Self-similar Solutions to a Parabolic System Modeling Chemotaxis

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Abstract. We study the forward self-similar solutions to a parabolic system modeling chemotaxis

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \quad \tau v_t = \Delta v + u$$

in the whole space $\mathbb{R}^2$, where τ is a positive constant. Using the Liouville type result and the method of moving planes, it is proved that self-similar solutions (u, v) must be radially symmetric about the oritgin. Then the structure of the set of self-similar solutions is investigated. As a consequence, it is shown that there exists a threshold in $\int_{\mathbb{R}^2} u$ for the existence of self-similar solutions. In particular, for $0 < \tau \leq 1/2$, there exists a self-similar solution (u, v) if and only if $\int_{\mathbb{R}^2} u < 8\pi$.

1. Introduction

We are concerned with the parabolic system of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v) \\ \tau \dfrac{\partial v}{\partial t} = \Delta v + u \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^N$ and $t > 0$, where $\tau > 0$ is a constant. This is a simplified system of the one given by Keller and Segel [16] describing chemotactic feature of cellular slime molds sensitive to the gradient of a chemical substance secreted by themselves. The functions $u(x,t) \geq 0$ and $v(x,t) \geq 0$ denote the cell density of cellular slime molds and the concentration of the chemical substance at the place x and the time t, respectively.

Backward self-similar solutions are studied in [12] for $\tau = 0$. The present paper is devoted to the forward self-similar solutions. Namely, this system is invariant under the similarity transformation

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$$

for $\lambda > 0$, that is, if (u, v) is a solution of (1.1) globally in time, then so is (u_λ, v_λ). A solution (u, v) is said to be *self-similar*, when the solution is invariant under this transformation,

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that is, $u(x,t) = u_\lambda(x,t)$ and $v(x,t) = v_\lambda(x,t)$ for all $\lambda > 0$. Letting $\lambda = 1/\sqrt{t}$, we see that (u,v) has the form

$$(1.2) \qquad u(x,t) = \frac{1}{t}\phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v(x,t) = \psi\left(\frac{x}{\sqrt{t}}\right)$$

for $x \in \mathbb{R}^N$ and $t > 0$. It follows that

$$(1.3) \qquad \int_{\mathbb{R}^N} u(x,t)dx = t^{(N-2)/2} \int_{\mathbb{R}^N} \phi(y)dy$$

for $\phi \in L^1(\mathbb{R}^N)$. Therefore, self-similar solution (u,v) preserves the mass $\|u(\cdot,t)\|_{L^1(\mathbb{R}^2)}$ if and only if $N = 2$. On the other hand, the mass conservation of $u(\cdot,t)$ follows formally in the original system (1.1) in any space dimensions. Regarding this fact, we study the case $N = 2$ in this paper.

By a direct computation it is shown that (u,v) in (1.2) satisfies (1.1) if and only if (ϕ,ψ) satisfies

$$(1.4) \qquad \begin{cases} \nabla \cdot (\nabla\phi - \phi\nabla\psi) + \frac{1}{2}x \cdot \nabla\phi + \phi = 0, & x \in \mathbb{R}^2, \\ \Delta\psi + \frac{\tau}{2}x \cdot \nabla\psi + \phi = 0, & x \in \mathbb{R}^2. \end{cases}$$

We are concerned with the classical solutions $(\phi,\psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ of (1.4) satisfying

$$(1.5) \qquad \phi, \psi \geq 0 \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \phi(x), \psi(x) \to 0 \quad \text{as } |x| \to \infty.$$

Define the solution set $\mathcal{S}$ of (1.4) as

$$(1.6) \qquad \mathcal{S} = \{(\phi,\psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2) : (\phi,\psi) \text{ is a solution of (1.4) with (1.5)}\}.$$

The existence of radial solutions $(\phi,\psi) \in \mathcal{S}$ has been known by [20, Theorem 1] and [22, Theorem 1.1]. We investigate the structure of the solution set $\mathcal{S}$.

