or she should reorganize the situation of construction into cause-effect or assumption-conclusion relationship. Didactically speaking, a teacher should activate the image schema of students in geometric situation to induce cognitive tools such as a rhombus, a kite, a rectangle and so on.

4.2. Noteworthy points in the justification of construction

There were several noteworthy points to be considered in the explanations of the students. They found out the known figure such as the isosceles triangle, the rhombus etc. in the construction process, and then made use of them as cognitive tools for their justification. One, however, failed in it and another succeeded in it in a sense.

In the case of Syun, he failed to expose the cognitive tool although his method was relevant. If he had referred to it as a rectangle and made the students pay attention to the central line in it, he might have succeeded in his explanation. His problem was summarized at the externalization of his ideas.

In the case of Taro, there would have been no problem if he had executed his method of isosceles triangle using the geometric software Cabri. He did it on the blackboard. The midpoint of graduations of a ruler became a practical problem. Such a perceptual issue prevented students from using their logical thinking.

In the case of Taka, he developed the if-then thinking. Such development of the logical form was brought about by a right triangle as a cognitive tool and reflecting steps of the construction as conditions for determining the figure. The teacher played an important role in this change of thought.

In the case of Yoshi, operative image schema seemed to work well in justifying his procedure although relational image schema was so effective to the other three cases mentioned above. His method based on a variable triangle was a kind of reduction to absurdity. In other words, fig.7 became right as construction through fig.8 and fig.9. In such justification, it seems that operative image schema gives several distinctions to the spatial continuous transformation and furthermore gives the temporal arrangement to several spatial distinctions in terms of geometric statements.

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DEVELOPING THE "OPERATIONAL APPROACH" TO ENCOURAGE STUDENTS' AWARENESS OF INCOMMENSURABILITY: What a Student Ta's Activity Suggest Us for Improving Our Teaching Design

Hiroshi IWASAKI (Joetsu University of Education)

ABSTRACT

The purpose of this study is to develop mathematics lessons, the "Operational Approach" where students are aware of incommensurability through their mathematical activity. In an attempt to do that, a cooperative teaching practice was conducted over a four-week period with 5 lessons given to a 9th grade class on June 2, 3, 4, 5, and 9, 2003. Several of the students were interviewed after the lessons.

In this paper we focused on and reported the last lesson, especially one target student Ta's activity there. To understand the Ta's activity, especially the construction process of incommensurability, we contrasted his activity in the last lesson with his activity in the interview.

As a result of this, Ta's activity seemed to be passive one under the teacher's guidance at a first glance. We, however, could find his active mathematics learning at meta-level. The interview revealed that Ta could make a hypothesis and the proof on incommensurability between a side of a square and the diagonal line by means of reflecting on his operational activity. This hypothesis and the proof which were constructed by him in the interview process were what we had not expected.

The "Operational Approach" could help students to be aware of incommensurability through their mathematical activity. For improving the lessons design, (1) we should provide the students with opportunities to explain their ideas or thoughts to someone, as well as to promote their activity. (2) The student Ta's natural construction process of incommensurability should constitute new epistemological elements of the lesson design.

BACKGROUND

For "we may say that an irrational number represents the length of a segment incommensurable with the unit" (Courant, R., Robbins, H., 1941, p.60), the concept of incommensurability is one of the most essential elements of irrational number. In this sense, if we know an irrational number $\sqrt{2}$, for example, then we have to know that $\sqrt{2}$ is incommensurable with 1.

Historically, there are two typical proofs of the incommensurability between a side of a square and the diagonal line. One is the indirect proof based on Pythagorean theorem, properties of integer and contradiction (reduction ad absurdum). This proof had been instructed in our country, as an advanced mathematical content of the square root. You can find the topics in several mathematics textbooks in the past in a junior high school, however, those are not in present mathematics textbooks.

Such an indirect proof seems extremely attractive to some, but impenetrable to others. It looks like technical one and has complicated argumentation structures of its own which makes many students difficult to approach. We have to constitute synthetic links when we understand the proof. These are essential differences between procedural manipulations in arithmetic and algebra and the more sophisticated thinking processes in mathematical proof (Barbard, T., Tall, D. 1997), however, this would be one reason why most of students could not understand it. Further more it would absolutely be difficult for the students to construct the proof through their mathematical activity, which would be ideal for lots of mathematics teachers in Japan. Seeking approximate values of $\sqrt{2}$ one after another, this inductive way is found in the present textbooks in Japan. It seems to be the only content on incommensurability.

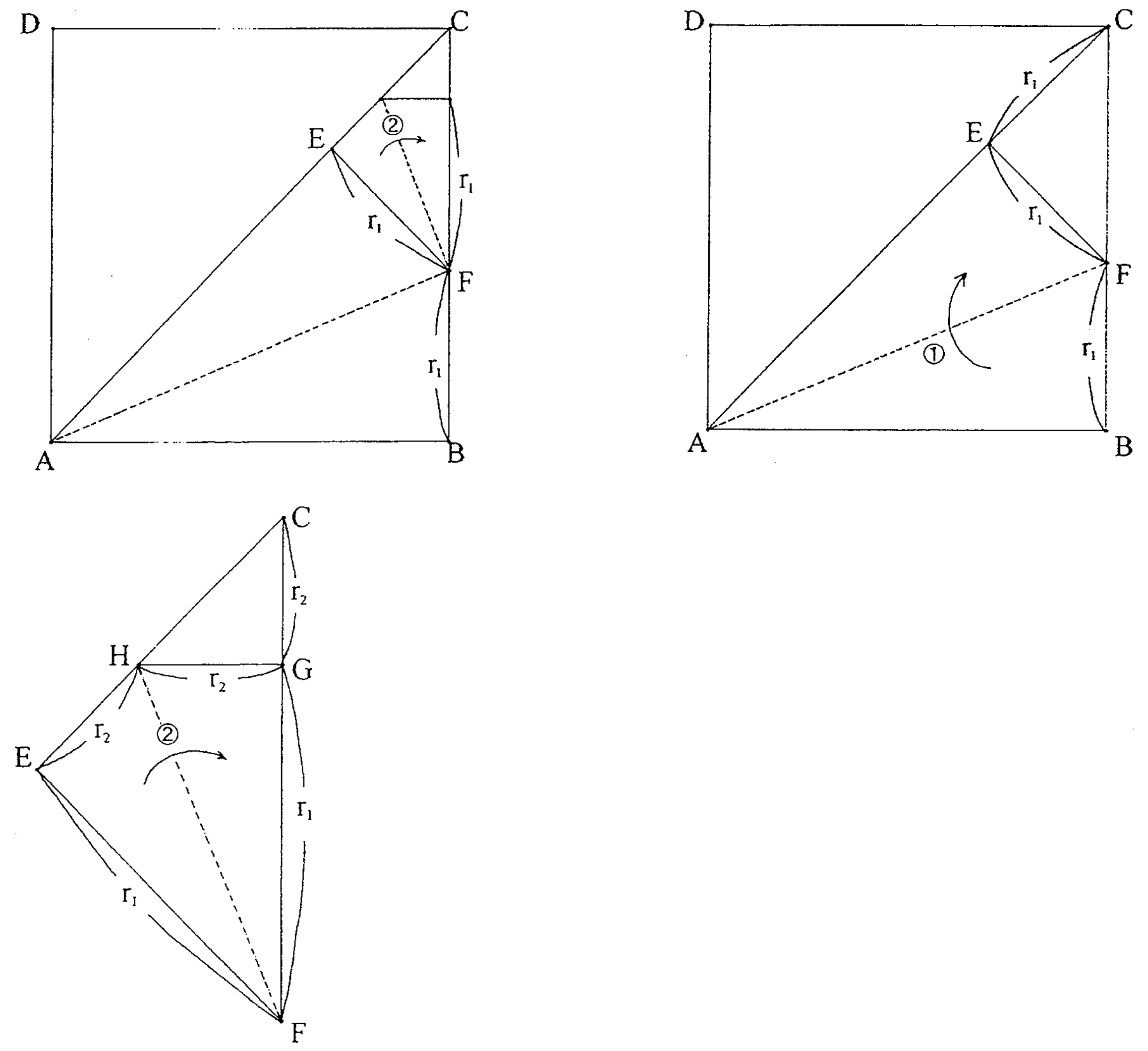
To improve this situation, we will adopt another typical proof which is based on Euclidean algorithm, especially one done by D. Kikuchi (1918) or Rademacher & Toeplitz (1933). According to Szabo (1969), the proof is a re-construction of proof, which is not in the folklore from the ancient times (Szabo, A. 1969 S.279). We shall use it as one of the epistemological elements for our teaching design.

From doing mathematics viewpoint, it has some advantages to design a teaching of mathematics based on this approach. Applying Euclidean algorithm to a side of square and the diagonal line forms a central of the proof, it could be done by folding and cutting a square sheet of paper. Reflecting the procedures, we can naturally get geometrical relationships that need for the proof. We shall call it "Operational Approach", which is illustrated as follows:

This study aims at developing mathematics lessons where the students understand incommensurability through their mathematical activity involving both empirical-inductive activity and logical-deductive one. In an attempt to do that, we conducted a cooperative teaching practice over a four-week period by using Operational Approach as an epistemological element for our lessons design.

The purpose of this paper is to investigate the following research question empirically:

How junior high school students can understand incommensurability through their mathematical activity?



METHODOLOGY

In an attempt to do that, the author and a teacher conducted a cooperative teaching practice a four-week period with 5 lessons given to an 9th grade class on June 2, 3, 4, 5, and 9, 2003. The teacher worked at a junior high school attached to Joetsu University of Education. He is an experienced one with a master's degree in education.

The author interviewed to several students after the lessons.

In this paper, I will focus on one student Ta's activity in the last lesson. He is one of the target students in the cooperative teaching practice. I will attempt to reconstruct Ta's understanding process of incommensurability in the lesson comparing with the same process presented in the interview.

Recent research concerning interactions in mathematics classroom suggests that a starting point for improving the everyday mathematics classroom is to have a better understanding of its unique cultural practice. (cf. Bauersfeld, H., Voigt, J., Kurmmheuer, 1988)

To gain a better understanding of mathematics classroom from students' perspectives is one of the effective research methods. It is a typical example that to try to understand the relationships between interaction and learning in the teaching process from one student viewpoint who participate in the class. (cf. Iwasaki, H., 2001; Nunokawa, K., 2002)

Teachers could get cues for improving their lessons from the students' activity there. When we visit a class, we usually try to report the students' activity there and give the information to the teacher after the lesson. Especially, in the process of our lesson study we discuss the teaching method in the meeting after the lesson, the effectiveness and the limitations of which are discussed frequently in light of the information of students' activity from the participants.

Cooperative Teaching Practice Study

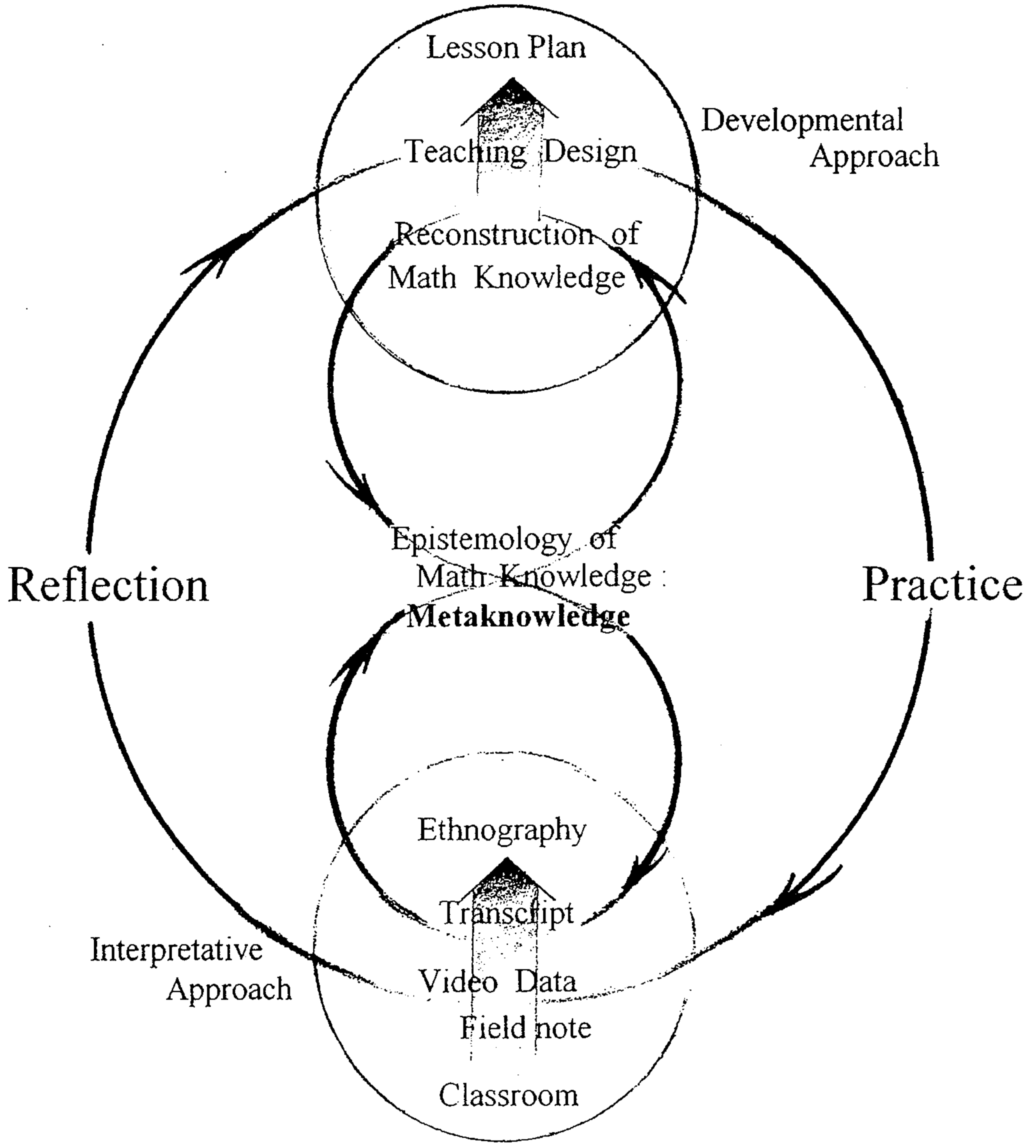
A cooperative teaching practice study aims at developing new design of mathematics lessons improving a present mathematics teaching, where researchers in mathematics education and practitioners in schools (teachers) work together to design the series of mathematics lessons.

- One of the most important characteristics of this study is found in its own holistic and cyclic process, which is organized with "epistemology of mathematics: metaknowledge" as a kernel and connects "a study in theoretical level" with "one in practical level" systematically.

Fig. 2 represents the concept of the cooperative teaching practice study. The outside circle in the figure represents a study in practical level. The inside one, which makes a figure like a letter `8', represents a study in theoretical level.

- Developmental approach and interpretative one are systematically linked together in the cyclic process --- this is another important characteristic of this study.

> It is promising that, as Kilpatrick (1992, p.31) has pointed out, research in mathematics education has gradually moved out of the library and the laboratory and into the school and classroom. More and more theory is becoming relevant to teaching practice and practice is getting the focus of theory. So the old and invalid dichotomy between theory and practice is disappearing. (Becker, J.P., Selter, C., 1996, p.550)



● We regard the developmental approach as important because of including "the execution of a thought experiment of teaching and learning both" (Steefland 1993, p.116). In this point, Wittmann (2000) described as follows:

This new approach was formulated and published as a connected series of SLEs in a handbook for practising skills (Wittmann ¥& M¥"uller 1990, Grade 1: chapters 1-3). It was based on a systematic epistemological analysis of arithmetic, on inspirations from the developmental research conducted at the Freudenthal Institute (cf., Treffers et al. 1989/1990, van den Heuvel-Panhuizen 1996) and on the intuitions of the designers. It was not based on empirical research conducted by professional researchers. Empirical studies which confirmed the holistic approach came only later (cf. Selter

1995, Hengartner 1999). So it was teachers who first tried it out in their practice and found that it works better than the traditional approach. (Wittmann, E. Ch., 2000, pp.12-13)

Further, we can list the followings as the characteristics of this study, the bases of which are compatible with the idea of "kritisch-konstruktiver Erziehungswissenschaft" in Germany (Klafki, 1976).

● The teacher is a full member in this study.

The teacher is not a simple practitioner who puts into practice a teaching plan designed by the researchers precisely. He/She is, rather, expected to present a new design or modify it from a practical point of view. Further more, he/she may modify or change the teaching plan according to the students' replies during the on-going teaching practice.

- The teaching in this study is not an `experiment' rather a `practice'.
- The process of this study itself should contribute for improving the present class.

● The researcher present some view points, ideas and plans for improving present mathematics education in general, however, this is not a start point of this study.

● The teacher confirms those in light of his/her experience in his/her present class. If we could share the importance of those, it would be the start point of the study, and the extent of importance for him/her would determine to what extent the study should be done. The extent of the study could be changed, however, it should be done according to whether or not it would be benefit for the students in the present class.

● The researchers' major role is to report what's happened in the class and to explain it for teachers.

Data

The data were collected in the course of the cooperative teaching practice study. Five lessons were conducted in a 9th grade class on June 2, 3, 4, 5, and 9,

2003. Those were done during the daily math's lessons. There was a time limitation to one school lesson. It takes usually 50 minute, sometimes 45 minute.

There were 19 female and 18 male students in the class. We selected several students for recording individual activity. We call them `target students'. The intention is, as mentioned above, to get a better understanding about what's happened in the lessons from a student's perspective. We selected the target students in the following viewpoints: they express their thinking clearly, they are in good locations for video cameras and they are not disagreeable for recording with video cameras.

Each lesson was recorded with 4 video cameras. One camera was used to record mainly the teacher's behavior and the whole-class discussion and the others were used to record the target students respectively. To record the target students throughout the lessons, the video camera sometimes zoom in on their working at hand.

Several of the students including the target ones were interviewed after the lessons. The interviews (the interviewer - one student) held during school in a separate room. The interview was done clinically so that we could have a better assessment of the students' understanding of the lessons. The interviewer was the author. The interviewer asked the students to solve tasks that presented in the lessons and explain about their solutions and ways of thinking.

Each interview was recorded with one or two video cameras, which sometimes zoom in on their working at hand. The interviews were designed for about 30min. – 50min. time period for each of the students.

Transcripts of all lessons and some of interviews were prepared from these data.

In this paper we will focus on the last lesson and one of the target students Ta. And we are going to attempt to understand Ta's construction process of incommensurability in the lesson through the clinical interview.

ANALYSIS AND DISCUSSION

The lessons, which have done as the Cooperative Teaching Practice, consist of four phases from a viewpoint how mathematical ideas about the incommensurability have occurred and developed in the lessons. The kernel task of the lessons was the following problem: "*When a rectangle is given, find the maximum square that tessellate it.*

In the phase (I), one student discovered an idea of early Euclidean algorithm. It was developed and formulated as an algorithm. It was called "mutual difference

Method" by the class.

In the phase (II), the class discussed the existence of a rectangle which you can't tessellate by same squares even if you made it smaller and smaller as possible you can.

In the phase (III), they tried to solve a problem such that "When an A3 paper is given, find the maximum square that tessellate it". They were thinking about how to apply the "mutual difference Method" to an A3 paper. To solve this new problem, the class folded and cut the given papers (empirical-inductive activity).

In the phase (IV), to approach the same problem more logically, the class applied the "mutual difference Method" to a side of a square and the diagonal line, which are equal to a shorter side and a longer one of A3 paper respectively. This operational procedure included the following activity: folding and cutting a square sheet of paper, reflecting the procedure and describing it in a figure sketch with literal expressions, and deducing some strange geometrical relations occurred in the process (logical-deductive activity).

Outline of the lessons

In phase (I), the first problem mentioned above was posed to the students as follows: "There is a rectangle board 30cm by 42cm. You want to cover it with square tiles, the size of which must be same and larger as possible as you can. Find the size of square tiles."

When *the rectangle board 155cm by 186cm was given,* one student Ha discovered the idea of early Euclidean algorithm. He said: "Taking 155 from 186, it gives 31, and the largest common divisor of the 2 numbers is, the largest, largest ⋯ You can narrows the largest common divisor of the 2 numbers then in the common divisors of 31, but, it has no divisor [except 1 and 31], applying the 31 [to 155 and 186], it was sufficient, so, I got [the answer] at once."

In phase (II), the problem was posed as follows: *When the rectangle 0.5 by 0.666⋯ is given, find the maximum square that tessellate it."* It gave rise to another problem on applying their algorithm "mutual difference Method". After their solutions were confirmed, with this as a turning point, the class discussed the existence of rectangles such as you can't tessellate by congruent squares: Are there such rectangles or not?

In phase (III), the problem was posed as follows: *There is a sheet of rectangle paper here. It is based on the A3 standard, the larger side of which is equal to the diagonal of the square with the smaller side. Find the squares that tessellate it."*

The structure of A3 was demonstrated with the teacher's figure construction on the blackboard in the front of the classroom. Finding the common divisor of two sides of the rectangle A3, each student was applying "mutual difference Method" in his/her own way with folding or cutting the A3 paper in his/her hand. They started to tackle this problem soon when it was posed. This fact serves as evidence of that they are able to apply the "mutual difference Method" in a geometric way as well as an algebraic way. Several students were surprised that it took a long time until the process was finished. The result seemed to have surpassed their expectations.

In the end of the 4th lesson, a student Ta who called by the teacher demonstrated a solution process with the A3 paper in front of the classroom. His solution could be described as follows:

(1) With the A3 rectangle paper, he moved one shorter side to the longer side and folded the paper, then cut off the remainder smaller rectangle (R_1), the longer side of which was equal to the smaller side of the beginning (first) rectangle [A3 paper]. (Ta folded one time in the operation.)
(2) With the smaller rectangle (R_1) which had been cut off, he moved the shorter side to the longer side in the same way and folded the paper. As a result of this, a smaller rectangle remained, further he moved one side of the right isosceles triangle to the longer side of the smaller rectangle (R_1), then cut off the new remainder rectangle (R_2), the longer side of which was equal to the smaller side of the next rectangle (R_1), he continued in this way.
(3) The rectangle (R_2) was folded 2 times, and smaller rectangle (R_3) remained in this operation.
(4) In the next process the rectangle (R_3) was folded 3 times and smaller rectangle (R_4) remained. The remainder rectangle (R_4) was too small to fold.

His solution process (1)–(4) described above represents the "mutual difference Method" (Euclidean algorithm). It was similar to the other students' one, however, their results were different each other. On the one hand, in his operation described above, the A3 rectangle paper was folded one time in above process (1), two times in (2), two times in (3) and three times in (4), the result was 1, 2, 2, 3. On the other hand, some students' one, for example, was 1,2,2,2,7.

The lesson had finished without confirming their results, so many students were working and talking each other. Several of the students was discussing their results around the teacher in front of the blackboard.

The first half of the 5th (last) lesson

In the beginning of the lesson, the teacher talked about the students' results in the 4th lesson: Some finished the folding operations, others did not finish. With regard to the results, some was 1, 2, 2, 3 and some was 1, 2, 2, 3, ⋯ , and others was

1,2,2,2,7 and so on. The teacher asked the students which was correct.

The main aim of the lesson was to ascertain the solutions all together in the class. The teacher suggested applying the "mutual difference Method" to a side of a square and the diagonal line instead of applying it to one shorter side and the longer side of the rectangle A3. When one side of the square was moved to the diagonal line, the teacher pointed to the triangle at the corner and asked the students what the triangle looked like. They answered it was a right isosceles triangle. At the same time, they had to explain the reason logically. After about 3 minutes small groups discussions, they pointed out two geometrical grounds: the vertical angle is right angle because it was resulted from folding the right angle of the square. As a result of this, sum of two base angles is equal to right angle. One of the base angles which is one of the corner of the original rectangle A3 paper is 45 degree, the other base is also 45 degree, so the base angles are congruent.

The latter half of the lesson

The students applied the "mutual difference Method" to a side of a square and the diagonal line according to the teacher's guidance. It included the following activity: folding and cutting a square sheet of paper, reflecting the operations procedures, drawing them in free hand figures, writing the results down in literal expressions with symbols, and deducing some geometrical relationships occurred in the process (logical-deductive activity).

The student Ta draw the figure as in Fig. 3 in the latter half of the lesson.

The teacher and students interaction in the latter half of the lesson could be re-structured as follows: With an A3 paper, ①The teacher asked the students to tell him which line segments you should select as the two concerned ones. The students should guess ahead how to fold the paper to apply the "mutual difference Method" to them. ②The teacher demonstrated carefully the folding operation using a large sheet of paper in front of the blackboard. After that, he asked the students to do the operation with a paper on their hand. ③The teacher wrote symbols on the segments to express the lengths respectively. ④The teacher wrote the results of the folding operations on the blackboard by using literal expressions.

The students activity seemed to be passive one simply followed the above pattern from ① to ④. Ta's activity also seemed to follow this pattern because he began to work after the teacher's instruction. However, the detailed analysis of Ta's activity in the lesson revealed that it was only a surface impression.

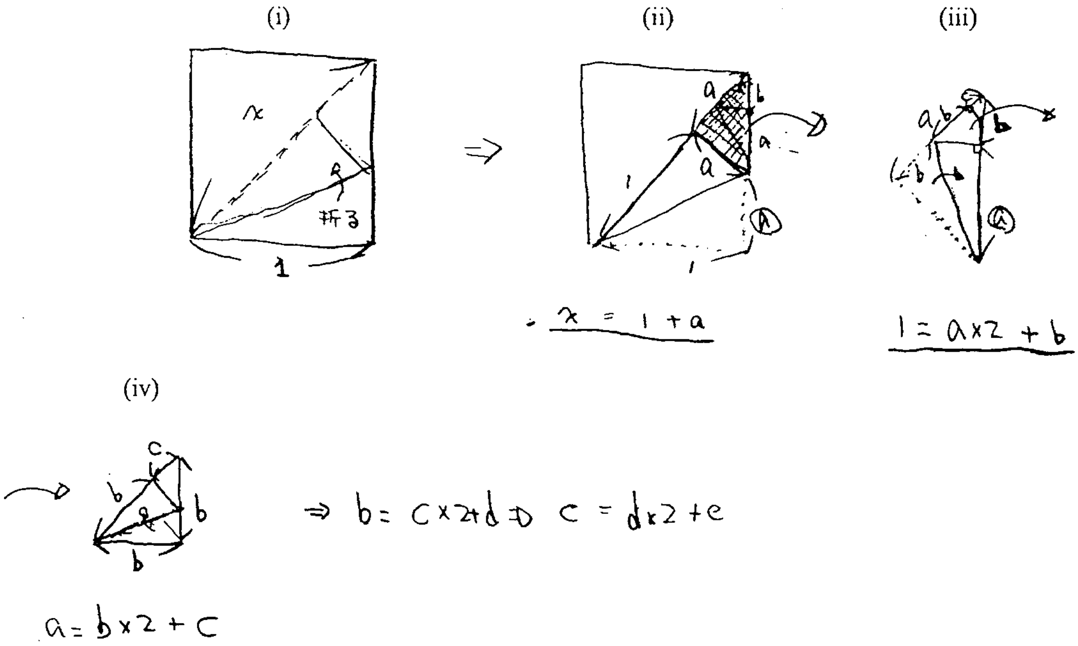


Fig. 3 Ta's descriptions of the "mutual difference Method" process (in the lesson)

On his descriptions of (i), (ii) in Fig. 3, Ta went into action followed the teacher's demonstrations. But, in except the descriptions of (i), (ii), he did before the teacher's demonstrations. If he looked the teacher's demonstrations and did the same way with the paper on his hand, then the aim would be nearly to confirm the results that he had done.

For example, in the case of applying the "mutual difference Method" to 1 and a, when the teacher demonstrated a was taken from 1 two times and asked the students did the same operation, then Ta folded one side of the small right isosceles triangle, which had been cut at one step before, to the hypotenuse of it. However, (iii) in Fig. 3 had been drawn already at that time. As soon as the teacher asked the students represented the new remainder with a symbol b, he wrote the symbol a,b appropriately in the small right isosceles triangle and wrote down the literal expression $1=a\times 2+b$ below. Then he encircled a in the drawing of one step before [Fig. 3 (ii)}] and a in the drawing of the small right isosceles triangle [Fig. 3 (iii)] respectively. He added an information which became in the (iii) to the right isosceles triangle of one step before [Fig. 3 (ii)}], the inside of which was painted out with slanting line.

Analyzing Ta's activity revealed that his activity was active one rather than passive one, and further he understood the relationships between the "mutual difference Method" and that folding process with A3 paper.

In the same way, he drew the small right isosceles triangle (iv) in Fig. 3 and wrote down the literal expression $a=b\times 2+c$ below. He did the equations $b=c\times 2+d$, $c=d\times 2+e$ without drawings. Those are compatible with what was done in the class interactions.

In the class interactions, the infinitely continuing process of the "mutual difference Method" have confirmed based on mainly the two facts, namely, the literal expression seemed to continue with regularity following the same pattern and the folding more smaller right isosceles triangles seemed to continue infinitely.

It is natural that you might guess Ta conjectured the infinite process of the "mutual difference Method". But, we could not find any evidence that Ta was aware of the fact. The interview mentioned below will makes clear that Ta was not aware of the fact, rather he thought the folding process should be finished.

Ta's activity in the Interview

The interview to Ta was done immediately after the 5th (last) lesson. The interviewer asked Ta what he was doing in the lesson and the aim of it.

> Ta : (*Drawing a rectangle roughly on a sheet of paper*), "Can you tessellate this paper (*pointing the rectangle on the paper*) by small squares? Then finding their size, or how to find them, – this would be what I was doing today."

This suggests that he had the aim which is compatible with the lessons. The interviewer asked Ta how he was doing in the lesson more concretely.

Before analyzing in detail, you should see Ta's descriptions in Figure 4 first. It represents the final one that he wrote in the interview. You could find immediately that those are similar to one in the lesson (Figure 3), where both geometrical and algebraic representations of the "mutual difference Method" process were described in the same way. Actually descriptions on (i), (ii), and (iii) in the Figure 4 are coincident with ones in (ii), (iii), and (iv) in Figure 3.

When he draw the descriptions in the lesson, he referred to the paper which prepared by the teacher. But he did not refer to it in the interview. This fact seems to be an only remarkable difference between them.

Through the interview, it became clear that Ta learned "when to do what and how to do it" with respect to figures, symbols, and literal expressions. In other words, Ta was aware of functions of those expressions. When he explained how he was doing in the lesson with those descriptions (Fig. 4), he used them appropriately. So, it is obvious that his descriptions in the interview are not a simple reappearance of ones in the lesson.

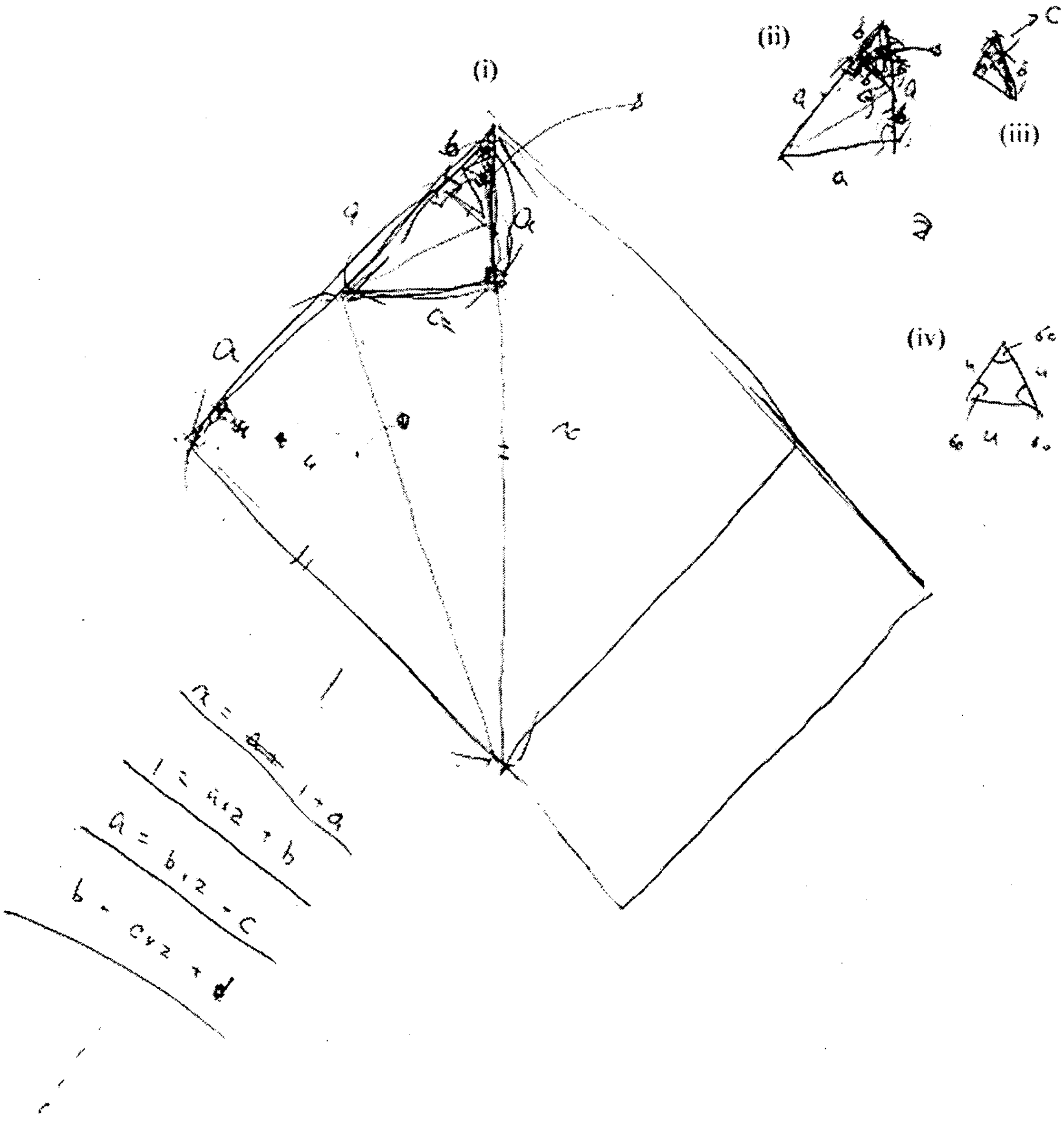


Fig. 4 Ta's final descriptions of the "mutual difference Method" process (in the interview)

For example, Ta explained a relation in the figure (Fig. 4 (i)) as follows, which serves as an evidence of that fact.

> Ta : After all, one is , one is equal to, two a and b, then two times of a plus b. (*at the same time, he wrote* $1 = a \times 2 + b$.)

It suggests that Ta was aware of a function of literal expressions as a tool for describing a result of the operation. This function is compatible with the usage in the later half of the lesson. Further, Ta derived a new equation $b = c \times 2 + d$ from the pattern in the sequence of literal expressions, which were written as descriptions of the results with no drawing figures. It suggests that he was aware of another function of literal expressions as a tool for looking for patterns. From interactionist perspective, learning is consistent with interactions, Sierpinska (1998) acutely pointed out:

> …if teachers and students engage in interactions of type A, then students are likely to develop ways of knowing and understanding of type $f(A)$. (Sierpinska, A., 1998, p.49)

Ta's learning described above illustrated the consistent relationships between interactions and learning, and is characterized as learning at meta-level.

> As it is with culture, the core of what is learned through participation is *when* to do *what* and *how* to do it. … The core part of school mathematics enculturation comes into effect on the meta-level and is "learned" indirectly. (Cobb, P., & Bauersfeld, H., 1995, p.9)

Therefore he also seemed to be aware of "infinity" as a natural consequence of the sequence of operations. He, on the contrary, clearly stated the sequence of the operations should be finished.

> Ta : … then, sometime, a number which can be divided, it becomes fit. And at last, this (*He was pointing the segment* b *in Fig. 4 (iii)*) may be congruent to this (*pointing the hypotenuse of the right isosceles triangle in Fig.4 (iii)*) …, then this (*pointing the right isosceles triangle in Fig. 4 (iii)*) would have to be equilateral triangle. I was thinking so, and if it can be divided, and if the size of a side of the triangle, for example become x, …, oh x is not so good, become u, if it becomes u or something, if it continues since u, then, the length of the side [a side of the triangle in (iii) of Fig.4] have to be u.

With a great deal of nodding, the interviewer turned Ta's attention to a sequence of literal expressions that he wrote, and continued as follows:

Interviewer: It is going to become fit at last.
Ta: Yes.
Interviewer: Is it your expectation or absolutely so?

Ta was beginning to examine it through the so-called "thought experiment" as follows:

> Ta : If it becomes a right angle, then this (*pointing where the right angle is folded in Fig. 4 (ii)*) should be a right angle, too. So, then a right angle continues infinitely.

Ta begins to be aware of that a sequence of the operations don't finish, rather continue infinitely through the "thought experiment" reflecting his own folding process.

> Ta : If it can be divided, then all, all of them must be congruent. Then it become a equilateral triangle, the sides of which are u, u, u. (*He drew a small triangle and described u in each of three sides of the triangle respectively. [see Fig. 4 (iv)]*), but, to be so, I'm wondering if the 90 degree must grow smaller.

He believes the operations must be finished on the one hand and the other, he recognizes the small triangle must become a equilateral triangle if the operations will finish, and further he began to be aware of that it was impossible through the "thought experiment" which based on geometrical relations reflecting his own folding process.

The former is conflicts with the latter.

> Ta : If I fold it, because of folding, then, the opposite must become 90 degree without doubt. If once it becomes 90 degree, it will be impossible … it continues more and more, then throughout 90. Folding back this, makes 90 … folding back again, it makes 90. … it must become 60 degree if it is an equilateral triangle.

In this way, Ta was only explaining his thought to the interviewer. The interviewer was only listening. Through this simple interaction, Ta was making a hypothesis concerned with incommensurability and proving it by himself based on some more simple geometrical relations. The hypothesis and the proof were what we had not expected. In other words we learned these from the student Ta. The Ta's natural construction process of incommensurability could help students are aware of incommensurability through doing mathematics involving logical-deductive activity.

CONCLUSION

In this paper we focused on and reported the last lesson, especially one target student Ta's activity there. To understand the Ta's activity, especially the construction process of incommensurability, we contrasted his activity in the last lesson with his activity in the interview.

As a result of this, at a first glance, Ta's activity in the lesson seemed to be passive one under the teacher's guidance apparently. We, however, found some active one when we analyzed it by interpretative approach in detail. On the one hand, the interview revealed that Ta's mathematics learning at meta-level, especially he learned functions of figures, symbols and literal expressions which was consistent with the process of the lesson. On the other hand, the interview also revealed that he had recognized what we had not expected on incommensurability. It suggest the limits of our teaching method, especially, what we described a pattern of the lesson, that is, fundamentally, the students' activity followed by the teacher's guidance.

Further, the interview revealed that Ta could make a hypothesis on incommensurability between a side of a square and the diagonal line by himself through reflecting his activity that did in the lesson. Ta could also ascertain the hypothesis logically based on the geometrical relations, where, as mentioned above, Ta was only explaining his thought to the interviewer and the interviewer was only listening.

As Kamii & DeClark (1985) suggestively pointed out from a constructivist point of view, students' explaining action itself could promote their knowledge construction by themselves. Ta's progress founded in the interview illustrated this assertion well. The conclusion derived from this for improving our teaching methods is that we should provide the students with opportunities where they explain their ideas or thoughts to someone, as well as to promote their activity.

Finally, we would like to point out that the student Ta's own construction process of incommensurability appeared here involves a new and more natural epistemological approach to incommensurability. It constitutes a new epistemological element of lessons design based on "Operational Approach" for junior high school students.

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WHAT HAPPENS WHEN THE MATHEMATICAL MEANING IS DEVELOPING IN THE CLASSROOM?: From One Student's Perspective

Hiroshi IWASAKI (Joetsu University of Education)

ABSTRACT

The research focus of this study is to have a better understanding of what happens in the mathematics class, especially the relationships between the interactions and the mathematics learning from a student's perspective. For the study, qualitative, hermeneutical and microethnographical case studies are utilized.

The mathematics class is taken from an 8th grade mathematics classroom. The context is the teaching of basic triangle congruence theorems. The class interactions seem fruitful from a metaknowledge perspective.

In this paper, we report on the class interactions from the perspective of one student: Yama. He was one of the active students in the mathematics class interactions.

Yama developed his mathematics understanding in the interactive manner in the class interactions with taking advice from the teacher and the students, especially one of active participants in the class: Yoshi. Yama developed his mathematical meaning and also ways of participation in the interaction processes.

The mico-ethnographic analysis of the class interactions from one student Yama's perspective revealed that Yama was developing his mathematical understanding through participating in the class interactions with discovering, sharing and formulating. The lessons with this type of communication and interactions give rise to Yama's meta-learning which was characterized as "ways of doing mathematics" by him.

The analysis also revealed that this type of communication and interactions was realized and maintained by various and mutually related elements: Classroom micro-culture, Characteristic of the participants, Existence of the others of importance and Discussions with focusing on concrete examples.

BACKGROUND

Recent research concerning interactions in mathematics classroom suggests that a starting point for improving the everyday mathematics classroom is to have a better understanding of its unique cultural practice. (cf. Bauersfeld, H., Voigt, J., Kurmmheuer, 1988)

To gain a better understanding of mathematics classroom from students' perspectives is one of the effective research methods. It is a typical example that to try to understand the relationships between interaction and learning in the teaching process from one student viewpoint who participate in the class. (cf. Iwasaki, H., 2001; Nunokawa, K., 2002) Teachers could get cues for improving their lessons from the students' activities there. When we visit a class, we usually try to report the students' activities there and give the information to the teacher after the lesson. Especially, in the process of our lesson study we discuss the teaching method in the meeting after the lesson. The effectiveness and the limitations of the teaching methods are discussed frequently in light of the information of students' activities from the participants.

The focus of this research is to gain a better understanding of mathematics classroom from students' perspectives. In particular, it aims to analysis the relationships between the nature of teacher-student or student-student interactions and the students' mathematics learning. *The research methodologies used in this qualitative study were hermeneutic and micro-ethnographic case studies.*

We shall survey some of the related studies which give us view points to have better understanding of what happens in the class, especially the relationships between the interaction and the mathematical learning.

The students who seem to participate in the class interactions successfully might not learn what the teacher wants. Voigt (1998) illustrates that relationship between interaction and mathematics learning through "the direct mathematization", which is one of the typical pattern of interaction in mathematics classrooms. In that case, according to Voigt (1998), answering all the teacher's questions correctly, the students may produce the mathematically incorrect statement $\cdots$.(cf. p.209) More generally, Voigt (1995) states that "in fact, several studies analyzing discourse processes in detail have concluded that students' attempts to participate in the constitution of traditional patterns of interaction can be an obstacle for learning mathematics." Therefore "we should take account of that a smoothly proceeding classroom interactions might not be interpreted as an indication of successful learning."(Voigt 1995, p. 166)

Bauersfeld (1995) explains why the gap between students learning and one that teacher wants occurs in a classroom.

> The failure of the classical methods of teaching can be interpreted as a manifestation of this classical principle's failure: The teacher knows and teaches the truth, using language as a representing object and means. Because there is no simple transmission of meanings through language, the students all too often learn to say

by routine what they are expected to say in certain defined situations. (Bauersfeld 1995, pp. 275-276)

From students' perspective their learning how to participate in the class interaction seems to be compatible with their learning mathematics. The point is whether it is what the teacher wants or not.

The students who participate in the class interaction learn how to participate in the interaction. From the interactionist point of view, if the participants engaged themselves in mathematical practice, how to participate is mathematics itself. Sierpinska(1998) points out " If what the student learns as mathematics is a certain discourse, then his or her way of knowing mathematics is a function of the characteristics of the communication and interactions in which he or she participates in the process of learning. " (Sierpinska 1998, p.54) "if teachers and students engage in interactions of type A, then students are likely to develop ways of knowing and understanding of type $f(A)$. "Teachers who are not happy with having their students develop this type of understanding and knowing may have to change the way in which they interact with their students. (cf. Steinbring 1993) " (Sierpinska 1998, p.49)

If the classroom interactions proceed smoothly, the participants would act based on what they regard as natural, which could be called microculture. Microculture represents "a dynamic system that is continually being constituted." (Voigt 1998, p.208.) The individual construction of meaning is done in a microculture while at the same time it contributes to the constitution of microculture. Ethnomethodologist call this relationship between the individual construction of meaning and the constitution of microculture "reflexivity". (Leiter 1980)

It is important for having better understanding of the lessons that "this notion of reflexivity, $\cdots$, implies that neither an individual student's mathematical activity nor the classroom microculture can be adequately accounted for without considering the other." (Cobb & Bauersfeld 1995, p.9-10)

Teachers who are not happy with having their students develop this type of understanding and knowing may have to change the way in which they interact with their students. (cf. Steinbring 1993; Sierpinska 1998, p.49)

What is learned by students participating mathematics class interaction? It would be ways of participating in the interaction and mathematical ways of knowing. Cobb & Bauersfeld (1995) make clear the nature of mathematics learning through participating in the class interactions:

As it is with culture, the core of what is learned through participation is when to do what and how to do it.... The core part of school mathematics enculturation comes into effect on the meta-level and is "learned" indirectly. (Cobb & Bauersfeld 1995, p.9)

Further more, Bauersfeld (1993) describes in detail:

> [T]he understanding of learning and teaching mathematics... support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using mathematics, or better: a culture of mathematizing as a practice. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks for the building, but the design for the house of mathematizing is processed on another level. As it is with cultures, the core of what is learned through participation is when to do what and how to do it. Knowledge (in a narrow sense) will be for nothing once the user cannot identify the adequateness of a situation for use. Knowledge, also, will not be of much help, if the learner is unable to flexibly relate and transform the necessary elements of knowing into his/her actual situation. This is to say, the core effects as emerging from the participation in the culture of a mathematics classroom will appear on the metalevel mainly and are "learned" indirectly. (p. 4)

METHODOLOGY

Cooperative Teaching Practice Study

A cooperative teaching practice study aims at developing new design of mathematics lessons improving a present mathematics teaching, where researchers in mathematics education and practitioners in schools (teachers) work together to design the series of mathematics lessons.

- One of the most important characteristics of this study is found in it's own holistic and cyclic process, which is organized with 'epistemology of mathematics: meta-knowledge' as a kernel and connects 'a study in theoretical level' with 'one in practical level' systematically.

  Figure 1 represents the concept of the cooperative teaching practice study.

  The outside circle in the figure represents a study in practical level. The inside one , which makes a figure like a letter '8', represents a study in theoretical level.

- Developmental approach and interpretative one are systematically linked together in the cyclic process — this is another important characteristic of this study.

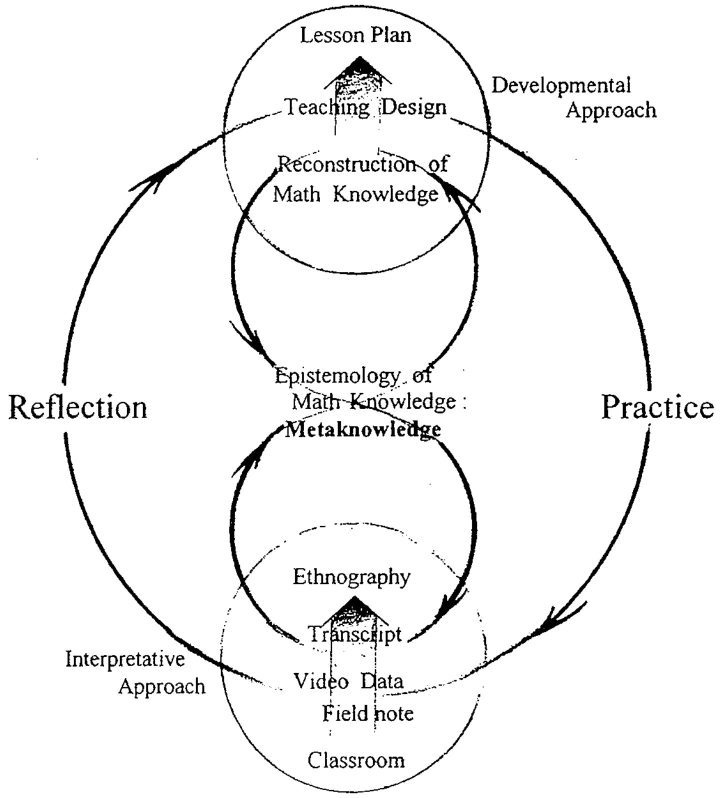

Figure 1: The concept of 'cooperative teaching practice study' in mathematics education

> It is promising that, as Kilpatrick (1992, p.31) has pointed out, research in mathematics education has gradually moved out of the library and the laboratory and into the school and classroom. More and more theory is becoming relevant to teaching practice and practice is getting the focus of theory. So the old and invalid dichotomy between theory and practice is disappearing. (Becker, J.P., Selter, C., 1996, p.550)

We regard the developmental approach as important because of including 'the execution of a thought experiment of teaching and learning both ' (Steefland 1993, p.116) . In this point, Wittmann (2000) described as follows:

> This new approach was formulated and published as a connected series of SLEs in a handbook for practicing skills (Wittmann & Müller 1990, Grade 1: chapters 1-3). It was based on a systematic epistemological analysis of arithmetic, on inspirations from the developmental research conducted at the Freudenthal Institute (cf., Treffers et al. 1989/1990, van den Heuvel-Panhuizen 1996) and on the intuitions of the designers. It was not based on empirical research conducted by

> professional researchers. Empirical studies which confirmed the holistic approach came only later (cf. Selter 1995, Hengartner 1999). So it was teachers who first tried it out in their practice and found that it works better than the traditional approach. (Wittmann, E. Ch., 2000, pp.12-13)

Further, we can list the followings as the characteristics of this study, the bases of which are compatible with the idea of 'kritisch-konstruktiver Erziehungswissenschaft' in Germany (Klafki, 1976).

- The teacher is a full member in this study.
  - The teacher is not a simple practitioner who puts into practice a teaching plan designed by the researchers precisely. He/She is, rather, expected to present a new design or modify it from a practical point of view. Further more, he/she may modify or change the teaching plan according to the students' replies during the on-going teaching practice.
- The teaching in this study is not merely a 'practice' rather an 'experiment'.
- The process of this study itself should contribute for improving the present class.
  - The researcher present some view points, ideas and plans for improving present mathematics education in general, however, this is not a start point of this study.
  - The teacher confirms those in light of his/her experience in his/her present class. If we could share the importance of those, it would be the start point of the study, and the extent of importance for him/her would determine to what extent the study should be done. The extent of the study could be changed; however, it should be done according to whether or not it would be benefit for the students in the present class.
- The researchers' major role is to report what's happened in the class and to explain it for teachers.

Data

To the purpose of this study, four mathematics lessons were taken from an 8th grade mathematics classroom. The context was the teaching of basic triangle congruence theorems. Five students were interviewed after the lessons.

The data were collected in the course of the cooperative teaching practice study described in detail above. Four lessons were conducted in an 8th grade class on June 9, 12, 13, and 16, 1995. The content focused on introducing triangle congruence theorems. All lessons were recorded with two video cameras in the front and rear of the classroom. The front camera was used to record the whole-class discussion and students' behavior and the rear camera was used to record the teacher's behavior. Transcripts of the lessons were prepared from these data and retranscriptions were done from the following point of view:

> When making the actual analysis one cannot simply use such a step-by-step procedure, $\cdots$one has to analyze carefully the epistemological status of the mathematical knowledge from phase to phase $\cdots$to explore the development and shifts of knowledge interpretation and understanding in the classroom interaction. The method of analysis is a dialectical one reflecting global and local aspects simultaneously. (Steinbring, 1993, p.38)

Several of the students were interviewed on September 22, 1995. The interviews (the interviewers - six students) held after school in a separate room. The interview was done clinically so that we could have a better assessment of the students' understanding of the lessons. The interviewers were the author and the teacher. The interviewers asked the students to talk about their thought to the lessons showing them the video-recorded their activity in the lessons, which was edited briefly but as including the main situations of the lessons. The interview was recorded with a video camera. Transcripts of the interview were prepared from the data.

This paper reports the class interactions from the perspective of one student, Yama, who was one of the active students in the mathematics class. Yama developed his mathematical understanding in the class interactions by taking advice from the teacher and the other students, especially one of the active participants in the class: Yoshi. Yama developed his mathematical understanding and ways of participating in class through the interaction processes.

In the interview, when he was asked to what he had learned in the lessons, he said "ways of doing mathematics". In the perspectives on enculturation school mathematics, it was considered to be his general comments on what he had learned "indirectly" through participating the lessons. We are very interested in the characteristics of the communication and interactions where he participated in the process of the lessons. What caused him to say that? What happened in the classes?

