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# KLEIN BOTTLE SURGERY AND GENERA OF KNOTS

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**In this paper, we study the creation of Klein bottles by surgery on knots in the 3-sphere. For non-cabled knots, it is known that the slope corresponding to such surgery is an integer. We give an upper bound for the slopes yielding Klein bottles in terms of the genera of knots.**

## 1. Introduction.

In this paper, we will study the creation of Klein bottles by surgery on knots in the 3-sphere $S^3$. Let $K$ be a knot in $S^3$, and let $E(K)$ be its exterior. A *slope* on $\partial E(K)$ is the isotopy class of an essential simple closed curve in $\partial E(K)$. As usual, the slopes on $\partial E(K)$ are parameterized by $\mathbb{Q} \cup \{1/0\}$, where $1/0$ corresponds to a meridian slope (see [**R**]). For a slope $r$, $K(r)$ denotes the closed 3-manifold obtained by $r$-*Dehn surgery* on $K$. That is, $K(r) = E(K) \cup V$, where $V$ is a solid torus glued to $E(K)$ along their boundaries in such a way that $r$ bounds a meridian disk in $V$.

Suppose that $K(r)$ contains a Klein bottle. Then $K(r)$ is shown to be reducible, toroidal or Seifert-fibered [**L**], and therefore it is non-hyperbolic. Gordon and Luecke [**GL**] showed that such a slope $r$ is integral when $K$ is hyperbolic. Furthermore, such a slope must be divisible by four in this case [**T1**]. These results together with the bound on exceptional surgeries [**A**, Theorem 8.1] imply that there are at most three surgeries creating Klein bottles on a hyperbolic knot in $S^3$.

However, unfortunately, there is no universal upper bound on the absolute values of such slopes. That is, for any positive number $N$, there exists a hyperbolic knot in $S^3$ which admits $r$-surgery creating a Klein bottle for $r > N$. See Section 5.

In [**T1**], we gave an upper bound on the absolute value of such a slope $r$ in terms of the genera of knots. That is, for a non-cabled knot $K$, $|r| \leq 12g(K)-8$, where $g(K)$ is the genus of $K$. Indeed, we had a better inequality $|r| \leq 8g(K) - 4$ if $r$ is not the boundary slope of a once-punctured Klein bottle spanned by $K$.

The main theorem of this paper greatly improves both estimations:

**Theorem 1.1.** *Let $K$ be a non-cabled knot in $S^3$. If $K(r)$ contains a Klein bottle, then $|r| \leq 4g(K) + 4$. Moreover, if $r$ is not the boundary slope of a once-punctured Klein bottle spanned by $K$, then $|r| \leq 4g(K) - 4$.*

We remark that such a slope can be non-integral for a cable knot. In fact, 16/3-surgery on the right-handed trefoil yields a prism manifold which contains a Klein bottle. Also we remark that, as far as we know, there is no example of the case that $r$ is not the boundary slope of a once-punctured Klein bottle spanned by $K$. (The knots of [**BH**, Propositions 18,19] are strong candidates.)

The extremal case $|r| = 4g(K) + 4$ can be described completely in the following:

**Theorem 1.2.** *Let $K$ be a non-cabled knot in $S^3$. Suppose that $K(r)$ contains a Klein bottle. If $|r| = 4g(K) + 4$, then $K$ is the connected sum of the $(2, m)$-torus knot and the $(2, n)$-torus knot, and $r = 2m + 2n$, where $m, n$ $(\neq \pm 1)$ are odd integers with the same sign.*

**Corollary 1.3.** *Let $K$ be a hyperbolic knot in $S^3$. If $K(r)$ contains a Klein bottle, then $|r| \leq 4g(K)$. Moreover, if $|r| = 4g(K)$, then $K$ bounds a once-punctured Klein bottle whose boundary slope is $r$.*

For example, $\pm 4$-surgery on the figure eight knot yield Klein bottles. Clearly, each slope bounds a once-punctured Klein bottle. (Consider a checkerboard surface of its standard diagram.) Since it has genus one, the above estimation is sharp. In Section 5, such a hyperbolic knot will be given for each genus.

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## 2. Preliminaries.

Throughout the paper, $K$ is assumed to be a non-cabled knot. We denote by $g$ the genus of $K$. Suppose that $K(r)$ contains a Klein bottle $\widehat{P}$ for a slope $r$. In general, 0-surgery can yield a Klein bottle, but we may assume $r \neq 0$ to prove Theorem 1.1. Thus we may assume $r > 0$. Let $k$ be the core of the attached solid torus $V$. We may assume that $k$ intersects $\widehat{P}$ transversely, and that $\widehat{P}$ is chosen to minimize $p = |\widehat{P} \cap k|$ among all Klein bottles in $K(r)$. Then $P = \widehat{P} \cap E(K)$ is a punctured Klein bottle properly embedded in $E(K)$ with $|\partial P| = p$. We note that $p \geq 1$ and $p$ is odd. Otherwise a closed non-orientable surface can be obtained by attaching suitable annuli along $\partial P$.

**Lemma 2.1.** *$P$ is incompressible and boundary incompressible in $E(K)$.*

*Proof.* See Lemmas 2.1 and 2.2 of [**T1**]. □

**Lemma 2.2.** *$r$ is an integer divisible by four.*

*Proof.* See Lemmas 2.3 and 2.4 of [**T1**]. □

Let $Q \subset E(K)$ be a minimal genus Seifert surface of $K$. Then $Q$ is incompressible and boundary incompressible in $E(K)$. Let $\widehat{Q}$ denote the closed surface obtained by capping $\partial Q$ off by a disk. We may assume that $P$ and $Q$ intersect transversely, and that $P \cap Q$ contains no circle component which bounds a disk in $P$ or $Q$ by the incompressibility of these surfaces. Also, we can assume that each component of $\partial P$ intersects $\partial Q$ in exactly $r$ points, since $\partial Q$ has slope 0.