Theorem 1. *Any $(\phi,\psi) \in \mathcal{S}$ is radially symmetric about the origin, and satisfies $\phi, \psi \in L^1(\mathbb{R}^2)$.*

Theorem 2. *The solution set $\mathcal{S}$ is expressed as a one parameter family:*

$$\mathcal{S} = \{(\phi(s), \psi(s)) : s \in \mathbb{R}\}.$$

If $\lambda(s) = \|\phi(s)\|_{L^1(\mathbb{R}^2)}$, then $(\phi(s),\psi(s))$ and $\lambda(s)$ satisfy the following properties:
(i) *$s \mapsto (\phi(s),\psi(s)) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ and $s \mapsto \lambda(s) \in \mathbb{R}$ are continuous;*
(ii) *$(\phi(s),\psi(s)) \to (0,0)$ in $C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$ and $\lambda(s) \to 0$ as $s \to -\infty$;*
(iii) *$\|\psi(s)\|_{L^\infty(\mathbb{R}^2)} \to \infty$,*

$$\lambda(s) \to 8\pi, \quad \textit{and} \quad \phi(s)dx \rightharpoonup 8\pi\delta_0(dx) \quad \textit{in the sense of measure as } s \to \infty,$$

where $\delta_0(dx)$ denotes Dirac's delta function with the support in origin;

(iv) $0 < \lambda(s) < 8\pi$ *for* $s \in \mathbb{R}$, *if* $0 < \tau \le 1/2$, *and* $0 < \lambda(s) \le \max\{4\pi^3/3, 4\pi^3\tau^2/3\}$ *for* $s \in \mathbb{R}$, *if* $\tau > 1/2$.

As a consequence of Theorem 2 we obtain the following:

Corollary. *There exists a constant* λ^* *satisfying* $\lambda^* = 8\pi$, *if* $0 < \tau \le 1/2$, *and* $8\pi \le \lambda^* \le \max\{4\pi^3/3, 4\pi^3\tau^2/3\}$, *if* $\tau > 1/2$, *such that*

(i) *for every* $\lambda \in (0, \lambda^*)$, *there exists a solution* $(\phi, \psi) \in \mathcal{S}$ *satisfying* $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$;

(ii) *for* $\lambda > \lambda^*$, *there exists no solution* $(\phi, \psi) \in \mathcal{S}$ *satisfying* $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$.

Remark. Biler [1] has shown that the system (1.4) with $\tau = 1$ has a radial solution (ϕ, ψ) satisfying $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$ for every $\lambda \in (0, 8\pi)$, and has no radial solutions (ϕ, ψ) satisfying $\|\phi\|_{L^1(\mathbb{R}^2)}/2\pi \ge 7.82\ldots$.

Theorem 1 is a consequence of the following:

Theorem 3. *Assume that* (ϕ, ψ) *is a nonnegative solution of* (1.4) *satisfying* ϕ, $\psi \in L^\infty(\mathbb{R}^2)$. *Then* ϕ *and* ψ *are positive, and there exists a constant* $\sigma > 0$ *such that*

$$\phi(x) = \sigma e^{-|x|^2/4} e^{\psi(x)}. \tag{1.7}$$

Assume furthermore that $\psi(x) \to 0$ *as* $|x| \to \infty$. *Then* ϕ *and* ψ *are radially symmetric about the origin, and satisfy* $\partial\phi/\partial r < 0$ *and* $\partial\psi/\partial r < 0$ *for* $r = |x| > 0$, *and*

$$\phi(x) = O(e^{-|x|^2/4}) \quad \textit{and} \quad \psi(x) = O(e^{-\min\{\tau,1\}|x|^2/4}) \quad \textit{as } |x| \to \infty.$$

The proof of Theorem 3 consists of two steps. First we show that (1.7) holds by employing the Liouville type result essentially due to Meyers and Serrin [19]. Then we show the radial symmetry of solutions by the method of moving planes. This device was first developed by Serrin [28] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [7, 8]. We will obtain a symmetry result for Eq. (1.8) below with a change of variables as in [23].

By Theorem 3 it follows that under the condition $\phi, \psi \in L^\infty(\mathbb{R}^2)$ the system (1.4) is reduced to the equation

$$\Delta\psi + \frac{\tau}{2} x \cdot \nabla\psi + \sigma e^{-|x|^2/4} e^{\psi} = 0 \qquad \text{in } \mathbb{R}^2 \tag{1.8}$$

for some positive constant σ. Moreover, $(\phi, \psi) \in \mathcal{S}$ if and only if ψ satisfies (1.8) with

$$\psi(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.9}$$

and ϕ is given by (1.7). Let $\lambda = \|\phi\|_{L^1(\mathbb{R}^2)}$. From (1.7) we see that

$$\lambda = \sigma \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy.$$

Then (1.8) is rewritten as the elliptic equation with nonlocal term,

$$\Delta\psi + \frac{T}{2} x \cdot \nabla\psi + \lambda e^{-|x|^2/4} e^{\psi} \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy = 0 \qquad \text{in } \mathbb{R}^2. \tag{1.10}$$

The proof of Theorem 2 is based on the ODE arguments to Eqs. (1.8) and (1.10). Furthermore, we employ the results by Brezis and Merle [2] concerning the asymptotic behavior of sequences of solutions of

$$-\Delta u_k = V_k(x) e^{u_k} \qquad \text{in } \Omega, \tag{1.11}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and V_k is a nonnegative continuous functions. We also need Theorem 4 below in order to prove Theorem 2. Here we recall Theorem 3 in [2].