ANALYSIS AND DISCUSSION

The Lessons

The lessons consisted of two main phases, Phase I and Phase II. In Phase I, students were able to use their intuitive ideas of triangle congruence in order to justify some geometrical relationships but they were not able to explain logically why the two triangles were congruent to each other. The teacher therefore tried to make the students notice that they had intuitively used the idea of congruence and had to use it more logically. In Phase II, to concretely shape the problem, the teacher formulated it as the present task: "*There are two triangles ABC and $A'B'C'$ where $AB = 5.6cm$, $BC = 5cm$, $CA = 4.1cm$, $\angle A = 60°$, $\angle B = 45°$, $\angle C = 75°$ and $A'B'C'$ are unknown. What conditions must be satisfied for the two triangles to be congruent to each other?*" (see Fig. 2)

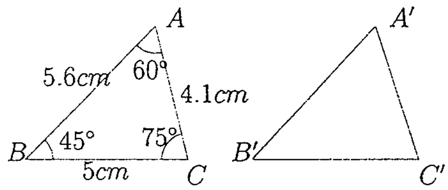

Figure 2: The present task

First of all, the teacher asked "In the case of one condition, for example, $AB = A'B' = 5.6cm$, are these triangles congruent to each other?" All of students said "No!" He asked "Why?" One of the students explained "There are many $A'B'C'$s which satisfy the condition. Point A' can move anywhere!" The teacher then asked "In the case of two conditions, for example, $AB = A'B' = 5.6cm, BC = B'C' = 5cm$, are these triangles congruent to each other?" All of the students said "Of course, no!" He asked again "Why?" One of the students explained "There are many $A'B'C'$s that satisfy these conditions. Point A' can move on a circle with the radius 5.6cm centered at B'." (see Fig 3)

In this way the teacher started making the students use construction as *a cognitive tool* to confirm whether the set of conditions could lead to the congruence theorem or not. He next asked "What do you want for $A'B'C'$or point A'?" Some of the students explained "We must have only one $A'B'C'$or point A'." In this way, students in the class came to share the idea that "if you can construct only one triangle when you construct it under certain conditions then the conditions must constitute a congruence theorem."

The teacher suggested that the students enumerate the conditions one by one according to the number of conditions. In the case of one condition, there are 6 sets of conditions. It was trivial for the students realize that all of these could not be congruence theorems. In the case of two conditions, there are 15 sets of conditions. This was also

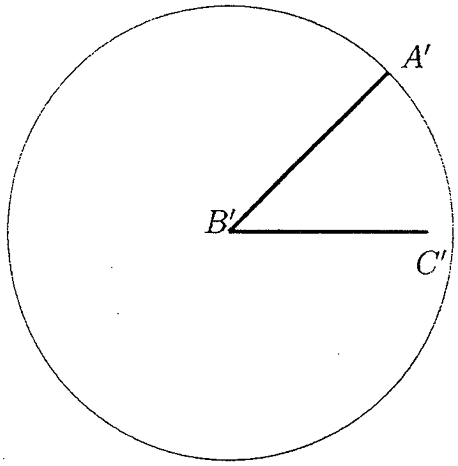

Figure 3: $AB = A'B' = 5.6cm, BC = B'C' = 5cm$

trivial for the students.

Surprising the Students

In the case of the three conditions, there are four cases: three sides (SSS), three angles (AAA), two sides and one angle (SSA), one side and two angles (SAA). First of all, the set of conditions "three sides (SSS)" was investigated. From the students' expressions, we could safely conjecture that they did not consider it necessary to investigate this, but the teacher offered to. Students accepted the proposal somewhat unwillingly. The teacher constructed a figure which satisfying the conditions ($A'B' = 5.6cm$, $B'C' = 5cm$, $A'C' = 4.1cm$) on the blackboard. The students helped him by voicing their opinions. Note the teacher's positive interventions in the interchange.

Students naturally accepted the figure which had two intersection points of two circles (see, Fig. 4). However, it was surprising that there were, as a natural result of the construction, two A' above and below side $B'C'$. The students had expected only one A' above the side $B'C'$.

[Situation 1]

*1 **T:** How many points do we have?

*2 **S:** One! One!

*3 **T:** Eh?

*4 **S:** (in haste)Oh! Two!

*5 **T:** One?

*6 **S:** Oh! Two! Two!

*7 **T:** What's this? (pointing at the point above)

*8 **S:** A'.

*9 **T:** A'. What's this?(pointing at a point below)

*10 **S:** A'.

*11 **T:** We have constructed it based on the conditions.(encircling the condition written on the blackboard with his yellow chalk)

*12 **S:** Yes.

*13 **S:** Oh! I see!

*14 **T:** Including the point (below), we have two points!

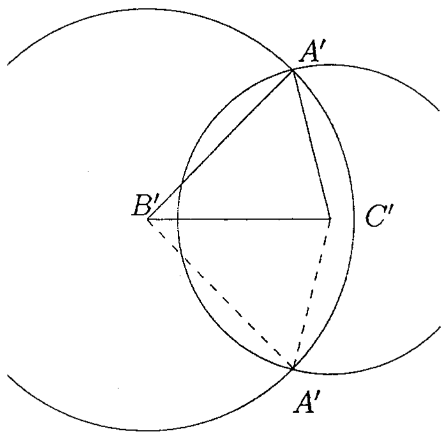

Figure 4: How many points?

[Situation 2]

*1 **T:** Oh?!, in that case, is this OK?
(drawing the side $A'B'$and $A'C'$above)

*2 **S:** A' hasn't been fixed!

*3 **S:** It seems to be right!

*4 **T:** So, here, we may include $\cdots$(drawing the side $A'B'$and $A'C'$below), Is this OK?

*5 **Yama:** Oh! I see what you mean!

*6 **Kusa:** Isn't that strange?

*7 **T:** Eh! Isn't that strange? Do you have only one (triangle)? Only one?

*8 **S:** It's not definite!

[Situation 2] shows that the students had not seen the other possible triangle $A'B'C'$ below the side $B'C'$although they have recognized two points which satisfy the conditions.

Namely, they tried to construct the same triangle as the triangle ABC. With this as a turning point, the students negotiated how to confirm whether the given conditions are congruence theorem or not. As a result, they changed their way of confirmation from "if you can construct *only one triangle* under a given set of conditions, the set of conditions must be a triangle congruence theorem" to "if you can have *only one sort of triangle* when you construct all triangles which satisfy a given set of conditions, the set of conditions must be a congruence theorem".

This suggests their growing awareness of "constructing all triangles which satisfy a given set of conditions" rather than "constructing the same triangle as triangle ABC, which satisfies a given set of conditions." Thus their manner of inquiry with construction, initiated by the teacher, became part of their cognitive tools for investigating congruence theorems. Finally, developing their cognitive tools enabled them to discover a new relation as described in the next section.

We will take up main phases in the lessons where Yama seems to have developed his mathematical understanding: Discovery, Sharing and Formulation.

Yama's Discovery

Yama made a discovery in the case of two sides and one angle (there are 9 sets of conditions, see Fig. 5), when the students were investigating the conditions through construction individually. Yama discovered that there is a case where you can not have only one sort of triangle when you construct all triangles which satisfy that conditions.

When Yama discovered that case, he shouted, "Say! ". Then he talked to his neighbor in a loud voice, "Try next to the center." The student devoted herself in her construction activity. She didn't respond to his suggestion. He shouted again looking at his notebook, "Something's wrong! Oh no! Hey!" The teacher asked the students, "If it is difficult for you to construct a figure which satisfies the conditions, you should draw the side which has the angle first. If you do so $\cdots$." Then Yama immediately said to him, "Is it difficult, isn't it? If anything, does it have two solutions? "

Then a student Yoshi who is behind Yama's seat said to him, "It is easy for me." He said to her , "It may be easy. But it has two solutions." showing his notebook. The interactions at that time was as follows:

[Situation 3]

*1 **Yama:** Extend the line with keeping 45 degree, then it has two there.

*2 **Yoshi:** 45 degree? Really?

*3 **Yama:** I can guarantee it is 45 degree.

*4 **Yoshi:** It seems to be 30 degree $\cdots$. Is it exactly true?

(1) $A'B' = 5.6cm$ $A'C' = 4.1cm$ $\angle \mathrm{A}' = 60°$	(2) $A'B' = 5.6cm$ $B'C' = 5cm$ $\angle \mathrm{A}' = 60°$	(3) $A'C' = 4.1cm$ $B'C' = 5cm$ $\angle \mathrm{A}' = 60°$
(4) $A'B' = 5.6cm$ $A'C' = 4.1cm$ $\angle \mathrm{B}' = 45°$	(5) $A'B' = 5.6cm$ $B'C' = 5cm$ $\angle \mathrm{B}' = 45°$	(6) $A'C' = 4.1cm$ $B'C' = 5cm$ $\angle \mathrm{B}' = 45°$
(7) $A'B' = 5.6cm$ $A'C' = 4.1cm$ $\angle \mathrm{C}' = 75°$	(8) $A'B' = 5.6cm$ $B'C' = 5cm$ $\angle \mathrm{C}' = 75°$	(9) $A'C' = 4.1cm$ $B'C' = 5cm$ $\angle \mathrm{C}' = 75°$

Figure 5: 2 sides and 1 angle: nine sets of conditions

*5 **Yama:** These are all 45 degree. [He is confirming these with his semicircular.] Oh no! Wait a moment.

*6 **Yoshi:** Is it 30 degree, isn't it?

*7 **Yama:** It must be 45 degree.[showing the consequences on his notebook with his semicircular on his notebook]

*8 **Yoshi:** It must be true.

*9 **Yama:** It is true, isn't it?

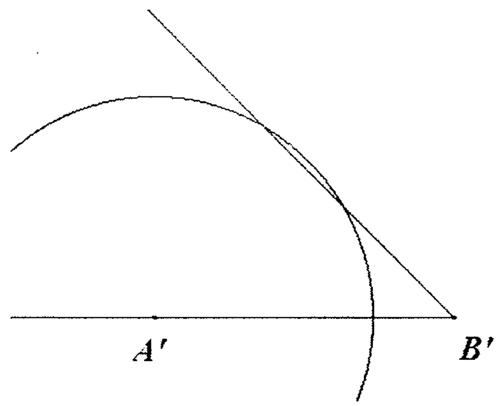

Figure 6: Yama's construction on his note

After the interaction, Yama said to the teacher with complete self-confidence, "It's not difficult, teacher! There are two solutions. It goes beyond the level of whether it is difficult or not!"

This Yama's utterances with complete self-confidence is a contrast to his utterances done before the interaction with Yoshi: "Is it difficult, isn't it ? If anything, does it have two solution?"

As the result of the social interactions with Yoshi, Yama have got more confident understanding of the mathematical relation which he discovered through his construction activity individually. Yoshi also have recognized the mathematical relation which is new to her.

Sharing the Discovery

[Situation 4]

*1 **Yama:** Yes, let me see, I will take the angle A first and make the sides expand. I draw the angle A first of all.

*2 **T:** Oh! Where do you take the base line?

*3 **Yama:** Nothing! I draw the angle A without the base line.

*4 **T:** How do you take angle A without the base line ?

*5 **Yoshi:** Draw the AB !

*6 **Yama:** Eh, all right, then draw the AB please.

*7 **T:** AB, AB. Is this right, isn't it?

Yama was trying to present his own ways of construction, (*1) however, the teacher didn't accept. (*2-*4) Yoshi then advised him. (*5) He hesitated, however, decided to accept her suggestion at once. (*6) This new proposal was accepted by the teacher.(*7) Accepting Yoshi's suggestion means that he has to reconstruct his own ways of construction he had done before. But he decided to accept her suggestion at once. He has 'flexibility'. After all, the teacher was demonstrating Yama's construction process on the blackboard according to his instruction. He could demonstrate his discovery to the class in this way and could contribute the class interaction. In the interview, he said that "announcing itself makes me happy". This is one reason why he makes him meet the teacher's request. As a result of this, he is trusted by the teacher and get many opportunities for presentation in the class.

In the nine conditions the students distinguished three conditions whose shape was SAS. It was trivial for the students realize that these conditions were all applicable and could be congruence theorems. Then the others (six sets of conditions) the other six conditions were investigated.

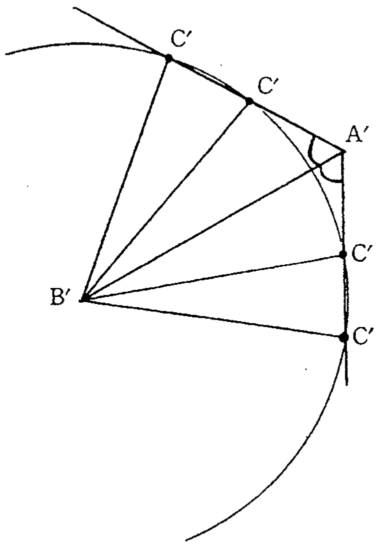

Figure 7: Demonstration of Yama's discovery

Yama raised his hand to try to explain why these conditions were applicable in some cases and not in other cases:

[Situation 5]

*1 **T:** What are you going to state, Yama?

*2 **Yama:** Yes. I tried to say at that time, well, let me see, er ··· then, I will take the center of the top.

*3 **T:** OK!

In this situation Yama decided the figure for his explanation; however, it had not been decided beforehand. "Thinking with announcing" rather than "announcing after thinking" represents his participation style. The interview with the students revealed that the "thinking with announcing" activity was also shared with some students in the classroom and was considered by them to be a natural process. The "thinking with announcing" represented the emergence of one of the classroom micro-culture. (See [Appendix] Interview)

Yama began to explain the reason why these conditions were applicable in some cases and not in other cases as follows:

[Situation 6]

*1 **Yama:** If the point A is longer upward, then it also met [the circle] at two points.

*2 **T:** (Putting his head little to one side, he asks to all students.) Do you understand?

*3 **Yoshi:** (She talks to Yama.) If $C'B'$ is shorter, isn't it ?

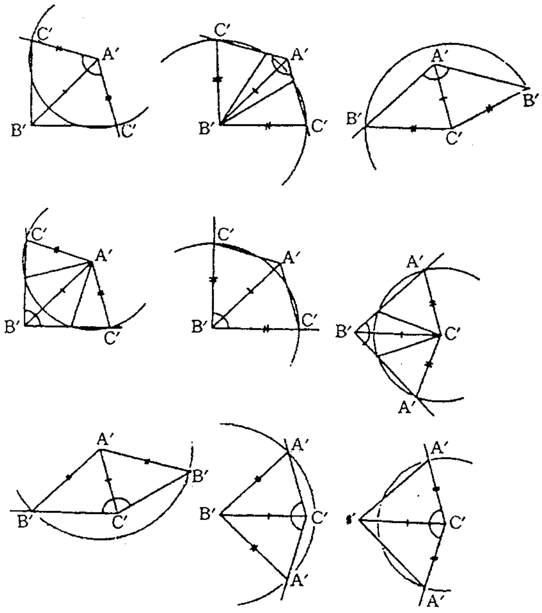

Figure 8: Nine constructions

*4 **Yoshi:** (She talks to Yama again.) If $C'B'$ is shorter.

*5 **Yama:** That's right! It might be more rough and ready.

*6 **Yama:** Could you construct once more with compasses when $C'B'$ is shorter ?

Yama began to explain. (*1) But his explanation was not accepted by the teacher. (*2) Then Yoshi who is his classmate gave him a productive suggestion: "If $C'B'$ is shorter". (*3-*4) Yama realized that her suggestion was convenient for his explanation, (*5) and he applied it immediately. (*6)

In this situation you can find Yama's flexibility again. Because he determined adopting the other student's suggestion, while at the same time he was trying to reconstruct his explanation.

[Situation 7]

*1 **T:** Shorter? The condition $C'B'$ is shorter? How long is it? Is this OK? (The teacher is setting his large compasses on the blackboard.)

*2 **Yama:** Make it longer a little.

*3 **T:** Make it longer a little? Is this OK? (The teacher is setting his large compasses on the blackboard again.)

*4 **Yama:** More, More!

*5 **T:** More ?

*6 **Yama:** I feel it is just. Then Please draw it once.

*7 **T:** (The teacher begins drawing.)

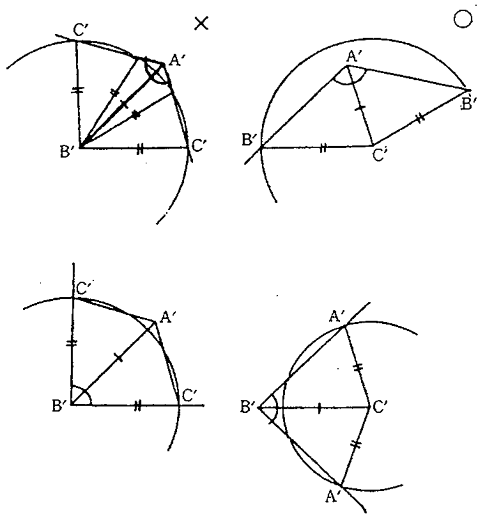

Figure 9: Applicable in some cases and not in other cases

*8 **Yama:** It might be short! Make it longer!

*9 **T:** Make it longer ? It's troublesome, isn't it? Is this OK? (The teacher is setting his large compasses on the blackboard again.)

*10 **Yama:** I think it would be just. (The teacher is setting his large compasses on the blackboard again.) Oh no! It might end in failure, $\cdots$ or all right ?

*11 **S:** It will go well.

*12 **S:** It will not go well.

*13 **Yama:** It has not done well. Teacher! shorter!

*14 **T:** Shorter ?

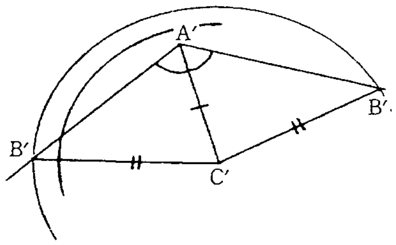

Figure 10: Make it longer a little.

On the one hand, Yama realized that if you make $B'C'$shorter, the circle with the radius $B'C'$ will meet the side $A'B'$ at two points. And he had realized the reason why the conditions were applicable in some cases and not in other cases, that is, he realized the reason depends on the length $B'C'$. On the other hand, his approach here seems to be a trial and error method. As a result of that, his attempts had all met with failure.

However, the interaction with the teacher made him a bright idea as follows.

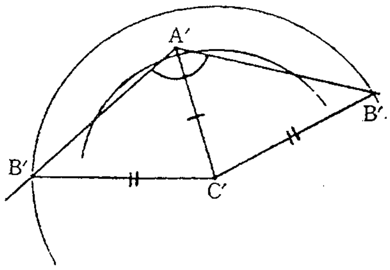

Figure 11: Please put it on A, teacher!

[Situation 8]

*1 **Yama:** Please put it on A, teacher!

*2 **T:** A ? A'? Here? (The teacher is setting his large compasses on the blackboard.)

*3 **Yama:** So then make it a little shorter.

*4 **T:** (The teacher is drawing on the blackboard by using his large compasses.)

*5 **Yama:** Then you can get two.

*6 **T:** (With nodding) Yes. You can get two in this way.

His attempt here is a complete contrast to one before. It is not a trial and error method but rather an intentional one. As mentioned above he had realized the reason depends on the length $B'C'$ already. This change to the intentional approach shows his growing mathematical understanding and the same time that he has realized it depends on not only the length $B'C'$ but also the length $A'C'$, namely, the relationships between the two length $B'C'$ and $A'C'$. It is essential for SsA congruence theorem which will be formulated by the students.

Yama explained that relationships in a concrete figure as follows.

[Situation 9]

*1 **T:** Is this figure ? OK? (The teacher is pointing the construction at the upper right in Fig. 8)

*2 **Yama:** Yes. If the length $C'B'$ is longer than $A'C'$,

*3 **T:** If longer ?

*4 **Yama:** It isn't be so, but If [the length $C'B'$] is shorter than $A'C'$, then you should have two.

*5 **T:** You should have two.

Formulating

Further the teacher called on that Yama make a clearer explanation by using two concrete constructions on the blackboard. (See Fig.9) Yama had selected the construction at the upper right, then he selected one just below for comparing. As you will find the following protocol, he developed his mathematical understanding through trying to meet the teacher's request.

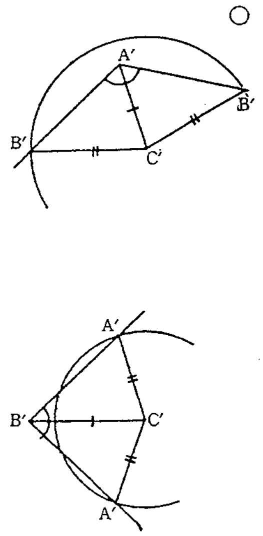

Figure 12: If comparing with this what ?

[Situation 10]

*1 **Yama:** One below.

*2 **T:** Below ? This? If comparing with this what ?

*3 **Yama:** Is the length of $B'C'$ longer than $A'C'$, isn't it ?

*4 **T:** $\cdots$ than $A'C'$?

*5 **Yama:** Oh, I see! Saying like that might be inadequate.

*6 **T:** Eh! What do you want to say ?

*7 **Yama:** Is there a center of an angle, isn't it? In the upper right [Fig.12], is A' the center, isn't it?

*8 **T:** Yes.

*9 **Yama:** The distance, $\cdots$ to the center of an angle, and,

*10 **T:** The distance from there ?

*11 **Yama:** When the other distance or distance of $B'C'$ is longer [than $A'C'$] , you will get one, when shorter, you should have two.

*12 **T:** You should have two, Yes.

*13 **Yama:** It is being so, isn't it ?

*14 **T:** It is being so, yes.

Meeting the teacher's request, Yama selected the construction below.(*1-*2) He represented a characteristic of that by using letters in it. 'Is the length of $B'C'$ longer than $A'C'$'(*3-*4) Next, when he tried to compare to the above, he became aware of that he could not compare by using letters in it. 'Oh, I see! Saying like that might be inadequate.' (*5) Characterizing the difference between them, he introduced a term 'a center of an angle' by giving an example: 'Is the upper right [Fig.12], A' the center, isn't it? (*7-*8)' And he represented one side in the conditions as 'the distance to the center of an angle'.(*9) Next he represented the other side of the conditions by using a term 'the other distance'. Then by using these terms he succeeded to characterize the difference beyond the concrete figures generally, (*9-*11) and to explain the reason why more general and explicitly. (*12-*14) His growing understanding was the result of the interaction where he was trying to meet the teacher's request. He was finding the representation to explain the reason by using the two concrete constructions that he had selected.

The teacher encouraged students to formulate the set of conditions. They formulated it as follows: S-S-A and "the side with an angle $\leqq$ the side with a side."

CONCLUSION

The mico-ethnographic analysis of the class interactions from one student Yama's perspective revealed that Yama was developing his mathematical understanding through participating in the class interactions with discovering, sharing and formulating.

This pattern of the lesson process represents a typical mathematical activity which compatible with social constructivist view of mathematics which embodied by "a creative cycle" where the relationship between objective and subjective knowledge of mathematics is realized. (Ernest, 1991, p.85)

The lessons with this type of communication and interactions give rise to Yama's meta-learning* which was characterized as "ways of doing mathematics" by him in the interview. [See Appendix:*7-*12] .

This fact gives evidence which empirically illustrates the interactionist view of mathematics learning, that is, "his or her way of knowing mathematics is a function of the characteristics of the communication and interactions in which he or she participates in the process of learning. " (Sierpinska 1998, p.54)

*Mellin-Olsen, S., *The Politics of Mathematics Education*, D. Reidel Publishing Company, Dordrecht, 1987.

It also suggests that to realize the class interactions with discovering, sharing and formulating is important for students' mathematics learning. The documents of the lessons described above give one of the typical reconstructions of the lessons which illustrate the mathematical communication and interactions which compatible with social constructivist view of mathematics.

The analysis also revealed that this type of communication and interactions was realized and maintained by various and mutually related elements: Classroom micro-culture, Characteristic of the participants, Existence of the others of importance and Discussions with focusing on concrete examples.

Classroom micro-culture

Yama's "thinking with announcing" acts contributed not only to developing his mathematical understanding through his participation in the class interactions but also to developing the fruitful interaction itself. The interview with the students revealed that the "thinking with announcing" was shared with the students in the classroom and was natural for them. The "thinking with announcing" represented the emergence of a classroom micro-culture. From the interactionist point of view, it is likely that the classroom micro-culture and his mathematical knowing were developed interactively.

Characteristic of the participants

Yama's 'flexibility', his attitude of meeting the teacher's request and his reactions contributed to realize the interactions as described above. (cf. Lo, J-J. *et al. 1994*)

Existence of the others of importance

It was very important for him to interact with another active student: Yoshi, who played the role of his adviser. They were learning from each other. Their fruitful relationship developed since they were able to complete the same task interactively in the process of the class interactions. Yama developed his mathematical understanding by reacting to Yoshi's suggestions. In this sense, the mathematical understanding itself that he developed was the result of the interactions. The fact that *their fruitful human relationship influenced the mathematically rich interactions* is worthy of attention.

Discussions with focusing on concrete examples

As showed in the [Situation 9, 10] typically, focusing on concrete example contributed to realize the interactions.

Then what is focused on is important for the fruitful interaction in mathematical point of view. Because [Situation 10] shows that comparing the *two* concrete constructions and characterizing the difference made Yama's mathematical understanding more general one. Indeed, by using these concrete examples and trying to justify the relationships he had discovered before and reflecting on it made his mathematical understanding. (cf. Steinbring 2000)

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[Appendix: Interview]

Interviewer asked the impression of the lessons to the students end of the interview.

*1 **Naka:** I coud understand the process untill coming to the conclusion.

*2 **Ko:** If you know the results only, you can't explain why. It seems to be great that we become to be able to explain why.

*3 **Yama:** I was knowing with announcing.

*4 **Naka :** I may have been raising my hand although I had understood by only half.

*5 **Yama:** Yes, Yes!

*6 **Ko:** I also feel so. Raising a hand and announcing what I understand by only half, then I am coming to know gradually what I had thought.

...

Interviewer asked the impression of the lessons to the students more concretely.

*7 **Int.:** *cdots* As Ko said at that time, if the lessons finished within five minutes, and I began the [next] lessons with saying "let's start the content 'proof' ", then ,you could compare them, couldn't you, what do you think?

*8 **Yoshi:** Finally, same as before the lessons.

*9 **Naka:** Same with respect to the amount of knowledge.

*10 **Yama:** No interesting!

*11 **Ko:** The amount of knowledge didn't change, but I seemed to see the process up to there.

*12 **Yama:** The ways of doing mathematics.

...

SEMANTICS OF 2D/3D FIGURE[1]

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ABSTRACT

The main purposes of this paper are to understand a student's knowledge system in the 2D/3D figure learning. It needs the support of natural languages for learners to interpret mathematics languages and understand mathematical concepts, and they construct their Inner mathematics which has an opened system, from the viewpoint of cognitive linguistics. Especially, meaning of 2D/3D figure learners construct has multi-aspect, influenced by Gestalt. So, according to the context, meaning of set of points is made as segment, polygon, or chord of a circle.

1. INTRODUCTION

The main purposes of this paper are to understand a student's knowledge system in a figure and space learning. While a knowledge system is researched from various approaches (philosophical, psychological, sociological, etc.), from a viewpoint in which there is the *Inner mathematics* constructed by students and the *External mathematics* by teachers or textbooks, I might describe the knowledge system as the *Inner mathematics* with student's language activities by the spoken language or a written language, drawing activities, etc. from the cognitive linguistics perspective in this paper.

The learner's present knowledge system has been constructed by coordinating learning experiences in school with ones in everyday life through reexaminating, revaluing, and making sense of it. In the mathematics classroom under the view of mathematics as a closed system (*External mathematics*), preferable knowledge is decided by the teacher and the textbook, and most of informal ways of thinking by learners are ignored or integrated into a preferable way of thinking. However, taking into consideration the nature of learner's knowledge system, what is called an *Inner mathematics* learners construct by themselves has an opened system and is not necessarily constructed in the mode assumed by teachers, even though a series of lessons with the intention with the teacher and the textbook are performed. When

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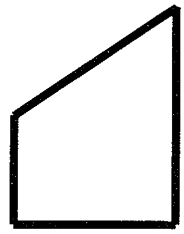
learners encounter a certain problem situation, the difference of how learners interpret and solve it depends on the difference of their knowledge system. So, it is important for teachers and researchers to understand the nature of learner's knowledge system.

2. ABOUT ISSUES IN THIS PAPER

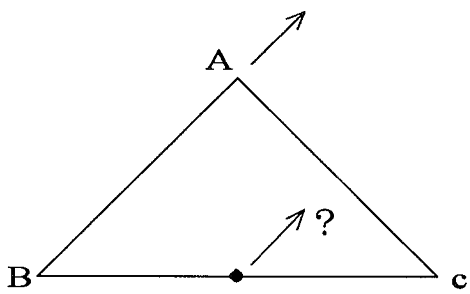
Below, this paper is advanced according to the following issues for me.

By intentional lessons with teachers or textbooks, students don't necessarily construct a knowledge system as intended.



We can see frequently such cases in elementary and junior high school. I think that there are *cognitive* and *conceptual levels* in them.

For example, in the former, in the classification of figures, while it is essential for figures to consist of straight lines and teachers and textbooks abstract the length and direction of them, children realize that a straight line is a concrete object which has the horizontal and perpendicular direction. Moreover, the "quadrangle" for children has referred to rectangular. Therefore, the quadrangle as shown in the above is not a quadrangle for children because "it is oblique!!". "Things like a quadrangle" in the environment which surrounds children is a television, a computer screen, a window frame, and a door, and cognitive experiences had by children have restricted the construction of concept which teachers expect.



△ ABC is a right-angled isosceles triangle. How does the circumcenter move according to the moving of point A?

In the latter, in the geometrical problem how the circumcenter moves when the right-angled point is moved after understanding that the circumcenter of a triangle is an intersection of the perpendicular bisector of three sides and, in a right-angled isosceles, it becomes the middle point of an oblique side, there is some students who are not permitted the motion on the concept although they accept that the circumcenter is defined by the construction. That is, they cannot control effects of cognition by thinking conceptually.

Both cases are influenced by cognitive and everyday experiences including a school life. On the other hand, I often feel the peculiarity of a classroom. As an example in function learning, in the problem "there is 160g paper by 50 sheets. How many sheets are they in the case of 1200g?", when calculating the amount of whole from quantity per one, students answer correctly by proportionality calculation, without feeling the "unnaturalness" of thinking the number of sheets per 1g. In the context of function learning, they adopt the algorithm for problem solving only to apply the quantity in the problem to proportional expression $y = ax$. In figure learning which treats the symbol which reminds the relation with reality in contrast with the algorithm act by such formula operation, the influence which cognition has on a concept becomes strong in many cases rather than a concept controls awareness. Cognitive influence has been ever recognized in figure teaching. Is it dependent simply on cognitive views? Does the cognitive influence have the same quality in all the grades and is it explained as Gestalt?

In mathematics learning, everyday languages often become the activator to realize mathematical concepts

One of the purposes of mathematics learning is to construct mathematical concepts through various mathematical languages and signs. Mathematics lessons are performed by only neither a mathematical language nor the sign. Teachers and textbooks have to define a mathematical language with everyday languages, and a communication between a teacher and students, students and students is developed using the defined mathematical and everyday language. I have supposed the relation of the everyday and mathematical language like that with the following diagram.

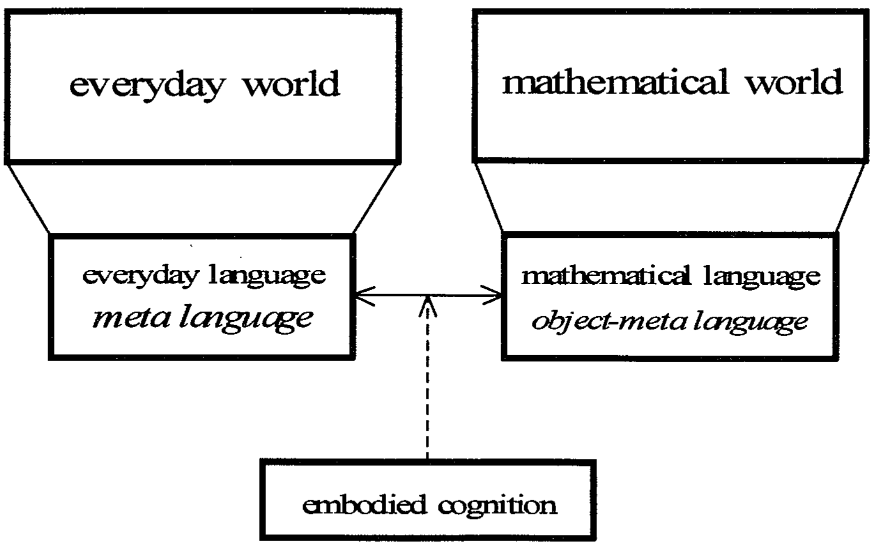
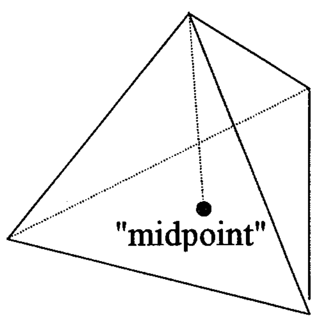


Diagram. The relation of the everyday and mathematical language

From the perspective of cognitive linguistics, languages are grounded in the bodily experiences (spatial cognition, the senses, motor senses, etc.), and either the literal interpretation for them or construction of mathematics are concerned with embodied cognition, for the mathematical languages are understood by everyday languages as **meta**-language, as Lakoff and his colleagues argued in their writing "*Where Mathematics Comes From?-How the Embodied Mind Brings Mathematics into Being-*" in 2000. Taking into consideration this mentioned above and the fact the mathematical world (mathematical conceptual field) is conceived through mathematical languages as sign, everyday languages and bodily experiences have the possibility to open the mathematical world.

Actually, learning of figure and space is performed not only with representations but also with various verbal and linguistic representations. The situation that latter representations (everyday and mathematical) support learning of 2D figure is not few either. This case is often observed when teachers and learners think of 3D figure. In these cases, everyday language functions as a *label* to symbolize the concept and as an *activator* to "bring out" naive concepts with reality. On early step of learning of figure, to search and present some concrete objects similar to abstract 2D/3D figural concepts is to make use of a label for naïve concepts leading to abstraction. And similarly, when learners call a perpendicular foot of positive tetrahedral "a midpoint" in everyday expression, they use this language as a label for it. If this label activates the other learner's concept images of the center of gravity, this is an activator, too.

The mathematical inquiry of the perpendicular foot is performed through constructing a perpendicular bisector of triangle. In that case "midpoint" may function effectively, or cause an incorrect concept. It is the former case to consider the intersection to be the center of gravity consisted of segments which are pulled in the "middle point" ("midpoint") of the opposite neighborhood from each vertex, and is the latter to consider the "middle point" ("midpoint") of the segment to be the center of gravity consisted of a segment pulled to the opposite neighborhood from one vertex.



3. COGNITIVE LINGUISTICS APPROACH AND MATHEMATICS LEARNING

About the Cognitive Linguistics

When as mentioned above, it becomes very various how to understand mathematics by each learner and community. Such a similar situation has been pointed out in linguistics. We cannot explain an actual language phenomenon from the standpoint of the system of sign that consists of rule and principle. To such a problem, after 1970's, some researches from the standpoint that these regard language as a system which is opened, and various rules are composed by language users by a bottom-up approach have come to be performed. These are called as **cognitive linguistics** in general, which stands at the following basic standpoints:

(a) Language is reflected by cognitive factors related to the interpretation of the external world, categorization and making sense of it by the subject.
(b) Language is regarded as having an opened system. The rule in this system is regarded as a part of the pattern (schema) that appears from the situation of actual language use. So, the rule itself is going to be transformed according to various situations.
(c) The research program stands at a standpoint of categorization from the theory of *prototype*.
(d) From semiotics, the research program takes an approach researching the relation between form and meaning of sign from the pre-linguistics cognitive ability enabling appearance of sign.

Even in the case of transmitting same situations, cognitive subjects choose different language forms according to the projection of the point of view by themselves and how to take a perspective. Therefore, by understanding the difference of language representation and mode of signification we are able to understand concretely how cognitive subjects make sense of a surrounding and interpret the external world.

When we see mathematics learning from a cognitive linguistics perspective, it will be said that mathematics has the opened system and signs and rules in the system may change flexibly by the user. This resembles the situation that just modern mathematics develops, and corresponds to the changing or expanding existing and new conceptual contents by prior means and idea. And as mentioned above, in the theory of *prototype*[2] that the adjacent field of class is ambiguous, even if definition of quadrangle is given by top-down or bottom-up approach, distinction of being a quadrangle or not by students is very ambiguous under cognitive influence or by effect made sense subjectively. For learners who cannot recognize the logicalness[3] which

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[2] To be concerned with this, I think that there are two roles of the definition of figure. On the one hand, mathematically, definition is the foundation of demonstrations and given by top-down approach. On the other hand, in initial learning of figure, by considering figures consisted of lines, definition has the role to expand and classify the world for inquiring figures by bottom-up approach.

[3] As mentioned above, all the attributes that are not essential are abstracted from mathematical language such as definition. Therefore, students cannot but make a judgment of the matter which is not expressed clearly by themselves. In definition of triangular, the

rigorous language includes implicitly, it may be a practical way of concept formation to strengthen correct class of quadrangle with a teacher or a textbook by showing many positive examples.

Cognitive Semantics of 2D/3D Figure

Generally, in linguistics, syntax and pragmatics other than semantics have been researched. In these disciplines, researchers are interested in the rule and the text syntax which constitutes the symbol system as language, and the way how they are used in an actual situation, while in semantics studying the meaning of sign itself and of the syntax made by it. If we view a sign including language as "something representing other things" broadly, all representations used in mathematics learning (letter, equation, graph, ···) are viewed as a kind of sign. This is similar about representations of 2D/3D figure, and these (symbolic) representations have some certain meanings.

What kind of system is considered as the system of the figure from a viewpoint of linguistics? What kind of rules and syntax are there in the system and how does it be used at in real situation?

Table. Comparison between figure and number from the viewpoint of concept, operation, and relation

	2D/3D figure	**Number**
Concept	Point, Line, Surface, Triangle, Quadrangle, Space figure	Integer, Decimal, Fraction Negative number Imaginary number
Operation	Shift, Turn (Parallel, Rotation, Congruent transformation) Turn over (Symmetrical transformation) Expansion, Reduction (Similar transformation) Equivalent transformation	Addition, Subtraction, Multiplication, Division
Relation	Spatial relationship (Parallel, Perpendicular, Twisted position) Inclusion relation (*logical*) Equality (Congruence, Similarity) Expansion, Reduction (Similarity)	Quantity (Equality) Proportional relation Divisor, Multiple

student has to realize it referring to only the shape, judge that the figure whose number of a straight line is four is not a triangle, and, on the other hand, judge that regardless of the length of segments and the size of angles they are a class of triangle.

To understand issues above, it is as follows when it makes a table (in the last page) contrastively from the viewpoint of concept, operation, and relation among the contents of learning in the figure and the number area focusing to the meaning (concept) of the sign itself and the relation (operation and relation) between signs.

Each concept is mathematical concept, and they are conceived only through certain representations. I would like to say that I am interested in the student's *Inner mathematics* in this paper, although many thinkers have had the philosophical and epistemological argument from ancient times as "Whether is concept referred to through representation a transcendent or a constructive entity?".

In an above table, it turns out that representation which represents each concept corresponds to each language in linguistics, and there are the operation (*dynamic*) and the relation (*static*) to those practical rules. We can point out in general about the feature of each area as following:

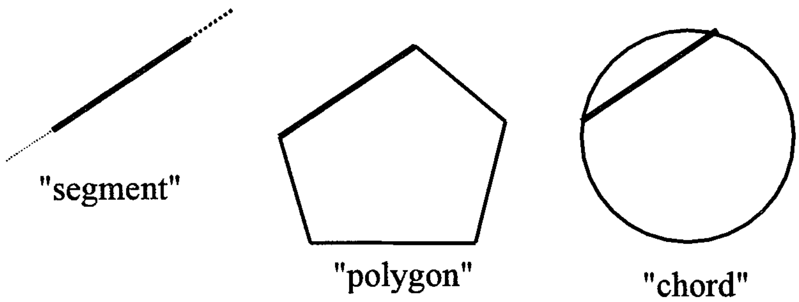
- In **Number**, each concept and operation is defined clearly in various ways mathematically. Representation (a number, a numeral) is arbitrary and not symbolic.
- In **2D/3D figure**, only concept and operation as research object are defined clearly. Representation (two- and three-dimensional, name,) makes it conceived of the relation with the concept itself and reality, and is symbolic strongly rather than arbitrary. The area of the figure itself is considered infinitely (a triangle, a quadrangle, a polygon, a polyhedron, a curve and a curved surface), and measurement of those is able to be found only approximately.
- The operation of **Number** is able to be represented by signs $+ - \times \div$ and is possible on paper.
- As for the operation of **2D/3D figure**, as compared with the operation of **Number**, linguistic representation and operational representation are connected strongly.
- In **Number** and **2D/3D figure**, composition of the operation is possible.
- 2D/3D (drawing) representation is symbolic from the viewpoint of nature of symbol, and, so, the relation with linguistic representation (name) is strong.

As there are many cases which show the metaphorical relation between everyday language and mathematical language about the last point, metaphorical properties are shown also in 2D/3D (drawing) representation. When we are going to represent the class of the figural concept in one 2D/3D (drawing) representation, it is regarded as synecdoche. And, 2D/3D (drawing) representation is dynamic metaphorical[4], which

[4] Although the drawn figure represents only the state of the last which it finished drawing, students have to see the construction order and conceive the partial-whole structure through

is greatly concerned with how a representation is seen. The difference between **2D/3D figure** and **Number** (and language) is to recognize reality in many (drawing) representations in spite of its peculiar rule, which is one of factors to be difficult for students.

Now, viewing the figure and space learning from the perspective of cognitive linguistics, what does the meaning of figure and space, that is, the subjective meaning made by cognitive subject mean? I suppose as mentioned above that the meaning has two levels – cognitive and conceptual –. Under the research program of cognitive linguistics, cognition of the external world is influenced by Gestalt, and a "meaning made sense" has *multi-aspect*, which constructs Gestalt which these aspects interact with each other in focusing on discussion about sense-making. In a word, we don't only view a representation of triangle as "something triangular" cognitively, but also analyze it as consisting of points, segments and angles, and regard it as the simplest figure or an entity having an inner relation "the sum of interior angles of a triangle is 180°" according to the problem situation conceptually. And, As Ronald W. Langacker (1987) who is one of researchers of cognitive grammar says with an idea of the FIGURE and the GROUND in cognitive science, the meaning of set of points is made as segment, polygon, or chord of a circle by background knowledge, as below.



4. FINAL REMARKS

In this paper, I consider the meaning about a conceptual object and relation between 2D/3D figure and everyday language. In particular, I have focused in the role and influence of everyday language for 2D/3D figure learning in this paper. The next issues are as following :

- What kind of the meaning of each 2D/3D figure does emergent through operation?
- To understand 2D/3D figure and operation from the viewpoint of image schema.
- …

As mentioned above, teachers have to take into consideration a polysemy of

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looking at it.

representation of 2D/3D figure and the use of language in figure and space learning in the mathematics classroom. Rigor of mathematics is not necessarily necessity for learners. As recognizing and accepting the existence of *Inner mathematics*, learner-centered learning of figure and space might be performed.

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TWO-AXES PROCESS MODEL OF UNDERSTANDING MATHEMATICS: A FRAMEWORK FOR THE TEACHING AND LEARNING MATHEMATICS IN A CLASSROOM

Masataka KOYAMA (Hiroshima University)

ABSTRACT

The purpose of this study is to make clear what kind of characteristics a model of understanding should have so as to be useful and effective in the teaching and learning of mathematics. In order to achieve this purpose, the previous studies related to models of understanding mathematics are summarized and the fundamental conception of understanding mathematics is described. Then, after discussing basic components substantially common to the process models of understanding mathematics, I present, as a theoretical framework, a process model that consists of two axes and call it the "two-axes process model" of understanding mathematics. Finally, we examine the validity of the "two-axes process model", especially the horizontal axis of the model by analyzing an elementary school mathematics class in Japan. Moreover, two important features of mathematics teacher's role are also suggested.

1. INTRODUCTION

The word "understanding" is very frequently used in both the descriptions of aims of teaching mathematics in the Course of Study (Ministry of Education, 1989) and in the teaching practices of mathematics in Japan. The putting emphasis on understanding mathematics should be desirable in mathematics education, but what it means is not clear. Moreover, it is an essential and critical problem that what mathematics teachers should do to help children understand mathematics. But it also has not been sufficiently made clear.

The key for the solution of these educational problems, in my opinion, is ultimately to capture what does it mean children understand mathematics and to make clear the mechanism which enables children's understanding of mathematics develop in the teaching and learning of mathematics. In other words, it might be said to "understand" understanding. It is, however, not easy and we need our great effort to do it. In fact, as Hirabayashi (1987) describes, the American history of researches in mathematics education seems to be the struggling with interpretations of understanding. The problem of understanding is still a main issue buckled down by some researchers, especially from the cognitive psychological point of view in the international group for the psychology of mathematics education (PME). As a result of their works, various models of understanding as the frameworks for describing aspects or processes of children's understanding of mathematics are presented (Skemp, 1976, 1979, 1982; Byers and Herscovics, 1977; Davis, 1978; Herscovics and Bergeron, 1983, 1984, 1985, 1988; Pirie and Kieren, 1989a, 1989b).

The purpose of this study is to make clear what kind of characteristics a model of understanding should have so as to be useful and effective in the teaching and learning of mathematics. In order to achieve this purpose, the previous studies related to models of understanding mathematics are summarized and the fundamental conception of understanding mathematics is described. Then, after discussing basic components substantially common to the process models of understanding mathematics, I present, as a theoretical framework, a process model that consists of two axes and call it the "two-axes process model" of understanding mathematics. Finally, we try to examine the validity of the "two-axes process model", especially the horizontal axis of the model by analyzing an elementary school mathematics class in Japan. Moreover, two important features of mathematics teacher's role are also suggested.

2. FUNDAMENTAL CONCEPTION OF UNDERSTANDING MATHEMATICS

What do we mean by understanding? According to Skemp (1971), to understand something means to assimilate it into an appropriate schema (p.43). Haylock (1982) answers this question in the following; a simple but useful model for discussing understanding in mathematics is that to understand something means to make (cognitive) connections (p.54). These explanations of understanding are (cognitive) psychological and imply that to understand something is to cognitively connect it to a previous one which is called a schema or a cognitive structure. We could say that a schema or cognitive structure is a model of a nerve net in the brain of our human beings. In this sense, to understand something is substantially an individual internal (mental) activity.

Moreover, comparing the Piagetian cognitive structures with the Kantian schemata and categories, Dubinsky and Lewin (1986) describe that the Piagetian cognitive structures are constructed from the outset and undergo systematic changes of increasing differentiation and hierarchic integration (p.59). This suggests us that the understanding defined above is not such a static activity as all-or-nothing but a complex dynamic phenomenon which could change in accordance with the construction and reconstruction of cognitive structures.

Therefore, accepting the conception of understanding mathematics as an internal (mental) dynamic activity, we necessarily need some methods to externalize children's understanding of mathematics. A retrospective method, an observation method, an interview method, and a combination of these methods are promising and useful methods for externalizing it. It is, however, almost impossible for us to directly see understanding when defined as the mental activity. Therefore, we need some theoretical framework. According to the definition of model by Gentner (1983), the theoretical framework for making clear aspects or processes of understanding mathematics could be called a ***model*** which has a mental activity of understanding as its prototype. In that sense, any model is indispensable for making clear understanding and the significance of building a model can be found in this point.

As already mentioned in the above section, the various models of understanding mathematics are presented in the previous studies. For example, there are such models as a discrimination of "relational and instrumental understanding" (Skemp, 1976), "a tetrahedral model" (Byers and Herscovics, 1977), "a 2×3 matrix model" (Skemp, 1979), "a 2×4 matrix model" (Skemp, 1982), "a constructivist model" (Herscovics and Bergeron, 1983), "a two-tiered model" (Herscovics and Bergeron, 1988), and "a transcendent recursive model" (Pirie and Kieren, 1989b). As Pirie and Kieren (1989b) point out, models can be classified into two large categories. The one is "aspect model" which focuses on the various kinds of understanding and the other is "process model" which focuses on the dynamic processes of understanding. The models presented in Skemp (1976, 1979, 1982) and Byers and Herscovics (1977) belong to the former and the models in Herscovics and Bergeron (1983, 1988) and Pirie and Kieren (1989a, 1989b) belong to the latter. We need both aspect model and process model to develop children's understanding in the teaching and learning of mathematics. They seem to be built mainly to describe the real aspects and processes of understanding as children learn mathematics. They are very useful for us to grasp them.

It is, however, not sufficient to describe the real aspects or processes of children's understanding, because mathematics education in its nature should be organized by both teaching activity and learning activity. Therefore, a model of understanding that is useful and effective in the teaching and learning of mathematics should have ***prescriptive*** as well as ***descriptive*** characteristic. Namely, the model is expected to have the prescriptive characteristic also in the sense that it can suggest us didactical principles regarding to the following questions; "What kind of didactical situations and how them should we set up to help children understand mathematics?" "To which direction should we guide children, in developing children's understanding of mathematics?"

3. BASIC COMPONENTS OF PROCESS MODEL

In order to build such a model of understanding mathematics, we must elucidate the processes of children's understanding in the teaching and learning of mathematics. In this section, focusing on a process model, we explore basic components of it. For theoretically exploring them, we examine process models of understanding (Herscovics and Bergeron, 1983, 1988; Pirie and Kieren, 1989a, 1989b) and a model of learning mathematics (van Hiele and van Hiele-Geldof, 1958; van Hiele, 1986).

Herscovics and Bergeron have been buckling down to the difficult task of building and modifying a model of understanding in the processes of mathematical concept formation. They built "a constructivist model" of understanding mathematical concepts basing on the constructivist assumption; children will construct mathematical concepts. The constructivist model consists of four levels of understanding; the first one, that of intuition, a second one involving procedures, the third dealing with abstraction, and a last level, that of formalization (Herscovics and Bergeron, 1983, p.77). Then they modified this model and presented an extended model of understanding. This extended model is called "a two-tiered model", one tier identifying three different levels of understanding of the preliminary physical concepts, and the other tier identifying three distinct constituent parts of the comprehension of mathematical concepts (Herscovics and Bergeron, 1988, p.15). Their fundamental conception underlying this model is that the understanding of a mathematical concept must rest on the understanding of the preliminary physical concept (p.20).

Pirie and Kieren (1989b) stress that what is needed is an incisive way of viewing the whole process of gaining understanding (p.7). And they present "a transcendent recursive model" of understanding that consists of eight levels; doing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. Their fundamental conception of understanding underlying the model and the important characteristic of the model are succinctly and clearly represented in the following quoted passage.

"Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without." (Pirie and Kieren, 1989b, p.8)

We can see that these models of understanding are process models that have prescriptive as well as descriptive characteristic and involve some levels of understanding. There is, however, an objection to the levels of understanding. In fact, examining the Herscovics and Bergeron model for understanding mathematical concepts, Sierpinska (1990) describes that therefore what is classified here, in fact, are the levels of children's mathematical knowledge, not their acts of understanding (p.28). This criticism is based on the different notion of understanding that understanding is an act (of grasping the meaning) and not a process or way of knowing. It is worth notice but in my opinion there must be some levels, even if those are levels of children's mathematical knowledge, in the process of understanding mathematics. The process model of understanding mathematics should involve some hierarchical levels so as to be useful and effective in the teaching and learning of mathematics.

The hierarchy of levels of understanding can be typically seen in a transcendent recursive model (Pirie and Kieren, 1989b). It reminds us of the van Hiele theory of levels of thinking in learning geometry which was presented in their doctoral dissertation (cf. van Hiele and van Hiele-Geldof, 1958). In the theory five hierarchical levels of thinking are identified and five learning stages for progressing thinking from a level to a higher level are involved (van Hiele, 1986). We notice that these models are very similar to each other in two respects. The one is each level set up and the other is the idea of progressing from one level to the outer (higher) level.

The first similarity can be recognized more clearly by illustrating the van Hiele model in the figure (Koyama, 1988). In fact, ignoring somewhat difference in the scope and domain of learning mathematics, each two levels indicated by a thick circle in the Pirie and Kieren model could be corresponded to each level in the van Hiele model respectively as follows.

Pirie and Kieren Model		***van Hiele Model***
(Doing, Image Making)	**⟷**	**(Concrete Object*, Geometrical Figure)**
(Image Having, Property Noticing)	**⟷**	**(Geometrical Figure*, Property)**
(Formalizing, Observing)	**⟷**	**(Property*, Proposition)**
(Structuring, Inventing)	**⟷**	**(Proposition*, Logic)**

[Note: The sign ⟷ indicates the correspondence between levels and the sign * indicates an object of thinking in each level.]

The second similarity is more important in a process model than the first, because it is concerned with the crucial idea of developing children's understanding of mathematics. The idea of developing children's understanding in the Pirie and Kieren model is "recursion", whereas in the van Hiele model it is "objectification" or "explicitation". These ideas seem to be substantially same and it might be called, in other words, reflective abstraction or reflective thinking. We can say that in the processes of understanding mathematics reflective thinking plays an important role to develop children's understanding, or to promote their thinking from a level to a higher level of understanding. Therefore these models suggest us that a process model should have learning stages involving reflective thinking.

After all, we identify such two basic components of a process model as ***hierarchical levels*** and ***learning stages***. In the next section, a process model with these two basic components is presented as a theoretical framework for developing children's understanding in the teaching and learning of mathematics.

4. THE "TWO-AXES PROCESS MODEL" OF UNDERSTANDING MATHEMATICS

In order to build the process model that can prescribe as well as describe how the process of children's understanding mathematics should progress, we must give serious consideration to the following questions. Through what levels should children's understanding progress? How should children develop mathematical thinking in any level of understanding? Relating to the first question, as already discussed, levels involved in the Pirie and Kieren model and the van Hiele model can be regarded as answers to it. Although we need to examine those levels and modify them in accordance with mathematical concepts intended in the teaching and learning of mathematics, they form a ***vertical axis*** of the process model of understanding.

Relating to the second question, learning stages involved in the van Hiele theory (van Hiele, 1986) and in the Dienes theory (Dienes, 1960, 1963, 1970) are very suggestive. On the one hand, in the van Hiele theory five stages in the learning process leading to a higher level are discerned; information, guided orientation, explicitation, free orientation, and integration (van Hiele, 1986, pp.53-54). On the other hand, in the Dienes theory six stages in the mathematics learning are set up basing on four principles of the dynamic, the constructive, mathematical variability and perceptual variability principle (Dienes, 1960, p.44); free play, rule-bound play, exploration of isomorphic structure, representation, symbolization, and formalization (Dienes, 1963, 1970). The stages in two models can be roughly corresponded like the followings; information to free play, guided orientation to rule-bound play, explicitation to exploration of isomorphic structure and representation, free orientation to symbolization, and integration to formalization respectively.