Let $G_Q$ be the graph in $\widehat{Q}$ obtained by taking as the fat vertex the disk $\widehat{Q} - \mathrm{Int}Q$ and as edges the arc components of $P \cap Q$. Similarly, $G_P$ is the graph in $\widehat{P}$ whose vertices are the $p$ disks $\widehat{P} - \mathrm{Int}P$ and whose edges are the arc components of $P \cap Q$. Thus the edges of $G_P$ and $G_Q$ are in one-one correspondence. When $p > 1$, number the components of $\partial P$, $1, 2, \dots, p$ in sequence along $\partial E(K)$. This induces a numbering of the vertices of $G_P$. Each endpoint of an edge in $G_Q$ has a label, namely the number of the corresponding component of $\partial P$. Thus the labels $1, 2 \dots, p$ appear in order around the vertex of $G_Q$ repeated $r$ times. An edge with labels $i$ and $j$ at its endpoints is called a *$(i, j)$-edge*. If an edge has a label $i$ at least one endpoint, it is called an $i$-edge. If both endpoints have label $i$, then it is called a *level $i$-edge*, or simply a *level edge*. Since $G_Q$ has just one vertex, the edges of $G_P$ have no labels. A *trivial loop* in a graph is a length one cycle which bounds a disk face of the graph.

**Lemma 2.3.** *Neither $G_P$ nor $G_Q$ contains trivial loops.*

*Proof.* This is Lemma 3.1 in [**T1**]. □

Although $P$ is non-orientable, we can establish a parity rule as a natural generalization of the usual one [**CGLS**]. Here we use a restricted form, because one graph has just one vertex. First orient all components of $\partial P$ so that they are mutually homologous on $\partial E(K)$. Also consider an orientation to $\partial Q$. Let $e$ be an edge of $G_P$ (and $G_Q$ simultaneously). Let $D$ be a regular neighborhood of $e$ on $P$. Then $D$ is a disk, and $\partial D = a \cup b \cup c \cup d$, where $a$ and $c$ are arcs in $\partial P$ with induced orientations from $\partial P$. If $a$ and $c$ have the same direction along $\partial D$, then $e$ is said to be *positive* in $G_P$, *negative* otherwise. See Figure 1. Similarly we define positive and negative edges in $G_Q$. Since $\partial E(K)$ is a torus and $E(K)$ is orientable, we have the following expression of the parity rule:

**Lemma 2.4** (Parity rule)**.** *Each edge of $G_Q$ ($G_P$, resp.) is positive (negative, resp.).*

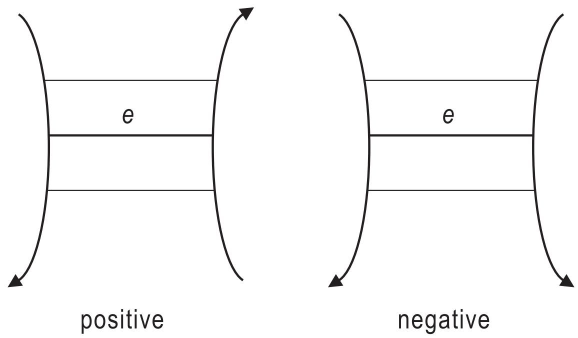


**Figure 1.**

## 3. Generic case.

Throughout this section, we assume $p > 1$. This means that $r$ is not the boundary slope of a once-punctured Klein bottle spanned by $K$.

A sequence of edges in $G_Q$ is called a *cycle*. Since $G_Q$ has a single vertex, this is not a cycle in a sense of graph theory. Let $D$ be a disk face of $G_Q$. Then $\partial D$ is an alternating sequence of edges and *corners* (subarcs of $\partial Q$). Thus we can regard that $\partial D$ defines a cycle. If $D$ is bounded by only $i$-edges, and all the $i$-edges have the same pair of labels $\{i, i+1\}$ at their endpoints, then the cycle defined by $\partial D$ is called a *Scharlemann cycle* with the label pair $\{i, i+1\}$. The number of edges in a Scharlemann cycle $\sigma$ is defined to be the *length* of $\sigma$. In particular, a Scharlemann cycle of length two is called an *$S$-cycle*. A triple of successive parallel edges $\{e_{-1}, e_0, e_1\}$ is called a *generalized $S$-cycle* if $e_0$ is a level $i$-edge and both $e_{-1}$ and $e_1$ are $(i-1, i+1)$-edges.

**Lemma 3.1.**

(i) *$G_Q$ does not contain a Scharlemann cycle.*
(ii) *$G_Q$ does not contain a generalized $S$-cycle.*

*Proof.* These are Lemmas 3.2 and 3.3 in [**T1**]. (In fact, [**T1**, Lemma 3.2] treats only an $S$-cycle, but the argument works in general.) □

**Lemma 3.2.** *At most two vertices of $G_P$ are incident to negative loops.*

*Proof.* Let $e$ be a negative loop at a vertex $v$ in $G_P$. Then $N(v \cup e)$ is a Möbius band in $\widehat{P}$. Since a Klein bottle contains at most two disjoint Möbius bands, the result follows. □

By Lemma 3.2, there are at most two vertices $u$ and $v$ of $G_P$ which are incident to negative loops. This means that $G_Q$ has at most two kinds of level edges. These are called level $u$-edges and level $v$-edges.

**Lemma 3.3.** *There are at most $r/2$ level $u$ $(v)$-edges in $G_Q$.*

*Proof.* Since $u$ has degree $r$ in $G_P$, there are at most $r/2$ negative loops at $u$. The result follows from the parity rule. □

Since $p \geq 3$, we can choose a vertex $x$ of $G_P$ which is not incident to a negative loop by Lemma 3.2. We fix this $x$ hereafter. Let $\Gamma_x$ be the subgraph consisting of all $x$-edges and the vertex of $G_Q$. Since $G_Q$ does not contain level $x$-edges, $\Gamma_x$ has just $r$ edges. A disk face of $\Gamma_x$ is called an *$x$-face*.