Theorem A [2]. *Suppose that*

$$0 \le V_k(x) \le C_0, \quad x \in \Omega, \tag{1.12}$$

for some positive constant C_0. Let $\{u_k\}$ be a sequence of solutions of (1.11) *satisfying*

$$\limsup_{k\to\infty} \int_{\Omega} e^{u_k} dx < \infty. \tag{1.13}$$

Then there exists a subsequence (still denoted by $\{u_k\}$) satisfying one of the following alternatives:

(i) *$\{u_k\}$ is bounded in $L^\infty_{\text{loc}}(\Omega)$;*

(ii) *$u_k \to -\infty$ uniformly on compact subset of Ω;*

(iii) *there exists a finite blow-up set $\mathcal{B} = \{a_1, \dots, a_\ell\} \subset \Omega$ such that, for any $1 \le i \le \ell$, there exists $\{x_k\} \subset \Omega$, $x_k \to a_i$, $v_k(x_k) \to \infty$, and $v_k \to -\infty$ uniformly on compact subsets of $\Omega \setminus \mathcal{B}$. Moreover, $V_k e^{u_k} dx \rightharpoonup \sum_{i=1}^{\ell} \alpha_i \delta_{a_i}(dx)$ in the sense of measure with $\alpha_i \ge 4\pi$, where $\delta_{a_i}(dx)$ is Dirac's delta function with the support in $x = a_i$.*

It was conjectured in [2] that each α_i can be written as $\alpha_i = 8\pi m_i$ for some positive integer m_i. This was established by Li and Shafrir in [18].. Chen has shown in [3] that any positive integer m_i can occur in the case $V \equiv 1$ and Ω is a unit disc. On the other hand, under more restrictive assumption that $V_k \in C^1(\Omega)$ we obtain the following theorem. It is related to Theorem 0.3 of Li [17] and is proven in the appendix of the present paper.

Theorem 4. *Suppose that $V_k \in C^1(\Omega)$ satisfies* (1.12) *and*

$$\|\nabla V_k\|_{L^\infty(\Omega)} \le C_1 \tag{1.14}$$

for some positive constants C_0 *and* C_1. *Let* $\{u_k\}$ *be a sequence of solutions of* (1.11) *satisfying* (1.13) *and*

$$\max_{\partial\Omega} u_k - \min_{\partial\Omega} u_k \leq C_2 \tag{1.15}$$

for some positive constant C_2. *Assume that the alternative* (iii) *in Theorem A holds. Then* $\alpha_i = 8\pi$ *for each* $i \in \{1, 2, \ldots, \ell\}$.

Recently, attentions have been paid to blowup problems for the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), & x \in \Omega,\ t > 0, \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - \gamma v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0, \quad v(x,0) = v_0, & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, τ and γ are positive constants, and ν is the outer normal unit vector. Childress and Percus [5] and Childress [4] have studied the stationary problem and have conjectured that there exists a threshold in $\|u_0\|_{L^1(\Omega)}$ for the blowup of the solution (u, v). Their arguments were heuristic, while recent studies are supporting their validity rigorously, see, [11], [13], [24], [26], and [27].

On the other hand, it is asserted that self-similar solutions take an important role in the asymptotic behavior of the solution to the Cauchy problem for the semilinear parabolic equation, see, e.g., [6], [14], and [15]. From Corollary, we are led to the following conjectures for the problem (1.1) subject to the initial condition $u(x,0) = u_0$ and $v(x,0) = v_0$ in $\mathbb{R}^2$.

For $0 < \tau \leq 1/2$, *if* $\|u_0\|_{L^1(\mathbb{R}^2)} < 8\pi$ *then the solution of the Cauchy problem to* (1.1) *exists globally in time, and if* $\|u_0\|_{L^1(\Omega)} > 8\pi$ *then the solution can blowup in a finite time.*

We organize this paper as follows. In Section 2 we show that (1.7) holds by employing the Liouville type result. In Section 3 we show the radial symmetry of solutions by the method of moving planes, and then give the proof of Theorem 3. In Section 4 we give the ODE arguments to investigate the properties of radial solutions of (1.8). We study the behavior of sequences $\{(\phi_k, \psi_k)\} \subset \mathcal{S}$ satisfying $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \to \infty$ in Section 5. In Section 6 we investigate the upper bounds of $\|\phi\|_{L^1(\mathbb{R}^2)}$. Finally, in Section 7 we prove Theorems 2 by using of the results in Sections 4-6. In the appendixes, we are concerned with the existence of solutions to the problem (1.8) and (1.9), and give the proof of Theorem 4.