According to Wittmann's idea (1981), these corresponding stages are classified into three categories. He emphasizes that three types of activities are necessary in order to develop a balance of intuitive, reflective and formal thinking, basing on the assumption that mathematics teaching should be modeled according to the processes of doing mathematics (Wittmann, 1981, p.395). I modify his definitions of three activities a little in order to form a ***horizontal axis*** of the process model. At any level of understanding, there are three stages, intuitive, reflective, and analytic stage.

Intuitive Stage: Children are provided opportunities for manipulating concrete objects, or operating mathematical concepts or relations acquired in a previous level. At this stage they do ***intuitive thinking***.

Reflective Stage: Children are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or languages. At this stage they do ***reflective thinking***.

Analytic Stage: Children elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage they do ***analytical thinking***.

Through these three stages, not necessarily linear, children's understanding could progress from a certain level to a next higher level in the teaching and learning of mathematics. As a result, a process model of understanding mathematics can be built theoretically and it is called the "two-axes process model" of understanding mathematics. The model consists of two axes, i.e. the vertical axis implying levels of understanding such as mathematical entities, relations of them, and general relations, and the horizontal axis implying at each level three learning stages, i.e. intuitive, reflective, and analytic stage.

There are two prominent characteristics in the "two-axes process model". First, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set up in the model. Second, the model could be a useful and effective one because it has both descriptive and prescriptive characteristic mentioned above. These prominent characteristics of the model are, however, only expected theoretically. Therefore, we must examine both ***validity*** and ***effectiveness*** of the model in light of practices of the teaching and learning of mathematics in a classroom.

5. COMPLEMENTARITY OF INTUITION AND LOGICAL THINKING IN THE PROCESS OF UNDERSTANDING MATHEMATICS

In Japan it is one of main objectives of school mathematics education to develop child's intuition and logical thinking. To realize this objective, many mathematics educators and researchers have made extensive efforts in various ways. However, we can not say that we have satisfactorily realized the expected result. In consideration of the existing state of things, we should capture the nature of children's thinking in the teaching and learning of mathematics.

In this section, we demonstrate the ***complementarity*** of intuition and logical thinking in a process of understanding mathematics basing on two basic notions of ***mental model*** and ***reflective thinking***. In more concrete terms, we try to examine and identify children's mental models of an abstract and mathematical concept in regard to intuition, and observe how children think reflectively on their mental models in a whole-class discussion in regard to logical thinking. To attain the purpose, in this paper, we try to examine the ***validity*** of the "two-axes process model," especially the horizontal axis of the model by analyzing an elementary school mathematics class in Japan.

A Sketch of Elementary School Mathematics Classes

The class to be analyzed is a part of four successive mathematics classes in a fifth grade (11 years old) classroom at the national elementary school attached to Hiroshima University in Japan. In February 1993, an elementary mathematics teacher of the classroom, Mr. Mori, planned and taught 36 children (18 boys and 18 girls) a topic named "Let's think with mathematical expressions". The children involved in those four classes are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other children in the local and public elementary schools.

The classroom teacher, Mr. Mori, has a vision of elementary school mathematics education. Mori (1994) states it as follows; "Students' learning by solving mathematical problems is a continuous process of solving their own problems. I believe such process is an ideal form of learning elementary school mathematics that the once solution of a problem produces a more expansive problem (p. 91)". He planned the topic named "Let's think with mathematical expressions" with this vision of mathematics education. The main objective of the topic is to help children appreciate thinking with mathematical expressions such as interpreting a mathematical expression expansively and insightfully.

To realize this teaching objective, he planned three sessions and four unit-hour (45 minutes) classes for the topic as follows.

First session; comparing lengths of two different semicircular roads (2 unit-hour classes)

Second session; comparing lengths of other geometrical figured roads (1 unit-hour class)

Third session; comparing areas of two different semicircular regions and summarizing the topic (1 unit-hour class)

The following is a rough sketch of an outline of four successive classes that actually developed in his classroom. In this sketch, children's activities are focused and picked up mainly.

First Class

1) Teacher set up the situation: "There are two places A and B. Let's make various roads between them". Children imagined and proposed their roads. Among them, semicircular roads were adopted and two different semicircular roads were drawn on a blackboard (Figure 1). One road L was

a semicircular road with the diameter AB. Another road M was a one made by two connected semicircular roads with the diameter AC and BC, where place C was located at a certain point on the segment AB.

2) Children predicted which road is shorter when comparing lengths of two roads L and M. At this point students had their own problem to be solved.

3) Children individually worked out the problem in their own ways. It must be noted that they had learned mathematical formulae for the length and area of a circle, and they know that circle ratio is about 3.14.

4) Children knew that two lengths of roads L and M are equal. Some children explained their own reasons of why two lengths are equal in the whole-class discussion. Children compared and interpreted those mathematical expressions written on a blackboard for the explanations.

5) Children compared lengths of two roads when place C had changed to be another point C' on the segment AB (Figure 2).

6) Children said their findings which they had been aware in this class and proposed their own problems to be worked on in the next class.

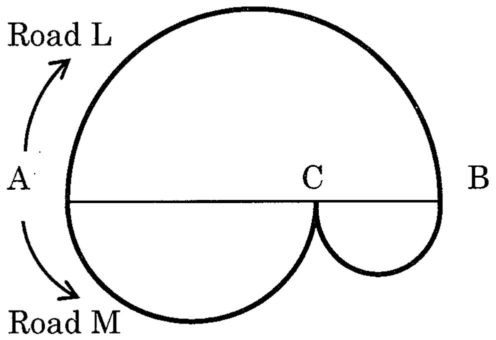

Figure 1.

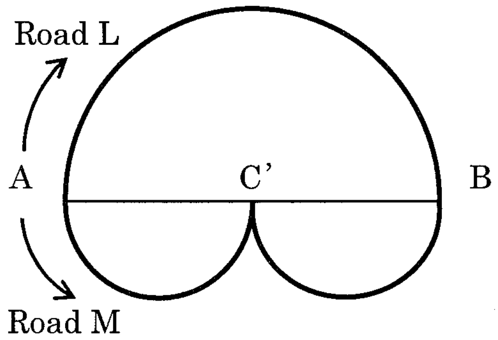

Figure 2.

Second Class

1) Children remembered what they had done in the first class.

2) Among the problems proposed at the end of the first class, children decided to work out the problem; "Compare lengths of two roads L and M when road M is changed to the one made by more than two small semicircular roads".

3) Children individually worked out the problem of comparing lengths when road M was changed to the one made by three small semicircular roads (Figure 3).

4) Children presented their own solutions and compared mathematical expressions written on a blackboard in the whole-class discussion.

5) Children worked out the more general problem of comparing lengths when the number of semicircular roads of M increased (Figure 4).
6) Children said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

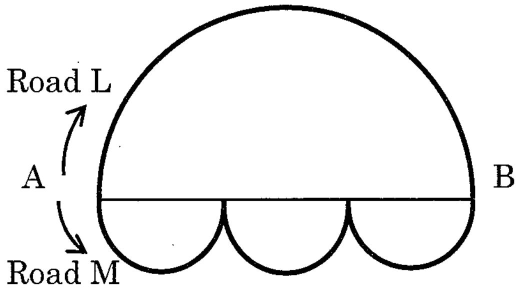

Figure 3.

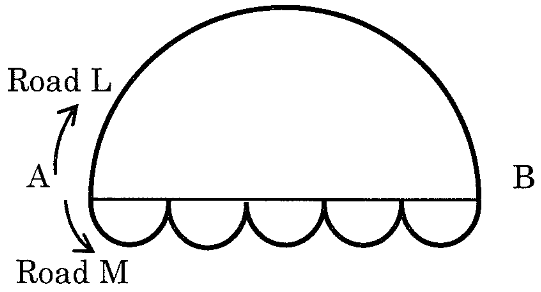

Figure 4.

Third Class

1) Children remembered what they had done in the second class.
2) Among the problems proposed at the end of the second class, children decided to work out the problem; "Seek for other geometrical figured roads which have a same rule as two semicircular roads".
3) Children individually investigated two quarter-circular roads (Figure5).
4) Children sought for other geometrical figured roads that have the same rule by means of mathematical expressions. Children checked, for example, two equilateral triangle roads (Figure 6) and two square roads (Figure 7).
5) Children said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

Fourth Class

1) Among the problems proposed at the end of the third class, children decided to work out the problem; "Compare areas of regions encircled by two semicircular roads (Figure 8)".
2) Children individually worked out the problem with their own predictions.
3) Some children explained their solutions of the problem.
4) Children thought about how the area of region encircled by the road M changes when a point C moves from A to B on the segment AB.
5) Children represented the change of area in a graph.
6) Children read and interpreted the graph and explained their own findings about the change of area in the whole-class discussion.

7) Children looked back what they had done in all four classes and summarized the content of the topic named "Let's think with mathematical expressions".

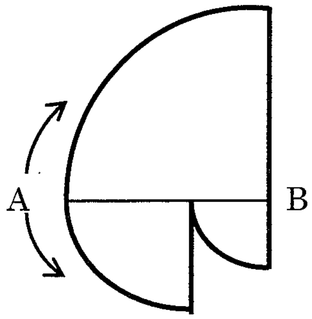

Figure 5.

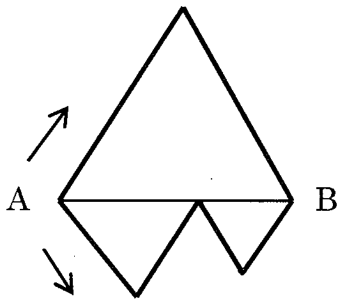

Figure 6.

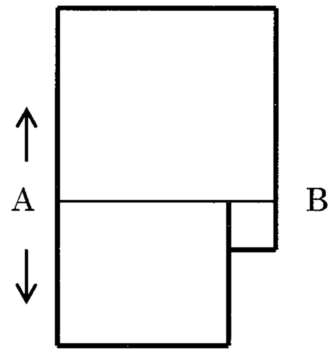

Figure 7.

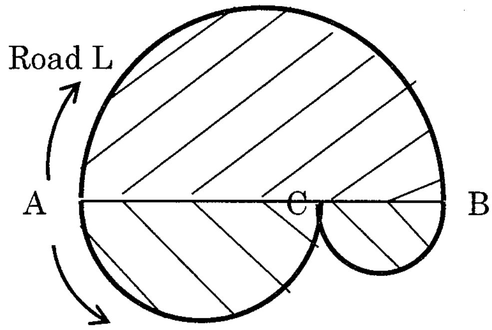

Figure 8.

Discussion by the Protocol Analysis of a Class

Four successive classes of the topic actually developed as shown in the above sketch. In this section, by analyzing the protocol of a class mainly in the first session, firstly we try to examine and identify children's mental models of length that lead to a misjudgment or a mathematically incorrect anticipatory intuition. Then we observe how their initial intuition has been changed under the control of children's reflective thinking in the whole-class discussion. Being based on this analysis of a class, we examine the validity of the horizontal axis, i.e. three learning stages of the "two-axes process model" of understanding mathematics.

Identification of Children's Mental Models of Length

In the first class, after teacher setting up a learning situation and children discussing about mathematical problems to be solved, the process of teaching and learning actually developed as follows. In the following protocol of a class, sign ***T*** and sign ***Sn*** mean the teacher's utterance and the ***n***th child's utterance respectively.

T: Today, we will try to work out the problem of comparing lengths of two semicircular roads L and M (Figure 1). How do you predict which is shorter, road L or road M?

S11: The length of road M is longer than that of road L, because the road M is bent at a point C.

S12: The road M encircles a smaller area than the road L does, so the length of road M is shorter than that of road L.

S13: The length of road M is shorter than that of road L, because the road M is closer to the straight line AB.

These three children's utterances of their prediction allow us to identify their mental models of length that they have initially at the class as products of their previous experiences of learning length. ***S11*** has a mental model like that when the both ends of two lines are trued up a curved line is longer than a straight line as shown in Figure 9. ***S13*** has a similar mental model to that of ***S11*** like that because the shortest line between two points is a straight line a line closer to the straight line is shorter as shown in Figure 10. On the other hand, noticing area, ***S12*** has a different kind of mental model like that the length of a closed geometrical figure is proportional to the area of it as shown in Figure 11.

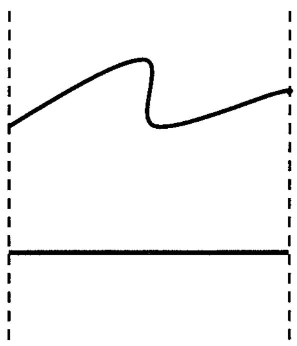

Figure 9.

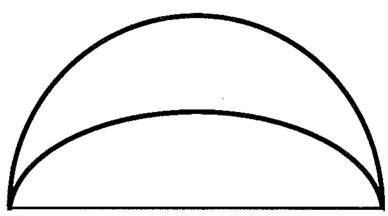

Figure 10.

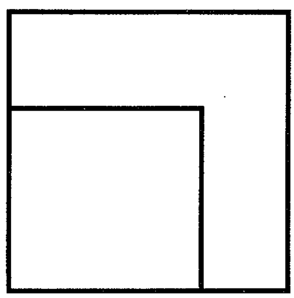

Figure 11.

All those mental models can lead to a mathematically correct judgment or prediction in some cases represented in figures 9, 10, and 11. However, in case of comparing lengths of two semicircular roads worked on in their class, their mental models produced a mathematically incorrect prediction. It might be said that they could not explicitly analyze the curvature (***S11***), closeness (***S13***), and similarity (***S12***). In any case, we could conclude that children's mental models of length that they constructed previously and had initially at the class have a negative effect on their anticipatory intuition (Fischbein, 1987; Koyama, 1991) without any explicit analysis of their mental models.

Examination of the Validity of Three Learning Stages in the "Two-Axes Process Model"

Next, we will observe how children's initial intuition has been changed under the control of their reflective thinking in the whole-class discussion. After predicting lengths, the process of teaching and learning in the classroom actually developed as follows.

T: You have different predictions and your own reasons. Which is longer, road L or road M? Let's make it clear. Work out the problem in your own way and write it down on notebooks.

S14: I can not do, because we have no information about the length of AB.

T: Do you need to know the actual length?

SS: (Many students say "Yes", but some students say "No".)

T: If you need to know it, use that AB is 10cm and AC is 6cm.

SS: (Students individually work out the problem by using the mathematical formula for a length of circle which they know.)

T: OK! Present your own work to your classmates. Anyone?

S15: I calculated the lengths as follows. Two answers are equal.

Road L; $10 \times 3.14 \div 2 = 15.7$ Road M; $6 \times 3.14 \div 2 = 9.42$

$4 \times 3.14 \div 2 = 6.28$

$9.42 + 6.28 = 15.7$

S16: I can calculate the length of road M with one mathematical expression like this.

Road M; $6 \times 3.14 \div 2 + 4 \times 3.14 \div 2 = 15.7$

S17: I can do it more easily by using parentheses like this. Two answers are equal.

Road M; $(6+4)\times3.14\div2=15.7$

S20: We do not need to calculate the lengths. The sum of AC and CB is equal to AB (looking at Figure 1), and we can see it apparently that both mathematical expressions for road L and road M is $10\times3.14\div2$.

So we can say that the lengths of two roads are equal.

T: You have explained your works with your own reasons well.
All of you seem to understand your classmates' explanations and be convinced them.

S21: Wait, Mr.! I have another idea. I used alphabetic letters. I thought about the problem when let the length of AB, AC, and BC be ***a***, ***c***, and ***b*** respectively. Then we can easily see that lengths of two roads are equal because two mathematical expressions are same like this.

Road L; $\boldsymbol{a}\times3.14\div2$ Road M; $\boldsymbol{b}\times3.14\div2+\boldsymbol{c}\times3.14\div2$

$=(\boldsymbol{b}+\boldsymbol{c})\times3.14\div2$

$=\boldsymbol{a}\times3.14\div2$

In this whole-class discussion, with the explanation of ***S15*** as a turning-point, children in this classroom reflect on their own calculating and thinking process and represent it in their own terms using mathematical expressions. This examination of the protocol allows us to conjecture that students do reflective thinking in their own ways. At this point, we should pay attention to the fact; ***S20*** and ***S21*** are explicitly aware that the mathematical expressions for lengths of two roads are same, while ***S15***, ***S16***, and ***S17*** put their eyes on only the fact that two answers are equal. In other words, for ***S15***, ***S16***, and ***S17*** a mathematical expression is a thinking method to calculate an answer for comparing lengths, but for ***S20*** and ***S21*** the mathematical expression itself is a thinking object. This difference must be significant from a point of view of the level of understanding mathematics, because, as van Hiele and van Hiele-Geldof (1958) suggest us, the objectification could push children's understanding of mathematics up to a mathematically higher level.

In fact, the explanation of ***S21*** stimulates other children and directs their understanding of this problem to a higher level, i.e. an understanding of the essential and mathematical structure of this problem.

T: It is a great idea. S21 used alphabetic letters. What can you see about the mathematical expressions explained by S21? Anyone?
S22: It does not depend on the actual lengths of AC and BC.
S23: They are expressed using alphabetic letters, so the lengths of two roads are equal even when a point C moves on the segment AB.
T: Is it true when a point C is close to the point A?
S24: Yes! As far as a point C is on the segment AB, two lengths are always equal.
T: Is it true? Please explain your reason in more detail.
(The following discussions are omitted.)

We can see in the above protocol that children do think about both the meaning of alphabetic letters and the structure of mathematical expressions. In other words, children in the classroom try to represent consequences of their reflective thinking more mathematically, analyze explicitly the structure of the problem, and integrate their findings as a whole. Therefore we might say that at this point of the class children do their analytic thinking.

As a result of this observation and protocol analysis of the class, we see that the process of teaching and learning mathematics in this classroom actually developed in line with the horizontal axis, i.e. three learning stages of the intuitive, reflective, and analytic. By the end of the first class, children in this classroom came to be able to control their mathematically incorrect anticipatory intuition by the logical thinking with mathematical expressions. It is saliently demonstrated by the fact that at the beginning of the second class, 34 out of 36 children could predict correctly. This fact allows us to insist that as a result of their learning experiences children had a fairly infallible intuition supported by the logical thinking with mathematical expressions including alphabetic letters. Therefore, we could conclude that the ***validity*** of three stages at a certain level of understanding mathematics has been demonstrated by the analysis of an elementary school mathematics class.

6. MATHEMATICS TEACHER'S ROLE AS A FACILITATOR

As another result of this research, we can characterize the teaching and learning of mathematics that enables children to actively construct mathematical knowledge in their meaningful way as the dialectic process of their individual and social constructions in the classroom as a society with their own learning history. It is compatible with "the wider orientation that views mathematics classroom teaching as controlling the organization and dynamics of the classroom for the purpose of sharing and developing mathematical meanings" (Bishop and Goffree, 1986, p.314).

Moreover, we can point out teacher's role as a facilitator for the dialectic process in the mathematics classroom. There are two important features of teacher's role as a facilitator. The one, related to children's individual construction, is to set a problematic situation in which children are able to be conscious of their own tasks and encourage them to have various mathematical ideas and ways. The other, related to children's social construction, is to encourage and allow them to make, explain, and discuss their various representations.

The teacher should not impose his/her authority on children's learning but make use of his/her professional ability and knowledge about both children and mathematics to be a facilitator for the dialectic process of children's individual and social constructions in the classroom.

7. CONCLUSION

In this paper, the so-called "two-axes process model" of understanding mathematics has been proposed as a perspective for teacher in teaching and learning of mathematics in a classroom. We have examined the validity of the horizontal axis that consists of three learning stages by analyzing an elementary school mathematics class. Moreover, two important features of teacher's role as a facilitator for the dialectic process in the mathematics classroom are suggested.

In doing so, we regarded children in a classroom as a whole and observed their process of understanding mathematics. It is, however, needless to say that we must also pay attention to an individual child's process of understanding mathematics. Moreover, we have to examine the ***effectiveness*** of the two-axes process model of understanding mathematics in a sense that we can really make a teaching plan by using this model and help children develop their understanding of mathematics to be an expected and higher level. These are difficult but important tasks to be faced and addressed in our future research.

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ANALYSIS OF PRIMARY MATHEMATICS IN BANGLADESH FROM PUPILS' AND TEACHERS PERSPECTIVES - FOCUSING ON FRACTION -[1]

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ABSTRACT

Most of the developing countries are now engaged with enlargement of pupils' participation in school under the initiative of UNESCO (1990). And it also emphasizes the quality of education which they receive at school. Bangladesh is also engaged with this task. This research sought it through research on the status of mathematics education from both angles of pupils and teachers. As a conclusion, three points were found out as follows: (1) uneven distribution of weak areas for pupils, and this tendency is not uniform among urban and rural schools, (2) very few pupils were able to explain the sentence of problem, and this is a manifestation of rote learning without proper understanding, and (3) teachers' less attention to pupils' difficulty and understanding level and teaching method should be tailored according to their needs.

1. INTRODUCTION

Currently UNESCO plays a leading role in promoting primary Education for All people (commonly termed as EFA) in the world. It started in 1990 with the conference which attracted hundreds of education-related people for the developing countries. There was a serious recognition that more than 100 million people, who were expected at their age, did not go to school. Since primary education promote individual human right and minimum capability to work in modern society, this situation was terrifying and so it has prompted them to take measures against it. Not only going to school but also having a quality education in school are very important, but the latter aspect is sometimes neglected in the discussion. Otherwise, they simply go to school physically but not psychologically. This research is to target at analyzing the qualitative aspect of primary mathematics education in Bangladesh under this sense of issue.

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Bangladesh is one of the most populous countries in the world, and occupies an important position for attainment of the above ideal, EFA. Historically, she regards the primary education as a means of reducing poverty and improving the quality of life. In the 1st drive, First Five Year Education Plan in Bangladesh (1973-78) proposed to establish 5000 new primary schools just after the independence in 1973. During this period, the government felt it necessary to make primary education compulsory with the basic goals of illiteracy eradication and realization of Universal Primary Education (UPE). In order to implement these policies, the government undertook various programs/ projects with her resources and also with assistance of development partners.

She continued this effort since then, and has been engaged in an innovative program 'Primary Education Development Program' (PEDP-I) between 1998 and 2003. The major objectives of the program were to improve school quality and system efficiency, to establish a sustainable, cost-effective and better-managed education system and to ensure universal coverage and equitable access to quality education. In consequence, amazing success was achieved in enrollment at primary level and it reached at 97% by 2002 (Directorate of Primary Education, 2002). In respect of qualitative aspect, however, the achievement was far away from the satisfaction. Therefore, by putting more focus on qualitative aspect of education, the Second Primary Education Development Program (PEDP-II, 2003-2008) is built on the achievements of the past decade, and especially on those of PEDP-I. In short, it reflects the government's goals for a quality primary education for all.

In this research, we would like to analyze the issues which she is grappled with. Therefore, the objective of this research is to identify the present status of mathematics education at primary level in Bangladesh from both pupils' and teachers' perspectives.

2. RESEARCH METHOD

In order to attain the above objective we developed three research tools to collect data regarding pupils and teachers. They are namely achievement test (ANNEX 1), interviews items (ANNEX 2) for pupils and questionnaires for teachers. All of them are translated into the national language, Bangla, and interview and explanation were conducted in that language. Period of field survey was 9th November, 2005 to 17th November, 2005.

Regarding selection of samples, we took the following procedure. In Bangladesh, Government Primary Schools (GPSs) are categorized into four levels such as A, B, C, and D. This categorization is done based on overall performance, consisting of such items as School Management Committee (SMC), Parent-Teachers

Association (PTA), Scholarship examination result, playground, plantation, scout group etc. About half of the schools belong to B category, and therefore it can be regarded as an average performance. So, two sample schools were chosen from this category with consultation of local authority. They are one GPS from an urban area and one GPS from a rural area in Mymensingh district. The urban school is situated in the center of Mymensingh and the rural one is situated about 30 kilometers away from the center. All grade 5 pupils were chosen as a sample in each school and four teachers of grade 5 out of eleven teachers within two schools were selected. Two teachers taught the grade 5 at that year and two more teachers taught at the previous year. There is no subject- based teaching at the primary level in Bangladesh. So basically these teachers have to teach all the subjects in different grades. Three teachers among them were female and one teacher was male. All of them have undergone C-in-Ed (Certificate in Education) training in PTI (Primary Training Institute) for one year.

Table 1 Number and distribution of sample pupils

	Grade	Boys	Girls	Age	Total
Urban School	5	8	16	10-13	24
Rural School	5	11	12	10-14	23
Total		19	28	10-14	47

Table 2 Number and characteristics of sample teachers

	Sex	Duration of teaching	Subjects to teach
Urban School	*Male Female	14 years 5 months 29 years 10 months	All subjects Mathematics, Bangla
Rural School	*Female Female	10 years 5 months 5 years 5 months	Mathematics, Social studies, Science Mathematics, English, Bangla

* At present mathematics class teacher at Grade 5.

3. RESULT OF RESEARCH AND ITS DISCUSSION

3-1 ACHIEVEMENT TEST

Result of achievement test is listed as follows. Table 3 shows the overall result of achievement test. The average score is 42.9%. Comparatively urban school pupils' performance (average score 51.7%) is better than rural school pupils' performance (average score 33.7%). The question-wise analysis (Table 4) reveals that in general, pupils' performance is comparatively better in Q1 (2), Q3 (1), Q4 (1), and Q6 (1). It seems that to some extent pupils are familiar with these types of question in the classroom test. In general the difficult problems are Q4 (2), Q5 (1), Q5 (2), Q5 (3), Q6 (3), and Q10. Since the sample size was small, we didn't conduct statistical test but did descriptive statistics.

Table 3. Results of the achievement test

	Average	Boys	Girls	Highest	Lowest
Urban School	51.7%	55.3%	49.9%	68%	34%
Rural School	33.7%	33.8%	33.6%	53%	11%
All	42.9%	42.8%	42.9%	68%	11%

Table 4. Question-wise achievements of pupils in percentage

	Coverage (Grade)	School in urban area			School in rural area		
		Combined	Boys	Girls	Combined	Boys	Girls
Q1 (1)	5	91.6%	100%	87.5%	30.4%	54.5%	8.3%
(2)	5	89.5%	93.7%	87.5%	60.8%	63.6%	58.3%
Q2	5	54.1%	37.5%	62.5%	0%	0%	0%
Q3 (1)	3	96.5%	100%	94.7%	40.5%	36.3%	44.4%
Q3 (2)	3	79.1%	75%	81.2%	26.8%	6.1%	45.8%
Q4 (1)	5	92.7%	100%	89%	69.5%	43.1%	93.7%
Q4 (2)	5	0%	0%	0%	18.4%	29.5%	8.3%
Q4 (3)	5	33.3%	37.5%	31.2%	16.3%	25%	8.3%
Q5 (1)	*	35.4%	50%	28.1%	13%	9.1%	16.6%
Q5 (2)	*	2.1%	0%	3.1%	21.7%	45.4%	0%
Q5 (3)	*	6.2%	6.2%	6.2%	19.5%	9.1%	29.1%
Q6 (1)	5	94.7%	96.8%	93.7%	38%	37.5%	38.5%
Q6 (2)	5	51.5%	98.4%	28.1%	28.2%	19.3%	36.4%
Q6 (3)	5	0%	0%	0%	13%	9.1%	16.6%
Q7	4	27.1%	18.7%	31.2%	15.2%	31.8%	0%
Q8	X	58.3%	50%	62.5%	69.5%	81.8%	58.3%
Q9	X	60.4%	62.5%	59.3%	58.6%	54.5%	62.5%
Q10	X	12.5%	0%	18.7%	8.6%	18.1%	0%

Legend: * it is covered but question pattern is different from Bangladesh pattern, X: it is not covered

The most difficult question for urban school pupils are Q4 (2), Q5 (2), Q 5(3), and Q6 (3), and their average scores are 0%, 2.1 %, 6.2 %, and 0% respectively below 10 %. On the other hand, the most difficult question for rural school pupils are Q2 and Q10, and the average score is 0% and 8.6% respectively. According to the teachers, some of the above question-patterns are not the same as those in textbook or classroom test. Rural school pupils' performance is comparatively better in Q4 (2) and Q6 (3) than urban school pupils' performance despite overall poor performance. The reasons behind this, however, might be that rural school pupils' help their parents for measuring in their agricultural works, and so they are familiar with these items. On the other hand, urban school pupils' performance is comparatively better in Q2 than rural school pupils' performance. The reasons behind this might be rural pupils' lack of conceptual understanding of this topic.

3-2 RESULT OF INTERVIEWS TO PUPILS USING NEWMAN PROCEDURES

In order to explore further the difficulties which pupils are facing, we employed Newman procedure (Natcha & Nakamura, 2006) for an interview. Here we modified the original Newman procedure in such a way that we proceeded to the next step of interview with pupils even if they were not able to explain the meaning or select mathematical operation. This is because some students showed correct answer without proper understanding of the problem, and this is believed as an important characteristic.

We have selected ten pupils for interview on the basis of their exam result of mathematics during the previous term. Five of them acquired good grade and the other five acquired poor grade. The interview items for pupils are divided into the following three levels (a) reading, (b) understanding of concept, (c) process and (d) specific mistakes (if any).

(a) Reading level (Simple recognition of words and symbols):
"Can you read the question?"
(b) Comprehension level (Linguistic interpretation of problems):
"Can you understand and explain the meaning of the question verbally?"
(c) Process skills level (Understanding and Execution of mathematical operations or procedures):
"Can you select and perform mathematical operations or procedures?"

Table 5 shows which level the pupils' error occurred at in urban and rural schools. According to the findings, all pupils could read Q5 (3), Q6 (1), Q8 somehow with a little difficulty, but none of them could understand the concept of Q5 (3), Q8 and few of them could understand the concept of Q6 (1). As for the process skill, in Q5 (3) most of them could not show the correct process, and in Q6 (1) and Q8 half of them could not show the correct process.

Table 6 shows which level the pupils' error occurred at according to their performance. In the process skill, high performers solved more problems than low performers in Q5 (3) and Q6 (1), but in Q8 poor performers answered 3/5 by simple combination of given numbers without much consideration.

In Q5 (3), the pupils of both schools felt difficulty to read the words "correct" and "appropriate". They did some mistakes in the process skill. For example, some of the reasons are such as "1/4 is greater than 1/3, since 4 is greater than 3" and "if a denominator is greater than other denominators then that fraction is bigger than other fractions". In Q6 (1), the pupils of rural school felt difficulty to read the word "process". They did some mistakes in the process skill. For example, they added the numerators and denominators respectively to find out the solution. In Q8, the pupils of both schools felt difficulty to read the words "relationship", "in terms of", "bracket". They did some mistakes in the process skill. For example, they only wrote the answer that was 5/3 instead of showing any process.

Thus, the number of correct answers does not tell us the exact number who really understood the meaning and followed correct solving process of the problem. In other words, there are certain numbers of correct answers by chance without proper thinking. This is a very important point to consider quality education.

Table 5 Pupils' level of errors by location

No.	No. of errors per solving level									No. of correct answer		
	(a) Reading			(b) Understanding of Concept			(c) Process					
	Urban	Rural	Total	Urban	Rural	Total	Urban	Rural	Total	Urban	Rural	Total
Q5 (3)	-	-	-	10	10	20	10	9	19	0	1	1
Q6 (1)	-	-	-	6	6	12	2	8	10	8	2	10
Q8	-	-	-	10	10	20	6	8	14	4	2	6

Table 6 Pupils' level of errors by performance

No.	No. of errors per solving level									No. of correct answer		
	(a) Reading			(b) Understanding of Concept			(c) Process					
	High performer	Low performer	Total	High performer	Low performer	Total	High performer	Low performer	Total	High performer	Low performer	Total
Q5 (3)	-	-	-	10	10	20	9	10	19	1	0	1
Q6 (1)	-	-	-	3	9	12	4	6	10	6	4	10
Q8	-	-	-	10	10	20	9	5	14	1	5	6

Table 7 Remarks by performance

	Q5 (3)		Q6 (1)		Q (8)	
	High	Low	High	Low	High	Low
Difficult words	"Correct", "appropriate"		"Correct", "appropriate"	"Process"	"Relation-ship", "in terms of", "Bracket"	"Bracket"
Regard-ing problem solving process	1. 1/4 is greater than 1/3, since 4 is greater than 3. 2. A: 1/4, B: 1/3: A=B	1. 1/4 is greater than 1/3, since 4 is greater than 3. 2. A: 1/4, B: 1/3: B=A	1. Added the numerators and denominators. He/she has no idea regarding fraction.	1. Calculation is accurate but could not explain the process. 2. Wrote the process but could not find out the solution.	1. Pupils have no idea how to find out the relationship.	1. Pupils have no idea how to find out the relationship. Only wrote the answer that was 5/3. 2. Only wrote the answer.

3-3 RESULTS OF QUESTIONNAIRES TO TEACHERS

We conducted questionnaire to teachers as well, regarding the pupils' understanding and their teaching in fraction. The questionnaire items are divided into the following five categories (a) test-evaluation (b) self-evaluation (c) pupils-evaluation (d) contents-evaluation and (e) teaching-methodology.

(a) Test-evaluation

According to the test evaluation items, urban and rural school teachers' estimations on the average score of pupils are 70% and 33% respectively. The former's high perception on pupils' performance does not match the pupils' actual score (42.9%) of achievement test.

Teachers in both schools mentioned that pupils are not accustomed to this type of question unlike their classroom test, which are almost the same as textbook contents. This tendency was, however, not confirmed as the Table 8 showed.

Table 8 Average score per grade

Grade	3	4	5	*	X
Average	61.3 %	21.2 %	43.7 %	16.3 %	44.6 %

Legend: "*" shows the item(s) covered but question pattern is different from Bangladesh pattern. "X" shows the item not covered in Bangladesh.

(b) Self-evaluation

In response to the item (4) on self-evaluation, the teachers stated that they feel difficulties in teaching "Fraction". Some of the reasons behind this are "pupils can easily understand the whole numbers but they find it is difficult to understand the concept of fraction", "practical use of fraction is difficult", "fraction related real/concrete materials are limited in the classroom". From a different angle, teachers reported in the item (5) that there are the most difficult topics to teach in Grade 5 such as geometry, simple fraction, decimal fraction, highest common factor and least common multiple. Some of the reasons for this are "lack of prior knowledge", "lack of proper reflection of pupils' demand in the textbook", "pupils cannot memorize the formula". Upon reflection on the answers together with those for the easiest topic, concept and relation with daily life are the key words.

(c) Pupils-evaluation

In response to the items (9) and (10) on pupils-evaluation, the teachers said, it is easy for the pupils to learn fractions although they also mentioned some difficult points to learn such as types of fractions, to find the least common multiple, and to convert into fractions with equal denominator. From these teachers seemed to pay attention to knowledge and skills aspect of fraction. Nevertheless, they said in item (11) that they are confident to teach fraction.

(d) Contents-evaluation

In response to the items (12) and (13) on contents-evaluation, the teachers' of both schools said, learning of fractions is an important topic among any other topics in mathematics. Some of the reasons behind this are application to practical life and to understand deeper mathematics. They mentioned that the main points in teaching "Fraction" are the concept of fraction, numerator and denominator, types of fraction, the relationship of numerical size of numerator & denominator with types of fractions. Again they repeated mostly knowledge and skills aspect.

(e) Teaching-methodology

There were two items (14) and (15) regarding teaching-methodology. As for the item (14) "how to teach, which is longer 1/4m or 1/3m?", teachers said that they would instruct the pupils at first to covert the above two fractions into the fractions with equal denominators and then show the pupils which one is longer, or explain it by using such drawings as meter scale, circle etc. Regarding the item (15) "how to teach, what is a half of 2m?" They said, they would explain them in class about the idea of half of 2m and 1/2m, or will ask the pupil to take a 2-meter long scale and then instruct the pupil to divide it into two equal parts.

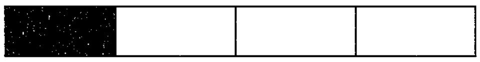
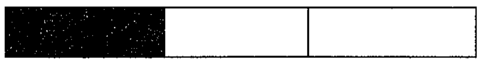
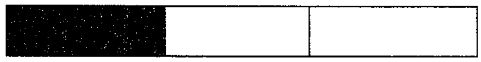
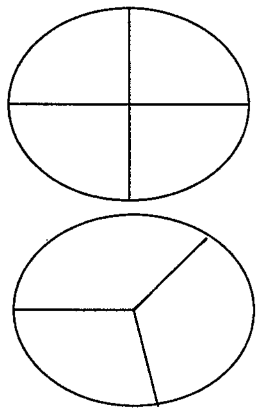
Table 9 Teachers' perception on the achievement test and teaching of Mathematics

Items No.	Average school in urban area	Average school in rural area
(5) The most difficult topic/s to teach in Grade 5.	Topic(s): Geometry, simple fraction and decimal fraction Reason/s: 1. Lack of prior knowledge 2. Lack of proper reflection of pupils' demand in the textbook. 3. Exercises are difficult to understand and to solve.	Topic(s): Highest common factor (2), least common multiple (2), measurement (2), simple fraction and decimal fraction (2) Reason/s: 1. Pupils cannot memorize the formula. 2. It is hard to explain about these topics without having the real/concrete materials.
(6) The easiest topic/s to teach in Grade 5.	Topic(s): Addition (2), subtraction (2), multiplication and division related problems; the highest number and the lowest number; average Reason/s: 1. Pupils are previously	Topic(s): Unitary method (2), simplification (2), average (2), percentage, time Reason/s: 1.These topics are more related to real life. 2. Easy to explain.

	familiar with these topics. 2. Pupils are more interested towards well-known topics. 3. These topics are not so difficult and therefore easy to understand.	
(10) Points of difficulty for the pupils to learn the concept of "Fractions"?	1. To differentiate the types of fractions 2. To convert into fraction 3. To fragment of fraction	1. To find out the least common multiple. 2. To convert into equal denominator. 3. To write the fraction on notebook.
(12) Importance of learning "Fractions" with comparison to any other topics in mathematics.	Yes (2) Reason/s: 1. If the idea of fraction is not clear at the initial level then it hampers the pupils' academic acquisition and practical life. 2. To solve the mathematical problems the use of fraction is noteworthy. 3. Pupils are able to know about larger, smaller, comparing things etc. through fraction.	Yes (2) Reason/s: 1. Fraction is an inseparable part of mathematics. Integer number is formed bringing together the fragmented parts. That's why it is important.
(13) Teacher's main point/s of concern to the pupils in teaching "Fractions"?	The main points is the concept of fraction, numerator and denominator, types of fraction, the relationship of numerical size of numerator & denominator with types of fractions	The main points is the numerator and denominator, different types of fractions

Legend: ( ) parenthesis denotes the number of respondent

Table 10 Teachers' strategies in fraction

Urban school	Rural school
(14)	
1. They will instruct the pupils at first to convert the above two fractions into the equal denominator fractions and then will show the pupils which one is longer. 2. Will explain it by using the following figures 1/4  1/3 	1. Will explain it by using the following figures 1/4  1/3  2. Will explain it by using the following figures 
(15)	
1. Will explain them about the idea of half of 2m and 1/2m. 2. Will explain it by using the meter scale.	1. Will ask the pupil to take a 2-meter long meter scale and then instruct the pupil to divide it into two equal parts. Therefore pupil will be able to understand the correct answer.

4. CONCLUSION

This research aimed at clarifying the present status of primary mathematics education in Bangladesh from both pupils' and teachers' perspectives. As a result of the field survey the following points were found out.

- As for weak areas for pupils, there is some biased tendency in their responses, and this tendency is not uniform among urban and rural schools although it is not possible to conclude so because of the small sample size. Teachers insisted that the type of question was different from the one which pupils encountered in a usual lesson or test. It is, however, not necessary true if pupils' performance would be better when they solve the usual type of questions. Besides, to promote the innovative thinking it may be necessary to give various types of problems.

・As for difficulty of pupils, through interviews it was noted that very few pupils were able to explain the sentence, but some were able to calculate and even to make correct answers. This is a manifestation of rote learning without proper understanding. There is no big difference between urban and rural, and high performer and low performer in this sense. So we can say generally they are weak in conceptual understanding.

・As for questionnaire, there found two very important and indicative points.

(Assessment of pupils' difficulty and understanding level) The first point is that teachers said they are confident to teach although it is difficult to teach, and it is easy for the pupils to learn. Besides this mismatch, some scores expected by teachers did not reflect actual performances of pupils. This inconsistency is very important for deliberation of the root cause. And it shows that their teaching activity is not fully based on understanding pupils' difficulty.

(Teaching method) They make good efforts to use drawing to explain the concept of fraction. They, however, didn't pay enough attention to "understanding the concept", which this research addressed to. With deeper understanding of pupils' difficulty, teaching method should be tailored according to their needs.

From these results, two points can be made as a conclusion. They are namely importance of grasping more precisely pupils' level and status of understanding of mathematical concepts, and of linking teaching activity of mathematical concept with pupils' level of understanding and daily life. Currently in many developing countries they are engaged in quality improvement of education by introducing child-centered approach. The approach, however, remains at the level of slogan in many cases, but does not fulfill the substance of this slogan yet. In this end, it is necessary to consider seriously strategies how to put the ideal into practice through realization of conceptual understanding.

Besides, this kind of research is to give implication for reflecting ourselves from the angles of educational practice and international cooperation in education.

・It provides with material to reflect on the situation in Japan.

New Public Management and Accountability are key words in the recent public sectors including education. Unless we are careful about the consequence of rote learning, which we found in the above, the reform movement may be effective in a short term, but in a medium or long term it will not produce the desirable outputs. Especially education requires a long span to see the effects of its endeavor, and it is important to substantiate the real understanding of pupils. For this purpose, both the Japanese "lesson study", which is now very popular among many countries and the recent emphasis on linguistic ability in primary education, are very important development in Japan.

・It provides with material to make educational cooperation efforts more effective.

For the Kananaskis Summit, 2002, Japanese Government proposed the BEGIN (Basic Education Growth Initiative) to the international community. She believes education made herself to today's position, and so systematization of her strengths is an important task. Lesson study and model lessons developed by many teachers for acquisition of basic linguistic ability and conceptual understanding can be good tools by which she will be able to contribute to the EFA movement.

Broadly speaking, in the first half of twentieth century, Japan yielded weapon to let other countries to obey her order, and in the latter half, she may look like yielding to economic power to do the similar. In the beginning of 21st century, she, however, will be almost overtaken by China, which forms a part of emerging economic superpowers, BRICs, in terms of total economic size, and may be overtaken by another emerging economic giant, India. This time we will not be able to yield either to weapon or to money. Our experience of building modern society through modern education system is still applicable to many developing countries (Japan International Cooperation Agency, 2004), and quality improvement of education through lesson study is given attention to by many countries, including USA (Stigler et. al., 1999). It is our task of 21st century to transmit this important educational message from our experience.

Acknowledgement: This field visit was enabled by the research fund from MEXT's Fundamental Research (B) "Comparative Research on Influence of Teachers' Views on Mathematics Lesson at Primary Schools in South-East and South Asian Countries" (Theme No. 16402045).

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Stigler, J.W. & Hiebert, J. (1999) *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*, The Free Press.

Q1. Work out the following calculations.

(1) $\frac{1}{5} + \frac{2}{5} =$

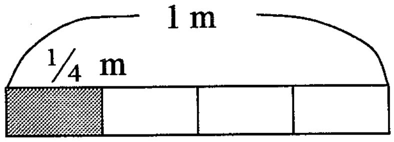
(2) $\frac{5}{7} - \frac{2}{7} =$

Q2. Express the answer in a fraction.

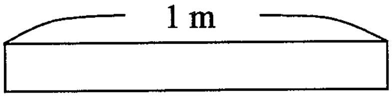
$3 \div 7 =$

Q3. The following figures show 1m of a bar. Shade a part of the figure to represent the following length.

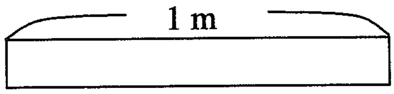
(Example) $\frac{1}{4}$ m



(1) $\frac{3}{4}$ m

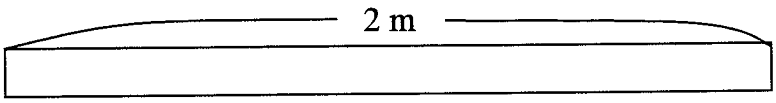


(2) $\frac{1}{3}$ m

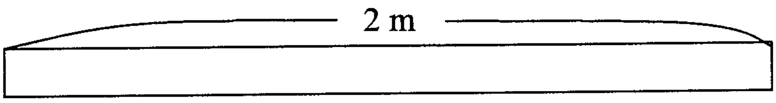


Q4. The following figures show 2m of a bar. Shade a part of the figure to represent the following length.

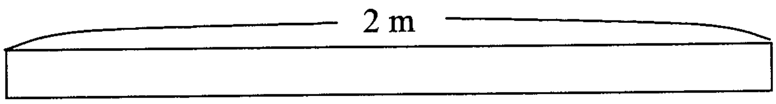
(1) $\frac{1}{2}$ of 2 m



(2) $\frac{1}{2}$ m



(3) $1\frac{1}{2}$ m



Q5. Which of the following is greater than the other? Write a correct choice (1, 2 or 3) in the bracket.

1. A is bigger than B. 2. B is bigger than A. 3. A = B

(1) A: $\frac{2}{5}$ B: $\frac{4}{5}$ Answer: ( )

(2) A: $\frac{3}{6}$ B: $\frac{1}{2}$ Answer: ( )

(3) A: $\frac{1}{4}$ B: $\frac{1}{3}$ Answer: ( )

Q6. Answer the following questions.

(Show your process to get the correct answer.)

(1) When you add $\frac{2}{5}$ ℓ of water to $\frac{1}{5}$ ℓ of water in a container, how much water is in the container?

(2) When you cut $\frac{2}{7}$ m away from $\frac{5}{7}$ m of a string, what is the remaining length?

(3) When you arrange 6 pieces of $\frac{1}{6}$ m paper in a line, what is the total length?

Q7. Change the decimal number " 0.7 " to a fraction.

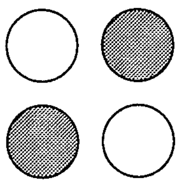
(Show your process to get the answer.)

Q8. Judy bought 5kg of meat. George bought 3kg of meat. Write a fraction in the bracket to show the relationship between Judy's meat and George's meat in terms of weight.

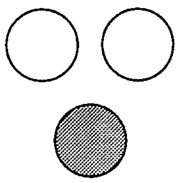
George's meat is ( ) of Judy's meat.

Q9. Which of the following represents "half" *for you*? Circle all the possible choice(s) in which "half" of circle(s) is shaded.

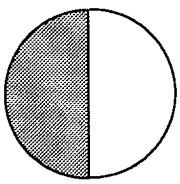
1.



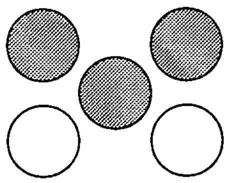
2.



3.



4.



Q10. Imagine and make a sentence problem in which the answer is $\frac{2}{3}$.

(You can express your sentence problem in words and/or any diagrams.)

ANNEX 2

Interview Items for Pupils

Item 1: Inquire what choice they selected and why in Q5 (3).

i) Ask them to read to you the sentence of Q5 (3). Can they correctly read it?

YES NO

ii) If there is any difficult term/expression for them to read, describe it below.

iii) Ask them to explain how they solve the question in words and/or any diagrams.

iv) Describe any specific mistakes, if there are any.

Item 2: Inquire how they solved Q6 (1).

i) Ask them to read to you the sentence of Q6 (1). Can they correctly read it?

YES NO

ii) If there is any difficult term/expression for them to read, describe below.

iii) Ask them to explain how they solved the question in words and/or any diagrams.

iv) Ask them to write their numerical expression to find the answer for Q6 (1).

v) Ask them to calculate the numerical expression. Can they correctly calculate it?

YES NO

vi) Describe any specific mistakes in the calculation, if there are any.

Item 3: Inquire how they solved the Q8.

i) Ask them to read to you the sentence of Q8. Can they correctly read it?

YES NO

ii) If there is any difficult term/expression for them to read, describe it below.

iii) Ask them to explain how they solve the question in words and/or any diagrams.

iv) Describe any specific mistakes, if there are any.

NOTE-TAKING AND METACOGNITION IN LEARNING MATHEMATICS: AN ANALYSIS IN TERMS OF SEMIOTIC CHAINING AND META-REPRESENTATION

Hiro Ninomiya (Saitama University)

ABSTRACT

In this paper, the theoretical frameworks of semiotic chaining and meta-representation are examined. Taking account of a Peircean model with three components, a representation of nested chaining, which consists of object, representamen, and interpretant, is identified. From Hirabayashi's (1987) implication, it is found that meta-representation can be either a method of learning mathematics or an explanation of the content of mathematics. Referring to the triadic model of chaining, both object and representamen are object-representation in Hirabayachi's terms, whereas the interpretant is a meta-representation. In this paper, a framework of reflexive writing is examined in terms of these constructs, and an example is analyzed using the nested (triadic) model of chaining. From this model, three types of note-taking are identified, and the importance of expressing an interpretant is emphasized.

1. DYADIC AND TRIADIC MODELS OF SEMIOTIC CHAINING

In an outline of semiotic chaining, Presmeg (2001) described briefly an interpretation of semiosis. Starting from Peirce's (1992) formulation of a semiotic model, there are three components, as follows.

firstness:	existing independently of anything else – an object
secondness:	a relation between the object and some sign which represents it – called the representamen in some of Peirce's writings
thirdness:	the interpretation of the sign, involving relationships among the object, the representamen, and a third component, called the "interpretant"

Fig.1: Three components of Peirce's (1992) semiotic model

Using examples from Driscoll (1994) and Whitson (1994), Presmeg (2001) presented the following.

object (o)	Sign (r) (representamen)	interpretant (i)
1. Sewing machine itself (design, function, model of operation)	**picture of sewing machine or part of it**	**apprehension of picture to reveal its function**
2. Idea, "Is the majority always right?"	**multiple signs, e.g., verbal description in textbook, example of a person whose minority religious views are a source of discrimination**	**interpretation that the majority is <u>not</u> always right**
3. Likelihood of rain	**falling barometer**	**decision to take an umbrella**

Fig. 2: Examples of three components of Peirce's triadic model (Presmeg, 2001, p.2)

Presmeg (2001) also presented another idea from Saussure, who defined the sign as a combination of a "signified" together with its "signifier". This constitutes a dyadic model, but "thirdness" (involving the interpretant) seems to be implicit in the interpretation of the sign. Lacan inverted Saussure's model, which gave priority to the *signified* over the signifier, to stress the *signifier* over the signified, and thus to recognize "far ranging autonomy for a dynamic and continuously productive play of signifiers that was not so easily recognized when it was assumed tacitly that a signifier was somehow constrained under domination by the signified" (Whitson, 1994, p.40). This formulation allows for a chaining process in which a signifier in a previous sign combination becomes the signified in a new sign combination, and so on. Chaining thus casts light on both processes as they are implicated in the construction of mathematical objects (Presmeg, 2001, pp. 2-3).

The example in figure 3 is from Walkerdine's (1988) description of a dialogue; in which a mother asks her daughter to name people they are pouring drinks for and to work out how many drinks by holding up one finger to correspond with each name.

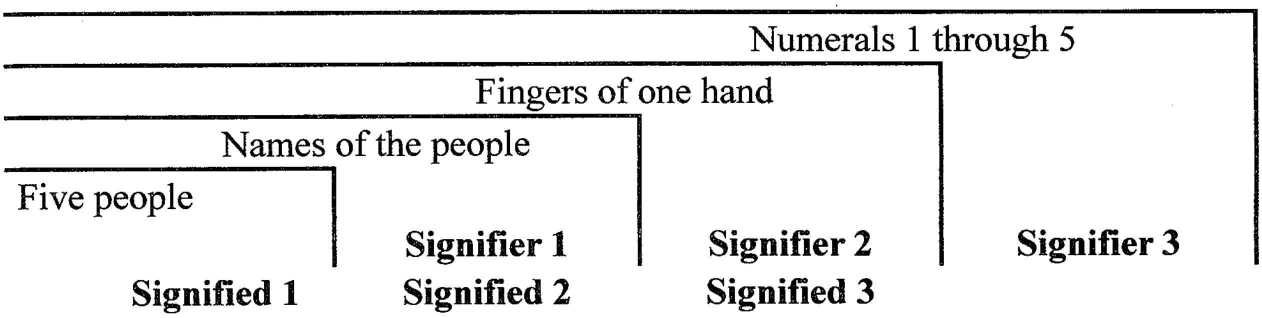


Fig. 3: An example of semiotic chaining (dyadic model)

In figure.3, "the names of the people" start as signifiers, but they quickly becomes signified in relation to new signifiers, "the fingers". Later the fingers become signified in turn, as spoken numerals become the signifiers.

Furthermore, Presmeg (2001) continued the discussion of the process of semiotic chaining, and examined a Peircean nested model that included the interpretant and therefore was more useful in interpreting her research data.

> Each of the rectangles in figure 4 represents a sign consisting of the triad of object, representamen, and interpretant, corresponding roughly to signified, signifier, and a third interpreted component, respectively. This interpretant involves meaning making: it is the result of trying to make sense of the relationship of the other two components, the object and the representamen. Note that the entire first sign with its three components constitutes the second object, and the entire second sign constitutes the third object, which thus includes both the first and the second signs. Each object may thus be thought of as the reification of the processes in the previous sign. Once this reification occurs, this new object may be represented and interpreted – or rather, resonating with the cyclic nature of the processes involved, the construction of symbolic notation and its interpretation also inform the creation of this object. (Presmeg, 2001, p.8)

Key: O = Object (signified)
R = Representamen (signifier)
I = Interpretant

Note: Presmeg is here using the term "sign" to stand for the totality of object, representamen, and interpretant in each case: thus each rectangle represents a sign, and there are three signs, nested like Russian dolls.