**Lemma 3.4.** *Any $x$-face contains at least one level edge of $G_Q$ in its interior.*

*Proof.* Assume that an $x$-face $D$ does not contain a level edge. Then $D$ contains a Scharlemann cycle by [**HM**, Lemma 5.2]. This contradicts Lemma 3.1. □

**Lemma 3.5.** *If $r > 4g-4$, then there are two $x$-faces $D_u$ and $D_v$ such that $D_u$ contains only level $u$-edges and $D_v$ contains only level $v$-edges.*

*Proof.* Let $X$ be the number of $x$-faces. Then an Euler characteristic calculation for $\Gamma_x$ gives

$$1 - r + \sum_{f:\ \text{a face of } \Gamma_x} \chi(f) = 2 - 2g.$$

Thus $X \geq \sum \chi(f) = 1 - 2g + r$. Since $r$ is divisible by four, $r \geq 4g$. Then $X \geq r/2 + 1$. The result follows from Lemmas 3.3 and 3.4. □

We show that the existence of the $x$-faces $D_u, D_v$ gives a contradiction. Let $D_\alpha = D_u$ or $D_v$. Thus $D_\alpha$ contains only level $\alpha$-edges.

Let $D$ be a disk face of $G_Q$. Recall that $\partial D$ is an alternating sequence of edges of $G_Q$ and corners. A corner with labels $\{i, i+1\}$ at its endpoints is denoted by $(i, i+1)$. If $\partial D$ contains only two kinds of corners $(\alpha, \alpha+1)$ and $(\alpha-1, \alpha)$, then $D$ is called a *two-cornered face*. Such a notion was first used in [**H**].

**Lemma 3.6.** *$D_\alpha$ contains a pair of two-cornered faces sharing a level $\alpha$-edge on their boundaries, such that at least one of such two-cornered faces contains only one level $\alpha$-edge.*

*Proof.* If $D_\alpha$ has no non-level diagonal edge (that is, each edge in $D_\alpha$ joining non-adacent corners along $\partial D_\alpha$ is level), then set $E = D_\alpha$. Suppose that $D_\alpha$ contains a diagonal edge $e$ which has distinct labels $\{a, b\}$. Note that $a \neq x, b \neq x$. Without loss of generality, assume that the labels appear in counterclockwise order around the corners of $\partial D_\alpha$, and that $a < b < x$. This means that these labels $a, b, x$ appear in this order around the vertex of $G_Q$. (Thus three inequalities $a < b < x$, $b < x < a$ and $x < a < b$ are equivalent.) Formally, we construct a new $x$-face $D'$ as follows: Consider that $e$ is oriented from the endpoint with label $a$ to the other. Discard the half (disk) of $D_\alpha$ right to $e$. Insert additional edges to the right of $e$ and parallel to $e$ until the label $x$ first appear at one end of this parallel family of edges. Possibly, the last edge has label $x$ at its both endpoints. But, except the last edge, there is no level edge among the additional edges. In fact, the label sets $I, J$ indicated in Figure 2 are disjoint, except the case where the last edge is a level $x$-edge. Let $D'$ be the union of the left side of $e$ and the bigons among these parallel family. Then $D'$ is an $x$-face. See Figure 2. There is no two-cornered face among the additional bigons. Repeat this process for every diagonal edge in $D'$ which is not a level $\alpha$-edge, then we finally get a new $x$-face $E$.

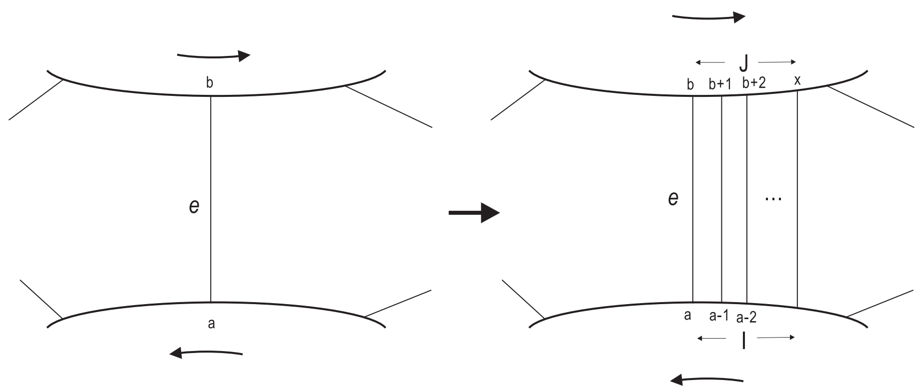


**Figure 2.**

Thus all diagonal edges in $E$ are level $\alpha$-edges, and $\partial E$ consists of $x$-edges. As remarked before, there may be level $x$-edges in $\partial E$. If $E$ does not contain a level $\alpha$-edge, then [**HM**, Lemma 5.2] says that there is a Scharlemann cycle $\sigma$ in $E$. By the construction of $E$, $\sigma$ lies in $D_\alpha$, and that is, $\sigma$ lies in $G_Q$. But this is impossible by Lemma 3.1. Hence $E$ contains a level $\alpha$-edge.

**Claim 3.7.** Any face adjacent to a level $\alpha$-edge in $E$ is two-cornered.

*Proof.* Let $e$ be a level $\alpha$-edge in $E$, and let $f$ be a face adjacent to $e$. Note that $\partial f$ may contain other level $\alpha$-edges. Let $(a_i, a_i+1)$ $(i = 1, 2, \dots, n)$ be the corners on $\partial f$ between successive level $\alpha$-edges (possibly, the same one) on $\partial f$, which appear in order around $f$ when we go around clockwise. Thus $a_1 = \alpha$ and $a_n = \alpha - 1$. See Figure 3.

Let $e_i$ be the edge on $\partial f$ connecting the points with labels $a_i + 1$ and $a_{i+1}$ for $i = 1, 2, \dots, n-1$. Note that $e_i$ is neither a level $x$-edge nor a diagonal edge in $E$. Also, $e_i$ can be an $x$-edge, otherwise it is parallel to an $x$-edge (which may be level) in $\partial E$.