2. Reduction to the single equation

In this section we show that the system (1.4) is reduced to Eq. (1.8) if $\phi,\ \psi \in L^\infty(\mathbb{R}^2)$. More precisely, we have the following:

Proposition 2.1. *Let* (ϕ, ψ) *be a nonnegative solution of* (1.4) *with* $\phi,\ \psi \in L^\infty(\mathbb{R}^2)$. *Then the relation* (1.7) *holds with some constant* $\sigma > 0$.

To prove this proposition we use the Liouville type result for second order elliptic inequalities essentially due to Meyers and Serrin [19].

Lemma 2.1. *Let* u *satisfy*

$$\Delta u + \nabla b \cdot \nabla u \geq 0 \qquad \text{in } \mathbb{R}^2. \tag{2.1}$$

Assume that $x \cdot \nabla b(x) \leq 0$ *for large* $|x|$. *If* $\sup_{x\in\mathbb{R}^2} u(x) < \infty$ *then* u *must be a constant function.*

Proof. Take a function μ as $\mu(r) = 1/\log(1+r)$. Then μ satisfies the Meyers-Serrin condition

$$\int_1^\infty \frac{k(t)}{t} dt = \infty, \qquad \text{where } k(t) = \exp\left(-\int_1^t \frac{\mu(s)}{s} ds\right).$$

Define v as

$$v(r) = \int_1^r \frac{k(t)}{t} dt, \quad r \geq 1.$$

Then $v(r)$ is positive and increasing for $r \in (1,\infty)$, and satisfies $v(r) \to \infty$ as $r \to \infty$. Furthermore $v = v(|x|)$ solves

$$\Delta v + \nabla b \cdot \nabla v = \frac{k(|x|)}{|x|^2}\left(-\mu(|x|) + x \cdot \nabla b(x)\right).$$

By the assumption, there exists a large $R > 0$ such that

$$\Delta v + \nabla b \cdot \nabla v < 0 \quad \text{for } |x| \geq R. \tag{2.2}$$

Now assume to the contrary that u is not a constant function. Without loss of generality we may assume that u is not a constant function in $|x| \leq R$. Define

$$U(r) = \sup\{\, u(x) \,:\, |x| = r \,\}.$$

Then $U(r)$ is strictly increasing for $r \geq R$. To see why, suppose $R \leq r_1 < r_2$ and $U(r_1) \geq U(r_2)$. Then u attains its maximum for $|x| \leq r_2$ at an interior point and by the strong maximum principle u is constant, which contradicts the assumption. Therefore $U(r)$ is strictly increasing, and we have $U(R+1) > U(R)$. Choose $\delta > 0$ so small that

$$0 < \delta < \frac{U(R+1) - U(R)}{v(R+1) - v(R)}. \tag{2.3}$$

Put $w(x) = u(x) - \delta v(|x|)$. Then it follows from (2.1) and (2.2) that

$$\Delta w + \nabla b \cdot \nabla w > 0 \qquad \text{for } |x| \geq R. \tag{2.4}$$

From (2.3) we obtain $U(R+1) - \delta v(R+1) > U(R) - \delta v(R)$. This implies

$$\sup_{|x|=R+1} w(x) > \sup_{|x|=R} w(x).$$

Since $w(x) \to -\infty$ as $|x| \to \infty$, w has the maximum at a point $x_0 \in \mathbb{R}^2$, $|x_0| > R$. Then we have $\Delta w + \nabla b \cdot \nabla w \leq 0$ at $x = x_0$. This contradicts (2.4). Hence, u must be a constant function. □

Lemma 2.2. *Let (ϕ, ψ) be a nonnegative solution of* (1.4) *with ϕ, $\psi \in L^\infty(\mathbb{R}^2)$. Then $\nabla\psi \in L^\infty(\mathbb{R}^2)$.*

Proof. Define u and v by (1.2), respectively. Then (u, v) solves (1.1), and it holds that

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{t}\|\phi\|_{L^\infty(\mathbb{R}^2)} \quad \text{and} \quad \|v(t)\|_{L^\infty(\mathbb{R}^2)} = \|\psi\|_{L^\infty(\mathbb{R}^2)}.$$