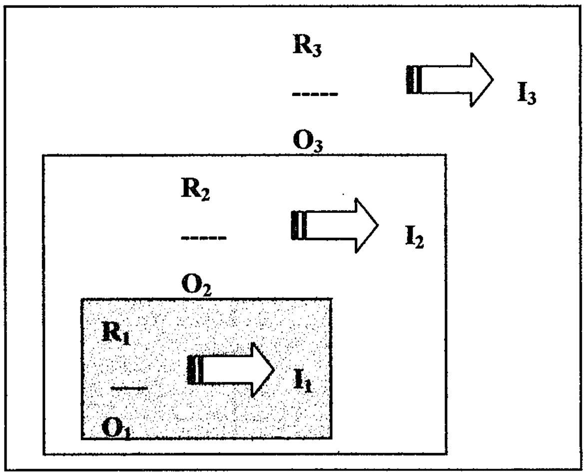


Fig. 4: A Peircean representation of a nested chaining (triadic model)

Examining the example of Walkerdine's "Mother and daughter pouring drinks" (fig. 3) in this Peircean representation of a nested chaining (fig. 4), gives another interpretation, as in figure 5.

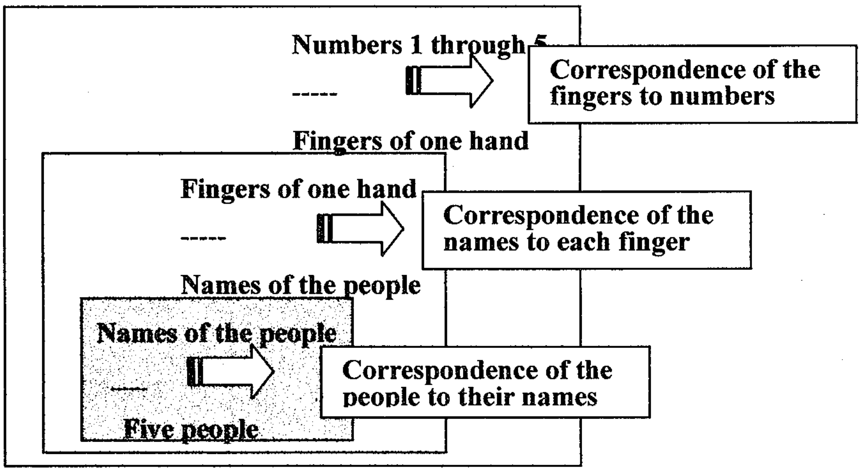


Fig. 5: An example of nested chaining model: Walkerdine's "Mother and daughter"

For each Interpretant in Fig.5, the following dialogue is illustrated from Walkerdine (1988). (M stands for mother, C stands for child.)

Correspondence of the people to their names

M: How many children are there?

M: There's Michelle.

C: Mark.

M: Mark.

C: Kirstie.

M: Kirstie.

Correspondence of the names to each fingers

M: Now have you got the right number of fingers? Michelle...put these down.

M: Michelle.

C: Yeah.

M: Mark.

C: Yeah.

M: Kirstie.

C: Kir-

M: Her little sister Katie came along and don't forget Stephanie.

C: Yeah.

Correspondence of the fingers to numbers

M: How many is that?

C: Four.

M: No.

C: Seven.

M: No. Count.

C: One, two, three...seven.

M: Four.

C: Four.

C: Five.

2. OBJECT-REPRESENTATION AND META- REPRESENTATION

Hirabayashi (1987) takes notice of the results from general linguistics, as it distinguish two types of languages, "object language" and "meta-language". The former is the type of language that corresponds to the "objects" of analysis or research, and the latter is the type of language that corresponds to the "method" of analysis or research. With regard to mathematics or mathematics education, Hirabayashi (1987) classifies expression of ideas into two types of mathematical representations or notations; which are "object-representations" and "meta- representations". The essential difference between them is the formality of their composition. Object-representation is composed with clear and definite rules of composition, whereas meta-representation can be composed without any kinds of rules and can be used easily. Another way to say this is that "object language" is a type of representation which itself is the aim or the object of learning mathematics. On the other hand, "meta-language" is another type of representation that is used for learning and teaching mathematics.

The important point is that we cannot distinguish the former from the latter easily, because the process of learning mathematics is complex, vital, and organic, and it is not mere logical constructive activity. Once an object-representation is learned and acquired by the learner, immediately it can be used as meta-representation for learning another object-representation. On the other hand, we may suddenly realize a meta-representation, which has been used roughly and easily, could be treated as an object-representation, with a manner of proper meaning and methodological validity.

We can summarize these idea from Hirabayashi (1987) as in figure 6.

> object-representation:
> - aim or object of learning mathematics
> - content of mathematics
> - can be used as meta-representation for another object-representation in a different context
>
> meta-representation
> - method of learning and teaching mathematics
> - explanation of the content of mathematics
> - can be conscious of being an object-representation in a different context

Fig. 6: Object-representation and meta-representation (Hirabayashi, 1987, pp. 388-391)

Referring to Peirce's semiotics in Figure 2, both Object (signified) and Representamen (signifier) seem to be regarded as object-representation, because both are the objectives in learning mathematics. On the other hand, the Interpretant seems to be meta-representation, because it is an interpretation of the signs and the interpretation might be used as a method of learning mathematics. The relations between components of the semiotics model and these representations are shown in figure 7.

> Object: object-representation
> Representamen: object-representation
> Interpretant: meta-representation

Fig. 7: Relations between components of semiotics model and representations

3. REFLEXIVE WRITING ACTIVITY

Referring to the concept of meta-representation, Hirabayashi & Shigematsu (1987) explained the concept of metacognition by using the analogy of person, "inner teacher", which means it works as if there is a teacher who makes comments within the inside of the students her/himself. This analogy seems to be very effective for understanding such internal operations as metacognition. By their analogy, such internal operations are regarded as being done by a person who is inside of the student her/himself, or by *another self*.

As one of the examples which use the idea of "inner teacher", Ninomiya (2001) proposed a framework of *Reflexive Writing Activities*, which is a kind of writing that has reflexive interaction with both learner and class activities. A student does *Reflexive Writing* from the viewpoint of either her/himself, or the viewpoint of other people, including *another self* or "inner teacher". Although *Reflexive Writing* is basically the reflection of a learner's own learning, she/he can never stop writing just with the answer or her/his own solution. Since students are encouraged to reflect on their own solutions, some reflective *internal operations* are needed and students are encouraged to show them in their writing. However, sometimes it is hard for students to distinguish metacognitive or other internal affairs from cognitive operations, so the concept of *another self* or "inner teacher" is introduced to the students. They are encouraged to watch their own learning processes from the view of *another self*, and have *another self* make some comments toward their own learning process as if they were told by teacher.

In this way, *Reflexive Writing* is formed with student's own answer or solution, and comments from *another self*. As mentioned above, the important point is that *Reflexive Writing* never ends with only one single statement but continue as if there are chain reactions. For example, when a student writes her/his own solution, she/he may be aware of something that is metacognitive or other internal operations, then the student may be able to add some more comments from *another self*. Further, because such comments stimulate the student's thinking, she/he can foster her/his own idea, and may get another solution. In such a way, writing activity and student's learning may develop their mutual interaction, and the nature of their relation is reflexive. Here, we can find a kind of chain action in the learning process.

For more investigation on two major parts of Reflexive Writing; "student's own answer or solution", and "comments from *another self*", we can see the relation to the framework of Peirce's triadic model. Since the former can be the object of their mathematics learning and the latter can be the method of learning or explanation of the contents, "student's own answer or solution" can be "object" or "representamen", whereas "comments from *another self*" can be "interpretant".

One of the best ways to promote *Reflexive Writing Activities* is "*Characters Method*" (Ninomiya, 2002). In this method, some characters such as persons, animals, etc. are used on purpose, in order to let students be aware of the existence of *another self*. Each character becomes either the student her/himself, or *another self*. During the instruction of *Characters Method,* the teacher never forces students to use characters, but just shows how to use characters in her/his blackboard writing. Most of the students spontaneously imitate their teacher's way, because they love to study in such a way. Although there is not special instruction for making characters, students learn the importance of the comments from *another self.* One example of *Reflexive Writing* is shown and analyzed in Ninomiya (2002).

4. CASE STUDY

Here is an example of student's reflexive writing in figure 8. This is a 5th grader's writing on the topic of "principle of place value in decimal numbers". The student has done this writing as voluntary homework in order to summarize what she had studied in school.

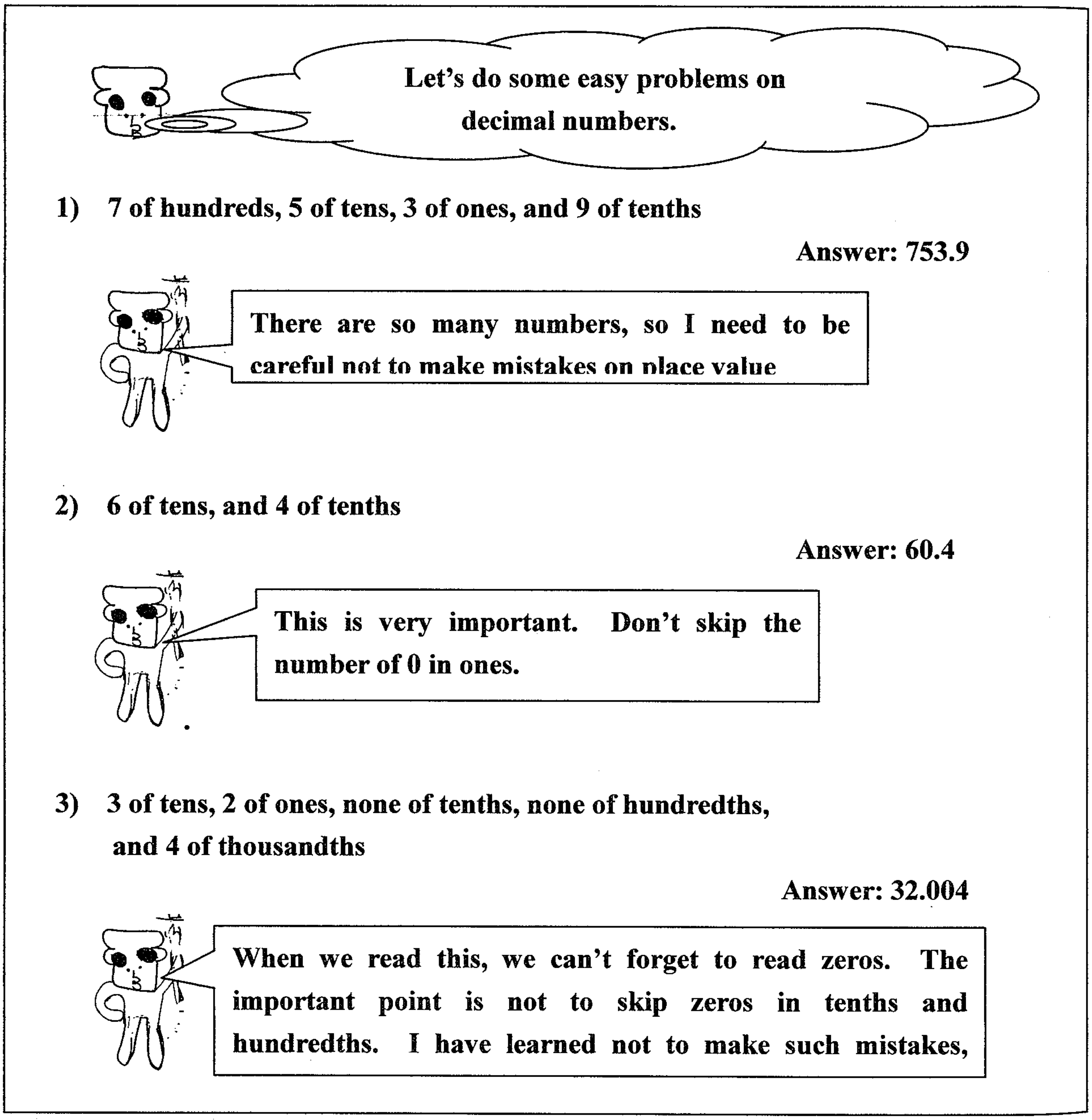


Fig. 8: Reflexive writing for the summary of decimal numbers

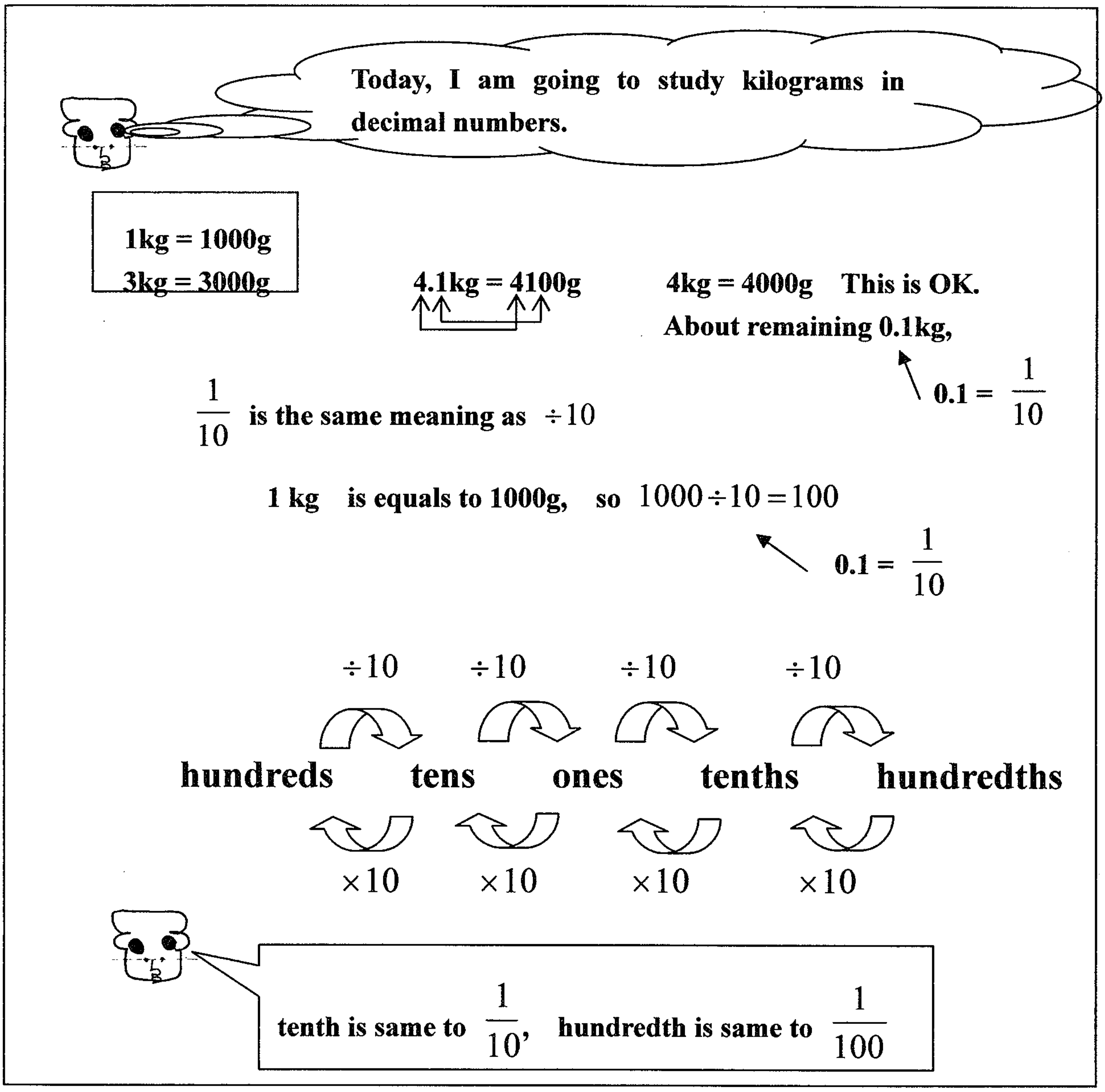


Fig. 8: Reflexive writing for the summary of decimal numbers (continued)

Now, these written representations are going to be applied to the framework of Peircean representation of a nested chaining of three signifiers, as in figure 4. At the very beginning of this written representation, as question #1, "7 hundreds, 5 tens, 3 ones, and 9 tenths", was presented. She wrote the answer, "753.9", and put some more comments for the answer with a character and a balloon. In this very beginning part, the question #1 is an Object (signified), because it is the meaning of the content, and signified by the answer. The answer is Representamen (signifier), because it signifies the content, or the question #1. The comment with a character and balloon is from *another self*, for it is presented from other person (the character). It also should also be regarded as meta-representation, because it is the explanation of the content. Since this comment is "meaning making", it should be considered as

"Interpretant".

Furthermore, just as Presmeg (2001) pointed out as "the entire first sign with its three components constitutes the second object (p.8)", there is another (second) Object. The next question, "6 of tens, and 4 of tenths", is going to be the second Object because it is constituted from the former signs. The rest of the process goes likewise.

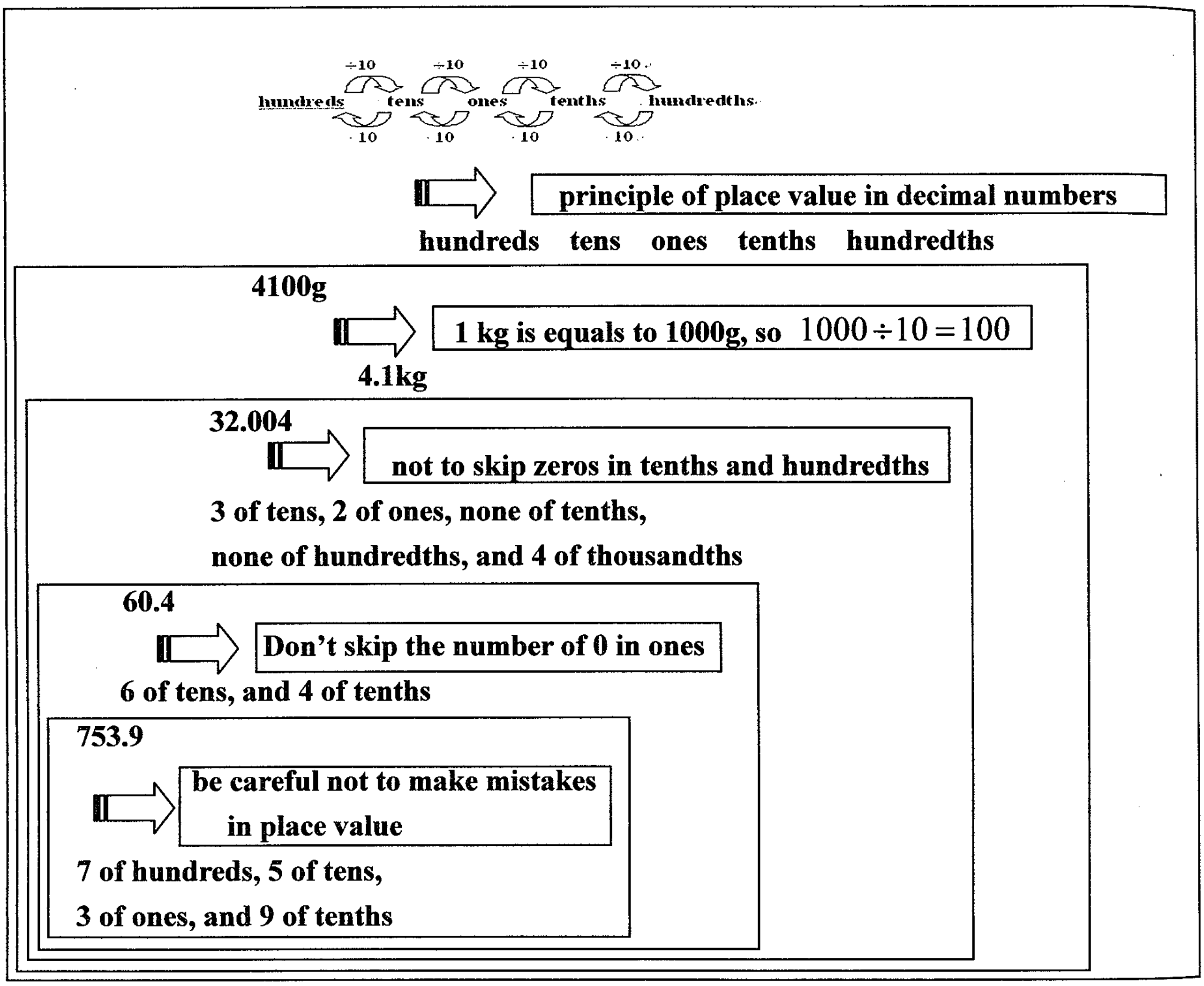


Fig. 9: Nested chaining model for the summary of decimal numbers

At the last stage of this nested chaining, figure 9 shows "principle of place value in decimal numbers" as for the "Interpretant", because the Representamen of student's writing shows that she had understood the major principle of place value in decimal numbers. Even though there is no written representation, we can conclude that she could do the "meaning making", which is the major role of Interpretant.

5. DISCUSSION

Reflexive Writing is one of the ways to present both object-representation and meta-representation. For example, figure 8 shows object-representation as both questions and answers, and meta-representation as the character's comment. According to the terms of Peirce's triadic model, the questions could be the Object, the solution could be the Representamen, and the comments from the characters could be the Interpretant. This is really good because it shows meta-representation or Interpretant as the comment from *another self*, and it helps students foster their understandings. Ninomiya (2001) indicates the positive influences of this learning method within both affective and cognitive domains; there are statistically significant differences between students who did the *Reflexive Writing* and those who did not. Ninomiya (2002) also shows the achievements of the students who have done an excellent job in *Reflexive Writing*. From the viewpoint of "meta-representation", it is suggested that a kind of activity that shows meta-representation or Interpretant, such as *Reflexive Writing*, is ideal during the learning of mathematics.

Considering the student's *Reflexive Writing* in figure 8, from the framework of Peirce's triadic model, we can identify either Object, Representamen, or Interpretant in her note. In the general case of note taking, three types of writing, or note-taking, could be identified from the viewpoint of semiotic chaining.

Type A: $R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_4 \rightarrow$
Type B: $O_1 \rightarrow R_1 \rightarrow O_2 \rightarrow R_2 \rightarrow O_3 \rightarrow R_3 \rightarrow$
Type C: $O_1 \rightarrow R_1 \rightarrow I_1 \rightarrow O_2 \rightarrow R_2 \rightarrow I_2 \rightarrow O_3 \rightarrow R_3 \rightarrow I_3 \rightarrow$

O = Object (signified)
R = Representamen (signifier)
I = Interpretant

Fig. 10: Three types of note-taking

"Type A" is a kind of note-taking which show only the Representamen, or the solutions. It could be possible when all of the problems are shown in the textbook and what is expected to the students is just the memorization of the rules or results of the problems. Of course this kind of note-taking leads to the poorest outcomes.

"Type B" expresses both Object and Representamen. This type of note-taking could be regarded as "ordinary and usual". It shows both Object(signified) and Representamen(signifier), or problems and solutions, and the semiotic chaining can be identified as figure 3. It seems good, but there are still some problems. From the viewpoint of representation (fig. 6), both Object and Representamen are "object-representation", which are just the contents / aims /objects of mathematics, but neither the explanation of the contents nor the methods of learning, which are needed for the ideal learning activities.

"Type C" is the case of *Reflexive Writing*, and it seems excellent. There are all three components of Peirce's semiotics model (fig. 1): Object, Representamen, and Interpretant. We can also see there is both object-representation and meta-representation, which means that this kind of note indicates the explanation of the contents and the methods of learning, as well as the contents / aims /objects of mathematics. *Reflexive Writing* and *Characters Method* are one of the good examples for Type C note-taking. Using characters is not the major issue, but expressing the metacognitions or other internal operations, or the comments from *another self*, is very important. Expressing the "Interpretant", students are able to be aware of metacognitions or other internal operations, and this helps them towards the ideal mathematics learning.

Analyzing the note-taking from the viewpoint of Peircean representation of nested chaining model seems effective. Especially, expressing "Interpretants" is very important, for they indicate the explanation of the contents and the methods of learning. Since written representation is the essential part of learning mathematics, using the framework of semiotic chaining in such manner is one of the effective ways to analyze the students' learning activities.

6. CONCLUDING REMARKS

First of all, the frameworks of semiotic chaining and meta-representation were examined in this paper. According to Peirce's semiotic model, three components; object, representamen, and interpretant, were identified. Considering Saussure's signified / signifier and Lacan's inversion, a dyadic model of chaining (fig.3) was provided. Taking account of Peirce's three components, a Peircean representation of a nested chaining (triadic model, fig. 4), which consist of three components (object, representamen, and interpretant), was also identified.

From the research of general linguistics, two types of mathematical representations, or notations (object-representation and meta-representation) were classified. From Hirabayashi (1987)'s implication, it was found that meta-representation can be either a method of learning mathematics and an explanation of the content of mathematics. Referring to the triadic model of chaining, both object and representamen were interpreted as object-representation, whereas interpretant is regarded as meta- representation.

Then, a framework of Reflexive Writing was examined, in order to identify a certain example in terms of the nested (triadic) model of chaining (fig. 9). From the nested model, three types of note-taking were identified, and the importance of expressing the interpretant was emphasized.

Further investigation is needed for the refinement of nested (triadic) model of chaining, and deeper analysis of various cases.

NOTE

This paper is rearranged from the author's presentation at The 10th International Congress on Mathematical Education, Topic Study Group 25, *Language and communication in mathematics education*, Semiotic aspects of mathematics learning. (http://www.icme-organisers.dk/tsg25/subgroups/ninomiya.doc)

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COMPLEMENTARITY OF THEORY AND PRACTICE IN IN-SERVICE TEACHER EDUCATION: PROSPECTS FOR MATHEMATICS EDUCATION IN KENYA

ATENG' OGWEL (HIROSHIMA UNIVERSITY)

> *"Why don't in-service programs teach us like they want us to teach our students? Why do in-service instructors so often start from what we don't know rather than from what we do know?"*
>
> ---
>
> Cooney and Krainer (1996, p. 1165)

ABSTRACT

This paper outlines some problems in secondary mathematics education in Kenya, and discusses achievements and challenges of an in-service programme. Prospects in the described situation are seen to include research in regular classrooms, appropriate theoretical frameworks, and institutional and cross-national collaboration. A brief analysis of an episode is used to reinforce these prospects in an effort towards complementarity of theory and practice in teacher education.

1. INTRODUCTION

The rhetoric in the story of a teacher in Cooney and Krainer (1996) suggests that in-service education is yet to significantly differ from pre-service, and exposes a need to re-conceptualize the tension between theory and practice. The teacher's desire for development programmes that reflect constraints in the classrooms as she *knows* them rather than hypothetical prescriptions, (see, Mason and Waywood, 1996; Sfard, 2005, for example), may, however, miss inherent reciprocal relation between theory and practice. The desired relationship is also emphasized in a special issue of the *Journal of Mathematics Teacher Education*: 'Inter-Relating Theory and Practice in Mathematics Teacher Education'. For example, in the editorial, *Petra Scherer* and *Heinz Steinbring* (2006) provide a schematic relationship among participants in mathematics education, and between theory and practice, and argue that "[no] finished, perfectly elaborated (pedagogical and mathematical knowledge) products are 'handed on' from the researchers to the teachers in practice, but the common, cooperative work is increasingly concentrated on the essential activities of 'learning and teaching mathematics'" (p. 105). Educators in developed countries are conscious of the tension between theory and practice and have made considerable progress in bridging it, unlike the situation in some developing countries where external research findings are adopted.

Similarly, professional development programmes independent from initial teacher education reflects some tension between theory and practice. For example, despite a three-month teaching practice--in which student-teachers are observed and graded in planning and teaching in public schools, they are overwhelmed in real practice and gradually surrender to *status quo* where little regard is given for lesson-planning, indicating that theory and practice are yet to be integrated (Jaworski and Gellert 2003). On the other hand, an emerging in-service training that places emphasis on planning is yet to significantly appreciate the complexity in ordinary secondary classrooms.

Consequently, in agreement with Steinbring, Bartolini Bussi and Sierpinska (1998), there is need for empirical research in typical mathematics classrooms backed by appropriate theoretical frameworks, in our case the *epistemological triangle* (Steinbring 2005a), to enhance in-service teacher education; narrow the gap between theory and practice and, harness the mutuality between pre-service and in-service teacher education. In the rest of this paper, we present some challenges in mathematics education in Kenya (section 2.1); enumerate some achievements of an in-service programme and outline its outstanding challenges (section 2.2).We offer some prospects (sections 3.1 and 3.2) with an illustration from an episode on sequences and series from a Grade 11 classroom (section 3.3).

2. MATHEMATICS EDUCATION IN KENYA

2.1 Curriculum and Instruction

Kenya has a centralized system of education, the 8-4-4 system, characterized by eight years of primary education, four years of secondary and at least four years of university education (see Figure 1). At inception, the 8-4-4 system set to promote self-reliance and expand middle-level vocational training (Hughes, 1992; Republic of Kenya, 1999) to counter soaring levels of unemployment. Currently, it also seeks to foster nationalism, promote social, economic, technological and industrial development; promote individual development; and enhance international consciousness (Kenya Institute of Education, 2002).

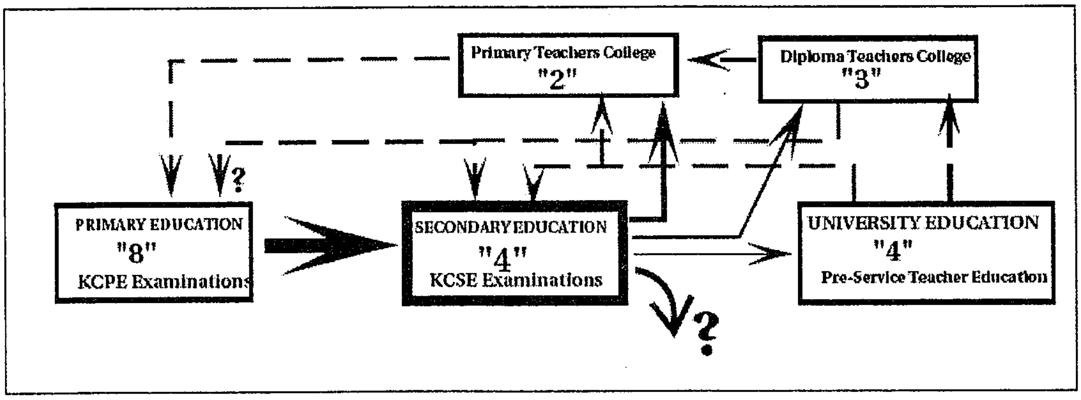

Figure 1: *A schematic transition within the 8-4-4 System of Education*

Within mathematics education, there are transitional problems evident in curriculum and instruction: while some topics have been pushed from primary to the secondary level or removed from the curriculum, mathematics curricula at the universities rarely reflect these changes.

Similarly, the congested secondary mathematics curriculum motivates a tendency to overlook topics recurring from primary schools or miss connection among some topics (Ogwel, 2006). Besides, suggested resources and notes in the secondary syllabus encourage the perception of the curriculum as set of maximum expectations, for example, it avoids "substitution" methods of integration or excludes "sum to infinity" in Sequences and Series (Kenya Institute of Education, 2002). On the other hand, multiple choices in the Kenya Certificate of Primary Examinations (KCPE) contrasts with structured questions in the Kenya Certificate of Secondary Education (KCSE).

Moreover, inefficiency in secondary mathematics education easily passes to the higher levels-- for example, while some prospective primary school teachers fail in mathematics, they ultimately teach it the primary schools (Republic of Kenya, 1999). Interactions in secondary schools are also markedly different from the primary and universities (see Steinbring, 2005a, for example). Therefore, secondary education faces a significant challenge in handling the transition from concrete-based reasoning in primary schools to formal mathematics at the universities.

2.2 In-Service Teacher Education

Some of the problems in secondary education are being addressed by a project-- the Strengthening of Mathematics and Sciences in Secondary Education (SMASSE Project) -- which seeks to upgrade the capability of students in Mathematics, Biology, Chemistry and Physics. Its ideals are reflected in a conception of Activities, Student-Centeredness, Experiments, and Improvisation (ASEI)-- believed to "enable pupils to develop an inquiry mind, develop the skill of making accurate observations, drawing conclusions, and holding discussions to enhance learning and development of skills"(SMASSE Project, 2002b, p. 70). The project conducts in-service training (INSET) in four thematic cycles: 'Attitude Change', 'Hands-on Activities', 'Classroom Actualization' and 'Student Growth Impact Transfer'. Moreover, successful ASEI lessons are enhanced by a Plan-Do-See-Improve (PDSI) cycle, and evaluated using observation instruments--ASEI/PDSI Checklists, which include items on resources, student activities and time management.

The SMASSE Project has had remarkable success in addressing negative attitude among teachers and other stake-holders in education (SMASSE Project, 2002a); it has popularized in-service education and led to establishment the Center for Mathematics, Science and Technology Education in Africa (CEMASTEA) to provide national and regional INSETs, through its strategic emphasis on INSET system construction (SMASSE Project 2002b). Among its acknowledged challenges, however, are sustainability, ownership by teachers and development of students' higher-order thinking. In addition, there are conflicting expectations which complicate a bridge between theory and practice. For example, professional development and additional certificates are expected to lead to promotion, but the national scope of SMASSE/ CEMASTEA complicates the expectation. Consequently, laxity by some teachers towards INSETs led to mandatory attendance of training sessions. Similarly, ASEI is uniformly applied in the four subjects, yet mathematics teachers fail to distinguish '*Activities*' from '*Experiments*' or develop "hands-on" activities in most topics. The dilemma is, whereas experimental skills in the sciences are examined in KCSE examinations, practical skills in mathematics appear to take time needed for completing the syllabus. Although observation instruments for 'successful' lessons are necessary, they give priority to theory over practice by focusing on *a priori* concept of ASEI; and constrains opportunity for 'metalearning' by teacher educators (Krainer and Goffree, 1999).

The Project's emphasis on hands-on activities may not address transition to higher education; is vulnerable to risks occasioned by 'pendulum shifts' in education, and may not be sustainable within the calls to reforms education to adequately prepare students for further education. For instance, while in 2006, students in Kenya began to use calculators in mathematics examinations, educators in the US, in an apparent truce on the *Standards* and ' math wars' (see also National Council of Teachers of Mathematics, 2006), were emphasizing "automatic recall of basic facts, the importance of abstract reasoning, the need to acquire a mastery of key algorithms, and the *judicious* use of calculators and real-world problems"(American Association for the Advancement of Science, 2006, p. 989, *added emphasis*). That is, Kenya may find herself adopting innovations that are out of phase in other countries without appropriate empirical research which align global trends with local contexts and constraints.

3. CONSIDERING THE PROSPECTS

3.1 Regularities of Mathematics Classrooms

To capture the complexity of regular mathematics classrooms, we conceptualize *Participants*, *Mathematics* and *Activities* to be interdependent elements (Figure 2). We acknowledge that teachers and students are the principal participants, but others including researchers, textbook authors and parents also affect the classrooms. It is the expectations, potentials and identities of the Participants that regulate the classroom (Ogwel, 2006). Mathematics on the other hand, is taken to involve relations and structural properties, consistent with Sierpinska's (1998) argument that "hat mathematics is not about things that can be shown, nor is it even about relations between such things. Most of the time, it is about *relations between relations*" (p. 38, *my emphasis*). This distinction enables us to include tasks, resources under Activities, in which they actually or potentially mediate Participants and Mathematics.

Whereas this characterization differs from the *didactical triangle* or Christiansen and Walther's (1986) schema, its assumptions of (a) observing lessons as they occur in the school calendar; (b) extended observation of lessons for acclimatization with the classroom environment and to minimize simulated lessons; and (c) a proactive view that classrooms provide learning opportunities for everyone (Ogwel, 2006) -- potentially narrows the gap between theory and practice. Video data collected from regular mathematics classrooms provide rich opportunities for researchers and teachers' reflection, but their analysis requires appropriate theoretical frameworks (Steinbring, Bartolini Bussi and Sierpinska, 1998).

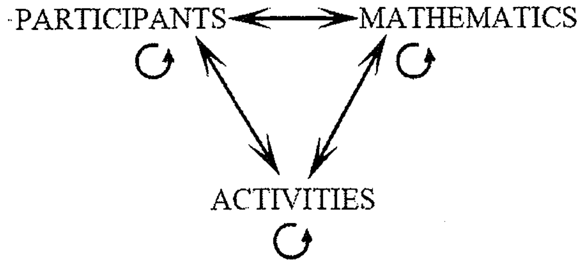

Figure 2: *An Interactive Model*

3.2 Epistemological Triangle

Challenges in mathematics education in Kenya call for perspectives that address progression towards advanced reasoning without trivializing knowledge constructed in lower levels. The *epistemological* triangle (Figure 3) offers such prospects, given its development through analysis of everyday classroom activities, balance between communication and epistemological aspects of mathematics in classroom interactions, mutual conception of its vertices and potential use in teacher education (Steinbring, 1998, 2005a, 2005b). Besides, it does not focus on the frequency of words used in a lesson (Bromme and Steinbring ,1994), but on the identification and correspondence between sings/ symbols and reference contexts, where new knowledge is seen as generalization of particular experiences (Otte, 2006; Steinbring, 2005a).

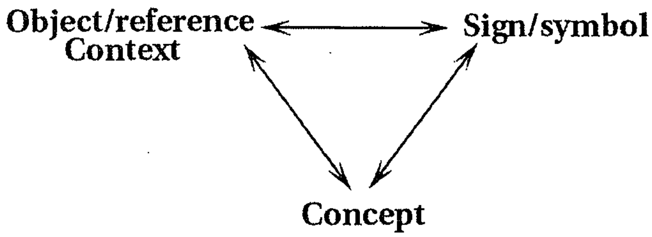

Figure 3: ***Epistemological Triangle***

Moreover, the epistemological triangle a parallel between historical developments of mathematics is used to distinguish scientific mathematics from school mathematics, each subject to social and theoretical constraints. However, this distinction does not trivialize professional mathematics, as Steinbring (2005a) argues that

> [I]n school-mathematical interaction one has to take into account the fact that students are not as familiar as mathematical experts with the communicative rules of the formal proof, and that they cannot use these rules as a mathematical medium in the same way [...] Instructional mathematical interaction is supposed to contribute to introducing individuals into this mathematical communication practice. (p.74)

Furthermore, instruction in secondary mathematics education is expected to reflect some 'active' interactions seen in elementary education, but there is need to distinguish discourse in mathematics from those in other disciplines, as Steinbring (2005a) argues that, the epistemological analysis is

> [o]pposed to an assessment of interactive events in mathematics instruction which rather remains at the surface of observing external phenomena [...] mathematical signs and symbols do not simply relate to concrete things, but that only through them relations and structures are expressed. (p. 5)

Nevertheless, this does not imply that elementary school mathematics is irrelevant in secondary education. For instance, in the Kenyan curriculum, the concept of place value appears to be used without linkage to other topics. Unfortunately, divisibility tests that require them are introduced as rules for students to memorize (see Ogwel, 2006). The following problems on decimal representation may enhance teachers' epistemological knowledge.

1. ½ = 0.1, 1/3 = 0.010101; ¼ =?
2. ½ = 0.1, 1/3 = 0.010101; 1/5 =?

These problems highlight the necessity for interactive construction of knowledge-- doubting the plausibility of Problem 1 may be followed by a challenge to find any relations among the numbers. Possibilities include treating the numbers as puzzles; sequences or relating $(½)^2$ = ¼ (= 0.01). The challenge therefore is to accept, contrary to the familiar conception, that ½ or 0.1 have multiple representations necessary for understanding properties of numbers. New knowledge may require re-interpreting place value and *remainders* in division or linking dividend, divisor and quotient through interactive development of new reference contexts, signs/symbols and concepts. That is, finding invariant *properties* in the problems. A comparison between the above problems and the following cases may provide a familiar reference (i.e., relating the dividend, divisor and quotient in a familiar base), and show the 'epistemological' significance of the '0'that is added whenever there is a remainder in division.

(a) 1/8 = 0.125 [10] '0' is added to the remainders on division to get 10, 20 & 40)
(b) 1/7 = 0.142857142857142857 [10]

Epistemological knowledge is necessary in teacher education, but requires a complementary phase of analyzing classroom interactions, taking into account the developmental nature of school mathematics. Although variety of analyses of different concepts illustrate the strength of the *epistemological triangle*, there is need to enhancing it by 'other users', and extend it into secondary classrooms. In the next section, we describe an episode from a Grade 11 Class, on Sequences and Series (Japan International Cooperation Agency, 2005). However, an analysis of the episode requires a caveat-- it is not about the teacher or any individual participant, but of the whole class in the context of *this* lesson (see also Stigler and Hiebert, 1999)That is, a need for the desired reciprocity between theory and practice prompts reflection of Professor Hirabayashi's ten-word philosophy: that *the best way to reform teaching is not to teach"* [1], which together with its converse-- *the best way to teach is not to reform teaching*-- illuminate preliminary dilemma in the prospects.

3.3 An Episode: The 20th Term of an Arithmetic Sequence

The episode is drawn from an ASEI lesson whose objective was to use matchsticks to introduce Sequences and Series in a Grade 11 class in Kenya. In the introduction of the

[1] A remark made during our presentation at the 39th meeting of the Japan Society for Mathematics Education, October 8, 2006, Hiroshima University:" Interactive Learning of Mathematics in Secondary Schools: The Concept of Similarity in School Mathematics"

lesson, two number sequences are given to students. This analysis is significant in outlining a necessary process of reflection between theory and practice-- where analytical frameworks cannot just be read, but have to be understood better through analysis of lessons (Steinbring 1998, 2005a). The episode is divided into four phases, incorporating students conjecture of the 20th term of a given sequence, group work, whole class discussion of the 20th term and extension of the 20th term problem. In the present analysis, we focus on the first phase given in the transcripts below. The following sequences of numbers were given during the introduction of the lesson:

(a) 4, 7, 10, 13, 16
(b) 3, 8, 13, 18,

Phase 1: Two Students' Solutions

In this phase, the students find the 20th term of the sequence "4, 7, 10, 13, 16..." as a prelude to an activity of using matchsticks. Their solutions reveal a similar relation between the 20th term and a common difference (20 ×3), but different "addends". The first student identifies the first term (4) and a common difference (3). She then knows that the required term depends on these two quantities. The second student similarly multiplies the common difference by the required term, but uses the sequence to find an addend. The teacher signals a need for further explanation, which leads to a consensus that the solution is correct. This is then followed by the introduction of the intended activity of developing a sequence of numbers from matchsticks.

3. T: Yes. And today we are going to continue with arithmetic sequence. (*Writing "Arithmetic Sequences").* And to start the lesson I would write two patterns on the chalkboard, examples of arithmetic sequence. *Writing "Arithmetic Sequence"* (1) 4, 7, 10, 13, 16... (2) 3, 8, 13, 18...

4. T: How would I get the 20th term?

5. S1: You get the difference between the *nth*, the two terms, then you multiply it by 20, and then you add to the first one.

6. T: Go and explain that on the chalkboard so that the others will understand (*some students giggle*).

7. S1: The 20th [term] and the difference in the first one (*first sequence)* is 3. So you do twenty times 3 that's sixty (20×3=60) (*speaking while writing*). Then the first term in the first sequence is 4, so you do 60 plus 4 "60 + 4 = 64"

8. T: & OK, that is how she would get the 20th term. Somebody else? (*some students raise hands*)

9. T: Yes (*selects a student*).

10. S2: &You do twenty times three you get sixty (20 ×3 = 60). Then you do 60 plus 1 you get 61 (60 + 1 = 61) (*attempts to give a piece of chalk to the teacher, the teacher extends hand then withdraw*).

11. T: (*Turns to the class*) A question?

12. C: Why?

13. T: Yes, explain why

14. S2: Um, You see, Um, you subtract, the difference here is 3, For example, now if we didn't have this 10. Um the ten, and I am told to look for the fourth, fourth term. I'll do three, three times four will give me 12 to get, I add 1 to

get 13 “3 ×4 + 1 = 13. The same with the fifth term, three times five is fifteen (3×5=15), plus 1 you get sixteen. That is how I will do with my pattern

15. T Did you *understand*?

16. C:& Yes

17. T: OK. Some would use the first method; others would use the second method. And the two methods are giving us different answers. Isn't it?

18. C: Yes

19. T: Now which one is correct?

20. C: Second/ first one (*most students say “second”)*

21. T: Second? And those who are for first they say the first answer is correct.

22. C: Yes

23. T: Now, we are going to do an activity, OK?

24. C: Yes

25. T: We are going to do an activity in groups. And this activity will help us, determine the 20th term, OK?

26. C: Yes

27. T: Of an arithmetic sequence or any other term like thirtieth, fifteenth, a hundredth, and so on, OK?

Phase 2: Arrangement of Matchsticks

In this phase, an arrangement of matchsticks was illustrated on charts (Figure 4) to guide students' activity. Figures 5a & 5b show arrangements by some group of students. The significance of this phase is that despite illustration of arrangements, Figure 5a diverges from the expected arrangements, and addition of matchsticks may contradict the objective of producing an arithmetic sequence. This phase was also characterized by group work and collaboration among the group members, where some arranged matchsticks while others completed Table 1 This table was used to predict the number of matchsticks necessary to make (a) 30 squares (b) 100 squares and (c) n squares.

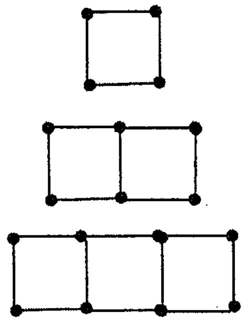

Figure 4: *An Illustration of Arrangement of Matchsticks*

Phase 3: Whole Class Discussion

Group discussion was followed by whole class discussion where Table 1 was used to obtain the 20th of 4, 7, 10, 13, 16... equivalent to the column of 'number of matchsticks'. The use of the table focused on the columns 'Number of Squares' and 'If d =3... .' where coefficients of 'd' were compared to the corresponding number of squares. That is 1 $\leftrightarrow$ 0; 2 $\leftrightarrow$ 1; 3 $\leftrightarrow$ 2..., leading to a prediction 20 $\leftrightarrow$19. Consequently, the 20th term was obtained as $4 + 19 \times 3 = 61$, then compared to the correct one given in section 3.4

Table 1: *A Table Completed During Whole Class Discussion*

NUMBER OF SQUARES	NUMBER OF MATCHSTICKS	CALCULATION	If d = 3 COMPLETE
1	4	4	4
2	7	4+3	4+d
3	10	4+3+3	4+2d
4	13	4+3+3+3	4+3d
5	16	4+3+3+3+3	4+4d
6	19	4+3+3+3+3+3	4+5d

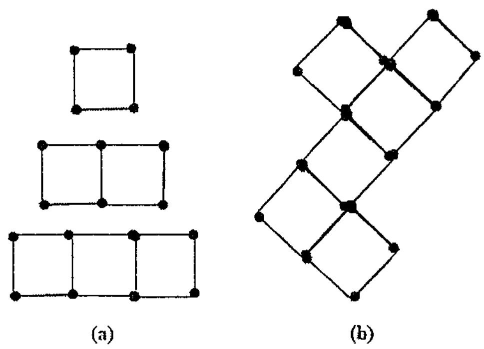

Figure 5: *Some Arrangements of Matchsticks*

Phase 4: Summary of the Lesson

In the final phase, a summary of the activity was used to solve the 20th term of the second sequence General expressions for respective sequences were obtained as a conclusion of the lesson: 4 + (n - 1) d for sequence; 3 + (n - 1) d for sequence.

3.4 Analysis of Two Students' Solutions

Student S1's Solution (5—7)

Student S1 seem to be introducing a generalization, using difference and nth and first. She uses the same approach when challenged to explain. Her use of signs takes the following forms:

1. 1 The difference in the first one (signifier 4, 7, 10, 13, …) ←→ (signified: 7 - 4 = 3)
2. The first term …. is 4 (signifier: 4, 7, 10, 13...) ←→ (signified: f = 4)

In reference to the task, the array symbolizes a general expression into which some numbers are input according to some rule. To an observer, the problem could be that the student does not apply the 'correct' term (n-1). However, a correct expression in this approach would still not satisfy the theoretical conditions for a general term of arithmetic sequence. The generalization has no deep connection to the problem, focusing only on two terms of the array (4, 7). In terms of the epistemological triangle, the student's solution is represented in Figure 6

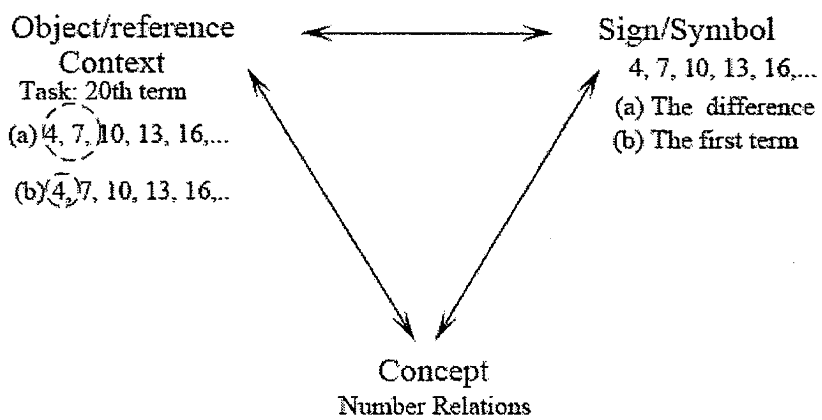

Figure 6: *Student S1's Solution*

Student S2's Solution (10—14)

To the second student, the sequence symbolizes relations in numbers involving terms, and common difference. She uses 20 in a similar manner to the first student, but interprets new symbols and references. She 'deletes' 10 in order to obtain an addend

‘1’, “If we didn’t” have this 10”: 4, 7,--, 13.... However, she is aware that this is a particular solution. Therefore, she confirms the addend using the fifth term: to obtain a pattern. The semiotic process in her argument proceeds as follows:

1. 1 You subtract, the difference here is ... (signifier: 4, 7, 10, 13,.... ←→ (signified: 7 - 4 = 3)
2. If we did’ t have ... (signifier: 4, 7, **10,** 13,...) ←→ (signified: consecutive terms of AS)
3. Look for the fourth term (signifier: 4, 7, --, 13, ...) ←→ (signified: fourth term of AS)
4. Same with fifth term (signifier: 4, 7, 10, 1316,... ←→ (signified: fifth term of AS)

The student used all the terms in the array, showing that her interpretation is connected to the problem given. Through these exchanges of signs and reference contexts, she applies the pattern to get the 20th term.

$t_4 = 4 \times 3 + 1$
$t_5 = 5 \times 3 + 1$
$t_{20} = 20 \times 3 + 1$
The solution shows that the student is moving from particular to general (Otte, 2006; Steinbring, 2005a), and relationally interpreting the array of numbers. In terms of the epistemological triangle, the student constructs new knowledge that is related to the problem as shown in Figure 7.

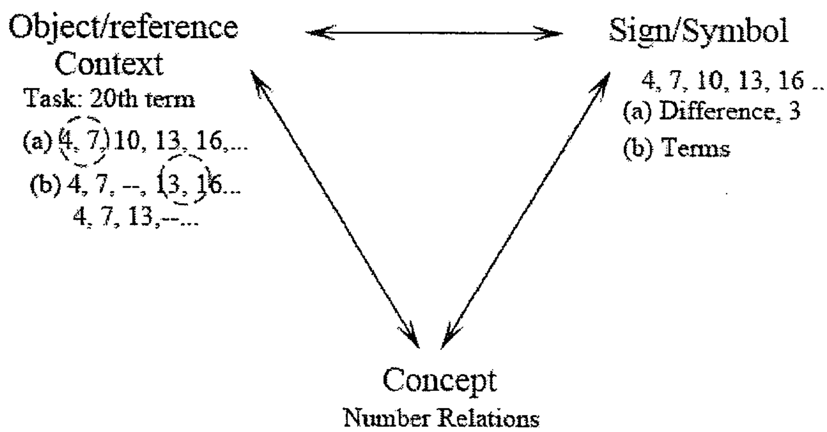

Figure 7: *Student S2’s Solution*

3.5 Discussion

The matchsticks arrangements in Figure 4 can only produce, in a direct way, the sequence 3, 7, 10, 13, 16... (See also Table 2) In addition, reference to the coefficients of an already known index; the‘d’, may promote isolated generalization seen in Student S1’s response. On the contrary, the student’s solution generalizes for the matchsticks and the second sequence also. The conventional solution that the activity of matchsticks produced is an equivalent form of the student's solution, where $k = a - d$. The solution, therefore, shows invariance that also solves the second sequence (b); 3, 8, 13, 18...: $t_{20} = 20 \times 5 + (-2)$. The student is actually working with the same sequence, finding, from the conventional expression, the $(n + 1)^{th}$ term, when the first term is $a^* = a - d$. That is, she is finding the 21st term of 1, 3, 7, 10...