First consider $e_{n-1}$. If $e_{n-1}$ is an $x$-edge, then $x = a_n < a_{n-1} + 1$ or $a_n < a_{n-1} + 1 = x$ (of course, for any two labels $a, b$, the inequalities $a < b$

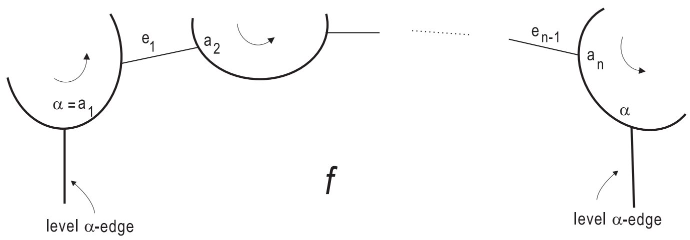


**Figure 3.**

and $b < a$ are equivalent). Hence $x \leq a_n < a_{n-1}+1 \leq x$ holds in any case. More precisely, when we go around the vertex of $G_Q$ (in counterclockwise direction), the two labels $a_n, a_{n-1}+1$ appear in this order between the successive $x$'s. If $e_{n-1}$ is not an $x$-edge, then it is parallel to an $x$-edge $e'$ (possibly, a level $x$-edge) in $\partial E$ by the construction of $E$. Let $F$ be the family of mutually parallel edges containing $e_{n-1}$ and $e'$. We refer to the end of $F$ containing the end point of $e_{n-1}$ with label $a_{n-1}+1$ as the left end.

Assume that the label $x$ appears at the left end of $F$ (in fact, at the "left end" of $e'$). By Lemma 3.1 and the construction of $E$, the label $a_n$ cannot appear at the left end of $F$. Hence three labels $a_{n-1}+1, x, a_n$ appear in this order, that is, $a_{n-1}+1 < x < a_n$, which is equivalent to $a_n < a_{n-1}+1 < x$. If $x$ appears at the right end of $F$, then we have the same inequality similarly. Hence $x \leq a_n < a_{n-1}+1 \leq x$ holds again. Thus we always have $x \leq a_n \leq a_{n-1} < x$.

Next, consider the edge $e_{n-2}$. By the same argument as above, we have $x \leq a_{n-1} \leq a_{n-2} < x$. Continuing in this way, we eventually get $x \leq a_n \leq a_{n-1} \leq \cdots \leq a_1 < x$. This means that the labels $a_n, a_{n-1}, \ldots, a_2, a_1$ appear in this order between the successive $x$'s. But recall that $a_n = \alpha - 1$ and $a_1 = \alpha$ are successive. Hence $a_1 = \cdots = a_j = \alpha$ and $a_{j+1} = \cdots = a_n = \alpha - 1$ for some $j$. Thus we have proved that $f$ is two-cornered face. $\square$

Choose a level $\alpha$-edge $e$ in $E$, which is outermost among level $\alpha$-edges in $E$. That is, $e$ cuts a disk $E'$ off from $E$ which contains no level $\alpha$-edge except $e$. Let $f$ and $g$ be the faces adjacent to $e$. Then these are a desired pair of two-cornered faces. Note that one of them can be a bigon, but then the other has at least three sides, since $G_Q$ does not contain a generalized $S$-cycle by Lemma 3.1. $\square$

Let $\widehat{T}$ be the torus which is the boundary of a thin regular neighborhood $N(\widehat{P})$ of $\widehat{P}$, and let $T$ be the intersection of $\widehat{T}$ with $E(K)$. Then $T \cap Q$ gives rise to a pair of graphs $\{G_T, G_Q^T\}$, where $G_T$ is a "double cover" of $G_P$, and each edge of $G_Q$ corresponds to a bigon of $G_Q^T$. Let $i1, i2$ be the vertices of $G_T$ corresponding to $i$-th vertex of $G_P$ such that they appear in the same order as the vertices of $G_P$. Thus $11, 12, 21, 22, \dots, p1, p2$ appear along $\partial E(K)$ in this order. In particular, each level $u$-edge ($v$-edge, resp.) of $G_Q$ yields an $S$-cycle in $G_Q^T$ with label-pair $\{u1, u2\}$ ($\{v1, v2\}$, resp.).

By Lemma 3.6, $D_\alpha$ contains a pair of two-cornered faces sharing a level $\alpha$-edge. These give an $S$-cycle $\sigma_\alpha$ and two faces $f_\alpha, g_\alpha$ adjacent to $\sigma_\alpha$. Note that $f_\alpha$ and $g_\alpha$ contain only $(\alpha 2, (\alpha+1)1)$ and $((\alpha-1)2, \alpha 1)$ corners. By Lemma 3.6, we may assume that $f_\alpha$ contains only one $(\alpha 1, \alpha 2)$-edge, which is an edge of $\sigma_\alpha$. Also remark that $f_\alpha$ contains only one $((\alpha+1)1, (\alpha-1)2)$-edge. By the construction of $\widehat{T}$, there are disjoint annuli in $\widehat{T}$, which contain the edges of $\sigma_u$ and $\sigma_v$ respectively. Note that the centerlines of these annuli are essential on $\widehat{T}$.

**Lemma 3.8.** *$u$ and $v$ are not adjacent on $\partial E(K)$.*

*Proof.* Assume that $u$ and $v$ are adjacent. For simplicity, let $u = 2$ and $v = 3$.

Let $X$ be the number of $x$-faces. As in the proof of Lemma 3.5, $X \geq r/2 + 1$. Thus $r \leq 2X - 2$.

Let $X_2, X_3$ denote the number of $x$-faces which contain only level 2 or 3-edges respectively, and let $X_1$ be the number of $x$-faces containing both kinds of level edges. By Lemma 3.4, $X = X_1 + X_2 + X_3$. Count the number of occurrence of label 3 in $G_Q$. Each $x$-face containing only level 2-edges contains at least two occurrences of label 3. The other $x$-faces contain level 3-edges, which do not lie on the boundaries. Hence $2X_1 + 2X_2 + 2X_3 \leq r$, since each label appear $r$ times around the vertex of $G_Q$. Thus $2X_1 + 2X_2 + 2X_3 \leq r \leq 2X - 2 = 2(X_1 + X_2 + X_3) - 2$, which is a contradiction. $\square$