Take $t_0 > 0$. From the second equation of (1.1) we have

$$v(t) = e^{((t-t_0)/\tau)\Delta}v(t_0) + \frac{1}{\tau}\int_{t_0}^{t} e^{((t-s)/\tau)\Delta}u(s)ds \equiv v_1(t) + v_2(t), \quad t > t_0,$$

where $\{e^{t\Delta}\}$ is the heat semigroup. We recall the L^p-L^q estimates for the linear heat equation,

$$\|\nabla e^{(t/\tau)\Delta}w\|_{L^q(\mathbb{R}^2)} \leq Ct^{1/q-1/p-1/2}\|w\|_{L^p(\mathbb{R}^2)} \tag{2.5}$$

for $t > 0$ with $1 \leq p \leq q \leq \infty$, where $C = C(\tau)$ is a positive constant. See, e.g., [10]. In particular we have

$$\|\nabla e^{(t/\tau)\Delta}w\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-1/2}\|w\|_{L^\infty(\mathbb{R}^2)} \quad \text{for } t > 0.$$

Then it follows that

$$\|\nabla v_1(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(t-t_0)^{-1/2}\|v(t_0)\|_{L^\infty(\mathbb{R}^2)} \leq C(t-t_0)^{-1/2}\|\psi\|_{L^\infty(\mathbb{R}^2)}$$

and

$$\|\nabla v_2(t)\|_{L^\infty(\mathbb{R}^2)} \leq C\int_{t_0}^{t}(t-s)^{-1/2}\|u(s)\|_{L^\infty}ds \leq C\|\phi\|_{L^\infty}\int_{t_0}^{t}(t-s)^{-1/2}s^{-1}ds$$

for $t > t_0$. Consequently, we obtain $\|\nabla v(t)\|_{L^\infty(\mathbb{R}^2)} < \infty$ for each $t > t_0$. By the definition of v it follows that $\|\nabla v(t)\|_{L^\infty(\mathbb{R}^2)} = t^{-1/2}\|\nabla\psi\|_{L^\infty(\mathbb{R}^2)}$. Thus we have $\nabla\psi \in L^\infty(\mathbb{R}^2)$. □

Proof of Proposition 2.1. Put $w(x) = -\phi(x)e^{|x|^2/4}e^{-\psi(x)} \leq 0$. Then $e^{-|x|^2/4}e^{\psi}\nabla w = -\nabla\phi - x\phi/2 + \phi\nabla\psi$. From the first equation of (1.4) we have

$$\nabla \cdot (e^{-|x|^2/4}e^{\psi}\nabla w) = 0, \quad \text{or} \quad \Delta w + \nabla b \cdot \nabla w = 0 \quad \text{in } \mathbb{R}^2,$$

where $\nabla b(x) = -x/2 + \nabla\psi(x)$. From Lemma 2.2 we have

$$x \cdot \nabla b(x) = \left(-\frac{|x|^2}{2} + x \cdot \nabla\psi(x)\right) \leq 0$$

for large $|x|$. As a consequence of Lemma 2.1, w must be a constant function. This completes the proof of Proposition 2.1. □

3. Radial symmetry: Proof of Theorem 3

In this section we investigate the radial symmetry of solutions to (1.8) and prove Theorem 3. Namely, we show the following:

Proposition 3.1. *Let $\psi \in C^2(\mathbb{R}^2)$ be a positive solution of* (1.8) *with* (1.9). *Then ψ must be radially symmetric about the origin.*

We prepare several lemmas.

Lemma 3.1. *We have*

$$\psi(x) \leq Ce^{-\min\{\tau,1\}|x|^2/4} \quad \textit{for } x \in \mathbb{R}^2 \tag{3.1}$$

with some constant $C > 0$.

Proof. Define

$$Lu = -\Delta u - \frac{\tau}{2}x \cdot \nabla u$$

and put $\kappa_\tau = \min\{1, \tau\}$. Let C be a positive constant and let $v(x) = Ce^{-\kappa_\tau|x|^2/4}$. Then

$$Lv = C\kappa_\tau\left(1 + \frac{(\tau - \kappa_\tau)}{4}|x|^2\right)e^{-\kappa_\tau|x|^2/4} \geq C\kappa_\tau e^{-\kappa_\tau|x|^2/4}.$$

Since $L\psi = \sigma e^{-|x|^2/4}e^{\psi}$, if we choose C so large that $C\kappa_\tau > \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}$, then $Lv > L\psi$ in $\mathbb{R}^2$. Since $v, \psi \to 0$ as $|x| \to \infty$, by the maximum principle we have $v \geq \psi$ in $\mathbb{R}^2$. This implies (3.1). □

We define $w(x,t)$ by

$$w(x,t) = t^{-\alpha}\psi\left(\frac{x}{\sqrt{t}}\right), \quad \text{where } \alpha = \frac{\sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}}{\tau}. \tag{3.2}$$