$t_n = (n - 1) d + a$
$t_n = nd + k$
$t_n = nd + a - d$

Consequently, there is a dilemma whether to continue with the activity, because of emphasis on 'hands-on' activity, or explore the equivalence in the students' work. On the other hand, there is a dilemma whether these tasks reflect the expectations of Grade 11, vis-à-vis progression to further education. The implications are, first a need for collaboration with mathematicians and educators in pre-service institutions to enhance content of mathematics and integrate situate current practices on established theoretical frameworks. Secondly, collaboration with those who develop resources appropriate to the needs of secondary schools, and teachers who use the resources. . The teacher's position in the preceding episode also needs to be appreciated-- how to proceed with a lesson where objective has been met by a student. This requires a mutual approach to encourage more teachers to volunteer their lessons for analysis and reflection, and to enhance understanding of epistemological status of mathematical knowledge.

4. CONCLUDING REMARKS

The tension between theory and practice require bridges which honour the constraints in regular classrooms and challenge existing practices. However, recognition of the existence of theory and practice, pre-service and in-service is necessary. Prospects in mathematics education in Kenya lie in the capacity of CEMASTEA to build a database of lessons, but it has to overcome institutional and technical challenges: while it is independent from pre-service and 'in-services' all mathematics and science teachers, its certificates are yet to be recognized by the employer. Besides, whereas its emphasis is on teaching methodology (ASEI Lesson-Planning), it is yet to significantly incorporate content in respective subjects.

The analysis of the "20th term" episode illustrates a need for caution in using hands-on activities, and reveals time and pedagogical constraints--where some students' solutions may pre-empt the content planned by the teacher. It further provides insights on some problems observed by teachers regarding ASEI lessons, and may initiate collaboration amongst participants in mathematics education. Although video data may be analyzed from alternative perspectives, the *epistemological triangle* may significantly address the transitional needs of secondary mathematics education in Kenya. Significant impediments to these prospects are in capacity development of those who teach, analyze, and research on regular classrooms--SMASSE/CEMASTEA has teachers and teacher educators and few researchers. In addition, existing visions of 'good' teaching and comparative analysis may an obstacle in accepting the real classroom situations as they are. Consequently, there is need for more studies in regular secondary schools and inter-institutional collaboration between pre-service and in-service in the area of further education; and cross-national collaboration in research could also be a prospect in terms of alternative theoretical and analytical frameworks. It is only then that we may respond to the opening rhetoric in Cooney and Krainer, (1996).

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SEMIOTIC CHAINING IN AN EXPRESSION CONSTRUCTING ACTIVITY AIMED AT THE TRANSITION FROM ARITHMETIC TO ALGEBRA[1]

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ABSTRACT

In the transition from arithmetic to algebra, it is important to create a learning environment which develops the way in which students view mathematical expressions. This paper reports how students may develop their views through an expression constructing activity. As the result of our analysis in terms of nested semiotic chaining, we identified four states of sign combinations and chaining that show how students progress in their view of mathematical expressions, and discussed the important role of the use of brackets in viewing an expression structurally.

1. PROBLEM OF THE TRANSITION FROM ARITHMETIC TO ALGEBRA

A number of studies have revealed that students' errors in school algebra may result from differences in viewing mathematical expressions between arithmetic and algebra (Sfard, 1991). For example, students have difficulty interpreting e.g. x + 7 as the result of a calculation, while they may recognize it as the operation of "7 added to x" (This problem is known as the gap between operational and structural conceptions). A focus of this study is to develop a learning environment that may help students to change their operational conceptions into the structural ones.

Research into school algebra has tended to focus on the teaching and learning of symbolic expressions (Kieran, 1992). However, it has recently been reported that we can promote students' algebraic ideas even in arithmetic (Carpenter et al., 2003). It may, therefore, be productive to examine how the learning of numerical expressions can be connected to symbolic expressions (Miwa, 1996) and also to examine the jump

[1] This paper is included in *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4*, 2006.

that students experience when the object of their thinking moves from numbers (quantities) to the relationships between numbers (quantities) (Koyama, 1988).

In this paper, we shall investigate students' learning of mathematical expressions in a teaching experiment designed for the unit "Four operations with positive and negative numbers", which is taught in Japan just before the unit "Algebraic expressions using letters". We shall examine how such learning may facilitate the students' transition from arithmetic to algebra, beyond just acquiring the calculation procedures.

2. THEORETICAL FRAMEWORK

2.1. Epistemological characteristics of constructing mathematical expressions

We believe that constructing composite expressions might provide good opportunities to develop the way in which students look at such expressions, since an algebraic expression usually consists of one composite expression and not a number of binary expressions, and a linear equation needs to be formulated in terms of the equivalence between composite expressions.

It is important to note that constructing composite expressions is consistent with the mathematical nature of the expression. A mathematical expression is defined as a finite sequence of symbols such that (a) the object symbols (1, 2, ... , a, x, ...), (b) the operation symbols (+, –, ×, ÷, ...) and (c) the brackets are arranged in accordance with the following rules (Hirabayashi, 1996): (Rule 1) The object symbol is an expression in itself, (Rule 2) If both A and B are expressions, then (A) + (B), (A) – (B), (A) × (B) and (A) ÷ (B) are also expressions, and (Rule 3) All that are constructed using (Rule 1) and (Rule 2) are single expressions. According to these rules, expressions like ((((1) + (2)) × (3)) – (5)) are constructed one after another. Of course, we can omit some brackets by applying supplementary rules such as the precedence of multiplication and division, and then get the normal representation (1 + 2) × 3 – 5. We may also notice that in rule 2 the language is spoken in terms of metalanguage (Allwood et al., 1977; Jakobson, 1973; Hirabayashi, 1987).

Our focus is that, in constructing the longer expression, each expression is conceived of as a unity, since the operation is carried out between two expressions rather than two numbers. Using Douady's (1997) terms, the expressions may then be regarded not as "tools", but as "objects".

2.2. Semiotic chaining as the framework for analysis

Presmeg (2001) proposed a model of nested semiotic chaining based on Lacan's inversion of Saussure's dyadic model (signifier and signified) and Peirce's triadic model (object, representamen, and interpretant) (Fig. 1). The model emphasizes the productive role of the signifier (R) (= the representamen), the chaining by which a signifier in a previous sign combination becomes the signified (O) (= the object) in a new sign combination, and that "each new signifier in the chain *stands for* everything that precedes it in the chain" (p.7).

We think that this gives a useful perspective for analysing the activity of constructing mathematical expressions. Namely, we expect that it may make it possible to describe the states of the sign combinations associated with the transition from the operational to the structural view of an expression.

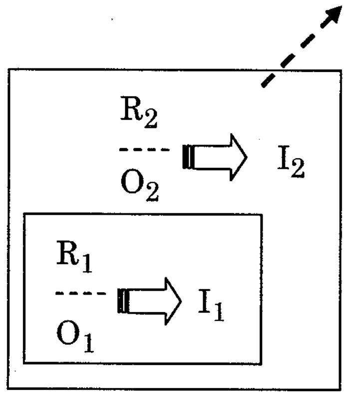

Figure 1. Presmeg's model of nested semiotic chaining.

3. TEACHING EXPERIMENT

3.1. Participants

The teaching experiment was performed with 28 seventh graders in a classroom of a public junior high school in Japan, with the collaboration of a teacher who had fifteen years experience and was interested in developing his lessons. Most students are not so willing to talk in a whole class situation but will talk with each other in small groups. Therefore, we needed to design lessons that would encourage them to participate in the class activities and so we chose a game as the basic format for our teaching experiment.

3.2. Task

We devised "The Expression Constructing Game" which is played by two groups. Each group is issued in advance with cards displaying various expressions. Suppose that group A has two cards $\boxed{(+3) \times (\quad)}$ and $\boxed{(\quad) \div (\quad)}$, and that group B has $\boxed{(-1) \times (-2)}$ and $\boxed{(-1)^3}$. The teacher announces the initial expression ("-12 = (+4) × (-3)"), and the students must then make a longer expression by replacing either of the numbers in the expression with an appropriate card. Any suitable number may be inserted in the empty brackets. The first group that manages to incorporate all of its cards into the expression is the winner. A sample record of a game is shown in Figure 2. If group A had replaced +4 with (+8) ÷ (+2), then group B could have used their (-1) × (-2); however, group A blocked it. This is a feature of the game.

$$\begin{aligned} -12 &= (+4) \times (-3) \\ &= (+4) \times \underline{(+3) \times (-1)} \\ &= (+4) \times (+3) \times \underline{(-1)^3} \\ &= \underline{(-8) \div (-2)} \times (+3) \times (-1)^3 \end{aligned}$$

Figure 2: A record of the expression constructing game.

Even when a card is incorporated, it can happen that the value of the whole expression may be incorrect if brackets have not been used. If the value is different from the original one, then points are not given. Thus, as well as checking the correspondence between the replacement expression and the number it replaces, the students must also check the correspondence between the whole expression and the original number. In other words, in this game both the construction and the calculation of the expression are being carried out at the same time.

3.3. Teaching Experiment

Our teaching experiment was conducted according to the methodologies of Confrey and Lachance and Cobb (described in Kelly and Lesh, 2000). Our conjecture was that in the act of constructing successive expressions students would upgrade their view of expressions from an operational to a structural one. We were interested in when and how this development might occur and what factors might sustain it.

The experiment continued for 12 hours during which the unit "Four operations with positive and negative numbers" was covered. During the first six hours, the students learned each of multiplication, division and involution with negative numbers (Addition and subtraction had already been taught before the unit). The data in this paper were obtained from the 7th to 11th lessons, in which the expression constructing games were conducted. The lessons were recorded on video camera, field notes were made, and transcripts were also made of the video data.

Two types of data analysis were conducted. First was the ongoing analysis after each lesson. Here we analysed what happened in the classroom in terms of the students' activities and utterances. Then we modified the subsequent lesson plan by taking into account both the original plan and our analysis of each lesson. Second was the retrospective analysis that occurred after all the classroom activities had finished. We first divided the classroom episodes into the meaningful entities chronologically in terms of what situations appeared to have made the students' conceptions change, and next analysed how they interpreted the situations based on the sign combinations. Finally, we made sense of the overall story of their learning by reviewing all the analyses in terms of semiotic chaining.

4. THE ACTIVITY OF THE EXPRESSION CONSTRUCTING GAME

4.1. Introducing the expression constructing game

After a brief explanation of the rules, four students were chosen to represent the two groups (A: Yoshi and Asa; B: Seki and Hoshi) and played a demonstration game on the blackboard. On this occasion the number of cards was limited to four and the cards were expressions of multiplication, division and involution (Fig. 3).

A. 1. $\boxed{(-4)\times(\ \ )}$ 3. $\boxed{(-1/3)\times(+12)}$ B. 1. $\boxed{(\ \ )\times(\ \ )}$ 3. $\boxed{(-1)\times(\ \ )}$

2. $\boxed{(\ \ )\times(\ \ )}$ 4. $\boxed{(-2)^2}$ 2. $\boxed{(-1)^3}$ 4. $\boxed{(-12)\div(+2)}$

Figure 3. The first situation for introducing the game.

T (teacher): Let's decide which team goes first. The team that answers ahead is first.
[He wrote "$-16 = (-2) \times (\ \ \ )$" on the board.]
Yoshi: +8
T: The game will start with team A. Please replace any one of your cards.
Yoshi: No. 1. [She wrote "$= (-2) \times \underline{(-4) \times (-2)}$."] (Underlining added by author.)
S (a student): I agree.
T: Well, now team B, please. Thirty minutes.
Hoshi: [He wrote "$= \underline{(-1) \times (+2)} \times (-4) \times (-2)$."]
Yoshi: [She wrote "$= (-1) \times \underline{(-2) \times (-1)} \times (-4) \times (-2)$."]
Seki: [He wrote "$= \underline{(-1)^3} \times (-2) \times (-1) \times (-4) \times (-2)$."]
S: It's wonderful!

The game ended in a draw as both teams completed the expressions successfully. After this, the teacher and the students together worked out the final expression to see whether it went back to the original number (-16). When the answer turned out to be -16, the students unanimously said "great", "wonderful" and clapped their hands. We found that they were surprised that they could make such a long expression and yet the result of the calculation coincided with the original number.

Then the teacher asked them what part of the fifth expression corresponded to -2 in the original expression, and the students confirmed that it was part of $(-1)^3 \times (-2) \times (-1)$. In so doing, he hoped to encourage them to think of the expression as a unity.

4.2. Student's difficulties and overcoming them using the brackets

In the 8th lesson, a problem occurred in one small group. After the group activities, the teacher let three groups present their records of the games. Of course this included the group that had experienced the problem. The record of this group is shown in Figure 4.

$$
\begin{array}{rll}
-8 & = (+32) \div (-4) & \\
 & = (+32) \div (-4) \times (+1) & [\ \boxed{(\quad)\times(\quad)}\ ] \\
 & = (+32) \div (-4) \times (-1) \times (-1) & [\ \boxed{(-1)\times(\quad)}\ ] \\
 & = (+32) \div (-8) \div (+2) \times (-1) \times (-1) & [\ \boxed{(\quad)\div(+2)}\ ] \\
 & = (+32) \div (-8) \div (+2) \times (-1)^3 \times (-1) & [\ \boxed{(-1)^3}\ ]
\end{array}
$$

Figure 4. The record of the game in one small group.

T: Well, Fuji, please tell us about the situation in your group.

Fuji: It is strange. [She pointed to the last expression] Here, 32 divided by -8 is -4. Then divide it by 2, and the answer is 2, because the rest of the numbers are all 1s.

Yoshi: Mr. Kuro, can I write on the board? It is not good from here to here. [She added the underlining.]

$$
\begin{array}{l}
[\ = (+32) \div \underline{(-4)} \times (-1) \times (-1) \\
\quad = (+32) \div \underline{(-8) \div (+2)} \times (-1) \times (-1)\ ]
\end{array}
$$

T: Please raise your hand if you can see their problem.

S: [All students raised their hands.]

This group was worried because the answer was not -8 once they had changed -4 into (-8) ÷ (+2). And, although they had discovered which replacement the mistake had resulted from, they could not see how to deal with it.

At this point, one student said "we can use (-2) ÷ (+2) instead of (-8) ÷ (+2)". He made this suggestion so that the value of the whole expression would be -8. However the idea was soon rejected by the other students because it violated the rule that the number must be replaced with an equivalent expression. After a while Jo said "Is it all right to add brackets? There!" The teacher asked her to write on the board.

Jo: [She wrote the brackets "$(+32) \div \underline{\{(-8) \div (+2)\}} \times (-1) \times (-1)$".]

S: Oh!

S: That's right! (with great surprise)

S: Yes, brackets!

T: The order is changed, isn't it? We do here, these brackets first. [He checked the calculation with the students.] What about the next expression?

S: Well, we add the brackets there.

T: Don't you think the brackets are great and powerful?

These exchanges were so influential that all the students now seemed to appreciate that brackets could make the order of the calculation change. In fact, in the next game, we could hear comments such as "The big bracket -4 plus the bracket -2 ... We cannot do without using the brackets" from many of the small groups.

4.3. The development in the students' views on the expressions

Though we could hear lots of students' comments on the use of the brackets by the end of the 9^{th} lesson, at the same time they sometimes used brackets unnecessarily, perhaps because they had been so strongly impressed by the use of the brackets in previous lessons. However, in checking the records reported on the board in the 10^{th} lesson, the students began to notice that there were unnecessary brackets, as a consequence of an implicit suggestion made by the teacher (Two out of the three records included unnecessary brackets).

T: (In checking the expression in which unnecessary brackets were not included) ... times, divided by, times and divided by. So, as no brackets are included, let's calculate it from the left.

S: Oh, I see. The brackets are not necessary in our expression!

S: It makes no sense.

S: Mr. Kuro, please delete those brackets. They make no sense. [He pointed to the expression "{(-2) × (-1)} × (-9) × (-9) ÷ (-3)".]

S: Mr. Kuro, please delete ours too. It is the top brackets. [He pointed to the expression "{(+36) ÷ (+2)} × (-3)".]

When the teacher asked them whether the brackets could be removed in checking the values of the whole expressions, they were able to answer well. But he did not ask them under what conditions the brackets could be omitted. If he had asked the conditions for the omission of the brackets, they would have had a further opportunity to think about the structure of the expression.

In the 11^{th} lesson, we observed another scenario where the students had not utilized brackets. One student changed "-10 = (-30) ÷ (+3)" into "= (-30) ÷ (-12) ÷ (-4)" on the board, and no one in her group remarked on the lack of brackets. However, it was soon refuted by the other group, with comments like "If we calculate it from the left, it doesn't go well". We concluded that they still didn't have a clear awareness of the usage of brackets and a similar state of affairs was the case in the 12th lesson too. However, through correcting these situations again and again, it seemed that they eventually became aware of the necessity for brackets, the order of calculation and the characteristics of operations. For example, they made the following expression as a final form.

"$= \{(+1) + (-31)\} \div \{(-8) \times (1/2) \times (+3) \div (-4)\}$"

When the teacher then asked the students whether the brackets ahead of (-8) could be removed, they could in unison state that it was impossible. This recognition seemed to indicate that they were now able to adopt a structural view of a complex expression, even one they had never previously met.

5. DISCUSSION

We may distinguish at least four states of the students' views of the expressions.

Expression A ----- ⇨ I_A Number

Fig. 5.1.

Longer Expression B' ----- ⇨ I_B Expression B

Fig. 5.2.

Expression with brackets C' ----- ⇨ I_C Expression C

Fig. 5.3.

Expression without or with brackets D' ------ ⇨ I_D Expression with brackets D

Fig. 5.4.

First, when the students replaced a number with an expression A as one step in the game, the expression may be seen as equivalent to the number (Fig. 5.1). In other words, it may not be conceived as something to be calculated but as a unity. We think this an important step in starting to view the expression structurally, and the game makes it emerge in a meaningful way.

Second, when we see the game as a whole, the longer expressions B' were constructed one after another based on the old expressions B so that their values were kept constant (Fig. 5.2). Here the expressions themselves were handled as objects (Douady, 1997). Also their recognition of each expression as a unity seemed to be facilitated through the teacher's navigation that led them to compare a certain part of the expression with the corresponding part of the other equivalent expression. We also found that successful completion of the expression constructing activity was often greeted by the students with surprise.

Third, the students found that the expression with brackets C' might regulate both the parts and the whole of expression C (Fig. 5.3). Namely, the idea of using brackets enabled them to resolve inconsistencies between the replacement of a number with a partial expression and the value of the whole expression, and again this was greeted with surprise. We think that the brackets contributed to making them see the expressions as unities.

Fourth, they modified the expression with brackets D into the one without brackets D' (Fig. 5.4), and through it were able to see whether adding or omitting brackets in the expression would change the order and structure of the whole expression. It should be noted that such recognition was gained after correcting some errors.

Overall, it is clear that these four states can be structured in terms of nested semiotic chaining. That is, we can see that the signifier in a previous sign combination became the signified in a new sign combination and that each new signifier in the chain stands for everything that precedes it in the chain (Presmeg, 2001). We may understand this chaining as both the process by which the view of the expression as a unity was developed and the process by which the role of brackets were recognized. Just as Radford (2003, p.62) stated that brackets "become essential because they help the students mark the rhythm and motion of the actions", it seemed to us that eventually the students could read the order and structure of the whole expression from the brackets. Prior to this experiment, all that the brackets had meant to the students was a command to indicate the precedence of calculation.

Namely, it seems that the brackets play a role not only in object language but also in the metalanguage for telling about it (Hirabayashi, 1987; Allwood et al. 1977). It may be similar to the way in which the plus and minus signs are used to show the meanings of adding and subtracting as well as positive and negative numbers and moreover the algebraic sum (Sfard, 1991). Thus we believe that providing students with an appropriate view of the role of brackets can be an important girder in the bridge from arithmetic to algebra, as a proper awareness of this is deeply related to the structural conception of expressions. However, we think it will be necessary to do a more detailed semiotic analysis, such as Radford (2003), in order to clarify the transition process, which is our future task.

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CHARACTERISTICS OF 5TH GRADERS' LOGICAL DEVELOPMENT THROUGH LEARNING DIVISION WITH DECIMALS[1]

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ABSTRACT

When we consider the gap between mathematics at elementary and secondary levels, and given the logical nature of the latter level, it can be seen as important that aspects of children's logical development in the upper grades in elementary school be clarified. We focused on 5th graders' learning "division with decimals" as its meaning is known to be difficult for children to understand because of certain conceptions which children have implicitly or explicitly of division. We discussed how children might develop logic beyond such difficulties in the process of making sense of division with decimals in the classroom setting. We suggested that children's explanations based on two kinds of reversibility (inversion and reciprocity) were effective in overcoming the difficulties, and that they enabled children to conceive multiplication and division as a system of operations.

1. INTRODUCTION

In learning operations with decimals or fractions, children tend to acquire only mechanistic procedures, like "invert and multiply" in division with fractions. However, there are gaps between mathematics at primary and secondary levels, and mindful of the logical nature of the latter, we think it necessary to encourage children to develop the logical reasoning required in upper grades while still at elementary school.[2] This study focuses on division with decimals and aims to clarify the process of extending the meaning of division beyond the integer domain.

Over a period of twenty years, a considerable body of theoretical and empirical research has shown that children and even adults have difficulties in solving multipli-

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[1] This paper has been published in *Educational Studies in Mathematics*, *60*, 2005.

[2] In Japanese educational system, the elementary school continues from 1st to 6th grades, lower secondary school the 7th through 9th, and upper secondary school the 10th through 12th.

cative word problems with decimals (e.g. Bell *et al*, 1989; Fischbein *et al*, 1985; Greer, 1992; Harel *et al.*, 1994). These reports have identified the difficult factors in students' backgrounds when they solved word problems. However, it is still not so clear how such difficulties might be overcome. In particular, we think it has not been clarified how children might learn division with decimals in introductory classroom lessons, because all subjects in the above noted reports were students or adults who had already learned multiplication and division with decimals.

Thus, this paper will be devoted to clarifying how children could overcome the difficulties, and how they might then develop logical reasoning in introductory lessons of division with decimals. We firstly examine the fundamental characteristics of division with decimals, both mathematically and logically. Next we review the psychological difficulties that previous studies have highlighted. After that, we present 5^{th} grade's classroom data designed implemented with a collaborating teacher. Finally, we discuss children's thought processes and the logical characteristics in extending the meaning of division beyond the integer domain.

2. THEORETICAL BACKGROUNDS

2.1. Fundamental characteristics of division with decimals and the possibilities of growth in children's logical reasoning

We begin by briefly examining two problems;

(A1) If 12 apples are fairly shared among 3 persons, how many apples does one person get? and

(A2) The price of 2.8 m of ribbon is 560 yen. How much does 1 m cost?

M_1	M_2
1	x = f (1)
a	b = f (a)

Ribbon (m)	Price (yen)
1	x
2.8	560

Figure 1. Vergnaud's schematic representation of isomorphism of measures.

Both have the same structure because each quotient is a quantity-per-unit and the both permit proportional reasoning, though they are very different in the psychological sense. Mathematically saying, "If (a, b) is any ordered pair of rational numbers and (a, b) ~ (c, d) is defined as ad = bc, the relation '~' is equivalent. Thus, (partitive) division then means to transform the element (a, b) into (quotient, 1) of the equivalent class (a: dividend, b: divisor)". Using Vergnaud's (1983) schematic representation would be useful in clarifying the relationships among elements (Fig. 1).

He termed "a structure that consists of a simple direct proportion between measure-spaces M_1 and M_2" as 'isomorphism of measures' (Vergnaud, 1983) where the relationship within each measure space is a scalar operator and the relationship between measure spaces is a function operator *f*. The schema enables us to conceive various multiplicative problems like equal sharing, constant price (goods and costs), and uniform speed (durations and distances). Here partitive division is conceived as finding the unit value f (1) in the schema. This class of problems can be solved by applying the scalar operator (÷ a) within measure space M_1 to the magnitude b.

However, we also need to pay attention to at least two replacements as previous steps or as useful strategies when children solve problems using their existing knowledge on division with integers. These can be represented as follows (Fig. 2).

Ribbon (m)	Price (yen)
2.8	560
28	560 x 10
1	x

Ribbon (m)	Price (yen)
2.8	560
0.1	560 ÷ 28
1	x

Figure 2. Two replacements of the division problem using Vergnaud's schema.

Namely, a pair of (560, 2.8) could once be transformed into other pairs of (5600, 28) by multiplying each by 10 or (20, 0.1) by dividing by 28, and successively into the pair of (200, 1) which is the quotient. These ideas are consistent with the mathematical view of division, where a pair of dividend and divisor forms a representative element of an equivalent class such that the quotient in any division is constant, rather than only two components of an operation (Schwarz, 1988).

We consider these views important for the growth of children's reasoning towards secondary mathematics, which seems to be conceived in terms of Piaget's formal operational thinking. Inhelder and Piaget (1958) used the notion of "formal operation" to characterize adolescent thinking starting from about 11 years of age. They noted that (A) it can proceed from some hypothesis or possibility, (B) it can be characterized as propositional logic by combining the statements p and q, p or q, not p, p implies q (Jannson, 1986), (C) the object for thinking is the generality of the law, the proposition, etc., and (D) it includes two kinds of reversibility. In exemplifying these characteristics, it may be helpful to consider such properties in division as:

(1) $a \div b = (a \times m) \div (b \times m)$; (4) $(a \div m) \div b = (a \div b) \div m$;
(2) $a \div b = (a \div m) \div (b \div m)$; (5) $a \div (b \times m) = (a \div b) \div m$;
(3) $(a \times m) \div b = (a \div b) \times m$; (6) $a \div (b \div m) = (a \div b) \times m$.

For example, what we transform it into $(560 \div 28) \times 10$ corresponds to the property (6) where a = 560, b = 28 and m = 10. We think it obvious that such transformation into another imaginary situation includes some hypothesis or possibility and the object for inquiry is then toward a general mechanism of division beyond just finding an answer. Furthermore, if why $560 \div 2.8$ may be 200 is justified by mediating another expression, the reasoning may have a syllogistic character.

In (D), one reversibility is inversion, which enables one to "return to the starting point by canceling an operation which has already been performed" (Inhelder & Piaget, 1958, p.272), and the other is reciprocity, which is related to "compensating a difference" (p.273) and is "required for equating operations which are oriented in opposite directions"(p.154). As to division, we think that inversion corresponds to the thinking which transforms division into multiplication as the reverse operation, and reciprocity to the thinking which regards division itself as multiplication as the equivalent operation.

Thus, we think the learning of division with decimals is considerably related to the growth of formal operational thinking. We will examine these properties in children's practical learning of division with decimals. Before that, we review previous researches on children's difficulties on division with decimals.

2.2. Children's difficulties regarding division with decimals

Here, we shall return to the problems (A1) and (A2) described above. Though we stated the both had same structure mathematically, it must be emphasized that in a child's mind they are greatly different with each other. That is, problem (A1) permits one to imagine the situation where something is divided into equal parts, however problem (A2) doesn't permit this thinking.

Since the early studies in this area researchers have identified several misconceptions which resulted from children's experiences in division with integers. The typical misconception is that "multiplication makes the answer bigger, and division makes it smaller" (Brown, 1981; Bell *et al.*, 1981). Fischbein (1985) considered that children's difficulties on multiplicative word problems involving decimals resulted from their primitive and implicit models, and explained that in partitive division model, there are such constraints as "the dividend must be larger than the divisor", "the divisor must be a whole number", and "the quotient must be smaller than the dividend", thus they had difficulty solving the problems which violated these constraints. Also, Fischbein (1989) stated that the models are robust and able to survive long after they no longer correspond to the formal knowledge acquired by the individual.

Several researchers have instead reported that numbers used or students' numerical preferences had intervening effects (Bell *et al.*, 1989; De Corte & Vershaffel, 1996; Harel *et al.*, 1994); though logically speaking, numbers don't influence the choice of operation (Schwarz, 1988). In particular, students' performances somewhat decrease when the multiplier or the divisor is a decimal, and still more drop when they are less than 1. In Bell *et al.*'s (1989) research, for example, students' (15-year-olds) percentages of correct answers were 40 to 50 % for division word problems with decimal divisors. It should be also noted that such numbers influence the choice of operation, but not the estimation of the answer. In fact, they achieved more than 30% high scores in the estimation-of-answer test.

As a phenomenon that remarkably shows the effects of the types of numbers, Greer (1987) proposed the notion of "non-conservation of operations"; that given two problems successively, which are equivalent except for the types of numbers involved, children don't recognize an invariance of the operations but changed the operation. This implies that the strategy of assuming an easier problem has little effect because they may change their operation while substituting integers for decimals.

Next, we describe and analyze 5th graders' learning of division with decimals.

3. METHODS

Our teaching experiment was implemented in collaboration with a teacher in a university attached primary school. The school has a character of an experimental school in which the teaching and learning of mathematics is practically developed. Classroom lessons were conducted for 38 fifth graders (20 boys and 18 girls). Before our teaching experiment, during their fourth grade, the children had already learned teaching units for "decimals" and "multiplying and dividing decimals by integers", and "multiplying by decimals" was learnt just before the experiment.

We designed three problems to be given in the following sequence to students during the teaching experiment.

Q1. The price of 2.5 meters of ribbon is 100 yen. How much does 1 meter cost?

Q2. The price of 2.4 meters of ribbon is 108 yen. How much does 1 meter cost?

Q3. The price of 0.8 liters of juice is 116 yen. How much does 1 liter cost?

We didn't include any divisions with a decimal dividend and positioned the problem in which the divisor is less than 1 as the final problem, because previous studies indicated that the influence of a divisor (especially less than 1) is more powerful than that of a dividend, and we considered that children could reconstruct the meaning of division in overcoming the influence of a divisor. Contrarily, we considered the problem in which the decimal part of the divisor is 5 relatively easy and accessible by using their knowledge of division with integers.

In solving such problems, a teacher's orientation is crucial for children to construct the meanings of division with decimals. The teacher's teaching in the lessons were planned to have three characteristics.

First, the teacher prompts children to conceive the division problem as having a lot of equivalent problem situations. Problem 1 worked well to construct those situations, for they could make a new situation using just two pieces of strip. We think it can be a way to avoid the phenomenon of "non-conservation of operations" in which they didn't conceive division by decimals less than 1 as division but multiplication.

Second, the symbolizing processes (Gravemeijer and Stephen, 2002) are assumed to develop the meaning of division (Fig.1). We think that mathematical understanding would progress along with the symbolizing processes (Dörfler, 1991). Then, we devised the operative material as mediating between the concrete materials and the number line ("operational number line"), which seems to connect the concrete view to the proportional view of division. Furthermore at a meeting after the third lesson, we decided to use the schema of proportion to promote the proportional view.

Third, the teacher often consciously formed some situations of cognitive disequilibrium (Piaget, 1985) that emerged from either intersubject or intrasubject conflicts (Cobb, Yackel, & Wood, 1993), for we expected children might develop their meaning of division with decimals in overcoming the disequilibrium. However, we decided not to give the conflicting data intentionally, but to form the opposed ideas they presented as a situation that they should resolve, because we attach as much importance as possible to their constructions, as is seen in the constructivist perspective.

The lessons were recorded by a video camera and field notes made; transcripts were also made of the video data. We could also use the children's writings in our analysis. Using these data we conducted qualitative analysis.

A characteristic of our analysis is to focus on the children's disequilibrium that emerges in classroom lessons. We analyze it as being the difference between the idea that they are newly conscious of and their existing conceptions. For that, we organize the meaningful episodes in the lessons chronologically in terms of in what conditions the disequilibrium may emerge and how children may grapple with it. Such disequilibrium may emerge as the intersubjective conflicts that students manage to resolve, because it is usually made explicit as the differences among ideas in the collective; we then regard the classroom practice reflexively related to children's mathematical conceptions and activity (Cobb, Yackel, and Wood, 1993).

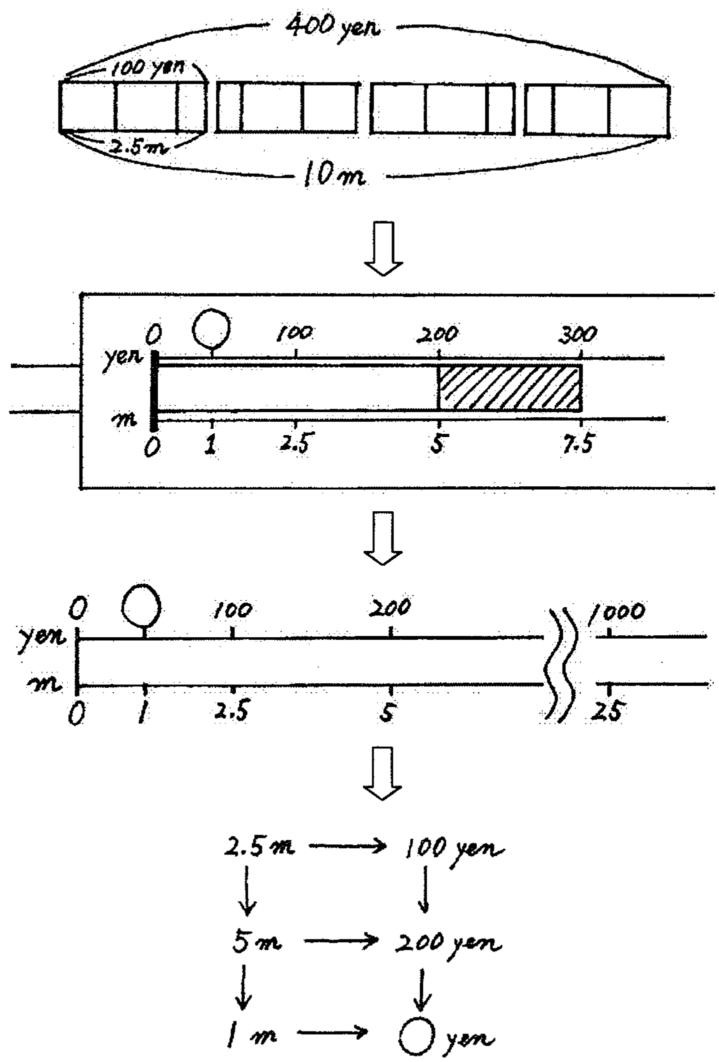

Figure 1. Symbolizing processes in developing the meaning of division with decimals; pieces of strip, operational number line, number line, and schema of proportion.[3]

4. RESULTS AND ANALYSES

The teaching experiment eventually needed six lessons, in which each lesson was 45 minutes, except for the fifth lesson which was just 30 minutes. However, basically four lessons were spent solving and discussing the three posited problems and constructing the meaning of division with decimals; and the last two lessons were used to confirm and deepen children's understanding of that which had been constructed. Thus we will mainly analyze the activities during the first four lessons.

4.1. Initial understanding of division with decimals

The first classroom lesson began with Question 1. The teacher prompted children by

[3] Operational number line is made by passing the long strip onto the paper on which the number line is written.

pairs to solve it in various ways using many pieces of strip each representing 2.5 meters. After the activities by pairs, they in turn presented their ways of solving in the whole class setting. Here three ways were eventually presented by the children.

Method 1 was to combine two pieces of 2.5 meters and to transform the problem into the division with integers (Fig. 2): For example, “Well, as this is 2.5 meters and 2.5 is a decimal, I add one more piece and change it into 5 meters. As I double (the length), I must also double the price. Then it becomes 200 yen. So, I divided 200 by 5 and get the price of 1 meter”. (Note: The figures in this section are copies from the blackboard and the Japanese translated into English by the authors.)

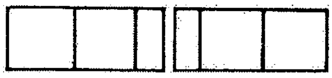

Figure 2. Method 1: To connect two pieces of 2.5 meters.

In method 2, one piece was divided into the same lengths as shown in Figure 3; “Well, first I divide it by 0.5 meters each (He drew the lines on the material on the blackboard). As the total is 100 yen, I divided 100 by 5 in order to get the price of 0.5 meters. It is 20 yen and it is the price of 0.5 meters. So, we can solve when we multiply 20 by 2 in order to change it into 1 meter”.

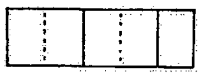

Figure 3. Method 2: To divide one piece into 5 equal parts.

Method 3 was to connect ten strips of 2.5 meters of ribbon.

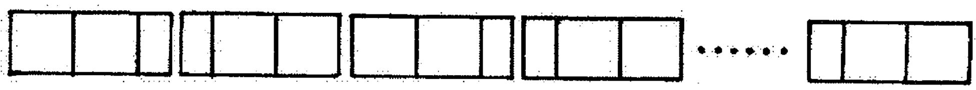

Figure 4. Method 3: To connect ten pieces of 2.5 meters.

Similar activities were subsequently undertaken, mainly using the operational number line for problem Q2 in the second half of the second lesson (Fig. 5). Through these two lessons in which the division problems with divisors bigger than 1 were solved and discussed, the following ideas were constructed and shared by children.

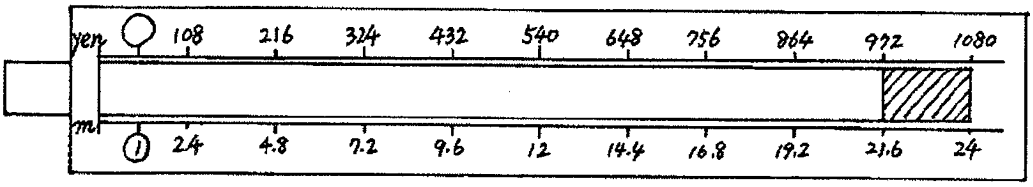

Figure 5. Traces of activities using the operational number line.

(a) There are many situations that are the same as what 108 yen is to 2.4 m.
(e.g., 216 yen is to 4.8 m; 324 yen is to 7.2 m; 432 yen is to 7.6 m; 540 yen is to 12 m; ... 1080 yen is to 24 m).

(b) If we multiply each number by 5 or 10, we can transform the problem into a division with integers.
(e.g., 108 (yen) ÷ 2.4 (m) = 1080 (yen) ÷ 24 (m) = 45).
(c) We can solve by first finding the price of 0.1 m.
(e.g., 108 (yen) ÷ 24 × 10 = 45).

In sum, the children's initial condition was to be able to consider the problem by transforming it into the equivalent divisions involving divisions with integers.

4.2. The birth of disequilibrium

In the third lesson, the teacher presented the problem Q3 in which the divisor was less than 1 ("The price of 0.8 liter of juice is 116 yen. How much does 1 liter cost? "). He encouraged children to solve the problem in pairs using the operational number line and to make mathematical expressions. They easily made expressions such as (116 × 10) ÷ (0.8 × 10) and found its answer utilizing (b) or (c) above.

However, when teacher then asked the children what the original expression was and a child answered "116 divided by 0.8" for that, another child questioned that expression. With this opinion as a start, they began to feel uncertain and so brought about the cognitive state of disequilibrium.

1 Arat: 116 divided by 0.8... Why is the expression right? It might not be 145.
2 Cs: It must be 145.
3 Arat: It might be 145, but the answer for the problem is not 145 yen.
4 Cs: Why? It must be 145 yen.
5 T: If 116 yen is to 0.8 liter, then 145 yen is to 1 liter. Is that wrong?
6 Naga: But, why do we get one liter when dividing by 0.8.
7 C: I think the answer is the price of 0.1 liter.

... ...

8 Yuki: I also think that if we do 116 ÷ 0.8, we get the price of 0.1 liter.
9 Cs: I agree!
10 Cs: No, it's wrong!

It was observed from the video data that seven children then considered the answer 145 as the price of 0.1 liter, though more children appeared to hold the same such belief. We consider this the influence of the implicit model or number effect; for such phenomena didn't come up previously. Analyses that follow are devoted to the stages and characteristics of children overcoming this difficulty.

4.3. Processes of overcoming the disequilibrium

4.3.1. Logical explanations and the robustness of the implicit model

The idea "the answer 145 yen to the problem is the price of 0.1 liter" was soon refuted. For example, the following opinions were presented.

11 Moto: If 145 yen were to 0.1 liter, then 0.1 liter was more expensive than 0.8 liter.

12 Mich: We must do $116 \div 8$ in order to work out the price of 0.1 liter.

13 Katt: (After pointing that both $116 \div 0.8$ and $580 \div 4$ have the same answer) If we divide by 4 liters, of course the answer is the price of 1 liter. The idea of 0.1 liter is strange.

14 Saku: 116 divided by 8 is 14.5, but 116 divided by 0.8 is... because of a tenth of eight, because the answer is 10 times 0.1 liters so-called 14.5, I think 116 divided by 0.8 remains as it is.

We can find the initial form of deductive reasoning in the above explanations. For example, Katt's utterance (13) is interpreted as the reasoning that statement 3 is deduced from statements 1 and 2 as follows.

Statement 1: If we divide 580 by 4, we get the price of 1 liter. (Agreed)

Statement 2: $580 \div 4$ can be equivalently changed into $116 \div 0.8$. (Agreed)

Statement 3: Therefore, $116 \div 0.8$ is the expression for finding the price of 1 liter.

Such discussion could temporally make them consent, for most of them judged what 116 divided by 0.8 should represent the price of 1 liter when the teacher checked the distribution at the end of the third lesson.

However, at the beginning of the fourth lesson when the teacher asked children whether they felt uncertain about the expression, 70 % acknowledged a feeling of uneasiness. Here again, children who were confident began to speak. For example;

15 Kawa: (Referring to the expression $116 \div 8 \times 10$) If we put this 10 in front, it is the same as $1160 \div 8$. So, I think this ($116 \div 0.8$) summarizes these expressions which were made to get the price of 1 liter of juice.

However after these explanations, when the teacher again asked children whether they could explain why the expression is the price of 1 liter, 80 % of them responded that it was difficult to explain.

16 Sasa: Though I don't know the reason, even with the division the quotient is bigger... than the dividend.

17 Naka: I can understand that if we divide something by 2, we get half. But I don't know how we get 145 when we divide by 0.8.

18 T: Do you think that "divided by 0.8" is a problem?

19 Cs: Yes. It's unclear and strange.

We find children implicitly experienced a cognitive state of disequilibrium. Here we can see that the opinions of Sasa (16) and Naka (17) were stated in terms of the violations of the partitive division model. This episode suggests that even if logical explanations are given, they aren't sufficient to overcome disequilibrium resulting from their experiences of division with integers. Though the expression 116 ÷ 0.8 was transformed into other expressions (e.g. 1160 ÷ 8), it seems that their difficulties do not resolve without discussing what "divided by 0.8" itself means.

4.3.2. The process of equilibration based on reversibility "inversion"

Equilibration was initiated from a child's utterance based on the inverse operation.

20 Saka: It's not good to consider 116 ÷ 0.8. By reversing it, if we think of the problem as "The price of 1 liter of juice is 145 yen. How much does 0.8 liter cost?", it will be 116 (He calculated it)... I got it. So, division means... even if the divisor may be a decimal or an integer, the answer is... to get 1 liter.

This opinion was very powerful for many children began to regard 116 ÷ 0.8 as being a valid expression for finding the price of 1 liter. Here it should be noted that this explanation included the meaning of division (the quantity-per-unit). But Saka's opinion was soon rejected because it had the character of checking after solving the division.

21 Arat: Saka said it's all right if we change it to a multiplication. However we used the division (the quotient) and so it's questionable whether it (the product) becomes 116. The explanation is not so meaningful.

22 T: Arat say that the multiplication is a story after finding the answer of division.

23 Saka: But, I proved it in such a way.

Here, Naka made the point more explicit.

24 Naka: Well, if it is 116 times 0.8, I can regard it to take 0.8 pieces of 116. But, please tell me how we do 116 ÷ 0.8. [4]

It seemed that Naka, and probably also other children, wanted to conceive division as a concrete operation. Saka's explanation based on the inverse operation was strong. However, children still somewhat remained in a concrete world, and they needed further explanations to reach a state of equilibrium.

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[4] In the Japanese notational system, we write 300×5 as the expression for the problem "The price of one apple is 300 yen. How much do 5 apples cost?" which is different from the English system.

4.3.3. The process of equilibration based on reversibility "reciprocity"

In the second half of the fourth lesson, the teacher proposed to rewrite the previous activities on the number line using the schema of proportion (Fig.6). After that, he asked to consider the meaning of "divided by 0.8" in the abbreviated schema (Fig.7).

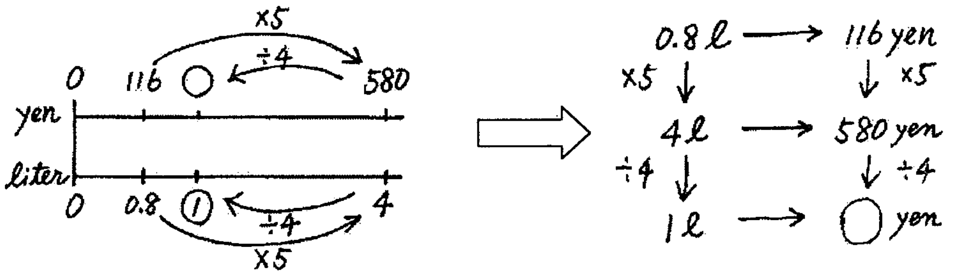

Figure 6. Activity on the number line and its translation to the schema of proportion

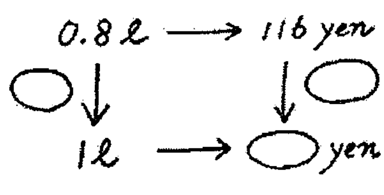

Figure 7. Abbreviated version of the schema of proportion.

25 T: Let us discuss by using these two parts.

26 Tach: (He wrote on the blackboard "× 1.25" beside the left blank)

27 T: Really? And what is it here?

28 Tach: (He wrote "× 1.25" beside the right blank.) 0.8 liter is 116 yen and 1 liter is 145 yen. I think some multiple of 0.8 liter is 1 liter. I calculated "1 divided by 0.8". I found 1.25.

29 T: Wait. 1 divided by 0.8? Oh, it's 1.25.

30 Tach: If we multiply 0.8 by 1.25, then of course we must also multiply the price by 1.25. So 116 times 1.25 is 145.

31 Cs: Yes. It's the same.

Though the teacher had expected them to put "÷ 0.8" into the left blank, actually "× 1.25" was natural for children. He then asked them to consider it without using the number 1.25 after he told that the number 1.25 was not written in the text of the problem. However, they persisted with the idea of 1.25.

Next, children reinterpreted the expressions familiar to them, e.g. 116 × 10 ÷ 8 as 116 × 1.25 and gained more confidence in the idea "× 1.25". Here, the teacher again tried to direct their focus to the relation between "× 1.25" and "÷ 0.8".

32 T: This is "times 1.25". Can you represent it by using division? By what do you

divide 0.8 liter in order to get 1 liter?

33 Naga: We divide it by 0.8.

34 T: If you divide 0.8 by 0.8, you get 1. Then by what do you multiply 1 liter in order to get 0.8?

35 Naka: We multiply it by 0.8.

36 T: If we multiply 145 by some number, we get 116. What is the number?

37 Sako: Oh, it's 0.8.

……………………………………………………………….

38 T: Is there anything you notice?

39 Arat: "× 1.25" and "÷ 0.8" are same.

40 T: Are they same? Everyone, check whether "÷ 0.8" is same to "× 1.25".

41 Cs: (They checked by some form of calculation.) Oh, they are the same.

……………………………………………………………….

42 T: (He wrote "A × 1.25 = _, A ÷ 0.8 = _".) I'm asking everyone whether multiplying some number by 1.25 is same as dividing it by 0.8 or not.

43 Cs: Yes. The same.

44 T: Do you think it's natural that the answer is bigger than the dividend?

45 Cs: Absolutely yes! It's natural.

Children made sense of "÷ 0.8" in terms of "× 1.25", in which they had confidence. It then seemed that children were more or less conscious of the reciprocal relations and understood why they should divide by 0.8 and why the answer would then be bigger than the dividend. In the fifth and six lessons teacher and children again discussed those relations fully, and summarized them as in Figure 8. It then seemed that most of them were clearly conscious of the relations.

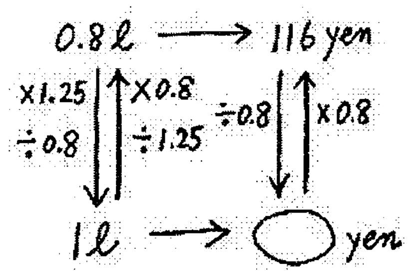

Figure 8. Reciprocal relations in the proportional schema of division.

5. DISCUSSION

When we look back on the lessons of division with decimals, children could make sense of it through experiencing the disequilibrium and repeatedly overcoming it, and they could develop their own logical reasoning in the process. In this section we first show the process as three developmental stages, followed by a discussion of the logical characteristics of children's reasoning at each stage.

5.1. Three stages in the development of making sense of division with decimals

We found that there were three stages in children's logical development as they made sense of division with decimals.

First, children conceived division with decimals by manipulating concrete objects. For example, they replaced the situation "100 yen per 2.5 m" with "200 yen per 5 m" by connecting 2 pieces of strip that represented 2.5 m of ribbon and solved it as division with integers ("200 ÷ 5"). We think that these activities had a character of the *concrete operational thinking*, but were important steps to connect division with decimals to their existing knowledge and to prepare their learning followed.

Second, children began to conceive division as the relationship between the equivalent expressions at the hypothetical-deductive level detached from the concrete one. Children's conceptions of division here were not so much concerning what answer could be found by the expressions $1160 \div 8$ or $116 \div 8 \times 10$, as the relationships among $116 \div 0.8$, the transformed expressions, and the answer. Thus, the expressions for them then were tools for justifying a statement, which was compared with the tools for finding the answer at the first stage.

However, in spite of these conceptions, they had the character of a concrete operation at the same time, like Naka's utterance (17). Children seemed to feel that the expression $116 \div 0.8 = 145$ was right, but instead interpreted it as representing the price per 0.1 liter. We think this a remarkable example that shows the influence of numbers (Bell et al., 1989; Harel et al, 1994), in particular the phenomenon of non-conservation of operations (Greer, 1987). That is, children didn't attain the cognitive state of equilibrium because of the obstinacy of the constraints of the implicit model.

Third, children constructed two explanations; each corresponded to two kinds of reversibility. One explanation was based on the inverse operation. It was when Saka (21) inversed the division into the form of multiplication that they firstly realized the correctness of the expression. However, more explanations were needed because the multiplication had the character of checking after solving the division problem. Next they made sense of the expression by using multiplication in another way. It was to consider "÷ 0.8" as equivalent to "× 1.25", which were two sides of the same coin. It was more natural for children to consider the operation changing 0.8 into 1 as "× 1.25" than as "÷ 0.8" because they had already learned multiplication with decimals and appreciated that "× 1.25" makes the answer bigger. We can deduce that this eventually led them to conceive multiplication and division as a system of operations, in other words to acquire *formal operational thinking*.

As Naka (24) stated "I can regard it to take 0.8 pieces of 116", children might conceive multiplication with decimals by extending their concrete operational thinking. But we think that division with decimals loses the meaning of a concrete operation in nature, since the divisor is no longer to divide in its operational sense. Thus, it seems natural with hindsight that children first contrasted division with decimals with multiplication, and next conceived it in terms of multiplication in two ways.

5.2. Logical characteristics in children's making sense of division

We find that children's reasoning was more or less mathematical and logical at the second stage. If we mathematically examine children's explanations in terms of the properties of division (1) to (6) described above, we can find properties (1) and (6) emerged frequently. Property (1) corresponds to transforming the expression $116 \div 0.8$ into $1160 \div 8$, and property (6) was used when $116 \div 0.8$ was changed into $116 \div 8 \times 10$. The other properties were also found, though maybe implicitly. For example, we can find properties (2) and (4) in Kawa's utterance (15). We think these show that children's reasoning had mathematical characteristics.

When we logically observe children's reasoning in the process of justifying why $116 \div 0.8 = 145$ represented the price of 1 liter, they had some characteristics of formal operational thinking. For example, Katt (13) developed syllogistic reasoning by combining some given facts, and Moto's reasoning (11) was at the hypothetical level, which may be related to the reductive absurdity. Furthermore, Kawa (19) implicitly used the commutative law and operated on the expression itself, as she stated "If we put this 10 in front, it ($116 \div 8 \times 10$) becomes the same as $1160 \div 8$". We can therefore say that at this point children's reasoning had some of the characteristics of formal operational thinking.

If we focus on the scalar operator to transform the elements (a, b), it was still integers in the children's reasoning; so they didn't attain equilibrium at the second stage. At the third stage children, however, recognized the role of decimal operator "$\div$ 0.8" that transformed 0.8 meters into 1 meter (quantity per unit) and at the same time 116 into the answer. Moreover, the operator "$\div$ 0.8" was reconceived both as the reverse operation of $\times$ 0.8 and the equivalent operation of $\times$ 1.25. We think at this point children attained a higher level of understanding of the equilibrated system of multiplication and division.

6. FINAL REMARKS

In this study, we tried to better clarify the processes by which children might overcome difficulties and develop logical reasoning through analyzing classroom lessons. The main outcome of this study was the clarification that reasoning based on two kinds of reversibility contributed to overcoming the difficulties. In particular, recognizing the reciprocal relations of operations made children's adherence to the constraints from the implicit model vanish. Then, we could describe these processes as three stages of making sense of division with decimals, in which they develop their reasoning logically and mathematically.

Here it should be noted that the above stages emerged not linearly, but as a process of attaining equilibrium in which temporal regressions (disequilibria) were often involved, and more coherent ideas were constructed by coordinating some ideas with each other every time a temporary state of equilibrium was achieved.

However, some future tasks remain; First, we couldn't discuss how the symbolizing processes (Gravemeijer and Stephen, 2002) assisted children's making sense of division with decimals. Second, though we dealt with the generalization of partitive division, logical development towards secondary mathematics are related with other proportional concepts which include the generalization of quotitive division, the ratio, the proportion, and the fraction and so on. Therefore, we need to examine the issue as being relationships in a web of related concepts. We believe addressing these tasks will be shed some light on ways to bridge the gap between elementary and secondary school mathematics.