Let $\Lambda_u$ ($\Lambda_v$, resp.) be the subgraph of $G_T$ consisting of four vertices $u1, u2, (u-1)2, (u+1)1$ ($v1, v2, (v-1)2, (v+1)1$, resp.) and the edges of $\sigma_u$, $\partial f_u$ and $\partial g_u$ ($\sigma_v, \partial f_v, \partial g_v$, resp.). As noted in the proof of Lemma 3.6, $g_u$ and $g_v$ have at least three sides, and hence $\Lambda_u$ and $\Lambda_v$ are connected. Hence there is an annulus $A_u$ ($A_v$, resp.) in $\widehat{T}$ which contains the edges of $\sigma_u$, $\partial f_u$ and $\partial g_u$ ($\sigma_v, \partial f_v, \partial g_v$, resp.). By Lemma 3.8, the vertices $u1, u2, v1, v2$, $(u-1)2, (u+1)1, (v-1)2, (v+1)1$ are distinct. We may assume that $A_u$ and $A_v$ are disjoint, and that one boundary component of $A_u$ ($A_v$, resp.) is very near to the edges of $\sigma_u$ ($\sigma_v$, resp.). Moreover, these boundary components bound Möbius bands $M_u$ and $M_v$, respectively, in $N(\widehat{P})$ meeting the core of the attached solid torus $V$ in a single point. (Consider the union of the bigon face of $\sigma_\alpha$ and the 1-handle $H$ bounded by the vertices $\alpha 1, \alpha 2$. By shrinking $H$ radially to its core, we obtain a Möbius band, and then perturb it to be transverse to the core of $V$.)

Let $q_u$ be the number of vertices contained in $A_u$. Let $H_1$ and $H_2$ are disjoint 1-handles on $V$ bounded by the vertices of $(u-1)2$ and $u1$, $u2$ and $(u+1)1$. Consider $W_u = N(A_u \cup H_1 \cup H_2 \cup f_u \cup g_u) \subset K(r) - \mathrm{Int}N(\widehat{P})$.

**Lemma 3.9.** *$\partial W_u$ consists of one or two tori.*

*Proof.* Let $W'_u = N(A_u \cup H_1 \cup H_2)$. Then $W'_u$ is a handlebody of genus three. Since $\partial f_u$ is non-separating on $\partial W'_u$, attaching a 2-handle $N(f_u)$ gives a genus two surface from $\partial W'_u$.

We claim that $\partial f_u$ and $\partial g_u$ are not parallel on $\partial W'_u$. If $\partial f_u$ and $\partial g_u$ are parallel on $\partial W'_u$, then these represent the same element of $\pi_1(W'_u)$. Taking as a base "point" a subdisk of $A_u$ as shown in Figure 4, we have $\pi_1(W'_u) = \langle x_1, x_2, y\rangle$, where $x_1$ ($x_2$, resp.) is represented by a core of $H_1$ ($H_2$, resp.) going from vertex $(u-1)2$ ($u2$, resp.) to vertex $u1$ ($(u+1)1$, resp.), and $y$ is represented by the edge of $\sigma_u$ not in the base "point" going from vertex $u1$ to vertex $u2$. We may assume that the $(u1, u2)$-edge on $\partial f_u$ is contained in the base "point". Then $\partial f_u$ never contain $x_1 y x_2$, although $\partial g_u$ contains it (in the appropriate directions). This is because $\partial f_u$ contains just one $(u1, u2)$-edge. Therefore $\partial f_u$ and $\partial g_u$ are not parallel on $\partial W'_u$, and then $\partial W_u$ cannot be a union of a 2-sphere and a genus two surface. Thus $\partial W_u$ is a torus or two tori according to whether the attaching sphere of $N(g_u)$ is non-separating or separating on $\partial(W'_u \cup N(f_u))$. $\square$

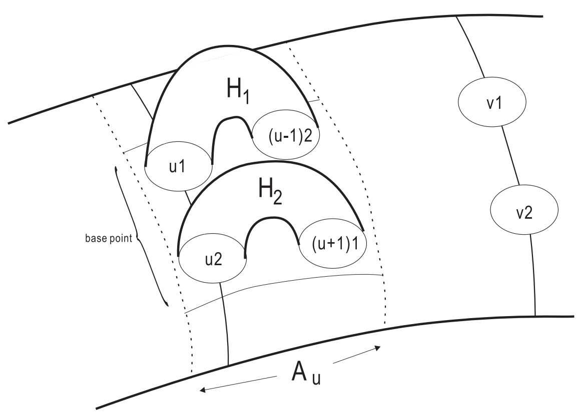


**Figure 4.**

Then $\partial W_u$ contains a torus $F$ containing $A_u$, since $W_u \cap \widehat{T} = A_u$. Note that the core of $A_u$ is essential on $F$. If not, one component of $\partial A_u$ bounds a disk $D$ in $K(r) - N(\widehat{P})$. Then $M_u \cup D$ or $M_u \cup A_u \cup D$ is a projective plane in $K(r)$. Take a parallel copy $D'$ of $D$ in $K(r) - N(\widehat{P})$, so that $D'$ is disjoint from $A_u$. Then the union of $D'$, $M_v$ and the annulus on $\widehat{T}$ bounded by $\partial D'$ and $\partial M_v$, which is disjoint from $A_u$, forms another projective plane in $K(r)$. Thus $K(r)$ contains two disjoint projective planes, but this is impossible, because $H_1(K(r))$ would be non-cyclic.

Let $A'_u$ be the remaining annulus of the torus. By the construction, $A'_u$ meets the core of $V$ in at most $q_u - 4$ points. Similarly, we obtain an annulus $A'_v$ by using $f_v, g_v$.

The edges of $\sigma_u$ and $\sigma_v$ separate $\widehat{T}$ into two annuli $B_1$ and $B_2$. Each $\mathrm{Int}B_i$ contains $p-2$ vertices, since $G_T$ is a double cover of $G_P$. We may assume that the edges of $\partial f_u$ and $\partial g_u$ are contained in $B_1$.

First assume that the edges of $\partial f_v, \partial g_v$ are contained in $B_1$. Let $B'_1 \subset B_1$ be the annulus region bounded by $\partial A'_u$ and $\partial A'_v$. Then the union $M_u \cup A'_u \cup B'_1 \cup A'_v \cup M_v$ gives a new Klein bottle in $K(r)$, which meets the core of $V$ in at most $p-4$ points. This contradicts the minimality of $p$.