Lemma 3.2 (i) *For every $T > 0$ we have $\sup_{0<t<T} w(x,t) \to 0$ as $|x| \to \infty$.*
(ii) *For every $\mu > 0$ we have $\sup_{|x|>\mu} w(x,t) \to 0$ as $t \to 0$.*

Proof. From Lemma 3.1 we have $|y|^{2\alpha}\psi(y) \to 0$ as $|y| \to \infty$, that is, for all $\varepsilon > 0$ there exists $R > 0$ such that

$$|y|^{2\alpha}\psi(y) < \varepsilon \qquad \text{for } |y| \geq R. \tag{3.3}$$

From (3.2) we have

$$|x|^{2\alpha} w(x,t) = \left(\frac{|x|}{\sqrt{t}}\right)^{2\alpha} \psi\left(\frac{x}{\sqrt{t}}\right). \tag{3.4}$$

(i) Fix $T > 0$. From (3.3) and (3.4) it follows that

$$\sup_{0<t<T} |x|^{2\alpha} w(x,t) < \varepsilon \qquad \text{for } |x| \geq R\sqrt{T}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\sup_{0<t<T} w(x,t) \to 0$ as $|x| \to \infty$.
(ii) From (3.3) and (3.4) it follows that

$$\mu^{2\alpha} \sup_{|x|>\mu} w(x,t) \leq \sup_{|x|>\mu} |x|^{2\alpha} w(x,t) < \varepsilon \qquad \text{for } 0 < t < (\mu/R)^2.$$

Then we have $\sup_{|x|>\mu} w(x,t) \to 0$ as $t \to 0$. □

For $\mu \in \mathbb{R}$ we define T_μ and Σ_μ by

$$T_\mu = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1 = \mu\} \quad \text{and} \quad \Sigma_\mu = \{x \in \mathbb{R}^2 | x_1 < \mu\},$$

respectively. For $x \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$ let x^μ be the reflection of x with respect to T_μ, that is, $x^\mu = (2\mu - x_1, x_2)$. It is easy to see that if $\mu > 0$,

$$|x^\mu| > |x| \text{ for } x \in \Sigma_\mu \quad \text{and} \quad \{x^\mu : x \in \Sigma_\mu\} = \{x : x_1 > \mu\} \subset \{x : |x| \geq \mu\}.$$

By Lemma 3.2 we have the following:

Lemma 3.3. (i) *For every $T > 0$ we have $\sup_{0<t<T} w(x^\mu, t) \to 0$ as $|x| \to \infty$, $x \in \Sigma_\mu$.*
(ii) *For every $\mu > 0$ we have $\sup_{x \in \Sigma_\mu} w(x^\mu, t) = 0$ as $t \to 0$.*

Lemma 3.4. *Let $\mu > 0$. Define $z(x,t) = w(x,t) - w(x^\mu, t)$. Then*

$$\tau z_t \geq \Delta z + c_\mu(x,t) z \quad \textit{in } \Sigma_\mu \times (0,\infty) \quad \textit{and} \quad z = 0 \quad \textit{on } T_\mu \times (0,\infty), \tag{3.5}$$

where

$$c_\mu(x,t) = \frac{1}{t}\left(-\alpha\tau + \sigma e^{-|x|^2/(4t)} \int_0^1 e^{s\psi(x/\sqrt{t}) + (1-s)\psi(x^\mu/\sqrt{t})} ds\right). \tag{3.6}$$

We have $c_\mu(x,t) \le 0$ *in* $\mathbb{R}^2 \times (0,\infty)$.

Proof. By virtue of (3.2) we have

$$\tau w_t = \Delta w - \frac{\alpha\tau}{t} w + \sigma t^{-\alpha-1} e^{-|x|^2/4t} e^{t^\alpha w}.$$

Let $w^\mu(x,t) = w(x^\mu, t)$. Then w^μ satisfies

$$\tau w_t^\mu = \Delta w^\mu - \frac{\alpha\tau}{t} w^\mu + \sigma t^{-\alpha-1} e^{-|x^\mu|^2/4t} e^{t^\alpha w^\mu}.$$

Since $|x^\mu| \ge |x|$, we obtain

$$\tau w_t^\mu \le \Delta w^\mu - \frac{\alpha\tau}{t} w^\mu + \sigma t^{-\alpha-1} e^{-|x|^2/4t} e^{t^\alpha w^\mu}.$$

Then we obtain $\tau z_t \ge \Delta z + c_\mu z$, where c_μ is the function in (3.6). Since α satisfies $\alpha\tau = \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}$, we have $tc_\mu(x,t) \le -\alpha\tau + \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}} = 0$ for $(x,t) \in \mathbb{R}^2 \times (0,\infty)$. □

Lemma 3.5. *Let* $\mu > 0$. *We have* $w(x,t) \ge w(x^\mu,t)$ *for* $(x,t) \in \Sigma_\mu \times (0,\infty)$.