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GEOMETRIC LANGUAGE IN THE GENERALIZATION OF NUMERICAL PATTERNS AS ALGEBRAIC THINKING

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ABSTRACT

Although algebraic thinking emerges as generalization of numerical patterns, the process is not necessarily clear. It is important problem to bridge the gap between number world and variable world in mathematics education. Thus we observe discourses in classroom teachings and focus on pupils' language as sign. We elaborate a lesson so that a researcher teach, which is said 'Jumping into class as teacher', in Japanese. We discuss it on the viewpoint of three types of language, natural language, language of geometry, language of algebra. It is clear that geometric language plays important roll. It is a criterion of the potential that pupils are able to generalize numeral pattern, whether they use geometric language which is the bridge between the island of numbers and of variables after all.

1. INTRODUCTION

It is important problem to bridge the gap between arithmetic and algebra, that is, number world and variable world in mathematics education. Algebraic thinking emerges as generalization of numerical patterns. The process is not necessarily clear, because it is long term and contains many contents. Wittmann (2004) shows the essential learning environment linking multiplication table, multiplicative calculation by writing, and the solution of quadratic equations. This is very interesting for us because we have not such a learning trajectory or material in Japan.

They say Japanese mathematics curriculum is the most coherent in the world. But even in Japan, elementary teachers teach multiplicative calculation by writing, without knowing how it works in high school, in order that children compute quickly and correctly. Also the high school teachers teach the formulas of literal expression and the quadratic equations without knowing what their children learned in elementary school arithmetic. Consequently most students compute literal expressions formally and don't recognize the network of number and variable world.

Also Wittmann (2004) mentions three types of language, natural language, language of geometry, language of algebra. I think that Dörfler (2000) names these languages as 'protocols'. The natural language is Japanese for us, dealing with number patterns. Radford (2000) analyses the feature of natural language in generalization of numerical patterns. The algebraic language deals with literal expressions and variables on algebraic thinking. In these languages, geometric language is the middle and the vaguest one. But I think it has the critical roll in generalization, and Dörfler (2000) call it 'prototypes'.

2. AIMS AND METHODOLOGY

The aims of this inquiry are as follows:

- To clarify the roll of geometric language in generalization of numerical pattern in a classroom.
- To consider the relationship of natural language and geometric language in classroom.
- To experience the essential learning circumstance as a teacher.

The methodology is 'Jumping into classroom as teacher', that is, I really teach 4th grade pupils of elementary school in a class. The theme is square number and pyramid number. 17th October in this year, I taught 27 pupils at the public school. This teaching is based on the lesson of Professor Kunimoto, Faculty of Education, Kohchi University, for 2nd grade pupils concerning quite same theme, which is the material in Wittmann's "Zahlen Buch" grade 2nd. I watched it several times on video tape and imitate it.

3. CASE STUDY OF CLASSROOM TEACHING

I taught 4th grade 17 pupils of a class in elementary school, 17th October 2006. This them and learning trajectory are based on the lesson of Professor Kunimoto, Faculty of Education, Kohchi University, for 2nd grade pupils. I watched it several times on video tape and imitate it.

The theme is square number and pyramid number. The material is in Wittmann's mathematics textbook "Zahlen Buch" grade 2nd.

Task 1:

Square number

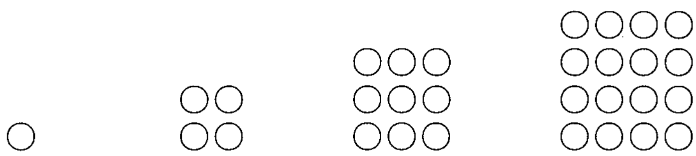

fig.1 fig.2 fig.3 fig.4

Leaning trajectory:

- The square numbers are introduced gradually from one on each side to four.

· Pupils infer the figure and number of 5 points on each side.
· Pupils infer the number of case 6, 7, 10 and 100 points.

Data:

101 Teacher ('I'): Now, I stick something on the board one by one. Please watch carefully, and think what these are. Later give out it. So, the first is this, the second is this. · · · And up to this (forth). What do you notice?
102 KI: A row increases one by one.
103 IC: Rows increase upward and sideways.
104 IS: The figure square keeps.
105 Teacher: [Writing 'square' on the board,] and let us call these circles as points. And how many points in each square?
106 SH: One.
107 Teacher: Please answer in seat order.
108 AI: Four.
109 WA: Nine.
110 KO: Sixteen.
111 Teacher: Then can you formulate them?
112 KU: 4×4 for 16.
113 Teacher: Is this right, everyone? Then, what is the expression?
114 IM: 3×3 is 9.
115 Teacher: So what is this, everyone?
116 All together: 2×2
117 Teacher: And this is 1×1. These are the first, the second, the third, the forth.
118 Some pupils: Oh, I understand the meaning.
119 Teacher: What is the next?
120 Pupils together: Fifth.
121 Teacher: What is the figure for it? Anyone wants to make it? Then, all together, come hear and do it.
122 Teacher: Friends who didn't come, is this right? OK, this is the fifth figure. Then how many are here?
123 KT: 25.
124 Teacher: What's the expression?
125 KB: 5×5
126 Teacher: How many are in the next?
127 KS: 36.
128 Teacher: What's the expression?
129 NO: 6×6.

130 Teacher: We continue them as they are. How many are the next?
131 KA: 49.
132 Teacher: Then we continue this, and how is the tenth? How many are here?
133 NI: 100.
134 Teacher: The expression is? Altogether, 10×10. And it is difficult, ・・・
135 KT: You may say thousandth.
136 Teacher: Oh, it is too difficult. How is the hundredth? These line up hundred.
137 SA: Thousand.
138 Teacher: How is this, others? Do you have another answer?
139 YA: Ten thousand.
140 Teacher: Which is right?
141 Pupils: Ten thousand.
142 Teacher: So, what is the expression?
143 IG: 100×100.
144 SH: I add the zeros.
145 Teacher: Why don't you explain to everyone?
146 SH: I move the zeros on the corner.
147 Teacher: Do you make sense. Do anyone explain SH's idea?
148 IS: First the zeros in one hundred are moved to another.
149 Teacher: These zeros are moved to here, so how many zeros are here?
150 A pupil: Four.
151 Teacher: We get ten thousand.
152 A pupil: Well the next question is thousandth.
153 Teacher: That's good, but shall we change the figure?
154 Some pupils: one hundred million, ten thousand.

Discussion:
We discussed some points in the lesson after. In my teaching, the first impressive utterance is "IS: The figure square keeps". This is very important remark, because awareness of invariance among varying things is a key of the cognition of generalization of patterns. In this case, the invariance is the pattern itself. Thus most pupils recognize the numeral pattern. The evidence is this;

119 Teacher: What is the next?
120 Pupils together: Fifth.
121 Teacher: What is the figure for it? Anyone wants to make it? Then, all together, come hear and do it.
155 Teacher: Friends who didn't come, is this right? OK, this is the fifth figure. Then how many are here?

156 KT: 25.
157 Teacher: What's the expression?
158 KB: 5×5
After all they may link the square figure to the square number. This shows us the importance of the geometric language.

Task 2:

Pyramid number

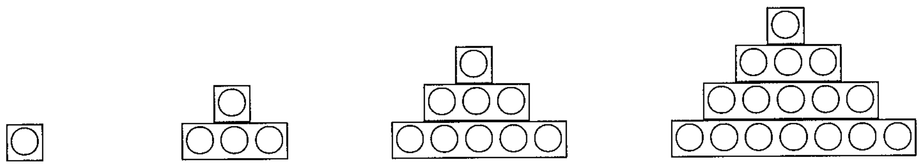

fig.1 fig.2 fig.3 fig.4

Leaning trajectory:

- The pyramid numbers are introduced gradually from fig.1 to fig. 4. But pupils can name these numbers.
- Pupils understand that the numbers of Square and Pyramid are same for each corresponding case, though the figures are different.
- Pupils make expressions for the numbers from fig.1 to 'fig.5'.

Data:
159 Teacher: Let us change the figure.
160 KT: Square and square, circle in it.
161 Teacher: [Sticking figures, from first to fourth.] What are these?
162 Some pupils: Tetris (Japanese computer game).
163 Teacher: Do you have other name?
164 KI: Going up the stairs.
165 IM: Pyramids become bigger and bigger.
166 Teacher: What name should we choose?
167 Some pupils: Pyramid, Tetris. ・・・.
168 Teacher: Let us decide by majority. Anyone choose Tetris? ・・・
Anyone choose Pyramid?
169 Teacher: So we call Pyramid. [Writing 'pyramid number' on the board.]
How many are there?
170 Pupils: 1, 4, 9, 16.
171 Teacher: What do you notice? ・・・KI, can you give us a hint?

172 KI: No, I can't.
173 Teacher: OK, then answer it.
174 KI: Except for 25, the left side numbers are same to the right.

Discussion:
In this time some pupils notice the Square and Pyramid have the same number. I should spread this view to other pupils. But Data lines 175-186 are omitted.

Data:
187 IS: Although the figure changes, the numbers are same in both sides.
188 Teacher: Do you make sense? Anyone who understand it!
189 SH: Same thing. Although the figures are square and triangle, the numbers in each figure are same.
190 Teacher: Although the figure changes, the numbers are same. Why? Can you reason it? This is so difficult that I prepare something. Do you know what it is? This is the miniature of this one. Please cut it with scissors and think how to transform it.
191 Pupils: [All the pupils manipulate the paper.]
192 Teacher: Anyone who can answer how you did?
193 IC: This figure, I cut it, and did like this.
194 Teacher: Thank you. The other way?
195 TU: I cut it in some pieces.
196 Teacher: The easiest way is to cut here and turn this. Why don't you try this?
197 KT: [The former pyramid figure, she did.]

Discussion:
This is a kind of 'operative proof (Wittmann, 2004)' by cutting the paper and removing the piece. Naturally on the languages, teacher and pupils used geometric language. It would be a naïve base of proof to use literal expressions in algebra.

Data:
198 Teacher: This is also transformed to square. Therefore although the figures seem different, the numbers are same. Do you understand clearly? Pyramid number and square number are same. Then I put back in this figure and write expressions in the board. Please watch carefully.
[Writing on the board, 1=1 1+2+1=4 1+2+3+2+1=9 1+2+3+4+3+2+1=16]
What is similar between these expressions and figures?
199 SI: [Coming before the board.] This number corresponds each other.

200 Teacher: How he regard these figures.

201 IS: Seeing lengthways.

202 Teacher: Yes, this is great discovery. Other opinions? · · · . No. So, What is the next expression? Please write on this paper. [Handing papers out.] · · · .

203 Teacher: First of all, what is the answer? [Pointing the right side of the lowest equality.]

204 KO: 25.

205 Teacher: Why is it 25?

206 KI: Before the magnets were 25.

207 IM: As SI said before, the next is calculated as 25.

208 TU: 1, 2, 3, 4, 5 and 1,2,3,4 and so on.

209 Teacher: I leave its expression until later. · · · .

210 Teacher: A little confused! So please answer the expression.

211 YM: $1+2+3+4+5$

212 Teacher: What is the next?

213 KA: $4+3+2+1$

214 SI: I have another way, $(1+2+3+4)\times 2+5$

215 Teacher: TU, what was your expression before? · · · Yes, this expression.

216 KU: $9+7+5+3+1$. I see it sideways.

217 Teacher: I think you have other ideas. In fact I thought these answer is

$1\times 1 \quad 2\times 2 \quad 3\times 3 \quad 4\times 4 \quad 5\times 5$

Do you notice this?

218 Next, you can see this figure sideway. And you get the other expressions.

219 Pupils: $1 \quad 1+3 \quad 1+3+5 \quad 1+3+5+7 \quad 1+3+5+7+9$

220 Teacher: The answers are same with these. So you can make different expressions following how you see the figure. Thank you very much.

Discussion:

The aim of this classroom activity is to make different expressions depending on a way of looking at the figure. Some pupils notice the sequence of odd number.

4. CONCLUSION

Pupils are very vivid and eager to present their opinion and in fact show me a variety of thinking. I have experienced the advantage of this material and learning trajectory. One of pupil's reflection for the lesson is as follows:

" I was very tired from today's lesson because I thought very much. However I am also surprised that there are many ways of mathematical thinking. In fact although the each expression for Square number is multiplications and that for Pyramid number is addition, the answer is same each other. Thus I feel mathematics is really great. Today I enjoyed very much."

We make sure that the geometric language plays important roll in generalization of numerical pattern as we already mentioned it in the discussion. It is a criterion of the potential that pupils are able to generalize numeral pattern, whether they use the geometric language. Dörfler (2000) call it 'prototypes'.

For the first task, we notice that pupils have the prototype as geometric language. They developed to 10×10 and 100×100. In addition, some pupils recognize the naïve law of exponent as "144 SH: I add the zeros".

For the second situation, I discuss with the class teacher which figure is better, only points or ones enclosed with square. She answers the latter as in the textbook. We notice this is right judgment. Pupils are very interested in naming those figures.

In this situation, they recognize the numeral pattern; "IS: Although the figure changes, the numbers are same in both sides". Moreover they link the expressions; 1=1 1+2+1=4 1+2+3+2+1=9 1+2+3+4+3+2+1=16 to the figure.

And "200 Teacher: How he regard these figures."

"201 IS: Seeing lengthways."

These are evidence of that most pupils generalize the expression in other cases. Our conclusion is that language of geometry is the bridge between the island of numbers and of variables.

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ON THE TEACHING SITUATION OF CONCEPTUAL CHANGE: EPISTEMOLOGICAL CONSIDERATIONS OF IRRATIONAL NUMBERS

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ABSTRACT

Generally we can point out two different ways in introducing new kinds of numbers as follows (e.g., Courant & Robbins, 1941/ 1996). The first is to represent a result of measurement. The second is to solve algebraic equations. However the relation between the two ways does not still seem to be clear. Although this issue might have been overlooked in any teaching situations, this can be didactically explicit in the teaching situation of irrational numbers from the conceptual change perspective. The purpose of this paper is to derive some didactical implications for a conceptual change situation by focusing on a knowing of "incommensurability" that can be an essential aspect of irrationals. For attaining this purpose, the epistemological considerations take place in three contexts: curricular contents, history and teaching experiment.

CONCEPTUAL CHANGE: A THEORITICAL PERSPECTIVE

Conceptual change theory has been widely used to explain students' understanding in a series of developmental studies referring to science education (e.g. Posner et al., 1982; Carey, 1985; Hashweh, 1986). This theory was developed by drawing on the philosophy and history of science, in particular Thomas Kuhn's account of theory change and Imre Lakatos's work of the scientific research programme. And it mainly used to explain knowledge acquisition in specific domain, with characterizing role of reorganization of existing knowledge in processes of learning. Vosniadou et al. (2001) argued that scientific explanation of the physical world often run counter to fundamental principles of intuitive knowledge, which are confirmed by our everyday experience. Consequently, in the process of learning, new information interferes with prior knowledge, resulting in the construction of synthetic model (or misconception). Similarly, when studying mathematics, in the course of accumulating mathematical knowledge, the students go through successive processes of generalization, while also experiencing the extension of various mathematical systems (Tirosh & Tsamir, 2006, p. 160); the most typical case of such kind of generalization or extension is the number concept (see, e.g., Merenluoto & Lehtinen, 2004). But, on the other hand, there is a general reluctance in philosophy and history of science circles to apply the conceptual change approach to mathematics (Vosniadou & Verschaffel, 2004). As has been discussed in mathematics education domain, we need to take *the specificity of mathematical knowledge* into account with a deep epistemological analysis of what the concepts considered consist of as mathematical concepts (Balacheff, 1990, p. 136).

Generally speaking, 'the term "conceptual change" embodies a first approximation of what constitutes the primary difficulty. ... Hence, there is the emphasis on "change" rather than on simple acquisition. ... The "conceptual" part of the conceptual change label must be treated less literally. Various theories locate the difficulty in such entities as "beliefs", "theories" or "ontologies," in addition to "concepts." '(diSessa, 2006, p. 265). Therefore we may need to identify what is special about the learning and teaching of mathematics in the conceptual change situation, analysing from the different dimensions of mathematical concepts/ knowledge.

The aim of this paper is to present didactical implications for designing the teaching situation of conceptual change by focusing on the irrational numbers as content. In fact, only a few researches on irrational numbers have been reported (Fischbein et al., 1994; Zazkis & Sirotic, 2004). On such background we argue the relation between two different ways in introducing new kinds of numbers: the first is to represent a result of measurement; the second is to solve algebraic equations. As will see later, a knowing of the incommensurability (no common unit between two magnitudes) can be crucial to bridge the two different ways. This issue will be considered or interpreted from the epistemological points of view, discussing three contexts: the curricular contents, history and teaching experiment. Then, in the final place, three items are derived as didactical implications with the help of such considerations.

EPISTEMOLOGICAL CONSIDERATIONS

Issues in the mathematics curricular contents relating to irrational numbers

The significance of irrational numbers as a subject matter can be described as follows: the existence of incommensurable quantity; its admittance and symbolism; curiosity about that the computational rules with infinite non-repeating decimals are available same as with rational numbers; and the rationale of the new number system, so on. Irrational numbers are introduced in the forms of "square root numbers" at lower secondary level (15-year-old students in the case of Japan). In the teaching situation of the square root, it is usually introduced in light of the practical need to express the concrete quantity (magnitude) as well as the teaching situations at the primary school level. For examples, it has been often taken the instructional way for finding out the length of the diagonal of the square, or the side of square having the double area of a given square. Indeed "quantity" is an object of measurement.

However a naïve practical conception cannot reach to the essential understanding of the square root because here we deal with "incommensurable quantity" in question. In addition, the teaching situation of irrational numbers can distinguish the situation dealing with the concrete quantity and the situation dealing with the computational rules following introduction of the symbol$\sqrt{\ }$. In doing so, it is not just the transition between situations but it is required to prepare mediated activities shifting from concrete/ practical conception to more theoretical/ formal one.

Students come to learn new kinds of numbers as school year advances. The introduction of new numbers must be a purposeful activity to respond to some necessities or overcome some limitations. For example, it is explained "the generalization from the natural to the rational numbers satisfies both the **theoretical need** for removing the restrictions on subtraction and division, and **the practical need** for numbers to express the results of measurement. It is the fact that the rational numbers fill this **two-fold need** that gives them their true significance" (my own emphasis) (Courant & Robbins, 1941/ 1996, p. 56). Since primary school year, new numbers emerge from some actions on quantities, that is, the practical need for numbers to represent the results of measurement. Although the need for introducing irrationals can also emerge from some actions on quantities, the object of the actions is "the length of a segment incommensurable with the unit" and its approach comes from responding to the situation that it cannot represent by sub-dividing the original unit. Here we can see the limitation on the measuring approach. Since the awareness of such kind of limitation can lead to the conception of incommensurability, it is necessary as its didactical orientation to prepare some effective activities.

Issues in a historical section

One of the most important dimension of epistemological considerations is to examine why the question of incommensurability arise in the course of history. In this paper the historical examination is to see "the history of mathematics as a kind of epistemological laboratory in which to explore the development of mathematical knowledge"(Radford, 1997, p. 26). This requires us to investigate status of human cognition in confronting with the question in a historical section.

The number theory in ancient Greek is concerning with the mathematics for handling discrete numbers world, such as "figural numbers". In such a primitive status it is no doubt to see that two segments are commensurable each other. The following statements are described in the modern manner about that (See, more details in Courant & Robbins (1941/1996, pp. 58-59)): In comparing the magnitudes of two line segments a and b, it may that while no integral multiple of a equals b, we can divide a into, say, n equal segments each of length a/n, such that some integral multiple m of the segment a/n is equal to b:

$$b = \frac{m}{n}a \tag{1}$$

When an equation of the form (1) holds we say that the two segments a and b are *commensurable*, since they have as a common measure the segment a/n which goes n times into a and m times into b. The totality of all segments commensurable with a will be those length can be expressed in the form (1) for some choice of integers m and n ($n \neq 0$).

The situation is, however, by no means so simple. It was getting to be doubtful to the existence of a kind of the segment, according to Boyer (1968), the Pythagorean's successors raised the question of incommensurability in earlier than B.C.410. The *Euclid's Elements Book X Def. 1* states that "*Those magnitude are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure*"(Heath, 1956, p.10). The discovery of the incommensurability is one of the most remarkable problems of history of mathematics regarding the disintegration of parallel between the (figural) number and quantity (magnitude) theories (*cf.* Eudoxus's theory). We human beings became aware of the world where we can reach only by thought purely (Szabó, 1969/ 1978), but it may be said that this was a product of the Greek intrinsic viewpoint of the academism towards mathematics. Thus it is pointed out that the concept of incommensurability did originate not from the practical source but from the theoretical one (Szabó, 1969/ 1978).

The following statement quoted from *Euclid's Elements Book X Prop. 2* forms a criterion of incommensurable relation: "*If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable*" (Heath, 1956, p. 17). It has to take into consideration that since the infinite continuable algorithm (so-called *Euclidean algorithm*) has a purely theoretical characteristic, it cannot be applied to two magnitudes as a *practical* criterion. Therefore the criterion had never used in any ancient literatures (Szabó, 1969/ 1978). In this context human cognition confronts the discontinuity that, in the case of two incommensurable magnitudes, the magnitudes must exist in theoretical, but they are never realized in practical because of the events only for thought. And it is also pointed out that the internal inspiration looking for a more rigorous mode of thinking arises (Wilder, 1968/ 1987). The new proof technique, namely *reductio ad absurdum*, was established in this context. Árpad Szabó refers to the proof of the incommensurability between a side of a square and the diagonal, and he emphasizes the connection between the establishment of the new proof technique and the shift to "anti-empirical and anti-intuitive tendency that underlying ancient Greek mathematics" (Szabó, 1969/ 1978).

Issues in teaching-experiment designed for the awareness of incommensurability

The teaching experiment was performed with 9 ninth grade students (15-year-old) in a classroom of a lower secondary school attached to national university in Japan in October 2005. The main question of this teaching experiment is to identify how students can become aware of incommensurability. In relation to such aim the teaching experiment consists of three phases: (i) introducing Euclidean algorithm; (ii) dealing with existence of common measure; (iii) justifying recursive or infinite process of operations. In this report we focus on the phase (iii) because it is the most crucial situation in terms of becoming aware of incommensurability.

The following tasks used in this experiment are relied on the earlier developmental research (Iwasaki, 2004).

Task1: *There is a rectangle board 30cm by 42cm. You want to cover it with square tiles, the size of which must be same and lager as possible as you can. Find the size of square tiles.*

Task2: *There is a sheet of the A3 standard here. Consider whether you can find the squares that tessellate the sheet.*

In the phase (i) and (ii), students worked on the task 1 and some extra tasks. They came to know a conception of Euclidean algorithm under a concrete situation of finding the GCD (greatest common divisor) of given two positive integers by folding a sheet and by showing algebraic expressions. In the phase (ii), students recognized the fact that if one finds a remainder then measure the previous measure by the remainder as a new measure, and if one finds no remainder then the algorithm terminates; common measure is found.

In the phase (iii), students worked on the task 2 by applying Euclidean algorithm to a side of square and the diagonal (i.e. in the A3 standard sheet, the larger side is equal to the diagonal of the square with the smaller side). Students developed gradually their activities with the help of some geometrical relationship, which can be illustrated as follows (Fig.1). In doing so, such operative activities could undergo a kind of *thought experiment*.

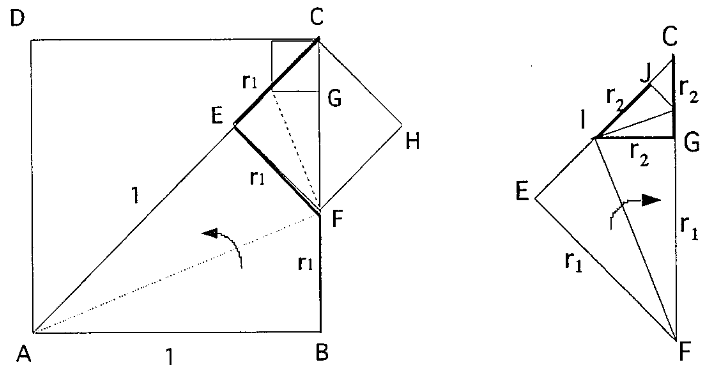


Fig 1: Measuring the diagonal of the square with the side

Consequently, we only need to remark the first three steps of the operative activity. Because, as we can see Fig.1, you start measuring the diagonal of the square (=AC) with its side (=AB), and repeat twice the procedure of subtracting small one from large one, then another smaller square and its diagonal (=IC) will appear. Under the thought experiment, it implies that the procedural can be recursive or infinite process.

1 T (teacher): How much is size of your finding square next?

2 S_1 (a student): ...[pointed the small square (right isosceles triangle)]

3 T: A side of the square may be 'c' following S4' expressions [See the Appendix]. So, now we found a small square, its side is 'c'. We don't prepare smaller sheet for folding anymore, but what does it imply?

4 SS (students): it continues endlessly.

5 T: Endlessly?

6 S_2: ...Surprising.

7 T: OK, let us reflect why you say it is endless. Explain in your own word.

8 S_3: Because the remainders are always made in the constant proportion.

9 T: Anything else?

10 S_4: The square...because if squares are found, then we can always find the right isosceles triangle.

All participating students became aware of the constancy of the procedure though above conversations. At the end, teacher suggested that the continued fraction might be useful for formalizing the operative processes. As a result, we obtained the development of the diagonal (=x) in the general form: (r: remainder)

$$x = 1 + r_1 = 1 + \frac{1}{\frac{1}{r_1}} = 1 + \frac{1}{2 + \frac{r_2}{r_1}} = \cdots = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{}{\ddots 2 + \frac{r_{n+1}}{r_n}}}}$$

It is well known that we can obtain an approximate value of the square root of 2 successively using the form above.

DIDACTICAL IMPLICATIONS

Let me summarize the main points that have made. Firstly the curricular contents show that new numbers have been introducing from the practical need in the course of learning, while irrationals tend to be introduced from theoretical need. But there are no didactical opportunities to relate two different ways. Secondly the historical context shows that the discovery of the incommensurability can lead to the theoretical nature of mathematics by establishing the reductio ad absurdum. Thirdly the teaching experiment shows that students can be minimally understood the conception of the incommensurability under the thought experiment. As a result of such consideration, it can be pointed out that as implications for designing the teaching in the conceptual change situation, at least the following three items have to be taken into account.

(1) Questioning, say, *is it possible to represent a result of measurement of incommensurable magnitudes?*

The numbers that students have already learned can be represented as a ratio of integers, but students may not always be aware of this explicitly. Paradoxically say, the "incommensurable" situation only enables them to be aware of “commensurability”. There is no situation for appreciating the idea of dividing of unit, except for the situation of introducing square root.

(2) Eliminating the tendency to cling to the “concrete”.

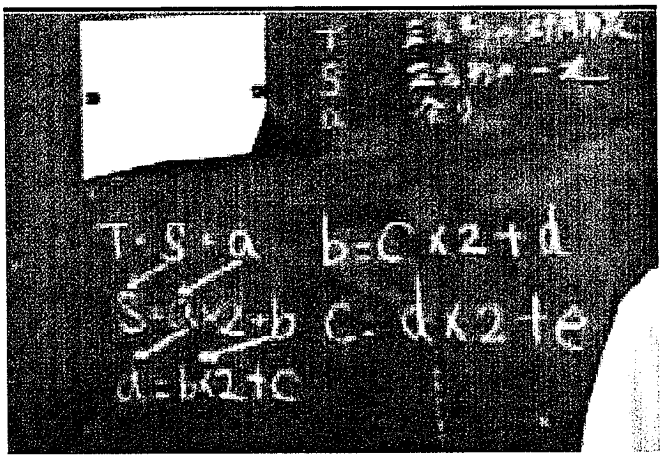
A conception of numbers clinging to the concrete has been well acting on the old numbers (rationals) in taking into consideration of its existence, and these numbers can become intuitive on the number line. However we should not overlook the following remarks: ‘Nothing in our “intuition” can help us to “see” the irrational points as distinct from the rational ones’ (Courant & Robbins, 1941/1996, p. 60). A practical conception of quantities (magnitudes) involving the concrete cannot be a position to make the incommensurability sense. It will be important to eliminate such a tendency ontologically (it is also discussed in the case of negative numbers in Hefendehl-Hebeker (1991)). It does not only suggest the instruction of square root numbers by approaching to the existence of solution of $x^2 = 2$. As a didactical implication, the tasks used in the teaching experiment can be effective settings for becoming aware of incommensurability. In short, context of justification in the history could be recontextualized into the context of discovery in the classroom.

(3) Shifting on value judgments toward the mathematical knowledge

More important point to note is, belonging to ‘meta-mathematical layer’ in Sierpinska & Lerman (1996)’ sense, what we aim at by developing Euclidean algorithm as a learning activity. The interactive activities of operating with folding a sheet and expressing its process have to lead to the activities by the thought-experiment. In doing so, Euclidean algorithm is primitively regarded as a practical method, for applying it to the material (*real*) objects, measuring the diagonal of square with its side. The view on the method can undergo changes though students’ applying the method and then deriving the theoretical conclusion from its infinite process. This implies students’ seeing as the *ideal object*. Under the thought-experiment it is expected or required for students to shift their value judgments toward the mathematical knowledge underlying item (1) and (2).

Appendix

The picture shows a student's writing on the blackboard (*T*: the diagonal of the square; *S*: the side of the square; *a*, *b*, *c*, *d*, *e*: remainders)



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DEVELOPMENTS OF A CHILD'S FRACTION CONCEPTS WITH THE HELP OF PSYCHOLOGICAL TOOLS: A VYGOTSKY'S CULTURAL-HISTORICAL PERSPECTIVE

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ABSTRACT

This paper aims to interpret Vygotsky's abstract theory in a concrete mathematical context. Based on cultural-historical perspective Vygotsky stresses the importance of psychological tools in the development of human behavior. Through the interviewing and observing fraction lessons the researcher draws two conclusions: Both of the learning material and the fraction symbols function as psychological tools but they have difference in some levels; Kanako, a third grader, developed concepts of equivalent fractions mediated by fraction signs in a class, but it is not a real concept.

1. VYGOTSKY'S CULTURAL-HISTORICAL PERSPECTIVE

According to Van der Veer and Valsiner (1991), Vygotsky's cultural-historical theory aimed at exploring where mental processes originated from and how they developed. In fact, Vygotsky and Luria (1930/1993) placed a great emphasis on "historical development" of human behavior, not only on "biological evolution" and "childhood development" (p.81). Based on four comparisons between behaviors of lower and higher forms such as between anthropoid apes and human beings (Van der Veer & Valsiner, 1991), Vygotsky and Luria (1930/1993) drew the following conclusions in reference to works of Köhler, Bühler, Engels, Lévy-Bruhl and so on (Yoshida, 2004a).

As regards nature, chimpanzees purely *use nature* with no intention, using tools as auxiliary. To make and use tools are inessential to survival for them. Using tools of labor, on the other hand, human beings *control nature* in accordance with their ends and plans. Hence these tools are essential to living for human.

Furthermore, regarding with psychological development, nonverbal communication and thought describe chimpanzees' behaviors whereas human beings invent artificial signs and behave relying on such signs and speech. This means that human beings *control their behavior itself using signs*. In short, Vygotsky recognized that the human ability to control behavior through sign systems was the key difference between anthropoid apes and humans.

Incidentally, what does it mean to control human behavior through sign systems? Vygotsky and Luria (1930/1993) illustrated it with comparisons between behaviors of *natural* people and *cultural* people. According to Roth, for instance, messengers of the North Queensland aborigines delivered a song repeating it from memory even though it took five nights to finish (as cited in Lévy-Bruhl, 1910/1966, p.94). Vygotsky gave an explanation of this as eidetic memory – an undifferentiated whole consisting of perception and memory – which people do *not control* but merely use.

Along with historical development of human beings, mnemotechnical aids became popular (Vygotsky & Luria, 1930/1993). For example, knot-based mnemotechnical systems for memorizing, or "quipu" were used to record important events or results of counting the number of animals (cf. p.104). Likewise, in Okinawa islands of Japan officers used to tie and interpret knots in a rope when collecting taxes (Ifrah, 1981/1988). And finally, human invented sings and letters for writing.

In conclusion, as sign systems developed, humans started to keep records with the help of mnemotechnical aids of knots and signs. In other words, human beings were freed from enormous amount of memories. As a result, it enabled humans to think abstractly, hypothetically, and logically (Vygotsky & Luria, 1930/1993). And this means that human beings control their behavior with the help of artificial signs.

2. PSYCHOLOGICAL TOOLS

> We have many reasons to assume that the cultural development consists in mastering methods of behavior which are based on the use of signs as a means of accomplishing any particular psychological operation. (Vygotsky, 1929, p.415)

This Vygotsky's description represents that self-control over behavior through sign systems is "the essence of the cultural development of man's behavior" (Vygotsky & Luria, 1930/1993, p.77). In this context *culture* has a special meaning for Vygotsky. As Van der Veer and Valsiner (1991) pointed out, using Barash's distinction of a cultural evolution, Vygotsky identified culture as sign systems – writing systems, counting systems, and language.

Such sign systems – the key to a cultural-historical development of human behavior – are explained as psychological tools by Vygotsky. Psychological tools are artificial instruments directed toward control over human behavior and they are the products of historical development of human behavior (Vygotsky, 1930/1997).

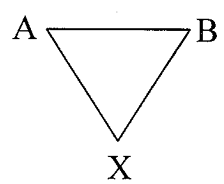

Figure 1: Schematic Triangle (Vygotsky, 1930/1997)

While natural memorization produces a direct associative connection A–B between stimuli A and B (see Figure 1), a psychological tool X makes a new path from A → X →B (Vygotsky, 1929, 1930/1997). At this time, X can play the *role of the object* which an act of behavior (ex. to memorize, to choose) for problem solving is directed toward and the *role of a means* of the psychological operations (ex. memorizing, comparing) to solve the problem.

Moreover, psychological tools are differentiated from technical tools because the technical tools change the object itself while the psychological tools effect no change in the object but influence human behavior or mind.

This could be explained using the example of the officers in Okinawa islands shown in the above as follows: To record taxes the officers tie knots and to report the taxes they interpret the knots, and here the knots are the object of their acts. In addition, the knots play the role of a means that enables the officers to reach their objectives. Furthermore, the knots do not change the rope but change the officers' behavior.

Although the discussion given above is important theoretically, for mathematics educators it is more important to reinterpret it in a context of a real world on mathematics education. Thus, the researcher presents the following surveys and notes on lesson observations to consider development of a child's fraction concepts with the help of psychological tools.

3. A SERIES OF RESEARCHES ON KANAKO'S FRACTION CONCEPTS

Survey 1 and survey 2 by interviewing: Before and after fraction lessons

The surveys 1 and 2 for six third graders were conducted respectively in January and March 2000 (Yoshida, 2000b, 2001, 2002). The purposes of the surveys were to clarify children's *everyday concepts* of fractions and to identify how children's concepts develop before and after fraction lessons.

Kanako, one of the children, solved each problem first by herself while the researcher was observing her problem solving activity, and then was interviewed. The problems given in the surveys before and after the fraction lessons were almost the same, but only the latter survey included a number line problem.

The first problem was to make a classification and a characterization. In fact, in the survey 1 Kanako classified seven figures and sentences into two gropes according to features that they have in common, and named the groups "1 out of 3" and "2 out of 6."

In the survey 2, after learning fractions in classes, she classified them into two groups at first and named them "1/3" and "2/6." After interviewing her about the idea, the researcher asked if it would be possible to reduce the number of groups she classified into.

Researcher: ... What do you think if you can reduce the number of the groups you categorized into?

... (omission of some sentences by several people)

Kanako: I think it is OK to put the two groups together in one.

Researcher: Why do you think so?

Kanako: Well, because when 1/3 is marked with an additional scale, it turns out 2/6. Again, it gets bigger and bigger. ... why I separated this (group of 1/3) from that (group of 2/6) is because the numbers for dividing were different from one another. ...

Researcher: What kind of name do you give to the new group you made?

...

Kanako: a group of '1/3 transformed into 2/6' and '2/6'

The second problem was to draw a picture showing "one fourth" and to describe a meaning of "one half" with words. Kanako gave Figures 2 and 3 respectively before and after the fraction lessons.

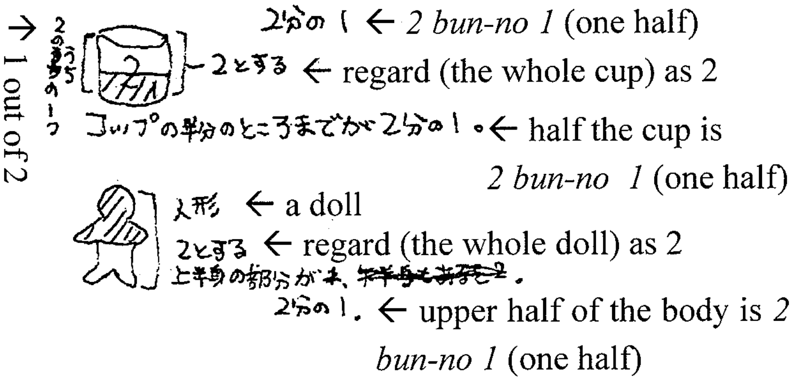

Figure 2: One half as "1 out of 2."

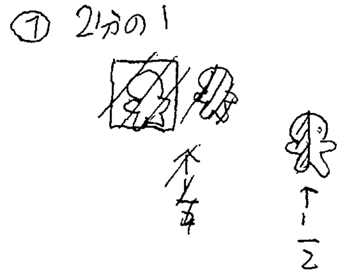

Figure 3: One half as "half a whole."

The third problem was to mark 4/10m, 1/2m, and 2/5m on a number line. Rika, one of the subjects in the surveys, gave an incorrect answer (see Appendix A) while Kanako gave a correct one (see Appendix B).

Observation on fraction lessons

A series of five fraction lessons for 39 third graders, including the six subjects in the surveys, were observed on March 1 – 7, 2000 in Hiroshima, Japan (Yoshida, 2002). Because of the official curriculum guidelines of that time, it was the first time for them to take fraction classes at school. A teacher specialized in mathematics set the following situations where children could learn fractions appropriately, according to his teaching experience.

In the first lesson, the teacher cut a piece of pink ribbon in two in front of the children and told them that the longer ribbon was 1m long. He asked how long the shorter one was. Through the lesson, they found out that triple of the length of the shorter ribbon was equivalent to that of the longer one. Moreover, a child raised a question; What would you do if the shorter ribbon did not correspond to the longer one entirely?

Therefore, in the second lesson the teacher asked the children to find out the length of a piece of new blue ribbon (45cm long) comparing to the longer pink ribbon (1m long) given in the first lesson. Through this lesson, the children realized that in this case it was the best way to fold the pink ribbon graduated in 1m. In short, they changed the benchmark for comparing from the ribbon in which the length was unknown to the ribbon with 1m-length.

In the next lesson, the teacher gave glass-shaped folding papers and asked, "This is a liter glass. How much juice is left in the glass?" Since the children had lots of experience of folding in the previous lesson, they started to solve this problem by folding the papers in some ways (see Figure 4). Through the folding activities, they found out that they could tell the amount of the juice in some ways depending on how many times they folded the paper. That means, some children gave the answer "2 out of 5 parts" while the other did "4 out of 10." After that, the teacher introduced a sign of fractions such as 2/5 and 4/10.

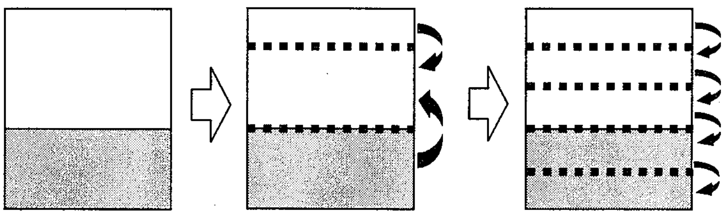

Figure 4: Glass-shaped folding papers.

The fourth lesson started with the same problem, but the amount of juice shaded on a glass-shaped folding paper was 3/5 liter. The focus of the children's interests changed from how to fold in the previous class to how to express the amount of the juice in this lesson, i.e. 6/10 l vs. 6/10 dl. Through an in-depth discussion on the issue, they achieved a consensus that it could be represented as 6/10 l and 3/5 l. The following conversation took place shortly after that.

Kazuo:	Mr., you can make it more, endlessly.
	… (omission of some sentences by several people)
Kanako:	Um, I got started with 3/5, and then, 6/10, 9/15, …
Teacher:	9/15? Wait. Just a second. I'm going to write them down here on the blackboard. 3/5, 9/15,
	…
Teacher:	12/20. Ha! Ha! [while he is writing it down.]
Kanako:	15/25, 18/30, …

At the end, Kanako notified that she gained 300/500, and so the other children looked into her notebook (see Appendix C) surrounding her desk.

4. DISCUSSION

The research suggests two findings as follows based on Vygotsky's theory.

First, we could regard both of the glass-shaped folding papers in the third lesson and fraction signs introduced in the fourth lesson as psychological tools to recognize the system of equivalent fractions, because both of them play the *role of a means* by which Kanako and the other children produced the idea of equivalent fractions as well as play the *role of the object* of their acts such as folding papers (to find out the amount of juice) and making fractions (to express the amount of juice in a variety of ways).

However, it is possible to distinguish the levels of the glass-shaped folding papers and fraction signs one another. That means the children and the officer who collected taxes depend on the concrete contexts using the glass-shaped folding papers and the knots, so that they cannot think and cannot solve their problems without the tools. On the other hand, after the teacher introduced the signs of fractions, Kanako developed expressions of equivalent fractions on her own motive without any concrete materials and not based on any practical situations. In short, the fraction signs as psychological tools could lead Kanako to a general, hypothetical, and/or abstract thinking and the ability of planning for the future. This corresponds with the following descriptions.

> With the aid of speech the child for the first time proves able to the mastering of its own behaviour, relating to itself as to another being, regarding itself as an object. Speech helps the child to master this object through the preliminary organization and planning of its own acts of behaviour. (Vygotsky & Luria, 1930/1994, p.111)

Second, we could conclude that Kanako's fraction concepts developed with the aid of psychological tools, i.e. fraction signs, through the lessons. For example, she modified her views on "one half" from "1 out of 2" (see Figure 2) to "half a whole" (see Figure 3). As Rika gave a wrong idea in the number line problem (see Appendix A), the idea of "1 out of 2" corresponding to each number of numerator and denominator for 1/2 is one of *everyday concepts* for fractions and causes the difficulty of learning fractions (cf. Yoshida, 2002, 2004b).

In addition, Kanako showed a remarkable development of fraction concepts in which she produced equivalent fractions from 3/5 to 300/500 during a class. However, the products made by Kanako (i.e. Appendix C) should be investigated with special attention because those equivalent fractions were probably made by adding 3 to the numerator and adding 5 to the denominator like 15/25 = (15+3)/(25+5) = 18/30, instead of multiplying the numerator and denominator of 3/5 by 6, taking her age (or her ability) and the hours of the lesson into account.

Such Kanako's thinking for equivalent fractions are regarded as *pseudoconcept* (Vygotsky, 1934/1987). In appearance a pseudoconcept and a real concept look alike, yet in reality a pseudoconcept is one of thinking in *complexes*, and besides it is in the highest level among five different types of complex (cf. Yoshida, 2000a). Sierpinska (1993) gives an example of the pseudoconcept in mathematics as follows: Children may select every triangle in similar manner to adults; however, it is based on the physical appearance of triangles and not based on a definition of a triangle.

As Berger (2005) describes, pseudoconcepts can lead "the transition from complexes to concepts" (p.158); in addition to this, children can communicate with adults and other advanced people because of pseudoconcepts. Therefore, it is regarded that a pseudoconcept takes an important role when children's concepts or thinking develop. In fact, the teacher communicated with Kanako in the class as if she could have understood equivalent fractions properly. Yet the results of the first problem in the surveys 1 and 2 showed that Kanako's thinking of equivalent fractions was not enough to reach a real concept because Kanako combined the group of 1/3 and 2/6 only after the researcher asked if it would be possible.

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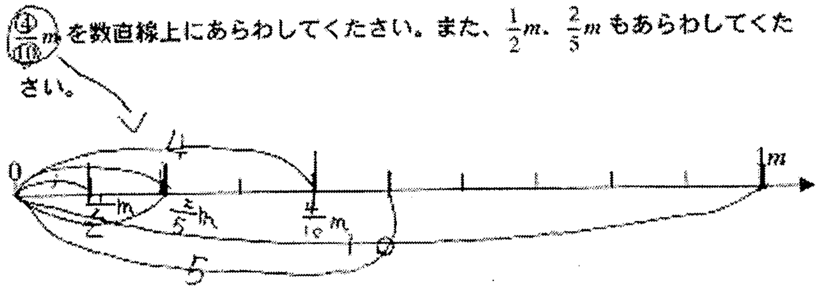
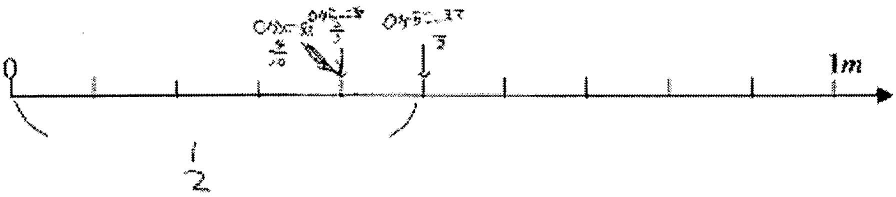
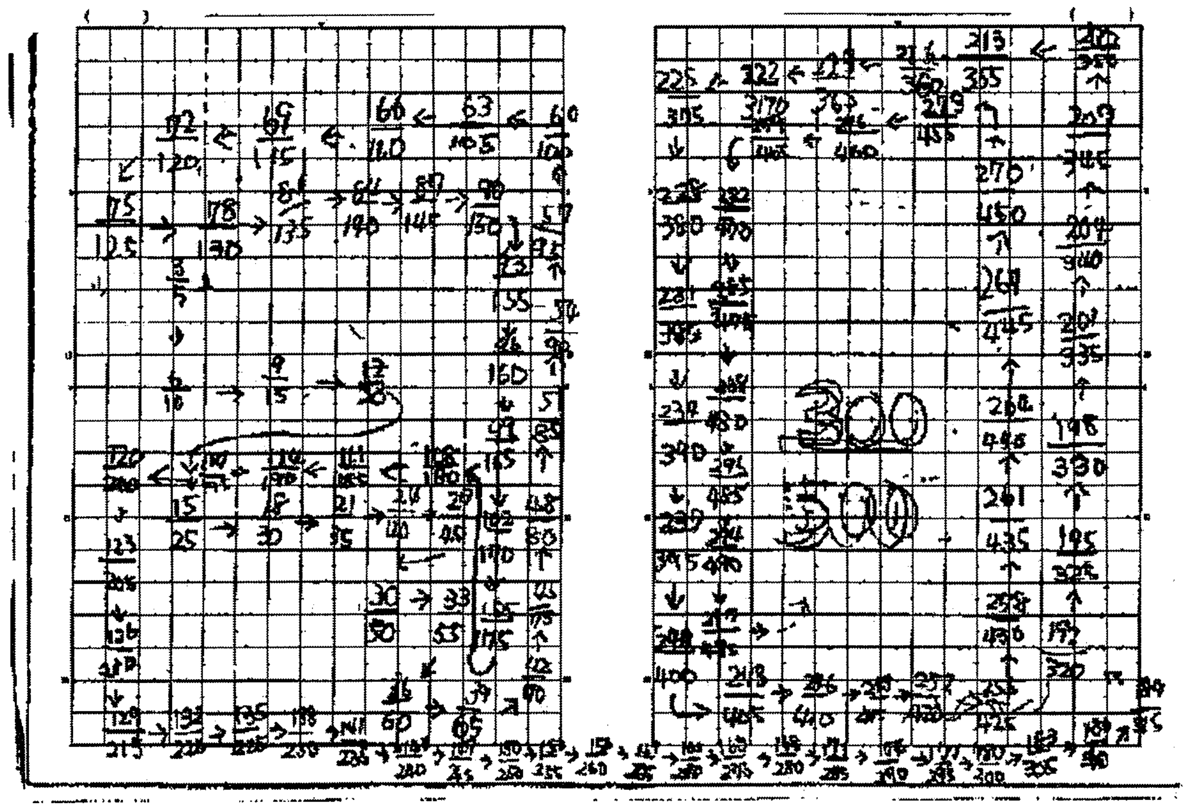
Appendix A: Rika's number line problem solving.



Appendix B: Kanako's number line problem solving.



Appendix C: Kanako created fractions starting with 3/5, on her own motive.



附記：本稿は下記に掲載されたものである。

Yoshida, K. (2006). Developments of a child's fraction concepts with the help of psychological tools: A Vygotsky's cultural-historical perspective. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol.5, pp.449-456). Prague, Czech Republic: PME.

第　3　部

招聘研究者論文編

LEARNING TRAJECTORIES BASED ON PROTOCOLS

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Abstract. It is nowadays undisputed that learning mathematics essentially amounts to an individual cognitive construction of mathematical concepts and processes. Yet, it is still to be investigated by which means the learner constructs knowledge. The notion of protocol is designed to integrate actions and their symbolizations as a constructive means. A hypothetical learning trajectory is presented using these ideas in an approach to the concepts of permutation and function.

Introduction

The notion of protocol is treated in Dörfler (2000) in a general and comprehensive way, yet pertinent examples there are only sketched. The purpose here is to trace the potential applicability of this notion in the specific case of permutations and the function concept. The method used thereby is the idea of hypothetical learning trajectory, see Gravemeijer at al. (2000) and Simon (1995). This means, a theoretically conceivable developmental process based on the formation of protocols is described in detail. Actual learning will deviate from it possibly in some or many stages but it is assumed that the overall development is realizable along this learning design. Of course, all that is subject to empirical research, in the sense of developmental research (compare Gravemeijer et al. 2000).

Protocols – a crashcourse

To make this contribution more self-contained a condensed presentation of the idea of protocol is given below.

It is widely accepted that many basic mathematical concepts have their genetic roots in human actions and in the reflective abstraction and schematization of those actions. One could call this a Piagetian view. What is not so clear is what mediates between the various activities (like counting, measuring, constructing, drawing, accociating) and the related mathematical ideas and concepts. In other words, what are the means and tools by which a reflective abstraction from actions can be accomplished by the individual learner. As a possible means for that goal it is proposed to take notes of the main steps, stages, phases and outcomes of the actions. The result of this process which is called here a protocol of the action might then be verbal statements, diagrams and symbolic inscriptions of any kind. In any case the resulting protocol reflects the focus of interest related to the actions and it will depend on the expressive means available to the respective actor (like, say, graphs, coordinate systems, tables). It is important to view a protocol as a cognitive process by which the individual describes his/her actions according to his/her goals and interests. Therefore the learner has to carry out the actions himself/herself and the notes taken have to be his/hers. In this sense the protocol expresses a personal view on the actions and what are their relevant aspects. The protocol then can be used to think about the actions, to restructure them, to organize them in different ways. The notes taken when establishing the protocol (some kind of inscriptions) are called the carrier of the protocol. The carrier can serve to replicate on itself or by itself the actions in an analogous form yet possibly in a more schematized form. The carrier even can get the object of novel actions leading to another level of protocols with their respective carriers. To start the process of protocol the learner has to be confronted with a problem context which potentially contains a motive for carrying out the intended actions or for imagining oneself doing so. As any learning process also the protocol cannot be fully determined and controled. Yet, it can and should be monitored and guided by a competent expert (e.g. a teacher). For instance, to the learner means for developing a protocol can be offered. The inexperienced learner will need to be prompted to make notes describing what he/she has done and accomplished in the course of the actions which commonly will manipulate some concrete material. Yet, gradually experience with taking protocols can develop and a growing supply of notational means will be available to the learner for that purpose.

Related work

The informed reader will realize that work by A.Sfard (e.g. Sfard (1991)), Ed Dubinsky (e.g. Dubinsky (1991)), D. Tall (e.g. Gray and Tall (1994)), and others (see Gray et al. (1999), Harel and Dubinsky (1992), and Janvier (1987)) has similar and related objectives. The common ground is the question how a learner builds mathematical knowledge about what are termed mathematical concepts or objects. The emphasis in this paper is on individual actions (as with Dubinsky) and on the symbolic/diagrammatic means to reflect on them for the purpose of schematization, abstraction and generalization. There is space neither to talk about the latter processes in their relation to protocols nor about the differences and commonalities with the important work of other authors.

A first view on permutations

As the starting point for a potential learning trajectory the following context is proposed: Arrange in some way 10 books on 10 places numbered 1 to 10 and describe what you do! (This is intended for learners without prior pertinent knowledge). Let S from now on stand for the learner whose learning trajectory is described here. S has at her disposal the 10 books and there are 10 marked places in front of her. Thus, S takes each of the books in turn and puts it on one of the places. Her description of this action first is a verbal one by saying which book she puts on which place. Then S is asked to report about the arrangement and to think about all possible and different ways of doing so. After several more arrangements S realizes that she needs some means of keeping evidence of what she has done. For this she gives names to the books like the capital letters A, B, C, ... and writes down lists of the form

A on 2, B on 8, C on 3, ...

In our terminology S has made a protocol of her actions of which the list is the carrier. After having used the lists several times to note the outcome of the arranging she realizes that instead of arranging the books she could equally well arrange the respective letters on places on a sheet of paper. This would be an instance of the important phenomenon that the signs or symbols which in a first run are used to describe a concrete situation are then substituted for the concrete objects acted upon. The relevant actions are then performed on or with the (carriers of the) symbols or signs. Such a transition will be gradual and S will be able to move back to the books if necessary for her. This is also an important shift of the focus of attention from the original actions to the carrier of the established protocol. At that phase for S the letters A, B, ... are still standing for the books yet manipulating the letters is more convenient than acting on the books. In the further process the letters for S will get more and more detached from their referents and the action of arranging the letters finally turns them into objects in their own right. This means, S is then arranging the letters themselves and not any more describing arrangements of books. This shift to the symbolic plane was prompted by the need to keep track ot the already done arrangements. Yet, the manipulation of the symbols on the paper for S keeps a referential meaning which can be evoked at any time. Working with the letters is kind of symbolic experiment or simulation which later on by S can be carried out purely mentally too. Thus, the used symbols are a means for what usually is termed interiorization of the action.