Next assume that the edges of $\partial f_v, \partial g_v$ are contained in $B_2$. Let $B''_1 \subset B_1$ be the annulus region bounded by $\partial A'_u$ and $\partial A_v$. Then the union $M_u \cup A'_u \cup B''_1 \cup M_v$ gives a new Klein bottle in $K(r)$, which meets the core of $V$ in at most $p-2$ points, a contradiction.

Thus we have proved Theorem 1.1 when $p > 1$.

## 4. Special case.

In this section, we consider the case where $p = 1$. Assign the points of $\partial P \cap \partial Q$ the labels $1, 2, \ldots, r$ along $\partial Q$ sequentially. Then the labels are also sequential along $\partial P$, since $r$ is integral.

**Lemma 4.1.** *If $G_Q$ has parallel edges, then $r = 4$.*

*Proof.* This is Lemma 4.2 in [**T1**]. □

Thus we may assume that $G_Q$ has no parallel edges hereafter.

**Lemma 4.2.** *If two edges of $G_P$ are parallel, then their endpoints appear alternately around the vertex of $G_P$.*

*Proof.* This follows from that all edges of $G_P$ are negative. □

**Lemma 4.3.** *Suppose that $G_Q$ contains a separating edge $e$. Then one component of $Q - e$ contains no edge of $G_Q$.*

*Proof.* Assume that each component of $Q - e$ contains an edge $e_1$ and $e_2$ respectively. Since $G_P$ consists of at most two parallel families of (negative) edges (cf. [**T1**, Section 4]), some two of $e, e_1, e_2$ are parallel in $G_P$. But this is impossible by Lemma 4.2. □

**Lemma 4.4.** *If $G_Q$ contains a separating edge, then $r \leq 4g$.*

*Proof.* Let $e$ be a separating edge in $G_Q$, and let $Q_1$ and $Q_2$ be the components of $Q - e$. By Lemma 4.3, we may assume that $Q_2$ contains no edge. If $Q_1$ contains a separating edge $e_1$, then $e$ and $e_1$ are not parallel in $G_P$ by Lemma 4.2. If $r > 4$, then $G_Q$ contains another edge $e_2$, which is parallel to $e$ or $e_1$ in $G_P$. But Lemma 4.2 gives a contradiction. Hence we may assume that $Q_1$ contains no separating edges. In fact, no edge in $Q_1$ is parallel to $e$ in $G_P$ by Lemma 4.2. Thus $G_P$ consists of $e$ and a parallel family of $r/2-1$ edges. By examining the labels of edges, we see that $G_Q$ has just three faces.

For $\widehat{Q}$, we have

$$1-\frac{r}{2}+\sum_{f\colon \text{ a face of } G_Q}\chi(f)=2-2g.$$

Thus $\sum\chi(f) = 1-2g+r/2$. Here $\sum\chi(f) = \sum_{f\neq Q_2}\chi(f)+\chi(Q_2) \leq \sum_{f\neq Q_2}\chi(f)-1$. Thus $2-2g+r/2 \leq \sum_{f\neq Q_2}\chi(f)$. Since $G_Q$ has at most two disk faces, $2-2g+r/2\leq 2$, and therefore $r\leq 4g$. □

**Lemma 4.5.** *If $G_Q$ contains no separating edges, then $r \leq 4g+4$.*

*Proof.* Recall that $G_P$ consists of at most two families $A$ and $B$ of parallel edges. Let $|A|, |B|$ denote the number of edges in $A$ and $B$ respectively. Since $|A|+|B| = r/2$ is even, $|A|$ and $|B|$ have the same parity.

If $|A|$ and $|B|$ are even, then we see that $G_Q$ has just one face by examining the labels of edges. See Figure 5.

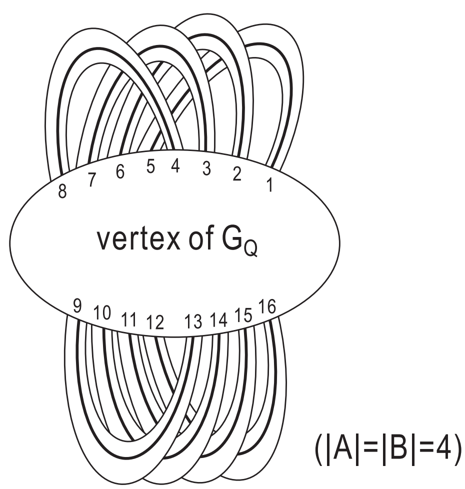


**Figure 5.**

Then $1-r/2+\sum\chi(f) = 2-2g$, and thus $1-2g+r/2 = \sum\chi(f)\leq 1$. Therefore $r\leq 4g$.

If $|A|$ and $|B|$ are odd, then $G_Q$ has just three faces. Thus $1-2g+r/2 = \sum\chi(f)\leq 3$, and then $r\leq 4g+4$. □

Lemmas 4.4 and 4.5 give the proof of Theorem 1.1 when $p = 1$. In fact, we can give the same upper bound $4g + 4$ by a 4-dimensional technique. We thank Seiichi Kamada for this suggestion. Consider $S^3 = \partial B^4$. The knot $K$ bounds $P$ and $Q$. Then pushing $P$ slightly into $B^4$ gives a closed non-orientable surface $P \cup Q$ embedded in $B^4$. Note $\chi(P \cup Q) = -2g$, where $g$ is the genus of $Q$. By Whitney-Massey Theorem (cf. [**K**]), the Euler number $e(P \cup Q)$ can vary between $2\chi(P \cup Q) - 4$ and $4 - 2\chi(P \cup Q)$. Thus $|e(P \cup Q)| \leq 4g + 4$. But $e(P \cup Q)$ is equal to the self-intersection number of $P \cup Q$, which is exactly the boundary slope of $P$ (see [**K**]).

## 5. Extremal case.

In this section, we examine the extremal case where $r = 4g + 4$, and prove Theorem 1.2. Recall that the points of $\partial P \cap \partial Q$ are labeled $1, 2, \ldots, r$ sequentially along $\partial P$ (and $\partial Q$) as in Section 4.