Proof. Let $z(x,t) = w(x,t) - w(x^\mu,t)$. We show that $z(x,t) \ge 0$ for $(x,t) \in \Sigma_\mu \times (0,\infty)$. Assume to the contrary that there exists a $(x_0,t_0) \in \Sigma_\mu \times (0,\infty)$ such that $z(x_0,t_0) < 0$. Take $\delta > 0$ so small that $z(x_0,t_0) < -\delta$. By (ii) of Lemma 3.3 we can take $T_0 \in (0,t_0)$ so that $w(x^\mu,T_0) < \delta$ for $x \in \Sigma_\mu$. Then it follows from $w(x,t) > 0$ that

$$z(x,T_0) \ge -\delta \qquad \text{for } x \in \Sigma_\mu. \tag{3.7}$$

Fix $T > t_0$. By (i) of Lemma 3.3 we can take $R > |x_0|$ so large that $w(x^\mu,t) < \delta$ for $|x| \ge R$, $x \in \Sigma_\mu$, $t \in [T_0,T]$. Then we obtain

$$z(x,t) \ge -\delta \quad \text{for } x \in \Sigma_\mu,\ |x| \ge R,\ t \in [T_0,T]. \tag{3.8}$$

Define $Q = \{x \in \Sigma_\mu : |x| < R\}$. Let Γ be a parabolic boundary of $Q \times (T_0,T)$, that is,

$$\Gamma = (Q \times \{T_0\}) \cup (\partial Q \times (T_0,T)).$$

From (3.5), (3.7), and (3.8) we have

$$\tau z_t \ge \Delta z + c(x,t) z \quad \text{in } Q \times (T_0,T) \qquad \text{and} \qquad z \ge -\delta \quad \text{on } \Gamma.$$

Put $Z = z + \delta$. Because $c_\mu(x,t) \le 0$, it follows from the above inequality that

$$\tau Z_t \ge \Delta Z + c_\mu(x,t) Z \quad \text{in } Q \times (T_0,T) \qquad \text{and} \qquad Z \ge 0 \quad \text{on } \Gamma.$$

By the maximum principle [25] we have $Z \ge 0$ on $\overline{Q} \times [T_0,T]$, which implies that

$$z(x,t) \ge -\delta \qquad \text{on } \overline{Q} \times [T_0,T]. \tag{3.9}$$

On the other hand $(x_0, t_0) \in Q \times (T_0, T)$ and $z(x_0, t_0) < -\delta$. This contradicts to (3.9). Hence $z(x,t) \geq 0$ for $(x,t) \in \Sigma_\mu \times (0, \infty)$. □

Proof of Proposition 3.1. From Lemma 3.5 we have $w(x,t) \geq w(x^\mu, t)$ for $\mu > 0$ and $(x,t) \in \Sigma_\mu \times (0,\infty)$. From the continuity of w we have $w(x,t) \geq w(x^0,t)$ for $(x,t) \in \Sigma_0 \times (0,\infty)$. We can repeat the previous arguments for the negative x_1-direction to conclude that $w(x,t) \leq w(x^0,t)$ for $(x,t) \in \Sigma_0 \times (0,\infty)$. Hence $w(x,t)$ is symmetric with respect to the plane $x_1 = 0$, which implies that ψ is symmetric with respect to the plane $x_1 = 0$. Since the equation (1.8) is invariant under the rotation, it follows that ψ is symmetric in every direction. Therefore ψ is radially symmetric with respect to the origin. □

Proof of Theorem 3. Let (ϕ, ψ) be a nonnegative solution of (1.4) with $\phi,\ \psi \in L^\infty(\mathbb{R}^2)$. Then ϕ is given by (1.7) for some constant $\sigma > 0$ from Proposition 2.1. It follows that $\phi > 0$ in $\mathbb{R}^2$, and $\phi(x) = O(e^{-|x|^2/4})$ as $|x| \to \infty$. From the second equation of (1.4), ψ satisfies the equation (1.8). By the strong maximum principle, $\psi > 0$ in $\mathbb{R}^2$.