To describe her putting the letters on the places S makes a diagrammatic sketch by writing down the letters and the numerals and by drawing kind of arrows from the former to latter. She ligns up the letters and numerals in their natural order and thus obtains an orderly organized graphical protocol for her arrangements. This diagram also enhances a shift from thinking about putting letters on places to associating a definite place with each letter according to the arrow drawn. Thus, the arrow for S initially stands for a concrete action and gradually gets detached from it by assuming in itself an operative character: associating a letter with a numeral. What an observer might term a mapping from letters to numerals for S is the drawing of the connecting arrows. This experience of drawing the arrows will later on be a basis for S's understanding of a mapping by metaphorically thinking of it (or of a correspondence, or of a function) as the (imagined) drawing of arrows. The importance of the bodily experience of manually drawing the arrows is to be emphasized. For a related approach see Lakoff and Núñez (2000). After this perspective to the future we return to the actual situation.

By now S has produced quite a few arrangements of the letters as lists and/or diagrams yet alas not in a systematic way and she is getting lost the more lists she writes down. Now she realizes that she needs not to repeat the letters in each of the arrangements and that it suffices to list the places to which the letters go.
This gives her a kind of table like

A	3	6	8
B	1	2	5
C	5	4	3
⋮			

or

A	B	C	...
3	1	5	
6	2	4	
8	5	3	

This more schematized protocol and its carrier now give S the opportunity to think about a systematic way to produce all possible arrangements. The idea is to keep the place for A fixed, say as 1, and arrange the other letters arbitrarily. To make it more clear to herself S decides to try her method first for a smaller number of books and places, say 4 each. She finds 6 ways of arranging the letters with A on 1 whereby 6 is the number of ways of arranging B, C, D on 2, 3, 4. The experience with the arrow diagrams now supports her insight that the same holds true with A on either of 2, 3 or 4 since whereever the arrow from A points to there are 3 places for B, C, D. This gives her $6 \cdot 4 = 24$ arrangements of 4 letters. Now the arrow diagram proves very powerful for S in that by adding stepwise one letter and one place she discovers that the numbers of arrangements get multiplied by the actual number of places in each step: the new letter has exactly those number of places to go to or to draw an arrow to.

This finally opens up the route to a complete solution of the original problem as 10.9.8.7.6.5.4.3.2.1.

Other trajectories

To continue on the learning trajectory for a further development of S's basic experiences related to the mathematical notion of function or mapping a great variety of directions is conceivable. Clearly, there are many other problem contexts conceivable out of which a protocol based first approach to the function concept can be developed. Dörfler (2000) mentions some of them, like noting the outcomes of certain measuring processes. As symbolic means for the carriers of the protocol again tables, arrow diagrams but also coordinate systems can be used the latter leading to the common cartesian graph. Possibly, for the learning trajectory of S a successive combination of several of those contexts will supply her with a broad experiential background for the function concept. To repeat it, in this way the so-called representations of functions emerge for S as ways of presenting outcomes of her actions to herself (and possibly to others). Thus, the tables, diagrams and graphs have a personal meaning for S. Here we restrict ourselves to another experience related to permutations and to some hints on generalizations from that.

Permutations: a second view

A bodily experience connected to permutations differing from the above one is offered to S by the following problem context. The books are already placed on their places on the shelf. Think about rearranging them in any possible order. The action of rearranging for S gives rise to successively refined protocols with their respective carriers: noting for each place i to which place j the book is to be moved; a list like 1 to 5, 2 to 6, 3 to 3, etc; a table condensing this description; a diagram with an arrow from, say, 3 to 1 if the corresponding shift is made. These protocols again provide the tool for finding the number of rearrangements but also an experiential basis for the notion of a mapping of a set into itself. The details follow a similar pattern as in the first view which led to a first experience with a case of a mapping or univalent correspondence of one set to another.

Generalizing

A possible further route focuses on the protocols and their (symbolic/graphic) carriers: First, by looking for other contexts and actions which can be treated and described by the established means. Second, by investigating the tables/diagrams in their own right: which are their properties, how are they related to each other, which alterations are sensible? The arrangement-problem might lead S to generally consider distributing objects into containers: what if a place holds more than one book? Such guiding questions can be asked also by the expert accompanying the learning trajectory. The carriers of the respective protocols are adapted easily by S and she even formulates their fundamental properties. Those by an observer would be viewed as the common definition of a mapping from one set into another. A similar progress occurs in the rearrangement case viewed as directions how to move around among certain places. The protocols are varied accordingly and their general type presents S's knowledge of mappings of a set into itself. S is now able to think in terms of the protocols and their carriers as structures in their own right which yet derive for S their meaning from the original context of diverse actions carried out by S. An observer could say that S's knowledge of the function concept consists in her ability to manipulate the protocol-carriers, to relate them to each other and to problem contexts. She can now talk about the tables and diagrams and is able to subsume other situations under these symbolic and graphic structures. For instance, she devises rules for traveling within the natural numbers giving rise to numerical functions. Much of S's expansion of her function knowledge is still geared to her basic metaphors of associating objects with places and of moving around among places. She gains more and more flexibility in using those metaphors for generalizing their target domains. An observer finally would ascribe to S that she has acquired the abstract function concept.

Conclusion

Of course the learning trajectory does not necessarily end here yet this paper has to come to its end. The learning journey of S started from her own actions and was strongly determined by how S presented those actions to herself by protocols and their carriers. We have not considered (beyond a guiding expert) the social context around this hypothetical learning trajectory which might have a profound influence. Cooperation and negotiation with others can provide the motive for the activities of S, can inform and correct them. But that would be a much more complex story.

Aknowledgement. This paper was written when the author held a Visiting Professorship at Rutgers University, New Brunswick, NJ.

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Mathematical Reasoning: Mental Activity or Practice with Diagrams

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Introduction

This contribution is concerned with a specific role for mathematical activities played by what has come to be termed now as inscriptions. Those are marks on paper, on a computer-screen or in any other medium. Depending on how inscriptions are used and viewed in relation to mathematics a variety of terms are used, like notation, representation, visualization, visual, diagram, graph, image. A great body of theoretical and empirical research documents the growing and widespread awareness of the indispensability of writing and drawing, or generally of the usage of inscriptions for any kind of mathematical activity be it learning or research. This is paralleled by similar developments in the philosophy and the sociology of science and in science education, see for instance Roth (2004) or Brown (o.J.).

Instead of tackling the fairly impossible task to give a comprehensive overview over the current state of the discussion about inscriptions in mathematics I will only point out some aspects which are related to the position taken here.

The term "visualization" generally is used in opposition to algebraic or (so-called) formal ways of notation. Thus, visualization uses a rather geometric and graphic-like mode and it is predominantly two-dimensional (i.e. non-linear and non-sequential). This opposition is reflected in the discussion about the reliability and admissibility of "visual" proofs (proofs without words, see Nelson, 1994) and the rigor of visualizations. Mostly, to visualizations is attributed a supportive role for understanding, for insight and for intuitive thinking but also for invention and detection. This is based on the assumption of a more direct accessibility and intelligibility of those visualizations: formal and algebraic-symbolic mathematics can purportedly be explained by visualizing it in a different graphic-geometric mode. The faculties of vision for detecting patterns and regularities will serve this purpose. To shed some doubt on such a too naive view on the power of visualization I just point to the conventional character of many of the visualizations which implies that one has to learn to read them as well. There is also a well-documented reluctance of learners to use visualizations (Dreyfus, 1994). A different use of visualizations is made in many branches of research mathematics for exploration and experimentation, see for instance (Borwein and Jorgensen, o.J.). Here in a way a reverse direction is pursued: one strives for algebraic-analytic explanations of the experimentally detected graphic or geometric patterns, a "formula" will show the reason for the regularities and relationships seen within the visualization. This possibly again reflects the attitude that genuine mathematics has to be done by algebraic-symbolic-formal means and notation systems. Is one neglecting with visualizations their symbolic quality then here there is a neglect, I think, of the graphic and visual properties of formulaic expressions. The latter as well have to be perceived and interpreted even where a more mechanistic calculus is available.

In mathematics education the term "representation" comprises all forms of mathematical inscriptions and notations and thus has a broader meaning than visualization. The central importance of representations for the learning and teaching of mathematics is underlined by publications like Goldin and Janvier (1998), Cuoco and Curcio (2001), Arcavi (2003), di Sessa (2000). An often made distinction is that between external (Darstellung) and internal (Vorstellung) representations whereby the former serves predominantly the purpose of enabling and developing the latter (the mental object). This points to a possible but mostly implicit bias which consists in viewing (external) representations rather (or only) as a means or a crutch for "constructing" the (mental) mathematical concept (or object). This latter one is conceived of as being "abstract" and outside of any sensual perceptions.

Also historically this strict separation of and dualistic distinction (in the Cartesian sense) between the mental-internal and the material-external (and perceivable) is well established. Mathematical thinking and reasoning was and is often considered to be a purely mental activity which is completely separated from any empirical investigations. The "remoteness from sense experiences" is emphasized by Gödel (1964) or for instance by Poincaré (1905) who speaks about the aesthetic character of mathematics "despite the senses are not at all taking part in it". The symbols and diagrams (in our sense) are considered rather as a crutch or as a means for expressing the genuine mathematical ideas and notions. An extreme position in this regard had been taken by Brouwer in his conception of intuitionism. For Brouwer, mathematics is completely independent of any representational means and language in particular. Those only serve the purpose to communicate the mathematical content and they do not have any influence on the latter. Symbols thus play only an auxiliary role and can in principle be dispensed with at least by the individual mathematician in his/her creative process, see for instance Brouwer (1907, 1912). In a Platonistic framework diagrams like the geometric figures are just instantiations of the general and abstract ideas which they permit to investigate by manipulating the representing diagrams. There are yet positions which emphasize the role of symbols and diagrams in their own right in a way similar as it is done here or by Peirce. An example is the so-called formal arithmetic which was developed by E. Heine and J. K. Thomae, cf. Epple (1994). Thereby the meaning of arithmetic is considered to reside in its operation or calculation rules rather than in its reference to the cardinalities of finite sets. The formalistic stance as developed by Hilbert takes a similar position: mathematics studies symbolic structures based on axiom systems defining the respective terms, operation and inference rules. But it should become clear from the examples given below that the stance taken here is much broader than classical formalism by comprising also what for Hilbert was the contentful mathematics, especially finite and discrete mathematics before and outside of any axiomatization. There is also a tradition of considering (part of) mathematics in analogy to rules based games like chess. This view was extended then by the later Wittgenstein in his concept of "language games". For all that see the exposition in Epple (1994). It is interesting that in all these discussions the viewpoint taken by Peirce (see later) is not taken into account at all. In any case, the topic treated here was in one way or the other always of interest to mathematicians and philosophers though the mainstream position was rather to confer on symbols and diagrams only an auxiliary role. In contrast to that in this paper the symbolic and diagrammatic structures are the genuine objects of mathematical, i.e. diagrammatic reasoning which results in theorems expressing general properties and relationships of those structures, i.e. the diagrams in the sense used here. For a related position see Detlefsen and Luker (1980) who stress the empirical character of any kind of computation which as they say (p. 813/814) "is always an experiment ... operating on symbols and not on the things for which the symbols may be taken to stand".

Internalism versus Externalism

Another dimension in all the investigations and discussions of representations, visualizations and the like is that those terms are suggesting that there is something different which is being represented or visualized by whatever inscriptions which have to be interpreted as signifiers of a certain signified, i.e. as signs. This view takes many different forms. Platonism considers representations as a means to investigate the timeless mathematical objects (Brown, 1998). Rationalism and idealism views the (external) representations as expressions of ideas, thoughts and mental conceptions which are the deep content of mathematics. Learning mathematics in this context is then described as a process of internalisation and interiorization which builds cognitive schemata incorporating those ideas and conceptions (Piaget and most of Cognitivism, but cf. Scaife and Rogers, 1996). Differences occur with regard to the means and ways of those mental constructive processes. One could dub by “internalism” all those positions which posit or postulate primacy of the mental, the cognitive over all external and material which are just secondary means for or expressions of mental processes like thinking or reasoning. A good example is the belief in a specific mathematical intuition which is viewed as the genetic origin of ideas and problem solutions prior to their externalisation by representations. I would subsume even under the heading of internalism all those endeavours which tie back (basic) mathematical notions to human bodily experience (as opposed to purely mentalistic views) since they still locate the origin and development of those notions within the boundaries of the body itself. A prominent example is Piagetean reflective abstraction which leads to cognitive schemata for the fundamental mathematical conceptions without any need for external or material means. In a similar vein, according to Lakoff and Nunez (2001) the substance of mathematical notions consists in image schemata and metaphoric projections which again use the mathematical notations just to express themselves. A weakness of internalism is its neglect of the social and of communication and its focus on the individual and its thinking. Within internalism teaching looks like an impossible task!

In many different variations one finds in the philosophy and education of mathematics a counter position which can be termed “externalism”. There the representations, the visualizations, the notation systems are the primary aspects of mathematics and the objects of mathematical activity. In an extreme and pure form externalism asserts that learning maths consists in acquiring expertise in handling what by internalism is viewed only as an external aid; that understanding is created by relating to each other various representations for the same notion (e.g. Duval, 1995); that mathematical reasoning is focused and dependent on operating with representations or diagrams; that intuition arises from intimate familiarity with a great variety of representations, visualizations, diagrams; that mathematical activity is primarily an activity with those sign systems (including their interpretations and applications). A position of this kind though possibly in a weaker form is taken by socio-cultural theories (Vygotski, Leontiev, Activity Theory) including situated cognition and situated learning. Within maths education one finds several authors stressing the fundamental role of sign systems, in the sense of externalism, for instance Duval (1995), Winslow (2004), Radford (2003), Rotman (2000), Sfard (2000), Bergsten (1999), di Sessa (2000), Meira (1998), Lennerstadt/Mouwitz (2002), Nemirovski (1994), Cobb et al. (1992). What they have in common, at least in my view , is that there is no talk about internal or mental representations but the focus is on the usage and production of external signs (systems) which is viewed and interpreted as the core of mathematical learning, thinking and understanding. A very pronounced externalistic position has taken the American philosopher Charles S. Peirce. His notion of diagram and diagrammatic thinking is treated next. For related discussions cf. Dörfler (2004a,b).

Diagrams and diagrammatic reasoning

Ch. S. Peirce (3.363 in Collected Papers) has made among others the following comment on a basic feature of mathematics:

“It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra."

For this way of thinking commonly the term diagrammatic reasoning (henceforth d.r.) is used. By analyzing various examples from different mathematical areas the potential of d.r. is investigated thereby to some extent substantiating the Peircean tenet. This, therefore, is an epistemological and theoretical analysis which yet is taken to suggest implications for all kinds of actual mathematical activity. It will then be the purpose of another paper, Dörfler (2004b), to point out inherent limitations to d.r. for which here was not enough space to be presented. I only indicate that those are cases where in principle the mathematical notion is not amenable to diagrammatic methods and one has to stick to a kind of conceptual reasoning based on linguistically or metaphorically prescribed properties.

I have chosen to stick to the term "diagram" as it has been used by Peirce and others though being aware that this term might cause some misunderstandings and arouse inadequate expectations. First of all, the reader should dismiss all geometric connotations of a more narrow flavor. This already can be seen from the above reference to Peirce who includes formulae of all kinds into his notion of diagram (or icon). What is important are the spatial structure of a diagram, the spatial relationships of its parts to one another and the operations and transformations of and with the diagrams. The constitutive parts of a diagram can be any kind of inscriptions like letters, numerals, special signs or geometric figures. This will be elaborated in more detail below.

Despite here the stance is taken that mathematical development to a great part consists in the design and intelligent manipulation of diagrams no general definition of the notion of diagram is given but rather several examples and descriptive features are presented. Generally speaking, diagrams are kind of inscriptions of some permanence in any kind of medium (paper, sand, screen, etc). Those inscriptions mostly are planar but some are 3-dimensional like the models of geometric solids or the manipulatives in school mathematics. Mathematics at all levels abounds with such inscriptions: Number line, Venn diagrams, geometric figures, Cartesian graphs, point-line graphs, arrow diagrams (mappings), arrows in the Gaussian plane or as vectors, commutative diagrams (category theory);

but there are inscriptions also with a less geometric flavor: arithmetic or algebraic terms, function terms, fractions, decimal fractions, algebraic formulas, polynomials, matrices, systems of linear equations, continued fractions and many more. There are common features to some of these inscriptions which contribute to their diagrammatic quality as understood here. But I emphasize that by far not all kinds of inscriptions which occur in mathematical reasoning, learning and teaching have a diagrammatic quality. Quite a few of what are taken as visualizations or representations of mathematical notions and ideas do not qualify as diagrams since they lack some of the essential features. Mostly this is the precise operative structure which for genuine (Peircean) diagrams permits and invites their investigation and exploration as mathematical objects. On the other hand, diagrams are of such a wide variety that a generic definition appears impossible and impracticable, as well. Accordingly, the various kinds of diagrams in a Wittgensteinean sense are connected by family resemblances and by the ways we use them. Some widely shared qualities of diagrams are proposed in the following:

- diagrammatic inscriptions have a structure consisting in a specific spatial arrangement of and spatial relationships among their parts and elements. This structure often is of a conventional character.
- based on this diagrammatic structure there are rule-governed operations on and with the inscriptions by transforming, composing, decomposing, combining them (calculations in arithmetic and algebra, constructions in geometry, derivations in formal logic). These operations and transformations could be called the internal meaning of the respective diagram.
- another type of conventionalized rules governs the application and interpretation of the diagram within and outside of mathematics, i.e. what the diagram can be taken to denote or model. These rules one could term the external or referential meaning (algebraic terms standing for calculations with numbers, a graph depicting a network or a social structure). The two meanings closely inform and depend on each other.
- diagrammatic inscriptions (can be viewed to) express relationships by their very structure from which those relationships must be inferred based on the given operation rules. Diagrams are not to be understood in a figurative but in a relational sense (like a circle expressing the relation of its peripheral points to the midpoint).
- diagrammatic inscriptions have a generic aspect which permits to construct arbitrary instances of the same type of diagram. This leads among others to consider the totality of all diagrams of a given type (like all triangles, all decimal numbers).

- there is a type-token relationship between the individual and specific material inscription and the diagram which it is an instance of (like between a written letter and the letter as such).
- operations with diagrammatic inscriptions are based on the perceptive activity of the individual (like pattern recognition) which turns mathematics as d.r. into a perceptive and material activity.
- diagrammatic reasoning is a rule – based but inventive and constructive manipulation of diagrams to investigate their properties and relationships.
- diagrammatic reasoning is not mechanistic or purely algorithmic, it is imaginative and creative. Analogy: the music by Bach is based on strict rules of counterpoint but yet is highly creative and variegated.
- many steps and arguments of diagrammatic reasoning have no referential meaning nor do they need any.
- in diagrammatic reasoning the focus is on the diagrammatic inscriptions irrespective of what their referential meaning might be. The objects of diagrammatic reasoning are the diagrams themselves and their already established properties.
- diagrammatic inscriptions arise from many sources and for many purposes: as models of structures and processes, by deliberate design and construction, by idealization and abstraction from experiential reality, etc. And they are used accordingly for many purposes.
- efficient and successful diagrammatic reasoning presupposes intensive and extensive experience with manipulating diagrams. A widespread "inventory" of diagrams, their properties and relationships supports and occasions the creative and inventive usage of diagrams. Analogy: an expect chess-player has command over a great supply of chess-diagrams which guide his or her strategic problem solving. Consequence: learning mathematics has to comprise diagrammatic knowledge of a great variety.
- diagrams can be viewed as ideograms like those in the Chinese writing systems. They are not translations from any natural language or abbreviations of names and definitions. By their diagrammatic structure they "directly" present (to the initiated user!) their intended meaning. The latter usually is a complementaristic system of relationships (between the elements or the parts of the diagram) and of operations and transformations. Diagrams can be taken to refer directly to the related concepts. This quality was explicitly intended for instance by Frege with his "Begriffsschrift". For a related discussion I refer to Krämer (1988).
- diagrams are composed of signs of different character in the sense of Peirce. There are icons, indices, and symbols as well. And a whole diagram has iconic and symbolic functionality if it in itself is considered as a sign in the

sense of Peirce. I do not discuss here the problem what can or should be taken as object and interpretant in the Peircean triad when viewing the mathematical diagram as its sign (or representamen) pole. For the sign conception by Peirce the reader is referred to Hoffmann (2002).

- diagrams for their being understood and used appropriately need to be described by natural language complemented by specific terms relating to the diagram. But these descriptions and explanations cannot be substituted for the diagram and its various uses. In relation to the diagram and the intended relations and operations this is a meta-language about the diagrams. That also focuses attention and interest on the relevant aspects and activities. It is similar to how the legend of a map of a city explains how to use that map appropriately. Generally thus diagrams are therefore imbedded in a complex context and discourse which better is viewed as a social practice.
- diagrams are extra-linguistic signs. One cannot speak the diagram, but one can speak about the diagrams. In this sense, diagrams are irreducible entities of mathematics (there is no mathematics without "formulae"). Yet, their properties can be named by words and formulated as theorems. Thus, on the other hand (specialized) language (as extension of natural language) is equally indispensable.

Another dimension of explicating the notion of diagram or of d.r. is which uses are made of them in mathematics. First, the most widespread usage is to use the admissible operations and transformations to solve a given task. This comprises calculating a numeric value, solving equations, constructing a proof in geometry, finding a derivation (in formal logic) and many others. Thereby one operates with the inscriptions by exploiting and observing their structure and its changes. Thus, this is a material and perceptive activity guided by the diagrammatic inscriptions. It is like in other material actions: to be successful one has to have acquired an intimate experience with the objects one is operating with which here are the inscriptions. This is required and not so much abstract or conceptual knowledge. There are algorithmic operations (consider the Gauss algorithm) but much of d.r. is highly creative because the appropriate operations with the diagrams have to be first of all devised and deployed.

This first type of use is the only one which I want to subsume as diagrammatic reasoning. It is essential that the diagrammatic inscriptions themselves are the objects of the activity which produces knowledge about and experience with the diagrams. Second, the other usages of diagrammatic inscriptions I will call representational. The first kind of representational use is when a diagrammatic inscription is taken as a model for some other material or virtual structure from any science including mathematics itself or from any practice. This is captured by terms like application of mathematics or mathematization. It is not the place here to discuss that any further. I only remark that therein lies an important source for the design of diagrams which then within mathematics become the topic of d.r., see Dörfler (2000). A second type of representational use is widespread in mathematics education: to use diagrams as representations for to be (by the learner) constructed abstract objects. The diagrams are taken as a means for mental or cognitive constructions and thus have little interest in themselves. They are then more kind of a methodological scaffold possibly unavoidable but to be dismissed when successful. This is diametrically opposed to d.r. where the focus is on the diagrams themselves as the objects of study and of operations and not on their doubtful mediation with virtual objects. In this representational view mathematics is a predominantly mental activity supported by diagrams whereas mathematics as d.r. essentially is a material and perceptual one. And this does not reduce mathematics to meaningless symbol manipulations since the diagrams have meaning through their structure, their operations and transformations and of course via their applications. This holds for all diagrams as considered here in a way completely analogous to how geometric figures can have meaning.

Diagrammatic Reasoning: Examples

For each of the mathematical topics referred to in the following the reader might consult any standard textbook on it. Browsing through any book on mathematics convincingly shows an abundance of diagrams but it does not show how they are used or exploited. The examples intend to highlight d.r. as a specific perspective on diagrams. This perspective takes the view that diagrams of many different kinds are the objects of mathematical activities. Those activities consist in exploring properties of the diagrams and of various operations with them, see also Dörfler (2001). In the sense of Peirce, an important aspect of this kind of mathematical activity then is the observation of the impact and outcome of empirical activity. The detected regularities then give rise to concepts which describe specific properties of the diagrams. Invariant relationships between those properties and the respective concepts are formulated as theorems. It should be mentioned that there is the (mostly realized) possibility of then arguing with the concepts and the already proved theorems without explicit recourse to the underlying diagrams.

In their famous book "Grundlagen der Mathematik" Hilbert and Bernays analyze operations with arrays of strokes (or points) the observation of which leads to much of what are taken to be properties of natural numbers. The natural numbers are interpreted as types of arrays of strokes two of which are of the same type if they can be matched one by one. Addition and multiplication appear as operations with those arrays which clearly show a diagrammatic character. Properties like evenness and oddness are observable qualities of those diagrams in the form of specific arrangements of the strokes. A good example of d.r. is the statement that the sum of two odd numbers (diagrams) is even. This results from observing the combining of two odd diagrams in an appropriate way. In this kind of d.r. that statement is a way of reporting one's observations (and not a statement about abstract objects):

********		*********		*****************
*******	plus	**********	gives	*****************

Here the generic character of the diagrams is an important feature which provides the generality of the assertions about the diagrams. Similarly, d.r. by inspection of the following diagram

********		*****		*************
********	plus	****	gives	************

implies the rule "even + odd = odd". In the same manner the corresponding rules for multiplication are obtained by d.r. with rectangular (product) arrangements. That any number either is prime or divisible by a prime number similarly results from d. r. using rectangular arrangements of the arrays of strokes. Another kind of d.r. is also possible here using algebraic expressions and their properties. For "odd + odd = even" this could be: $(2m+1)+(2n+1)=2m+2n+2=2(m+n+1)$. Again, this d.r. is a manipulation of diagrams and an observation of the outcome by knowing that a diagram of the form $2(\ldots)$ is equivalent to evenness. Here we have the very common phenomenon that diagrams of very different kinds describe or model each other. This is also termed as "isomorphic representations" in the representational view of diagrams. Here this simply means that one can translate between the two kinds of diagrams such that the operations and properties of both uniquely match each other.

In this vein, a Ferrer's graph is a diagram for a partition like $12=2+2+3+5$ in the form of an array of lines of points:

**	transposed	****
**		****
***		**
*****		*
		*

By inspection various relations can be observed to be formulated then as properties of partitions, see Liu (1968). For instance, one can exchange the lines and columns in a Ferrer's graph which in the example corresponds to $12 = 4 + 4 + 2 + 1 + 1$. A case of d. r. then is the observation that to any partition with at most m parts corresponds one with no part greater than m and vice versa. Therefore the number of those two kinds of partitions is the same.

The young Gauss is reported to have found the sum of the first 100 positive integers by thinking of those numbers being written down in the following way:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \dots & 49 & 50 \\ 100 & 99 & 98 & 97 & \dots & 52 & 51 \end{array}$$

and adding the two numbers to get 101 in each of the 50 columns. Thus the sum is 50 times 101 which 5050. I consider this as a good example of d.r. since it depends on the structure of the rectangular array which can be used for every even number 2n in the form:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 2n & 2n-1 & 2n-2 & 2n-3 & \dots & n+2 & n+1 \end{array}$$

where each of the n columns adds up to $2n+1$ such that the sum total is $n(2n+1)$. This is gleaned from the diagram by observing certain regularities and relationships. This type of diagram permits in an analogous way to sum the integers from $k+1$ up to $k+2n$:

$$\begin{array}{cccc} k+1 & k+2 & \dots & k+n \\ k+2n & k+(2n-1) & \dots & k+n+1 \end{array}$$

Here each column adds up to $2n+1+2k$ and thus the sum is $n(2n+1+2k)$ where $k=0$ corresponds to the original diagram.

The creative act consists in inventing this diagram and very likely results from an extensive experience with number relations of all kind. There are other diagrams which can be used in this case and also for any number of integers. This is the following pattern:

$$1+2+3+\dots+(m-1)+m+$$

$$+m+(m-1)+(m-2)+\ldots+2+1$$

where each of the m column gives $(m+1)$ and thus the sum $1+2+3+\ldots+m=$ $(1/2)m(m+1)$. One could say that the above diagram results from the diagram $1+2+3+\ldots+m$ by an appropriate transformation. This might even become more diagrammatic when imagining the whole process as being carried out with collections of, say, pebbles which for $m=6$ leads to the diagram:

	*	*	**	**	***
*	*	**	**	***	***
***	**	**	*	*	*
***	***	**	**	*	

Combining column-wise gives a union of 6 sets of 7 elements which has twice the number of elements which one is looking for. The main activity is thereby the experimental investigation of diagrams whereby one has to stick to the operational rules (either for finite sets or numerals). The latter diagram can also be organized in a well-known pattern with 6 rows of 7 items each:

o	*	*	*	*	*	*
o	o	*	*	*	*	*
o	o	o	*	*	*	*
o	o	o	o	*	*	*
o	o	o	o	o	*	*
o	o	o	o	o	o	*

Very important is the generic and general quality of all those diagrams which permits to consider the general case in a specific example. But even an inductive argument will use (algebraic) d.r. of a form like:

$$(1+2+\ldots+n)+(n+1)=(1/2)n(n+1)+(n+1)$$
$$=(1/2)(n(n+1)+2(n+1))=1/2(n+1)(n+2)$$

What is in my opinion not valued appropriately is the role of perception, of the material activity (with the inscriptions) and of pattern recognition in all of that. Too much emphasis is laid on the "ideas" and on purely mental activity. But I assert that the ideas emerge from the observation and manipulation of the diagrams, of course always under a specific perspective. This perspective is expressed by the operation and transformation rules which are admitted for manipulating the inscriptions. And I think, one should resist the temptation to view that kind of d.r. only as a means to discover or justify properties of – in this case – natural numbers. I prefer to say that the discourse of natural numbers expresses the observations made about the diagrams, that it describes properties of, and relationships among, the latter. And for that great many diagrams of different kinds are used and investigated.

Any deduction in formal logic furnishes a perfect example for d.r.: it is a chain of formulae-diagrams transformed into each other according to the deduction rules of the respective logical calculus. The quality of a term is determined by its diagrammatic structure ("well-formed" formulas). Each step in a logical deduction depends on the structural or diagrammatic properties of already derived formulae which permit the application of one of the deduction rules. The latter consists then in a transformation or manipulation of formulae (logical terms). All this demands on part of the learner a great deal of perceptual activities like pattern recognition, pattern matching, rule based material actions with inscriptions. Thus (formal) logical reasoning as a case of d.r. rather appears to be a kind of craft which presupposes a great deal of experience, exercise and a stock of available and deployable formulae (in the role of tools).

Elementary Linear Algebra offers many striking examples of d.r. Consider the following formula $(A\alpha,\beta)=(\alpha,A\beta)$ where α,β are (column-)vectors of R^n, A is an $n\times n$ Matrix, A' its transpose, and (. , .) is the usual inner product. I present two different kinds of d.r. which demonstrate the formula by observing transformations and patterns of diagrams of linear algebra. The first takes recourse to the components a_i and b_i of α and β and the elements a_{ij} of A. Then the left side gives rise to the following diagram:

$$\begin{pmatrix} a_{11}a_1 & + & a_{12}a_2 & + & \dots & + & a_{1n}a_n \\ a_{21}a_1 & + & a_{22}a_2 & + & \dots & + & a_{2n}a_n \\ \dots & & \dots & & \dots & & \dots \\ a_{n1}a_1 & + & a_{n2}a_2 & + & \dots & + & a_{nn}a_n \end{pmatrix} \text{ inner product with } \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

which according to the rules transforms into:

$$\begin{matrix} (a_{11}a_1 & + & a_{12}a_2 & + & \dots & + & a_{1n}a_n)b_1 & + \\ (a_{21}a_1 & + & a_{22}a_2 & + & \dots & + & a_{2n}a_n)b_2 & + \end{matrix}$$

$$\dots$$

$$(a_{n1}a_1 \quad + \quad a_{n2}a_2 \quad + \quad \dots \quad + \quad a_{nn}a_n)b_n$$

which can be transformed according to the rules of elementary algebra or by reading the rectangular schema columnwise into the diagram:

$$\begin{array}{l} (a_{11}b_1 \quad + \quad a_{21}b_2 \quad + \quad \dots \quad + \quad a_{n1}b_n)\cdot a_1 + \\ (a_{12}b_1 \quad + \quad a_{22}b_2 \quad + \quad \dots \quad + \quad a_{n2}b_n)\cdot a_2 + \\ \qquad\qquad\qquad \dots \\ (a_{1n}b_1 \quad + \quad a_{2n}b_2 \quad + \quad \dots \quad + \quad a_{nn}b_n)\cdot a_n \end{array}$$

which is a pattern resulting from

$$\begin{pmatrix} a_{11}b_1 & + & a_{21}b_2 & + & \dots & + & a_{n1}b_n \\ a_{12}b_1 & + & a_{22}b_2 & + & \dots & + & a_{n2}b_n \\ & & & & \dots & & \\ a_{1n}b_1 & + & a_{2n}b_2 & + & \dots & + & a_{nn}b_n \end{pmatrix} \text{ inner product with } \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

After exchanging left and right side the right-hand part of this diagram results from the matrix product

$$\begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = A'\beta$$

I hope it comes clear from this that the proof of the formula is a sequence of manipulations of diagrams which heavily rely on perceiving structural properties of the latter. Nowhere the (external or referential) meaning of the diagrams is needed. But what is needed is experience and fluency with manipulating them and the ability to recognize patterns in the diagrams.

Another kind of d.r. proceeds as follows by exploiting the formula $(\gamma,\delta)=\gamma'\delta$ for (column-)vectors γ,δ where on the right the product is matrix multiplication:

$$(A\alpha,\beta)=(A\alpha)'\beta=(\alpha'A')\beta=$$

$$\alpha'(A'\beta)=(\alpha,A'\beta)$$

Again, this consist in not more (but also not less) than effective manipulation of diagrams (= formulae here). We know that this is difficult for students. Yet, possibly not because of lack in conceptual understanding but because of lack of experience with manipulating diagrams. This is very much hands-on experience which presupposes concrete and material activity with the diagrams by investigating their properties through "calculations". What also transpires from these observations is that the reliability and security of mathematical reasoning (partly) resides in the perception of structural properties of diagrams.

Many proofs in Calculus show a great deal of d.r. The main steps consist in transformations and combinations of inequalities according to the usual ε-δ-definitions of limits, continuity, etc. The premises and the conclusion likewise correspond to diagrams with a specific interpretation and the idea of the proof is how to transform the former into the latter. And this is a creative and at least partly perceptual activity. The diagrams which occur in these cases are like the following:

$$|f(x)-f(a)|<e/2 \quad |g(x)-g(a)|<e/2$$

and

$$|(f+g)(x)-(f+g)(a)|=|f(x)-f(a)+g(x)-g(a)|<$$
$$<|f(x)-f(a)|+|g(x)-g(a)|<e/2+e/2=e$$

Here d.r. is the manipulation and combination of what are considered as inequalities and this d. r. is known to be dependent on much (empirical) experience with those diagrams. Thus d.r. is based on sort of manipulative skill with diagrams (and possibly less on understanding of abstract objects). Finding a proof often consists in exploiting diagrammatic relationships which is very similar to geometric proofs using auxiliary elements and various transformations of figures.

There are mathematical theories like graph theory (see Bondy and Murty, 1976) the objects of which are (types of) diagrams. Of course, these theories furnish prominent examples for d.r. Thus, the objects of graph theory are diagrams made up by points (vertices) and lines (edges) between pairs of them like the following:

Choose seven points a,b,c,d,e,f,g in the plane and draw edges (connecting lines not necessarily straight) between the following pairs: a and b, a and c, a and e, b and d, c and d, e and f, f and g, g and e, g and a.

Many diagrammatic characteristics and operations can then be observed and executed like: degree of a vertex (number of edges attached to it), levels of connectedness (deleting vertices or edges may or may not separate a diagram into unconnected parts), or existence (or not) of cycles. The proofs found in textbooks can be read as verbal descriptions of d.r. by translating the latter into a more conceptual language. Yet, to follow the verbal reasoning a translation back to d.r. (with generic graphs) is almost unavoidable (and sometimes is offered by the authors themselves).

In many cases d.r. does not actually manipulate material inscriptions but relies on imagined and verbally described diagrams and their properties. Many proofs in Graph Theory, cf. Bondy and Murty (1976), are very good examples for that. As an example I consider the theorem that in a graph G on n vertices without triangles the number of edges is at most $n^2/4$. For that choose an independent set (i.e. without edges) A with the maximal number α of vertices and put $\beta = n - \alpha$. Since no two neighbors of a vertex can be joined by an edge (G has no triangles) the degree $d(v)$ of every vertex v is at most α. Every edge has one end-point in $B = G - A$ and therefore the number of all edges of G is not greater than the sum of the degrees $d(v), v \in B$, which again is at most $\alpha\beta$ (since each $d(v) \leq \alpha$). But $\alpha\beta$ is less or equal to $(1/4)(\alpha+\beta)^2 = n^2/4$. It should be clear how this sequence of arguments is based on the observation of imagined diagrams. It also can only be understood by following such observations in the way of a kind of thought-experiment

The diagrammatic character might be less evident in the case of polynomials (over some ring R). A polynomial like, say, $5x^3 - 2x^2 + 8x + 1$ can be considered as a diagram with a conventional structure. Indeed, the spatial relations of its parts are crucial for its being a polynomial and all the operations with polynomials depend on these structural and diagrammatic relations within the polynomial as a diagram. The results of Kirshner (1989) indicate that the operations with algebraic terms for many students are based on visual (i.e. diagrammatic) characteristics of those terms.

The proofs of properties of operations with polynomials consist in transformations of the polynomial-diagrams according to the agreed upon rules. For instance, that the degree of the product of two polynomials is the sum of the two respective degrees results from observing the structure of the product polynomial, i.e. from d.r. It is also by skilfully manipulating the polynomial diagrams that one realizes: If $p(a) = 0$ then $(x - a)$ divides $p(x)$, as is visible from the standard proof in any textbook. This underlines the material, experiential and observational aspect of doing mathematics.

These rules can be changed, for instance by "calculating" modulo a fixed polynomial p. This results in a collection of specific diagrams (the polynomials of degree less than $\deg p$). Those enjoy many diagrammatic properties, among them a specific kind of product which gives rise to a ring-structure for these diagrams. If the ring R is finite, say $R = Z_4$, then all these diagrams can be listed and their algebraic behavior be observed by inspection. Thus, this is not "abstract" algebra but diagrammatic algebra! Take $p(x) = x^2 + 2$ (over Z_4), then all polynomials modulo p essentially are the constants and all $ax + b$ with a from $\{1,2,3\}$ and b from $\{0,1,2,3\}$, i.e. we have 16 diagrams. For instance as a sum and a product using $x^2 = -2 = 2$ one obtains: $(2x+3)+(x+2)=3x+1$ and $(2x+3)*(x+2)=2x^2+6+7x=2*(x^2+2)+2+3x=3x+2$. Again all this is manipulation of diagrams according to conventional rules and the mathematics of it state regularities and invariants of those operations and manipulations.

Mathematics textbooks abound with instances where the creative invention of a diagram is the main step in a proof. Elementary Euclidean geometry presents a great variety of such inventions. Another is for instance in the standard proof of the Cauchy-Schwarz-Inequality for the inner product which reads as

$$(\alpha,\beta)^2 \leq (\alpha,\alpha)(\beta,\beta)$$

The new diagram used for the proof is $(\alpha + x\beta, \alpha + x\beta)$, x any real number, which runs as follows:

$$0 \leq (\alpha + x\beta, \alpha + x\beta) = (\alpha,\alpha) + 2x(\alpha,\beta) + (\beta,\beta)x^2$$

Therefore $4(\alpha,\beta)^2 - 4(\alpha,\alpha)(\beta,\beta) \leq 0$ because of the previously established diagrammatic property that $ax^2 + bx + c \geq 0$ for all x implies $b^2 - 4ac \leq 0$.

Other examples for similar inventions can be found in the proofs that the sum, product and quotient of continuous (or differentiable) functions are continuous (or differentiable). One of those is:

$$f(x)g(x) - f(x_0)g(x_0) = f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)$$

This often is called a “trick” but it really is the purposeful design of a diagram.

As another example consider the theorem that any polynomial $p(x)$ irreducible over a given field K has a root in a field $K_1 \supset K$ (see Cigler, 1995). This sounds as being about abstract objects but in fact it is about the possibility of designing diagrams of a specific sort. This design runs as follows: $K[x]$ designates all diagrams which are polynomials over K. Calculating in $K[x]$ modulo $p(x)$ amounts to reducing any polynomial h in $K[x]$ by (polynomial) multiples of p (for instance by dividing h by p to get the remainder as the polynomial equivalent to h modulo p - again a diagrammatic operation). When calculating in $K[x]$ modulo $p(x)$ the polynomials of degree less than $\deg p$ enjoy all the properties of a field which we denote by K_1. It contains K in form of the polynomials of degree 0. According to the rules for K_1 we have for the diagram x as an element of K_1 that $p(x)=0$ (p is simultaneously a polynomial over K_1). The crucial point is not that there is a symbol x with $p(x)=0$ but that this x belongs to a collection of diagrams with which we can calculate according to the rules laid down for a field. The "existence" of a root is the design of a diagram with specific properties and as a member of a collection of diagrams (all the polynomial diagrams of degree less than $\deg p$) for which addition and multiplication are realized as diagrammatic manipulations and which show all the properties of a field like the real numbers.

A special case of this construction are of course the complex numbers. Here a more direct construction of diagrams is common by introducing the "diagram" i which satisfies $i^2+1=0$, i.e. this i is the x of the general design with $p(x)=x^2+1$ and $K=R$ (real numbers). And $K[x]$ modulo (x^2+1) are just the diagrams (polynomials) $a+ib$ $(a,b \in R)$ with the usual well-known operations which are the operations modulo x^2+1, i.e. exploiting $i^2+1=0$. Thus, the essential point is the design of diagrams $a+bi$ and operations with them which satisfy a set of rules: mathematics as diagrammatic design (and not as the invention of abstract objects). This design can be carried out also if a and b are taken from any ring with unit. In the case of, say, Z_4 then all the designs $a+ib$ can be listed and their properties visually inspected. The resulting diagrams (i.e. the complex numbers over Z_4) are the following ones: $0,1,2,3,i,2i,3i,1+i,1+2i,1+3i,2+i,2+2i,2+3i,3+i,3+2i,3+3i$. Using $i^2+1=0$ or $i^2=-1$ or $i^2=3$, one can easily "calculate" which is a form of d.r. For instance: $(1+i)*(2+3i)=2+2i+3i+3i^2=2+5i+9=3+i$. This because we follow the diagrammatic rules for Z_4 and the generally accepted rules from elementary algebra. That completely resonates with the statement by Peirce.

To analyze the importance of the available and accessible stock of knowledge of and about diagrams I have chosen a proof of the famous theorem by Cayley on the number of different labeled trees on n vertices which is n^{n-2}, cf. Aigner and Ziegler (1998). Thereby a labeled tree is a connected graph on the vertex set $\{1,2,\ldots,n\}$ without cycles, cf. Bondy and Murty (1976). The labelling discriminates between trees which otherwise abstractly are isomorphic. Clearly those trees are diagrams. For the proof one invents a new kind of diagram by choosing a "leftmost" and a "rightmost" vertex which possible coincide. Having n choices for each of them gives $n^2 n^{n-2}=n^n$ as the to be proven number of the new diagrams with n vertices. Here now knowledge about other diagrams is helpful: there are n^n mappings f from $\{1,2,\ldots,n\}$ into itself. Those mappings are diagrams on $\{1,2,\ldots,n\}$ with arrows from i to $f(i)$. To each such mapping now in a constructive way corresponds a tree with the two distinguished vertices and vice versa in a one-to-one fashion. The establishment of this correspondence, cf. Aigner and Ziegler (1998), is an impressive example of d.r. since it proceeds by manipulating and transforming the mapping-diagrams into the tree-diagrams. Many other proofs presented by Aigner and Ziegler (1998) are similarly amenable to an analysis with regard to d.r.

Conclusion

Based on these examples a tentative response to Peirce's statement could be that by d.r. mathematicians (and learners of mathematics as well) investigate designed diagrams (in the form of inscriptions) as empirical and material objects. This leads to the detection of (sometimes even surprising) properties, relations, regularities and invariants. Some of those properties and relations (the axioms) are distinguished as characterizing the respective class of diagrams and taken as the basis for deductive reasoning. The latter, in this view, is another way of talking about the diagrams by using concepts incorporating their various properties and relations.

Much more could and should be said about d.r. For instance, how diagrams are designed to describe certain relations of or operations with other kinds of diagrams which results in a layered system of d.r. Further, how mathematics develops a conceptual language to talk about the diagrams and how reasoning then occurs in this language. And finally, which should or could be the consequences for learning and understanding mathematics: Does the idea of d.r. point to the necessity of the acquisition of manipulative skills in operating with diagrams, i.e. a kind of material and observational experience as a prerequisite for doing mathematics? Will this constitute a new but fundamental role for "calculations" as a basis for d.r.? All this will be the topic of further research.

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LESSON STUDY CHARACTERIZED AS A MULTI-TIERED TEACHING EXPERIMENT[1]

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Japanese lesson study, in various adapted forms, is becoming increasingly significant in professional development of mathematics teachers in the USA. Our goal in the research reported in this paper was to investigate, in a three-tiered teaching experiment, the cycles of learning of two researchers, six teachers, and the students in three grade 4 classes and three grade 5 classes, in a six-month long lesson study project in the first half of 2002. The learning processes evolved in each of the three tiers (students, teachers, and researchers) over the course of three iterations of a lesson on measurement taught respectively in grades 4 and 5 by the three grade 4 and three grade 5 teachers. This paper documents some of these shifts in learning in each of the three tiers and assesses this form of lesson study for professional development, through the eyes of the teachers.

IMPROVING THE TEACHING OF MATHEMATICS THROUGH LESSON STUDY

Although it is recognized that adaptations are necessary in using traditional Japanese lesson study in a different country where the culture and values may not be congruent with those of Japanese society (Stigler & Hiebert, 1999), the assumption is sometimes made that this form of professional development will be universally beneficial. In our research we set out to investigate in a systematic way the processes that take place over the course of several iterations of the same lesson, with planning and debriefing sessions preceding and following each iteration, for the purpose of assessing what is learned by teachers in this form of professional development. As researchers, our own learning was a central element in the study. And for both teachers and researchers, the learning of the grade 4 and grade 5 students in the study was the reason for the project in the first place. Thus a multi-tiered teaching experiment (Lesh & Kelly, 2000), which takes account of the learning of students, teachers, and researchers, was an appropriate choice of methodology, as will be elaborated in the following sections.

Conceptual framework.

The conceptual framework of the research is drawn from theoretical and empirical fields (Brown & Dowling, 1998). In the theoretical domain, our literature base includes the books by Stigler and Hiebert (1999) and Liping Ma (1999), both of which were supplied to and studied by the team of teachers prior to the commencement of the lesson study. The research was also informed by the growing lesson study literature in the USA

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(Fernandez et al., 2001; Lewis, 2000; Murata & Takahashi, 2002). The conceptual framework embraced "six principles for gradual, measurable improvement through lesson study" (Stigler & Heibert, 1999):

1. Expect improvement to be continual, gradual, and incremental.
2. Maintain a constant focus on student learning goals.
3. Focus on teaching, not teachers.
4. Make improvements in context.
5. Make improvements the work of teachers.
6. Build a system that can learn from its own experience (pp. 132-136).

Some of these principles will be re-visited in discussing the results of the study.

In the empirical field, a three-tiered teaching experiment (fig. 1) was structured as follows (adapted from Lesh & Kelly, 2000, p. 211).

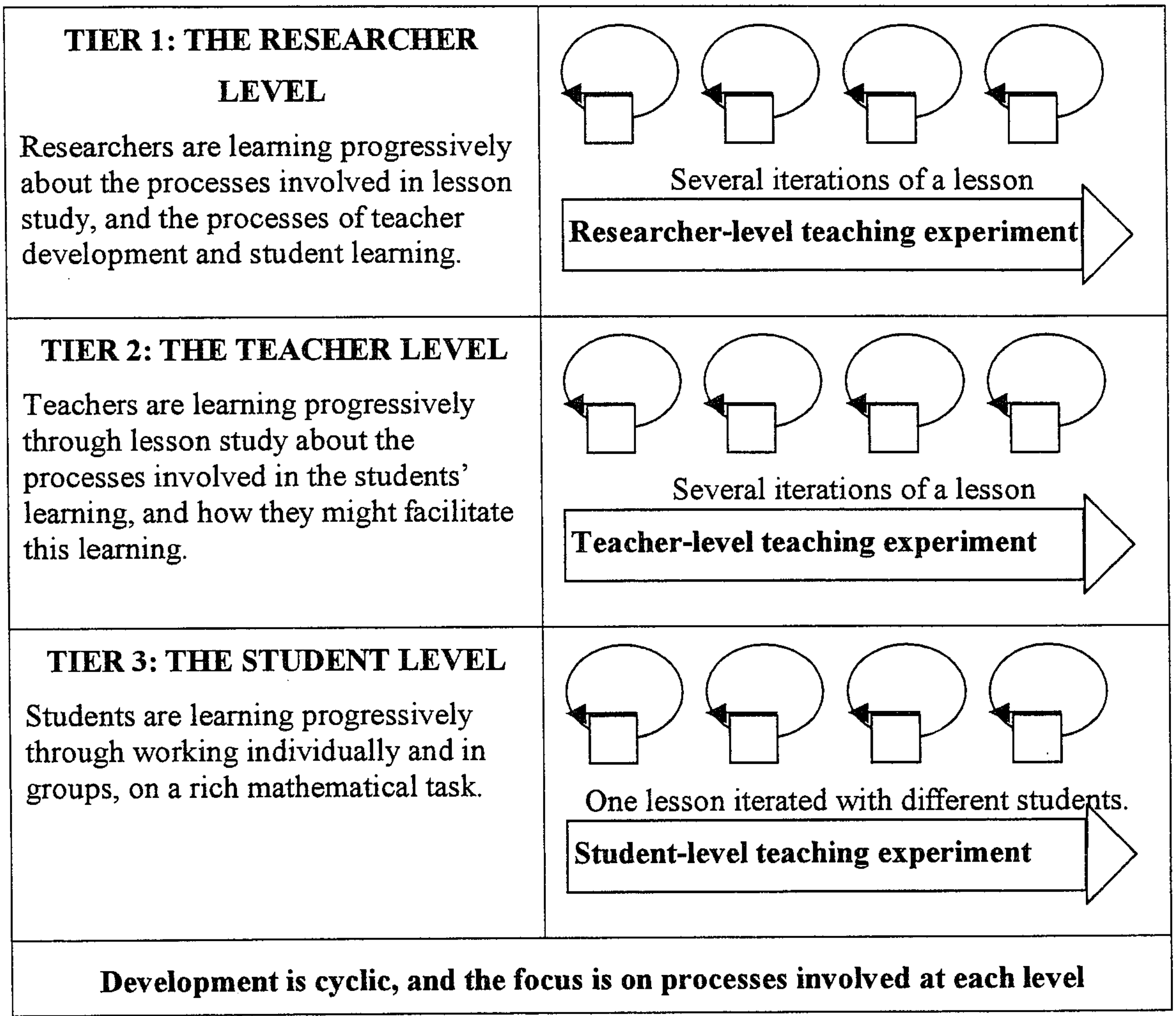

Figure 1: Lesson study as a three-tiered teaching experiment.

A three-tiered teaching experiment was a natural choice for the methodology of the research, because this form of research design involves teams of teachers and researchers working together, investigating a research question in the natural setting of the classroom, developing learning trajectories for the students with regard to the research question, meeting and reflecting on the outcomes of the experiment, and then repeating the whole process several times. The cyclic nature of the iterations of teaching a lesson on measurement, resonates also with the developmental research process described by Gravemeijer (1994), and the research is also consonant with the teacher development experiment of Simon (2000). Because space restrictions prohibit the reporting of details of learning in all three tiers, the specific research questions addressed in this paper are as follows.

1. What learning was reported by the teachers through participation in the lesson study?
2. Did the teachers judge this experience to have contributed to their professional development, and if so, in what ways?

Thus we are concentrating on the second tier in this paper. Because all three tiers are intimately connected, we shall also report on some of the learning that took place in the third tier, that of the students, and this whole paper details learning at tier 1 because it is the researchers who are reporting. But the focus is on tier 2, the learning of the teachers. These six teachers (three of grade 4 and three of grade 5) were chosen because they were known to one of the researchers, and because of their interest in participating in the lesson study.

Criteria of quality were addressed in the research by two forms of triangulation, namely that of multiple observers, and that of multiple data sources. The whole team of six teachers and two researchers reflected on the experiences of each iteration, and of participating in the lesson study. The results of the research are the negotiated interpretation by the whole team, of the data collected. There were six data sources, namely, notes taken during nine planning and debriefing meetings, audio recordings of some of these meetings, transcriptions of video recordings of lessons, lesson study grids drawn up by the grade 4 and grade 5 teachers in two teams, artifacts of students' work in lessons, and finally, field notes of presentations in which all but one of the team of six teachers and two researchers participated, at two conferences, one local and one regional.

The empirical setting (Brown & Dowling, 1998) and the choice of a lesson topic and problem, are elaborated in the next section.

Three iterations of a lesson on measurement.

Steps in the Japanese lesson study process are as follows (Stigler and Hiebert, 1999).

1. Defining the problem.
2. Planning the lesson.
3. Teaching the lesson (cycle 1).
4. Assessing the lesson and reflecting on its effect.

5. Revising the lesson.
6. Teaching the revised lesson (cycle 2).
7. Assessing and reflecting again.
8. Sharing the results (pp. 112-115).