Assume $r = 4g + 4$. By the proof of Lemma 4.5, $G_P$ consists of two parallel families $A$ and $B$ such that $m = |A|$ and $n = |B|$ are odd. In fact, $|A|, |B| > 1$, otherwise $G_Q$ contains a trivial loop. Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the edges of $A$ and $B$ respectively, where they are numbered successively. That is, $a_i$ has labels $i$ and $i + m$, and $b_j$ has $2m + j$ and $2m + n + j$. See Figure 6, where the two end circles of the cylinder are identified with a suitable involution to form a Klein bottle $\widehat{P}$.

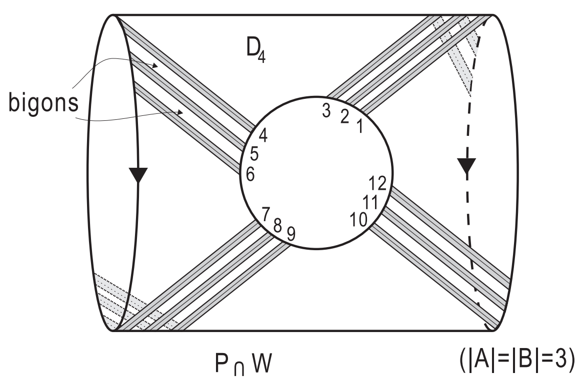


**Figure 6.**

**Lemma 5.1.** *$K$ is fibered.*

*Proof.* Let $W = E(K) - \mathrm{Int}N(Q)$. Then $\partial W$ consists of two copies of $Q$, $Q_0$ and $Q_1$ say, and an annulus $\delta$. We show that $W$ has a product structure $Q \times [0, 1]$ such that $Q \times \{0\} = Q_0$ and $Q \times \{1\} = Q_1$. Then the result immediately follows from this.

We see that $P \cap W$ consists of a 4-gon $D_4$ and $(m-1)+(n-1)$ bigons. See Figure 6. For such a bigon $D$, $\partial D \cap Q_0$ and $\partial D \cap Q_1$ correspond to edges of $G_Q$ and $\partial D \cap \delta$ is two spanning arcs of $\delta$. For example, if $D$ corresponds to the bigon face of $G_P$ between $a_1$ and $a_2$, then $\partial D \cap Q_0$ and $\partial D \cap Q_1$ correspond to $a_1$ and $a_2$, respectively. Cut $W$ along these bigons. Let $W'$ be the resulting manifold. That process cuts $Q_i$ into a disk for each $i$, since $Q_0$ is cut along arcs corresponding to $a_1, a_2, \dots, a_{m-1}, b_1, b_2, \dots, b_{n-1}$, and $Q_1$ is cut along $a_2, a_3, \dots, a_m, b_2, b_3, \dots, b_n$. By the irreducibility of $E(K)$, $W'$ is a 3-ball. Thus $W$ has a product structure as desired. $\square$

Thus $W$ is identified with $Q \times [0,1]$, and $E(K)$ is identified with a mapping torus $Q \times [0,1]/(x,1) = (f(x),0)$, where $f$ is a homeomorphism of $Q$. Let $Q_i$ denote $Q \times \{i\}$ in $W = Q \times [0,1]$. In fact, it is convenient to regard $f$ as the map from $Q_1$ to $Q_0$.

Let us keep using the notation in the proof of Lemma 5.1. To clarify the argument, we regulate $P$ in $E(K)$ up to isotopy. In the same way as [**FH**, Proposition 2.1], $P$ can be isotoped to be monotone except for just one saddle point in Int$P$ with respect to the bundle structure of $E(K)$. Furthermore, we may assume that $2m+2n$ arcs on $\delta$ coming from $\partial P$ and $m+n-2$ disks in $W = Q \times [0,1]$ coming from the bigon faces of $G_P$ are all saturated (that is, the unions of $I$-fibers) with respect to the product structure of $W$. Finally, we isotope the 4-gon $D_4$ of $P \cap W$ such that $\pi|_{D_4}$ is an embedding except for four arcs on $\delta$, where $\pi : W = Q \times [0,1] \to Q_1$ denotes the natural projection.

Hereafter, we regard the edges $a_i, b_j$ of $G_Q$ as the arcs on $Q_0$. This means that each $a_i$, $1 \le i \le m-1$, appears as the intersection of $Q_0$ and the disk which corresponds to a bigon face of $G_P$ between $a_i$ and $a_{i+1}$, and the arc $a_m$ is one of the arcs of $D_4 \cap Q_0$. Further, we set $a_i' = \pi(a_i)$ on $Q_1$. Then $a_{i+1} = f(a_i')$ holds for each $i = 1, 2, \dots, m-1$.

**Lemma 5.2.** *$K$ is composite.*

*Proof.* Let us introduce two more arcs on $Q_0$ as follows.

First, let $a_{m+1} = f(a_m')$. Recall that the endpoints of $a_i$ are labeled by $i$ and $m+i$ and those of $b_j$ are labeled by $2m+j$ and $2m+n+j$. Thus the action of $f$ is cyclic on the set of the endpoints of $a_i, b_j$, and so the labels of the endpoints of $a_{m+1}$ are $m+1$ and $2m+1$.

**Claim 5.3.** $a_{m+1}$ is disjoint from $a_2, a_3, \dots, a_m$ and meets $a_1$ in only the endpoint with label $m+1$.

*Proof.* Clearly, $a_m$ is disjoint from $a_i$, and so $a_m'$ is disjoint from $a_i'$, $1 \le i \le m-1$. This implies that $a_{m+1} = f(a_m')$ is disjoint from $a_{i+1} = f(a_i')$ for $1 \le i \le m-1$. Furthermore, since $D_4 \cap Q_0 = a_m \cup b_n$, $D_4 \cap Q_1 = f^{-1}(a_1) \cup f^{-1}(b_1)$ and $\pi|_{D_4}$ is embedding except for four arcs on $\delta$, $a_m'$ meets $f^{-1}(a_1)$

in a single point. Hence $a_{m+1} = f(a'_m)$ meets $a_1$ in only the endpoint with label $m+1$. □

Next, we give an orientation to each edge of $G_Q$ so that it runs from the endpoint with smaller label to the other, and let $e$ be the arc on $Q_0$ obtained as the product $a_1 * a_{m+1}$. By the observations above, the endpoints of $e$ have the labels of 1 and $2m+1$, and $e$ is disjoint from $a_2, \dots, a_m$.