While the team recognized that it might be necessary to adapt the process to US culture, these eight steps were all part of the study. The first seven steps were followed by a third cycle of revising, teaching and reflecting on the lesson, which was taught once by each of the six teachers (see Table 1). Presentations at a local conference on August 15, 2002, and at a regional conference on October 18, 2002, completed the eighth step.

Our program began with preparatory studies of two books (Ma, 1999; Stigler & Hiebert, 1999). After gaining familiarity with the lesson study approach, the group began, in February, meeting on a regular basis at the university for the purpose of selecting a topic and task and planning to teach a lesson by mid-April. In all, there were nine planning and debriefing meetings in addition to the three iterations of teaching the lesson.

Grade Level	**First Iteration**	**Second Iteration**	**Third Iteration**
Fourth Grade	April 18, 2002	May 8, 2002	June 12, 2002
Fifth Grade	April 17, 2002	May 8, 2002	June 12, 2002

Table 1: Dates of the teaching of the measurement lesson

At the meeting on March 6, all the participants had brought suggestions for possible mathematical content areas and specific topics suitable for a lesson that could be taught both in grade 4 and grade 5. Content areas that were suggested included geometry (perimeter, area, volume), fractions (equivalent fraction, multiplication of fractions), probability, and measurement of length. The researchers recognized that ownership by the teachers was important, and the group finally agreed on the following measurement task, set in the context of an imaginary radio competition, which seemed open and rich enough to facilitate students' learning of measurement in grades 4 and 5. The competitive nature of this formulation of the task may already be a departure from a Japanese cultural value of cooperation.

> "Walking in Sunshine" (Play a portion of this song, and fade out).
>
> Hey! Hey! Hey! You can be walking in sunshine when you enter out contest, "Steppin' to Cash Contest." 4th and 5th grade students, how would you like to walk into 500 dollars? You heard me right! 500 dollars! Well, you can if you enter the "Steppin' to Cash Contest." Here's all you have to do! Figure out how many footsteps it will take to walk from Normal to Peoria. If your estimate matches ours or is the closest, you could win 500 dollars! So come on, step up to win!
>
> "Walking in Sunshine" (and fade out).

The task lent itself to exploring issues of mathematical models for real situations, and allowed us to teach measurement within the context of problem solving.

Planning, preparing, and predicting.

By April 10, the planning had progressed to the following basic structure for the first iteration of the lesson, to be taught in grade 4 by Kelly and in grade 5 by Barry[2].

1. Announcement of "Radio Station Contest" by the teacher.
2. Whole-class discussion of ideas by students and teacher.
3. Work by individual students, each writing on a big yellow sheet of paper, deciding whether and how they wanted to take actual physical steps, to mark these out on the paper, to represent their thinking concerning the problem.
4. Work in groups of four students, again representing on a big white sheet of paper the results of the sharing of ideas and group activities.
5. Whole-class presentations and discussion of the results of small-group work.

Materials such as yardsticks and calculators would be made available. Students also had access to the information that there are 5,280 feet in a mile, and the distance from Normal to Peoria was taken to be 42 miles (or re-negotiated to be 40 miles in Barry's class).

A large part of the team preparation had involved negotiation of meanings of elements of the problem itself. What is a step? Is it different from a pace? How is it measured? It was foreseen that students might ask some of the questions they in fact did ask: Does it matter who does the walking? Does it matter if the walker is happy or sad? Does one have to walk in a straight line, and does this make a difference? Because the aspect of predicting student responses was known to be important in Japanese lesson study, the team worked out two grids, one for each of grades 4 and 5, consisting of four columns with the following headings: learning activity; expected student reaction; guidance/advice (to be provided by the teacher); and finally the actual reaction of the students (to be filled in after the teaching of the lesson). As an example, a small part of a grid for grade 5 is presented in table 2. A final grid was completed by the grade 5 teachers as a group after the third iteration, that is, after all three grade 5 teachers had taught the lesson, as a summary of "what happened" in all three iterations, taken in order. Thus the final grid shows (indirectly) the changes that took place between iterations as a result of reflections and debriefing by the whole group. Because of space restrictions, only two sections – introduction and small-group work - of Barry's lesson (that is, the first iteration) are presented in table 2. (Stages omitted are the "yellow sheet" work of individual students prior to the session in small groups, and the whole-class presentations and questions that followed the "white-sheet" work in groups.)

Issues that arose in the teachers' reflections were the role of questioning, the structuring influence of the tools that are provided (including the calculator and the yardstick), students learning through their mistakes, "allowing students to struggle with a process, rather than a focus on one correct answer or desired destination" (Barry, August 15). Some of these issues are discussed in the next section.

[2] All names of teachers are pseudonyms.

Learning Activities	**Expected Student Reaction**	**Guidance/Advice**	**Actual Reaction**
Introduction Pose question to entire class: "How many steps is it from Normal to Peoria?" Have class clarify things they need to know in order to solve problem What do you know? What do you need to know in order to solve this problem?	Wonder if it's a real contest How many steps are in a mile? How many miles is it to Peoria? How big is a step?	Write question on the board. Write "40 miles" on the board We're going to assume that's it	*Barry's Class:* How many steps are in a mile? What is the exact number of miles to Peoria?
Group work Students take yellow paper to the pre-assigned groups and are asked to develop a strategy to solve the problem Instructed to ask questions, share information… Materials: large pieces of white paper and a yardstick At some point during the process, teacher may want to reconvene the class to share questions that are being asked (not strategies)	Ask whose steps to measure What is a step? Some students will measure feet, rather than steps Expect students to watch other groups Actually take steps and begin to measure Some computation Begin talking about an "average" step	Redirect the original question Would __________ make a difference? What do you think a step is and why7? Show me how you're going to walk to Peoria We want to see a visual representation, or a drawing Is that what your picture represents? Watch for inconsistencies in what they're physically doing and how they're representing it	How many yards are in a mile? I have a math book. What do we need to look up? How many feet in a mile? We're going to estimate the steps in a mile Groups began taking steps Students performing calculations on their yellow papers Students counting steps on yardstick Resources being used or requested: (textbook, rulers, calculators, floor tiles) Began drawing on big sheets of paper Discussed measuring toe to toe, heel to heel and toe to heel Students decide to find an average step for their group Students jumped to simple calculations in an effort to solve quickly "It'd be easier to walk to Peoria than to go through this."

Table 2: Part of Barry's grade 5 lesson study grid.

CONCLUSIONS FOR PROFESSIONAL DEVELOPMENT

The teachers made numerous reflective comments about the value of meeting with other teachers for the purpose of promoting student's knowledge and problem solving abilities. All six agreed that they had never had another educator in their classroom to offer them constructive ideas about helping children understand and reason through mathematics (comments from a meeting on June 12, 2002). Barry elaborated on this (August 15), "You had other colleagues there in the room with you. Usually that means they are there to watch me, and critique. But now, these others were watching what I was watching." All six teachers were encouraged that they could study the ways their students were learning within the immediate situations of their classrooms. This point illustrates Stigler and Hiebert's (1999) fourth principle for lesson study, "Make improvements in context."

In a related observation, the teachers shifted the way they participated in classroom observation. Beginning with the first round of the lesson in mid-April, the teachers who were not presenting were observing students by moving rapidly from group to group, much as they might have done were they responsible for the lesson themselves. In contrast, one of the researchers focused her observations on only one of the groups of students in the classroom throughout the entire lesson. The teachers noticed this contrast in approaches, and we discussed the different purposes for each at the meeting on May 1. As a group, they decided to use this "teaching experiment" style for questioning and following one or two students closely through the duration of a lesson. After meeting on May 8 (after the second iteration), they commented on how much more they could learn about the lesson as a dynamic process by watching a specific group progress through the entire process, and chose to use the technique again on June 12 (third iteration). Barry reflected as follows,

> Instead of me figuring out all the students, we each watched a 'pocket' of that class. Here is what I saw, this is what the other teachers helped me do: I shifted from accomplishing a particular goal. I moved instead to look at what the kids are thinking and how I could help them grow. The different environment [of the lesson study approach] shifted my focus.

He attributed the growing ability to see what children need to grow in their mathematics to this particular research environment. This kind of observation can form a critical part of teachers' classroom practice, supporting and extending an "informal assessment" component of their pedagogy.

One barrier the team had to overcome was the difficulty of changing from a typical emphasis on classroom routines, and on the sequencing of student exercises into the substantive issues for lesson study. We came very slowly to this latter emphasis. It took a long time and much effort to ask new questions: how do children think about a mathematical idea, how does that idea fit in the curriculum, and what kind of strategies do children use, or need to use to investigate that mathematical idea? The six teachers in our group were initially focused on crafting a lesson together. But our group progressed quite slowly into the substantive work of anticipating students' reasoning and strategies related to the mathematical concepts. Resonating with Stigler and Hiebert's second and third principles, "Maintain a constant focus on student learning goals," and

"Focus on teaching, not teachers," lesson study only succeeds where teachers genuinely shift to assessment of the students' thinking within a classroom where a lesson is being taught without so much attention to the words and actions of the teacher. In a collaborative teaching experiment such as this, the lesson comes to be seen as belonging to the entire group, not to any one individual teacher: critique is then not of an individual, but an attempt to improve the lesson that then belongs to all.

Final word: where are we going?

Barry voiced it well (notes, August 15 presentation):

> A practical area that arose was that of how this process and these changes in lesson preparation and presentation impact classroom management, especially in the areas of timing and assessment. As teachers we want to work towards a point where we are less focused on "neatly wrapping up the lesson" in the allotted time, and more focused on the process and what the students are learning through that process of mistakes, conversation, questioning, self-evaluation, etc. We also want to work towards a point where we can find ways of assessing this process and find ways of making that assessment work within the boundaries and confines of our current evaluation system.

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Appendix: Teachers' Report

I. THE MATHEMATICS OF THE LESSON ("Where We Were")

- Our idea for our lesson came from a lesson we originally found in the Everyday Mathematics 5th grade Math Journal. We focused our lesson on the problem/question: "How many footsteps is it from Normal to Peoria?"
- We identified several areas of mathematics and mathematical thinking that were addressed through the lesson we developed. These included:
    - problem solving
    - calculations/computation
    - conversion
    - reasonable answers
    - measurement: benchmarks, standard units, iteration

II. CHILDREN'S THINKING ("Where We 'Are'")

- Through the presentation of the lesson and the subsequent "changes" and developments of the lesson itself, it became clear to us that one of our main areas of focus was children's thinking. We observed several levels of student thinking through the process which aided in our lesson development and our professional growth. Some areas and examples of student thinking include:
    - children often started their thinking and problem solving process where they were "comfortable" - on a practical level this translated into many students starting with basic calculation and computation (i.e. one student states "I know, there are 1000 feet in a mile and 40 miles to Peoria, so it's 40,000!"). While that statement was oversimplified, many students started with similar statements, though some with more accurate figures.
    - through the process students began to move from calculation and computation to "conversation". Students began to discuss the value of a footstep, what information they would need to solve the problem, how to best arrive at a reasonable solution, etc. This was partly accomplished through teacher questioning of student thinking. Students were asked to evaluate and defend their ideas that they had negotiated as a group. They were asked to defend their statements with pictures, summaries, detailed explanations, etc.
    - Many students encountered the conflict of personal benchmarks and standard units of measurements. Students were forced to reconcile differences in the measurements of their own feet and steps, what they

already knew about the relationship of feet to miles, what they had already decided as to the distance in miles from Normal to Peoria...

- eventually most students turned from the numbers and calculations to physical measurements and/or visual representations. Most students/groups began taking steps. Some students were simply measuring their foot and equating that to a footstep. Other groups physically took "steps", while some groups took natural steps, marked them, and measured the iterations of their steps 9from "toe to toe" for example), while other groups "forced" their steps to fit within a yardstick for example.

III. PROFESSIONAL DEVELOPMENT ("Where We Are Going")

- Through this process, in which we originally believed our focus would be on the development of the lesson itself, we came to a point where two other areas of focus seemed to take priority, the first being children's thinking and the second being our growth and development as professionals, leading us to the point where we began thinking "what do we do now and where do we go from here?"
- One area of professional development we focused on was "questioning". What types of questions do we ask/should we ask? What are the results of the questions we ask? Does our questioning allow for student mistakes that lead to learning? Does our questioning create an environment that allows for students to struggle through a process and independently think without leading them to "our" desired destination before they are ready, etc.
- The Role of the Teacher also became an area we focused on professionally and is directly related to the idea of questioning. Students need a balance of Teacher-directed instruction and student directed learning. Well-developed teacher questioning and well-planned student learning activities can help create a balanced learning approach (between totally teacher directed and "raw", complete constructivism). As teachers we began to see the need for a focus more on the student learning and letting that help direct instruction, rather than a focus on a set of goals that "have to be covered".
- As professionals we also saw a need to create an environment that allows for learning through mistakes, allows for student questions, allows for struggle with a process, rather than a focus on one set correct answer or desired destination.
- A practical area that arose was that of how this process and these changes in lesson preparation and presentation impact classroom management, especially in the areas of timing and assessment. As teachers we want to work towards a point where we are less focused on "neatly wrapping up the lesson" in the allotted time, and more focused on the process and what the students are learning through that process of mistakes, conversation, questioning, self-

evaluation, etc. We also want to work towards a point where we can find ways of assessing this process and find ways of making that assessment work within the boundaries and confines of our current evaluation system.

- As professionals we see real potential for growth in what happens next - In seeing what the students do in the next step of the process, how they reconcile the conflicts in their thinking, how their thinking changes, how they deal with a variety of reasonable and acceptable solutions to a single problem. We believe that the real growth for students occurs during the heart of the lesson, as they work through the process, as they are asked probing questions, as they struggle and make mistakes, as they work through the frustration and negotiate ideas with their groups. As teachers we often see the most potential for professional growth in the "results" and therefore see real potential for professional growth in evaluating what we see in the students' thinking in "day 2", in the process of seeing them evaluate their progress, in seeing the "closure" of the lesson and how the students deal with that.
- We are also interested in what direction this process would take if it were to continue. If we continued to work on this lesson would our focus change again? Would we get back more to the lesson development itself? What other changes would we make to the lesson? What if we started on another lesson? How would we apply what we have learned? Would we be able to apply what we know about student thinking and professional development in crafting our second lesson so that it would start out better and develop farther?

Children's Construction of New Mathematical Knowledge – Analysis of Classroom Episodes from Primary Teaching

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Abstract. The acquisition of new knowledge represents a theoretical and practical problem for the learning and understanding of mathematics. The central meaning of the interactive construction of actual new knowledge will be discussed referring to the theory of "collective learning processes" (Max Miller). This concept will be used for a qualitative analysis of episodes [of elementary teaching] under an epistemological perspective. The analysis will lead to three forms of interactive construction of new mathematical knowledge.

1. Introduction: Concrete Visualizability or Structural Universality as the Basis of Mathematical Understanding?

The common conception is that elementary school mathematics should be taught in a visual manner, with concrete material and direct references to the students' experiences. It is naturally proceeded on the assumption that the concrete visualizability of the learning material is the very thing to stimulate a sensible understanding of mathematics. But which particularities can be observed when carefully analyzing the actual course of interactive learning and teaching processes and when comparing them with regard to "concrete visualizability" and "arithmetic-structural universality"?

An important question of a research project*) tries to understand elementary school students, possibilities of constructing new mathematical knowledge in common interactions, of accomplishing exemplary generalizations as well as of explaining new mathematical relations with their own words.

Elementary school students are not able to construct new mathematical knowledge including the necessary generalizing reasons with the common concepts of elementary algebra and to operate with the yet unknown knowledge. For them, the new mathematical knowledge is tied to situative learning and experience contexts (Bauersfeld 1983). When trying to generalize arithmetical relations, the children have to develop their own situative descriptions. Still, they are able to see the general in the particular and to name it with their own words (Mason & Pimm 1983).

Geometrical-intuitive and arithmetical-structural learning environments (with questions to "figurate numbers" (triangular, rectangular, and square numbers) as well as, among others, to "crossing out – squares" and "number walls") have been conceived in order to stimulate these interactive construction processes. Both types are substantial, i. e. profound problems that are rich in relations (cf. Wittmann 1995), which make many activities and forms of reasoning possible for the children. The (eleven) teachers participating in the research have constructed teaching material of their choice, out of the conceived learning environments, and have carried out short teaching units (of maximally 5 lessons; 47 lessons have been documented in 3rd and 4th grades in the area of Dortmund).

2. New Mathematical Knowledge in Interactive Learning Processes

For every theory of learning and acquiring knowledge, the question what the core of the new knowledge consists of respectively how the newness of the knowledge yet to learn can be characterized is of central importance. The sociologist Max Miller (1986) makes this problem of the emergence of the new the central starting point of his learning theory of "collective argumentation". He says: "Any learning or developmental theory can ... legitimately be expected to give an answer to the question how the new in the development can emerge. ... Every answer to the question ... is... bound to the following validity criterion: it has to show that the new in the development presupposes the old in the development and still systematically exceeds it, otherwise there can be no new respectively the new is already an old, and the term »learning« or »development« loses any sense" (Miller 1986, p. 18).

Miller adds three important questions: "How can ... the validity of his already acquired (old) knowledge ... be shaken or relativized for the single individual? How can the single individual make new experiences that systematically exceed his present knowledge? And how ... can there be an obligation for the single individual to further develop his knowledge ...?" (Miller 1986, p. 18/19). On the one hand, these questions aim at the problem of actual new knowledge which is not yet present in the old knowledge – for example as a mere schematic or logic deduction -, and on the other hand, they trace the social conditions under which it is possible for young people to construct this pretentious knowledge.

In entirely individualized "learning processes" that are limited to the single student and his cognition, the individual can but add knowledge derived from his present knowledge basis, which – according to Miller – therefore is no new knowledge, but identical with the old. Only interactive processes make it possible to potentially develop new knowledge by contrastings, contradictions and new interpretations. "Only in the social group and because of the social interaction processes between the members of a group, the single individual can make those experiences that make fundamental learning steps possible" (Miller 1986, p. 20/21).

The problem of the relation between new and old knowledge thematized by Miller for *social* learning processes plays a very central role for mathematics also from an *epistemological* perspective. Isn't all correct mathematical knowledge logically consistently connected and thus deductible? How can new knowledge thus systematically exceed old knowledge if it results logically from it? Here the following paradox appears: On the one hand, all mathematical knowledge is logically consistent and hierarchically organized; hence new knowledge is deductible from the given foundations and does not systematically exceed the old knowledge. On the other hand there exist actually new and so far unknown insights in mathematics, e.g. by means of solving problems or by means of proving assumptions.

The example of the mathematical proof illustrates this paradox best. Rotman gives the following description to this: „...a proof is a logically correct sequence of implications .... Proofs are arguments, and as Peirce has indicated it impressively, every argument has an underlying idea - which he called leading principle, which changes the otherwise irreproachable sequence of logical steps to an instrument of conviction. ... It is absolutely possible to follow a proof without such an idea in a restricted sense by means of agreeing with every logical step. ... Nevertheless a leading principle is always present ... and without this, proofs could not *be* proofs“ (Rotman, 1988, pp. 14/15)

In mathematics a true new construction of mathematical knowledge means in the core the production of a *new relation* between elements in the present knowledge which is then brought into a consistent connection with the familiar knowledge. In doing so, this new and more general knowledge relation mainly changes the interpretation of the old knowledge: The new knowledge is no deduction from the old knowledge, but the new knowledge provides a new interpretation of the already familiar knowledge and modifies it. Thus, for example, the introduction of negative numbers is no logical continuation of the familiar natural numbers, but reversibly, the old, familiar numbers are now interpreted from the view of the new conceptual aspects of negative numbers in a generalizing way, namely as so-called „whole numbers“.

In the following it will be represented, with reference to three exemplary teaching episodes, how, in interactive knowledge constructions, socially constituted epistemological qualities of mathematical knowledge have consequences to the effect if really new knowledge is constructed or if, after all, it is a matter of knowledge identical to the old one. The central problem lies in Miller‘s statement: " ... it must show that the new in the development presupposes the old in the development and still systematically exceeds it ..." (Miller 1986, p. 18).

3. Differences of Interactive Knowledge Constructions in the Frame of Geometric and Arithmetical Learning Environments – Exemplary Episodes

Visual material plays a central role for the acquisition of mathematical knowledge in elementary school. Attention must be paid to the fact that it does not work automatically, but has to be actively interpreted and structured by the children. "There is no direct way from the visual material to the student‘s thinking, at best different difficult detours. The property of the number 3 is not visible at three smarties or three Lego bars, as if the child could derive it by means of simple contemplative observation. It is an abstraction, which ... does not work by merely leaving out the supposedly unimportant" (Lorenz 1995, p. 10). This abstraction has to be managed by the child itself after all, by reading new – yet invisible – relations and structures into the visual material. Thus by constructing new knowledge.

In traditional mathematics teaching, arithmetical exercises are mainly assessed under the perspective of a mathematically correct procedure (Winter 1982). In the frame of substantial learning environments, not the algorithmic procedure, but the underlying algebraic structure is stressed as the essential basis of the development of arithmetical relations. Separated packets of calculation tasks or "multi-colored dogs" (Wittmann 1990) do not contribute to that; productive exercises in which an operative structure or a substantial problem organizes the arithmetical structure are needed. Then arithmetical exercises can be filled with meaning and illustrated in exemplary relations by a wired arithmetical structure.

The sketched orientation to the essential aspects of geometric–intuitive and arithmetic –structural learning environments constitutes an important starting point for the analysis of the interactive construction of new knowledge in exemplary episodes.

3.1 Pierre and Caroline Determine the Ninth Square Number with Dot Patterns

Dot Patterns are to be used structurally as geometrical-intuitive reference domains for the interpretation and development of arithmetical relations. In contrast to this, the following empirical procedure of the children could be observed in many episodes to the topic of "figurate numbers": the dots usually served the children as concrete marks to directly count numbers. The collection of many single dots dominated, and it was counted step by step with partly more effective strategies; the intended structure was not actively constructed.

In the following episode the children have obtained an exercise sheet on which the dot pattern given until the third position was among other things to be continued and the according quantity was to be noted. The exercise is worked on at the school desks; the camera observes the co-operation between Pierre and Carolin at one desk. Before this short scene begins (see appendix), the two of them have completely created the 5th, 6th, 7th and 8th dot field together and eventually always determined the wanted quantity by means of counting dot by dot.

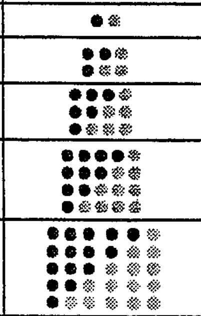
The determination of the quantity of chips in the following patterns shows a refinement of the strategy from a „direct counting of all chips in the completely drawn pattern“ to an *„additive* determination of the quantity of chips out of the quantity of the inner (old) chips and the outer (new) border chips“.

In the course of the entire episode, one can observe a modification of the counting determination of the 7th, 8th and 9th square number - as well as bigger numbers -, but this change takes place exclusively in the frame of the counting strategies and does not contain any conceptual insights. Only the counting, direct determination procedure for the wanted quantity is internally made shorter and more effective. One can detect an „optimization“ of the strategy to determine further square numbers which can be classified as a „concrete empirically-counting way of preceding“. The use of the dot field in this episode eventually remains bound to the old knowledge; exclusively single dots are counted directly. No actual new relation is constructed, neither in the numbers nor in the dot field. This knowledge does not systematically exceed the old knowledge.

3.2 Christopher and Nico Determine the Sixth Rectangular and Triangular Numbers

In this lesson, the children examine the relation between triangular and rectangular (oblong) numbers. The dot fields for the first five rectangular numbers (divided by color into two triangular configurations) have been noted on a big poster on the blackboard together with the numbers of each triangular or rectangular number. Now it is about determining the numbers and the configuration for the sixth position.

		Picture	
1.	2		1
2	6		3
3.	12		6
4.	20		10
5.	30		15



91 Ch I noticed something.

92 Ch Up there it goes four. Then it goes six. Then it goes eight. And then it goes ten. [*At this point, T indicates on the left side of the board first the number 20 and then the number 30.*] Then it goes twelve. [*T now points at the empty field below the number 30.*] Therefore there should be thirty-two on the other seventeen [*2 seconds break*] ohm, forty-two should be on that and on the one twenty-seven.

Christopher names the number series 4, 6, 8, 10, 12. He apparently refers to the second column [which the teacher points at]. He seems to have in mind the respective growth between the numbers, out of which he deduces the new numbers 32 and 17 (93). In the left number column, he has constructed a number bigger by 2, in the same way as he does in the right column: from 15 to 17. He corrects the numbers to 42 and 27; both numbers are raised by 12.

94 T I see. You mean ..., that's quite an interesting idea, Christopher. You mean, here should be a forty-two? [*points at the empty field below the number 30*]

The teacher confirms and enters this number.

96 T Yes. And there? [*points at the empty field below the number 15 at the right side of the table*]

97 Ch Twenty-seven.

Christopher repeats once again that the "27" should go in the other spot. Then the teacher asks for a justification, but Christopher cannot give one. Then the teacher calls Nico.

100 T No? ... Nico.

101 N Twenty-one.

102 T Why do you think twenty-one?

103 N Because twenty and twenty are forty [*points at the ten's place of "42"*] and one and one are two [*points at the unit's place of "42"*]

Nico claims "21" and reasons with a "calculation": 20 plus 20 equal 40, and 1 plus 1 equal 2.

The analysis of Christopher's knowledge construction shows that he develops a continuation principle for the 6^{th} rectangular and triangular number out of the present arithmetic pattern. By enumerating the series 4, 6, 8, 10, 12 it is meant exemplarily that the difference between the rectangular numbers always grows by "2", and therefore "12" has to be added to the 5^{th} number now. This addition of "12" is transferred to the triangular number, and "27" is determined as the 6^{th} triangular number. Christopher constructs a general arithmetic relation between the rectangular numbers and transfers it directly to the triangular numbers. This connection is inferred solely from the arithmetic structure; it is not proved, for instance by referring to the geometric pattern of the rectangular numbers.

In Nico's knowledge construction, the rectangular number "42" is halved exemplarily, first the ten and then the one. The connected intention – the triangular number is half of the matching rectangular number – is pronounced explicitly; the argument is limited in a tight form only to the procedure of the arithmetic halving.

Both students actually construct new knowledge, which cannot be derived from present one; they produce new arithmetic, structural relations, which were not used or known in this form before. It is striking that these knowledge relations are limited to the arithmetic number symbols, without using a connection to the geometric configurations.

Using the geometric relation between triangular and rectangular numbers, the question why the locally observed structure is really general and why it could be "always" continued could be answered. The produced new knowledge constructions are largely detached from the geometric problem; it is apparently only about arithmetic particularities and structures.

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

3.3 Kim Justifies why the Magic Number is Always "66"

In this teaching episode, the children work in an arithmetic – structural learning environment on "crossing out number squares" that are constructed by adding given border numbers of a table; they have the following quality: one can circle three arbitrary numbers so that there is one and only one circled number in each row and each column. The sum of such three numbers, no matter which choice, is constant. In this example, the constant "magic number" is "66". The children are to do a worksheet and try to give a reason why the "magic number" is always "66".

Magic Formula 1:

Calculate and fill the table!

+	13	9	4
10	23	19	14
12	25	21	16
18	31	27	22

Compare with your magic square!

10 Ki [*shows the teacher the first page of her copy-book with the "magic formulas" and explains the solution*] Mhm, we know it now -, now. One can divide the six-, one can divide the sixty-six into three different things #

12 Ki # Always. And one can do that very many times.

14 Ki So that there are nine solutions. And if one, like, takes the twenty-two, the twenty-one and the, whatever, then the result is always sixty-six. Or if one takes the thirty-one, sixteen and the nine -, ... nineteen, then the result is sixty-six again. [*Kim, her partner, and the teacher appear on the screen*]

Kim comes to the teacher and tells her reasons. She imagines that the "66" is divided into three different "things"; this could be done several times, until one had nine "things". The teacher agrees and then asks why the result is not 77 or 100. Kim answers that in this particular square, there were the very numbers to obtain the 66; and if one wanted the magic number to be 77, one would have to choose other numbers for the square. Later, she exemplarily gives numbers to prove this.

Kim's construction of the particular number square contains the idea of an "inversion": The magic number is not the "subsequent" result of an arithmetic rule; it is put at the beginning of the construction of a number square. Therefore the sum of three circled numbers is always "66". The "equation" $\bigcirc+\bigcirc+\bigcirc=66$ was only understood as a rule to calculate the magic number so far. But Kim now interprets this connection so that the circled numbers can be found out by dividing the number "66" into three numbers.

This construction represents really new knowledge. Even if the construction done by Kim is not directly realizable, and requires the division of the magic number into 6 border numbers, it can be interpreted as a construction of really new knowledge which is not merely detached from the given problem but which extends the arithmetic structure of the given magic square and makes it more understandable.

4. Different Forms of Constructing Mathematical Knowledge: Rule or Fact Knowledge, Problem-Oriented Relational Knowledge and Problem-Separated Relational Knowledge

In the research project, 39 comparable teaching episodes with interactive constructions, intended meanings as well as generalizations of mathematical knowledge in elementary school were analyzed thoroughly. In doing so, three different types of interactive knowledge constructions were determined: (1) knowledge constructions which occur almost exclusively in the frame of already present fact knowledge in situative, empirical contexts; (2) constructions of really new knowledge, maintaining an interrelation with the mathematical problem context; (3) constructions of new general knowledge relations which are detached from the present problem connection.

In the first example, Pierre and Caroline do not produce really new knowledge; they draw bigger and bigger concrete dot fields and count the amounts of dots; in doing so they produce further fact knowledge on the basis of present fact knowledge [type (1)].

In the second example, Christopher and Nico develop new arithmetic relations in the frame of existing number series; they leave the geometric-structural context in which a justification would have been possible. They produce problem-separated relational knowledge [type (3)].

The third episode exemplarily shows the construction of new, problem-related knowledge [type (2)]. Kim constructs a new relation between the numbers in the magic square and the magic number, which is based on the present problem context, produces a new structure and therefore conveys new insights.

The constructions of new knowledge by students in elementary school take place in the frame of two essential dimensions : They require the *social interaction* with other children as well as with the teacher and they take place in *context-bound descriptions and exemplary generalizations*. These central aspects of knowledge constructions - interactive resp. collective argumentation and situative, exemplary necessity- emphasize especially that mathematical knowledge is no totally objective, thus no subject- and socially-independent product, but that it is essentially determined in its development by social interactions and subjectively influenced, exemplary descriptions and interpretations.

With Miller's central orientation to the problem of actually new knowledge which „... presupposes the old in the development and yet systematically exceeds it ..." the following questions appear: Does the necessary exemplary and situative representation of the knowledge not by force prevent a construction of actually new knowledge since the children are referred back to the old familiar and already often employed means of description? Or: Can the course of the interaction really lead to systematically new knowledge because the constructions of the knowledge have to take place one after another in the temporal action and thus seemingly are a consequence of the preceding old knowledge?

The connection between old and systematically new knowledge is neither determined immediately by the kind of forms of description and representation nor by the outer interaction connection. This means on the one hand that mathematical knowledge and thus particularly the new knowledge cannot be equated with the means of representing and describing this knowledge, hence that it does not correspond with the way in which this knowledge is „expressed". The knowledge is not directly hidden in the signs and symbols; it develops from the intentions meant by these signs. In addition, successful constructions of new mathematical knowledge have to be distinguished from the interaction connections, since interactive relations and processes are subject to their own logic, their own independent mechanism of proceeding. „We should resist the temptation of identifying the learning of mathematics with the student's successful participation in interaction patterns" (Voigt 1994, p. 82). In order to do so, it is also necessary that certain intentional interpretations - which one tries to reconstruct within the analysis - are linked with the interactive constructions expressed in exemplary, situative forms.

In his interactive knowledge construction, Christopher has released himself from the geometrical structure, he has re-constructed the arithmetical structure in the sequence of rectangular numbers. The sequences of numbers mentioned by Christopher allow the interpretation that he means and intends that the difference between the rectangular numbers constantly increases by 2, and this interpretation is confirmed with the construction of the 6^{th} rectangular number. Then he formally transfers the arithmetical structure of the rectangular numbers to the triangular numbers. The construction meant in this way indeed systematically oversteps the old knowledge. Also the old knowledge is connected with the new knowledge by means of the temporal interactive succession. But from an epistemological point of view this new knowledge does not presuppose the old knowledge: With that, it is not of importance in the first place that a mathematically correct result is obtained with this construction, but rather that such a connection does not become possible with the old knowledge, in the course of which the old knowledge is generalised from the point of view of the new knowledge relation. This new knowledge does not supply an interpretation for the old knowledge. In didactics, this kind of knowledge construction is for example called „overgeneralisation“ or also „blind induction“ (Karaschewski 1966).

In her interactive knowledge construction, Kim employs a multitude of exemplary and situative descriptions, mixed with phrases such as „always, very often, different things“ which are supposed to suggest an intended generality of her construction. For instance, the employed magic number „66“ is meanwhile hardly seen as a calculation result, but interpreted in the sense of an exemplary „numeric variable“. Even though it is visible how much Kim is dependant on situative descriptions, the analysis makes it clear at the same time that Kim intentionally constructs a new knowledge relation with her exemplary representation by means of interpreting the calculation „circled number + circled number + circled number = magic number“ in a new way, for example in the sense of an equation, and by deducing a potential construction of crossing out number squares from the magic number. Thus the new knowledge relation constructed in this way also contains the possibility of generalising the familiar knowledge about crossing out number squares and of interpreting it in a modified way. Knowledge which systematically exceeds the old knowledge and at the same time presupposes it is constructed.

From an epistemological perspective, the construction of real new mathematical knowledge presupposes the production of a new *relation* or *structure* between elements of already familiar knowledge (cf. Steinbring 1999, 2000). This new relation is necessarily constructed and formulated in respectively exemplary descriptions and with situative references by employing the familiar means of description. The intended, meant relation has to be distinguished from the exemplary necessity of the means of description and of the current situation. This makes it necessary to analyze such an interactively constituted knowledge description thoroughly with regard to a contained, intended construction of a new knowledge relation. Furthermore, it has to be examined whether this new intended knowledge relation is at the same time a generalized interpretation of the old knowledge - perhaps also communicated in the current interaction.

If an interactively constructed and intended new relation in the old knowledge in the analysis can be reconstructed which at the same time allows a modified interpretation of the old knowledge, then essential conditions of Miller's validity criterion for mathematical knowledge are fulfilled: „... it must be shown that the new in the development presupposes the old in the development and yet systematically exceeds it ...". For mathematical knowledge - epistemologically seen consisting of structures and relations - this dualism of a „presupposition with simultaneous exceeding" can be realized by means of the social constitution of a new knowledge relation in the old knowledge context.

Remark

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Mathematics as the Science of Patterns – A Guideline for Developing Mathematical Education from Childhood to Adulthood

Erich Ch. Wittmann

T1

It is a great pleasure and honor for me to contribute to this conference, and I would like to express my sincere thanks for the invitation. During the past 10 years I have been enjoying a growing exchange with Japanese colleagues, and this exchange has been very inspiring for me. I am very grateful for the new contacts during my present visit.

In my talk I would like to describe one important feature of the developmental research which we have been conducting in our project "mathe 2000" since its inception in 1987. We consider mathematics education or didactics of mathematics as a design science like engineering.

T 2

Therefore the design, the empirical research and the implementation of substantial learning environments are at the very core of "mathe 2000". Of course we are well aware that research in mathematics education has to consider a variety of interdisciplinary aspects. Nevertheless, in our work the priority is clearly on mathematics, that is, more precisely, on elementary mathematics.

Our research is based also on another conviction: the reform of mathematics teaching can only be successful if mathematical education from childhood to adulthood is addressed as one whole. There are quite different social norms and preferences[TB1] of educators and teachers at the different levels: teachers at the kindergarten, primary level, secondary level and university, all have their own views. In order to overcome the differences and to co-ordinate education at the different levels and to shape **one** coherent conception of mathematical education we need some overall view, some global guideline which is acceptable and feasible at all levels.

In my talk I would like to explain how and why we in "mathe 2000" have adopted the view of mathematics as a science of patterns as such a guideline.

My talk has three parts:

In the first part I will present three trajectories of learning environments which we have developed. The emphasis is on early childhood and on the primary level, but I will at least indicate how these trajectories can be continued to higher levels.
These examples will provide illustrations for the second part of my talk in which our guideline will be analyzed in various directions. However, I should emphasize that in our understanding learning environments have a dignity of their own, independently of the theoretical considerations for which they may be used as illustrations.
In the third part I will say a few words about the consequences for teacher education.

Part 1

"mathe 2000" has adopted the view of mathematics as the science of patterns with an important accent: we are not primarily interested in ready-made and static patterns but in dynamic patterns which can be developed globally in the curriculum as well as explored, continued, re-shaped, and invented locally by the learners themselves. For us mathematical processes count much more than the final products. According to our conviction it is mathematical activity which forms the common link of learning mathematics across the various levels.

T 3

We have to understand children's active and playful approach to learning as the natural approach to mathematics at all levels. William Clifford, one of the great English mathematicians of the 19th century, expressed this fundamental point very convincingly.
The four so-called "general objectives" stated by Heinrich Winter, the German Freudenthal, in the mid-seventies of the past century have been particularly useful for our work as they reflect the four phases of a mathematical investigation:

Mathematizing
Exploring
Reasoning
Communicating

When constructing a learning environment we follow the natural flow of mathematical activity as expressed in these objectives: We look for situations which we think interesting for teaching, mathematize them, explore the emerging structures, look for proofs and check the opportunities for communication. In doing so we restrict ourselves to means of expression which are accessible to students. In addition we always try to combine the investigation of patterns with the practice of basic skills. I will explain the reason for this crucial decision later.

1. The race to 10 and some variations

T 4 Vol. 1 of our "Little Book of Numbers" which we have developed for pre-school education contains a well-known game of strategy: the race to some given target number. **T 5**

We start with the target number 10. The rules are as follows: A line of circles is numbered from 1 to 10. The first player starts by putting 1 or 2 counters on the first circle or the first two circles, the second player follows by putting 1 or 2 counters on the next circles similarly. Continuing in this way the players take turns until one of them arrives at the target and in doing so wins the game.

T 6 -12 A possible sequence of moves is the following one – here the red player wins.

T 13 - 20 Here is a second sequence - this time the blue player wins.

While playing the game repeatedly children get more and more familiar not only with the number line but also with the mathematical structure of the game.

I would like to share with you a short section from an interview with two five-year-olds who play the game. The language is German. However, there is a lot of nonverbal communication. So you can get at least some impression of the freshness with which the children are acting and you can get even some idea of what is going on in their minds.

((Video 5 minutes))

T 21 As a proud grandfather I remember very well what happened when my wife played the game for the first time with our granddaughter Isabelle, at that time 4 years old. In the third round grandma consciously placed her red counter on position 7. Isabelle hesitated a moment, looked out of the window and when she returned to the game she said: "Grandma, it's your turn".

T 22 The naïve approach of kindergarten children can be elaborated to a systematic analysis: By pursuing the moves backwards one can classify the positions in positive and negative ones: 10 is a positive position as the player who reaches it wins. 9 and 8 are negative positions as the player who reaches them leaves the opponent the chance to move to the positive position 10. Position 7 is positive as the player who reaches it forces the opponent to move to a negative position. The same argument shows that 4 and 1 are winning positions.

As a consequence the first player has a winning strategy: In the first move of the race to 10 she puts down one counter and then responds to a 2-counters move of the second player with a 1-counter move and to a 1-counter move of her opponent with a 2-counters move.

As our empirical studies show 4 to 5year-old children play this game with great pleasure and develop some first insight into the winning strategy. However, this knowledge is fairly instable. A few days later most children have to re-discover what they seemed to have mastered before.

In grade 1 the game is re-visited with targets up to 20.

T 23 In grade 2 a variation is played on the hundred chart:

The players take turns in adding 1, 2, 10 or 20 and move the counter correspondingly upwards. The player who reaches first 100 is the winner. An analysis shows that winning positions are the numbers which are congruent 1 mod 3. So the first player has a winning strategy.

There is a variation which is essentially equivalent from the mathematical point of view, but provides a better understanding of the 100 chart: One player starts with a red counter on number 1, the other one with a blue counter on number 100. The red counter is moved upwards +1, + 2, +10, + 20, the blue counter downwards –1, –2, –10, –20. The game ends when the counters meet. As there are 98 numbers between 1 and 100 the first player has a winning strategy as the number 98 is congruent 2 mod 3.

T 24 In grade 3 another variation of the game is played on the "thousand book": The allowed moves are +1, +2, +10, +20, + 100 or + 200.

The winning strategy is analogous to the winning strategy for the hundred chart.

T 25 In grade 5 all games will be revisited and analyzed by means of the divisibility arguments mentioned above.

1.2 Arithmogons

T 26

As the name suggests "arithmogons" connect arithmetic to geometry. **T 27** In the simplest case a triangle is divided in three fields. We put counters or write numbers in these fields. The rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side.

T 28 We can also start with two numbers inside and one outside and calculate the missing numbers, **T 29** or we can start from two numbers outside and one inside.

T 30 When the three numbers outside are given, we have a problem that does not allow for an immediate calculation but requires some thinking. The solution can be found by varying the number of counters in the inner fields (more or less) systematically.

T 31- 37 For example, we start by decomposing 9 into 5 and 4. Then we have to enter a 2 on top. However, 5+2 is 7, and not 11 as required. We change 5 into 6. In order to preserve 9 [TB3]the number 4 has to be replaced by 3. Now 3 + 2 = 5, that is we need 3 on top. 6 + 3 = 9 is closer to 11 – an encouragement to continue: 7 + 2 = 9, 2 + 4 = 6, 7 + 4 = 11, the solution.

Some children notice the dependencies in an arithmogon: If one number is increased by 1, then the number in the adjacent field has to be decreased by 1, which leads to the augmentation of the number on top by 1. As a result the number on the left side is increased by 2.

T 38 – 40 In grade 2 arithmogons are revisited with bigger numbers and again solved by systematically varying the numbers.

At this point I would like to report on an instructive e-mail correspondence which I had some time ago. The episode shows what reservations against reform exist in the educational system.

T 41 The e-mail which I received reads as follows:

Mathematics for little professors?

Ladies and gentleman,

the little son of a friend of mine is attending the second grade of a primary school in Berlin. Three weeks ago he came home with a very interesting homework. The problems which he had to solve were taken from the textbook „Das Zahlenbuch". These problems involve triangles with three numbers attached to the three sides, the three fields inside are free and have to be filled with the solution numbers.
Although the boy belongs to the group of best students of the class, he gave up in despair after some time, and the parents took up the problems. After about 40 minutes the father succeeded in solving them.
At school the students were told that they should just "try". The father happens to know a professor from the department of mathematics at the Free University of Berlin who after filling 1 ½ pages of DIN A 4 paper in 45 minutes solved the problems by using the Gaussian Algorithm, the determinant formula and Cramer's rule.
At the end he found a graphic illustration which includes the following solution formulas: $x = (a+b-c)/2$.
We suspect that not all primary teachers are familiar with this solution. The boy's teacher was not able to comment on it.
Now we have a question: When two adults rack their brains in trying to solve these problems, how will a second grader feel? We believe that failure, frustration and the loss of fun for mathematics could be the consequence.

We would greatly appreciate receiving your response.

Best regards
Bettina Göhrke und Olaf Koppen (father)

T 42 – 43 Here are the two sheets of paper which were sent to me at my request. They reveal the state of the art of Linear Algebra.

T 44 In my reply I first pointed out that one variable was sufficient in solving the arithmogon by algebraic means: We start with x in the left box and get $55 - x$ and $x - 11$ in the right and the top box.

The relationship on the left side is fulfilled if $2x - 11 = 33$.

This equation has the solution $x = 22$, from which we derive 11 and 33 for the other inner numbers. In fact $11 + 33 = 44$ and $22 + 33 = 55$. **T 45** The same method leads to the general formulas which were given by the mathematician in Berlin.

This is exactly the way in which arithmogons can and should be continued at the lower secondary level. Please note, that the primitive method of systematically varying the numbers is a very good help for finding the equation.

T 46 My next comment was that the relationships expressed in the three formulas can be analyzed at a more elementary level. A slight transformation reveals a hidden structure: Each inner number is the difference of half of the sum of the numbers outside and the opposite number outside.

This can be explained quite easily:

(1) The sum of the inner numbers is equal to half of the sum of the outer numbers as in calculating the outer numbers each inner number is used twice.

(2) If we subtract an outer number from the sum of the inner numbers we get the inner number opposite to the outer number.

These facts pave the way for a systematic solution in grade 4. **T 47** You see here a page from volume 4 of our textbook where the students are guided to the solution in three steps.

T 48 - 49 The first subtask presents easy arithmogons: the children have to calculate the numbers outside. Then they are asked to calculate the sums of the inner and outer numbers. The children discover that the latter sum is double as much as the first one and prove this fact by stating that each inside number contributes to two outside numbers. In the third step students are asked to subtract an outside number from the sum of the inside numbers and discover that the result is the inside number opposite to the outside number. This explanation is completely within the range of fourth graders[TB9].

The three steps can be summarized and allow for determining the inside numbers from the outside numbers directly.

T 50 At higher levels Arithmogons can be extended to polygons with more than three sides. With quadrangles new phenomena arise: Either we have more than one solution or no solution.

T 51 The mathematics behind arithmogons is quite advanced: the inner and outer numbers can be written as vectors, and the relationship between them is a linear mapping from R^n into R^n. Interestingly the corresponding matrix is non-singular for odd n and has the rank (n–1) for even n.

The two-dimensional structure of arithmogons can be generalized to arithmohedra in an obvious way: Numbers are assigned to vertices and faces of a polyhedron such that the numbers assigned to the faces are the sum of the numbers assigned to the vertices of this face. In this way a rich variety of examples can be created.

This material can be explored by College students and student teachers in a course on Linear Algebra with great gains. All phenomena and all concepts relevant for the theory of systems of linear equations occur and can be explained in this context, up to Steinitz' theorem and the dimension theorem. For student teachers it is most important to experience their own mathematical training in the professional context. We will come back to this point in section 2.

1.3 Fitting polygons

A fundamental idea of elementary geometry is „Fitting". Freudenthal describes it as follows:

> *In paving a floor with congruent tiles there is a leading idea - fitting. It is the same as in space and it is realized as concretely. Fitting is a motor sensation. Psychologists can tell you how strongly the motor component of the personality is marked at a young age, how important motor apprehension and memory may be. Things fit. Do children ask why? Apart from rare exceptions young children do not. All these miracles of our space do not seem to make any impression on them. But they grind as millstones. The highest pedagogical virtue is patience. One day the child will ask why, and there is no use of starting systematic geometry before that day has come. Even worse: it can really do harm.*

T 53 In developing „fitting" over the grades the "mathe 2000" curriculum starts in grade 1 with paper and scissors activities). A square is decomposed into two or four isosceles right triangles and the parts are recombined to make other figures.

In grade 2 this activity is extended: square paper is folded and cut such that equilateral triangles and halves of equilateral triangles are obtained. One of the figures which can be made by these forms is well-known from a proof of the Pythagorean theorem.

T 54 In grade 3 squares, regular triangles, pentagons, hexagons and octagons with the same side length are constructed by means of a template. Children can explore experimentally which figures fit which way. They realize that there are only three regular tessellations and discover some semi-regular tessellations.

T 55 In grade 4 children make regular polygons from cardboard by means of the „drawing clock“ and build the five Platonic solids. The name „drawing clock“ is derived from the fact that a circle is divided into 60 equal parts. As 60 is divisible by 3, 4, 5 and 6 the drawing clock allows for a convenient construction of squares, regular triangles, pentagons, hexagons and octagons. For example, to draw a regular pentagon[TB11] one has to divide the circumference in five equal parts of "12 minutes" each and connect the points. When drawing clocks of different sizes are used polygons of different sizes are obtained. The shapes are copied on cardboard. The circular segments attached to the sides of the polygons can be folded down and used for pasting the polygons together. In this way children can make stable models of all five Platonic solids. The proof of the existence of at most five Platonic solids at the end of book 13 of Euclid's "Elements of Mathematics" is fully in line with children's arguments.

T 56 In grade 5 cutting and fitting experiments with polygons will be used to find[TB12] the concept and the measure of an angle what again corresponds to the historic development. In the following grades cutting polygons into pieces and re-combining these pieces is the usual way to the area formulas and to the Pythagorean theorem.

Part 2 Theoretical Reflections

T 57

1. The quasi-empirical nature of mathematics

First of all the activities in exploring patterns and finding proofs depend on appropriate representations of the mathematical objects in question. It was I. Lakatos who in his masterpiece "Proofs and Refutations" first pointed to the fact that mathematical theories are always developed in close relationship with the construction of the objects to which they refer. Ring theory grows with the construction of rings, topology grows with the construction of topological spaces, and so on. These mathematical objects form a kind of "quasi-reality" which permits the researcher to conduct experiments similar to experiments in science. The race to a target number is based on the number line, arithmogons are based on a geometric structure and operations with counters and numbers, relationships between geometric figures are based on operations with forms. In all cases the students have something to operate on and to observe phenomena.

In general education [TB15] *informal* representations of mathematical objects are indispensable as they provide a hands-on "reality" which is much easier to grasp and to work on than symbolic representations.

2 The double role of representations

T 58

Representations of mathematical objects form an interface between pure and applied mathematics. They can be seen as concretizations of abstract mathematical concepts and at the same time as representations of real objects. Compared with the abstract objects which they represent these representations are more concrete, compared with the real objects which they model they are more abstract.

T 59 Counters are a simple example. On the one hand collections of counters can be considered as *concrete models of abstract numbers*. Operating with counters allows for proving relationships between numbers. On the other hand counters can be used *to model real situations*, as word problems show.

Conics provide another example for the amphibian-like role of representations: On the one hand a drawing of an ellipse is a concrete representation of the mathematical object "ellipse" and on the other hand a schematic representation of any real elliptic shape, for example a lithotripter for breaking up kidney stones, a mirror of a telescope or the orbit of a planet.

So studying mathematical objects via representations is the best preparation for mathematical applications. Contrary to real objects or models of real situations which are charged with various constraints abstract representations of mathematical objects allow for free operations and for establishing theoretical knowledge. Such representations are much more applicable than knowledge directly derived from mathematizing real situations.

From this perspective the widely held view that mathematics which is taught in general education, in particular at the lower levels, should in principle be derived from and closely related to applications is an educational error which is bound to undermine mathematics teaching instead of supporting it.

3 Operative proofs

T 60

By working with informal representations of mathematical objects sound proofs of general statements become possible. In the race to a target number the winning strategy depends on looking at pairs of moves and identifying a general pattern, independently of the special positions. The relationship of the inner and outer numbers of an arithmogon does not depend on special numbers but only on the rule for calculating the numbers outside from the numbers inside. Proofs of the Pythagorean theorem are based on operations with polygons.

The crucial point of this type of proof has been clarified by Jean Piaget in his epistemological analysis of mathematics: Mathematical knowledge is not derived from objects, but *from operations with objects* in the process of reflective abstraction: When it is intuitively clear that the operations applied to a special object can be transferred to all objects of a certain class to which the special object belongs then the relationship based on these operations is recognized as generally valid. As these proofs draw from the effects of operations on the objects under consideration they are called "operative proofs".

T 61 The advantages of operative proofs in the context of education are obvious: These proofs are embedded in the investigation of problems, closely related to the quasi-experimental investigation of patterns, based on the effects of operations and can be communicated in a simple problem-oriented language.

4 The epistemological triangle

T 62

In his empirical research into teaching/learning processes Heinz Steinbring has introduced the epistemological triangle. This triangle is an expression of the fact that learners cannot grasp a mathematical structure solely at the symbolic level, especially, if it is presented in a ready-made form. **T 63** The mathematical concepts carrying the structure become meaningful only via reference contexts in which the structure is embodied and which allow for conducting experiments. The experiences gathered in working within reference contexts form the basis for social exchange with the teacher and among the students. Along this exchange mathematical meaning is being established.

Arithmogons are a good illustration. Students cannot capture the conceptual relationships in their symbolic setting. They need operations with counters and calculations with numbers as a reference context in order to explore, explain and understand the arising patterns.

Part 3

The reform of mathematics teaching as indicated in the first two parts requires a re-construction of teacher education for an obvious reason:

Teachers who have acquired first-hand experiences with mathematical processes during their studies are much more likely to carry the reform of mathematics teaching forward than teachers who have not.

The question is how teacher education can be organized in order to provide these experiences best. Around the world there is a widely shared view among mathematics educators that mathematics education proper (didactics of mathematics) is the key to reforming teacher education, and therefore the main emphasis is put on courses in mathematics education. However, there are good reasons to see the mathematical courses as the key to reform. Unfortunately, this is not yet understood properly: Mathematical courses for student teachers are the blind spot of educational reform. **T 64** Our experience shows that teacher education can effectively be improved by linking mathematical courses to substantial learning environments as far as both subject matter and method are concerned. To develop such courses is a challenging problem for the future. Within "mathe 2000" a special series *Mathematics as a Process* has been started which tries to serve the purpose. As the title of the series indicates the emphasis is on mathematical processes. **T 67** The first volume "Arithmetik als Prozess" is available. In this book *informal representations* are preferred to formal ones wherever possible. So teachers are systematically given the opportunity to learn the professional language which they need for communicating with students.

In conclusion let me come back to the quotation at the beginning of my talk.

T 68

What is special about little children? They are curious, open, active and they have another most important trait: Small children do not know what a mistake is. When one-year-olds learn to walk they fall down very often – without having the feeling that something is wrong. When they start talking they are not aware that their first attempts are only approximations. No adult would blame them for doing something wrong. On the contrary: children are encouraged to carry on.

This is an important lesson we have to learn for the teaching of mathematics at all levels. Learning mathematics should be seen as learning top walk mathematically – and the adults should show the same patience and the same understanding which they show for little children.

Thank you very much for your attention.