**Claim 5.4.** $e$ is separating and essential in $Q_0$.

*Proof.* Recall that $G_Q$ has just three disk faces. One of the disk faces, denoted by $D_a$, is bounded by the edges $a_1, a_2, \dots, a_m$ together with subarcs of $\partial Q_0$. Another one $D_b$ is bounded by the edges $b_1, b_2, \dots, b_n$ and the other one $D_{ab}$ is bounded by the edges $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$ together with subarcs of $\partial Q_0$.

Since the labels of the endpoints of $a_{m+1}$ are $m+1$ and $2m+1$, the arc $a_{m+1}$ is contained in the disk face $D_{ab}$. Thus $e$ is also contained in $D_{ab}$, and so it is a diagonal arc which separates $D_{ab}$. Moreover, since the labels of the endpoints of $e$ are 1 and $2m+1$, the endpoints separate $\partial D_{ab}$ into two parts one of which contains $a_i$'s only and the other contains $b_j$'s only. This indicates that $e$ is separating and essential on $Q_0$. □

Now, we consider the closed surface $\overline{Q_i}$ which is obtained by shrinking $\partial Q_i$ to a single point $y_i$ for $i = 0, 1$. We abuse the notations for the arcs and the faces on $\overline{Q_i}$ corresponding to those on $Q_i$. Let $\bar{f}$ be the homeomorphism from $\overline{Q_1}$ to $\overline{Q_0}$ induced from $f$.

**Claim 5.5.** $\bar{f}(e')$ is isotopic to $e$ fixing $y_0$, where $e' = \pi(e)$ in $Q_1$.

*Proof.* Let $[a_i]$, $[a'_i]$, $1 \le i \le m+1$, and $[e]$, $[e']$ be the elements of $\pi_1(\overline{Q_0}, y_0)$ and $\pi_1(\overline{Q_1}, y_1)$ represented by the corresponding ones.

Let $R$ be the polygon bounded by $e$ and $a_1, a_2, \dots, a_m$ in $D_{ab}$. Then, under the above setting, $\partial R$ is represented as

$$a_1 * {a_2}^{-1} * a_3 * {a_4}^{-1} * \cdots * a_m * e^{-1}.$$

Then, this gives the relation

$$[e] = [a_1][a_2]^{-1}[a_3][a_4]^{-1} \dots [a_m],$$

and so we have

$$[e'] = [a'_1][a'_2]^{-1}[a'_3][a'_4]^{-1} \dots [a'_m].$$

Also $\partial D_a$ is represented as

$$a_1 * a_m * {a_{m-1}}^{-1} * a_{m-2} * {a_{m-3}}^{-1} * \cdots * {a_2}^{-1}.$$

This gives

$$[a_1] = [a_2][a_3]^{-1}[a_4][a_5]^{-1} \dots [a_m]^{-1}.$$

Let $\overline{f}_* : \pi_1(\overline{Q_1}, y_1) \to \pi_1(\overline{Q_0}, y_0)$ be the homomorphism induced from $\overline{f}$. Then

$$\begin{aligned} f_*([e']) &= f_*([a'_1][a'_2]^{-1}[a'_3][a'_4]^{-1} \dots [a'_m]) \\ &= [a_2][a_3]^{-1}[a_4][a_5]^{-1} \dots [a_m]^{-1}[a_{m+1}] \\ &= [a_1][a_{m+1}] = [e]. \end{aligned}$$

Therefore two loops $e$ and $f(e')$ are homotopic on $\overline{Q_0}$ fixing $y_0$, and hence isotopic. □

As a result, we can obtain an essential, separating annulus in $E(K)$, each of whose boundary circles meets the longitude of $K$ in a single point, from $e \times [0, 1] \subset W$. By [**BZ**, Lemma 15.26], such an annulus comes from a decomposing sphere or a cabling annulus for $K$. This concludes that $K$ is composite. □

*Proof of Theorem* 1.2. By Lemma 5.2, $K$ is composite. Then by [**T2**], $K$ is the connected sum of two 2-cabled knots $K_1$ and $K_2$. Let $K_i$ be the $(2, m_i)$-cable of a knot $\widetilde{K}_i$ for $i = 1, 2$. Then $r = 2m_1 + 2m_2$ [**T2**]. Also,

$$g(K) = \frac{|m_1| - 1}{2} + \frac{|m_2| - 1}{2} + 2g(\widetilde{K}_1) + 2g(\widetilde{K}_2)$$

by [**S**]. Since $|r| = 4g(K) + 4$,

$$2|m_1 + m_2| = 2|m_1| + 2|m_2| + 8g(\widetilde{K}_1) + 8g(\widetilde{K}_2).$$

Thus $m_1$ and $m_2$ have the same sign and $g(\widetilde{K}_1) = g(\widetilde{K}_2) = 0$, and hence $K_i$ is the $(2, m_i)$-torus knot. This completes the Proof of Theorem 1.2. □

Finally, we give the examples of hyperbolic knots which show that the estimation of Corollary 1.3 is sharp for each genus $g$.

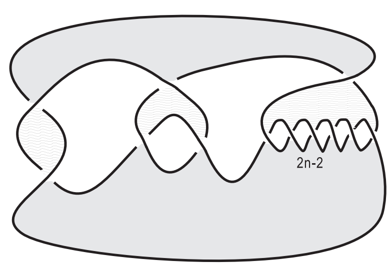


**Figure 7.**

**Example 5.6.** For genus one case, the figure eight knot is such an example as mentioned in Section 1. Let $n \geq 2$ and let $K$ be the $(2, 3, 2n - 3)$-pretzel knot. (When $n = 2$, $K$ is 2-bridge, and it is $6_2$ in the knot table [**R**].) Then

$K$ is hyperbolic [**Kw**], and it obviously bounds a once-punctured Klein bottle whose boundary has the slope $4n$. Also, $K$ has genus $n$, since the Seifert surface shown in Figure 7 has minimal genus by [**G1**, **G2**].

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