Assume furthermore that $\psi(x) \to 0$ as $|x| \to \infty$. Then, by Proposition 3.1, ψ must be radially symmetric about the origin. Hence $\psi = \psi(r)$, $r = |x|$, satisfies the ordinary differential equation

$$\psi_{rr} + \left(\frac{1}{r} + \frac{\tau}{2}r\right)\psi_r + \sigma e^{-r^2/4}e^\psi = 0, \quad \text{or} \quad (re^{\tau r^2/4}\psi_r)_r + \sigma r e^{(\tau-1)r^2/4}e^\psi = 0 \quad \text{for } r > 0.$$

From $\psi_r(0) = 0$, we have

$$re^{\tau r^2/4}\psi_r = -\sigma \int_0^r s e^{(\tau-1)s^2/4} e^\psi ds < 0 \quad \text{for } r > 0.$$

This implies that $\psi_r(r) < 0$ for $r > 0$. From Lemma 3.1 we obtain $\psi(r) = O(e^{-\min\{\tau,1\}r^2/4})$ as $r \to \infty$. This completes the proof of Theorem 3. □

4. Structure of the solutions set to (1.8) with (1.9)

From Theorem 3 the solution ψ of (1.8) with (1.9) must be radially symmetric about the origin. Then the study of the solutions is reduced to the problem:

$$(4.1)_\sigma \qquad \begin{cases} \psi_{rr} + \left(\frac{1}{r} + \frac{\tau}{2}r\right)\psi_r + \sigma e^{-r^2/4}e^\psi = 0, & r > 0, \\ \psi_r(0) = 0 \quad \text{and} \quad \lim_{r\to\infty}\psi(r) = 0, \end{cases}$$

where $\sigma > 0$. In this section we investigate the structure of the pair (σ, ψ) of a parameter and a solution. Define the set $\mathcal{C}$ as

$$(4.2) \qquad \mathcal{C} = \{(\sigma, \psi) : \sigma > 0 \text{ and } \psi \in C^2(0,\infty) \cap C^1[0,\infty) \text{ is a solution of } (4.1)_\sigma\}.$$

For $(\sigma, \psi) \in \mathcal{C}$ we have $\psi \in C^2[0,\infty)$ by Lemma 4.1 below.

Proposition 4.1. *The set C is written by one parameter families $(\sigma(s), \psi(r; s))$ on $s \in \mathbb{R}$, that is, $C = \{(\sigma(s), \psi(r; s)) : s \in \mathbb{R}\}$. The pairs $(\sigma(s), \psi(r; s))$ satisfy the following properties:*

(i) $s \mapsto (\sigma(s), \psi(\cdot; s)) \in (0, \infty) \times C^2[0, \infty)$ *is continuous;*

(ii) $\lim_{s \to -\infty} \sigma(s) = 0$ *and* $\lim_{s \to -\infty} \psi(\cdot; s) = 0$ *in* $C^2[0, \infty)$;

(iii) $\lim_{s \to \infty} \|\psi(\cdot; s)\|_{L^\infty[0,\infty)} = \lim_{s \to \infty} \psi(0; s) = \infty$.

First we show the following:

Lemma 4.1. *Let $\psi \in C^2(0, \infty) \cap C^1[0, \infty)$ be a solution to $(4.1)_\sigma$. Then $\psi \in C^2[0, \infty)$ and $\sup_{r \geq 0} \psi(r) = \psi(0)$. Moreover we have*

$$\sup_{r \geq 0} |\psi_r(r)| \leq \pi^{1/2} \sigma e^{\psi(0)} \quad \textit{and} \quad \sup_{r \geq 0} |\psi_{rr}(r)| \leq \frac{3 + 2\tau}{2} \sigma e^{\psi(0)}. \tag{4.3}$$

Proof. From $(4.1)_\sigma$ we have $(re^{\tau r^2/4} \psi_r)_r + \sigma r e^{(\tau-1)r^2/4} e^{\psi} = 0$ for $r > 0$. From $\psi_r(0) = 0$, it follows that

$$\psi_r(r) = -\frac{\sigma}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{\psi(\xi)} d\xi. \tag{4.4}$$

By using the L'Hospital's rule we obtain

$$\lim_{r \to 0} \frac{\psi_r(r)}{r} = \lim_{r \to 0} -\frac{\sigma}{r^2 e^{\tau r^2/4}} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{\psi(\xi)} d\xi = -\frac{\sigma e^{\psi(0)}}{2},$$

which implies $\psi \in C^2[0, \infty)$. Since $\psi_r(r) < 0$ for $r > 0$ from (4.4), we have $\sup_{r \geq 0} \psi(r) = \psi(0)$.

From (4.4) we have

$$|\psi_r(r)| \leq \left(\frac{1}{r} \int_0^r \xi e^{-\xi^2/4} d\xi \right) \sigma e^{\psi(0)}. \tag{4.5}$$

We see that $(1/r) \int_0^r \xi e^{-\xi^2/4} d\xi \leq \int_0^\infty e^{-\xi^2/4} d\xi = \pi^{1/2}$. Then the left hand side of (4.3) holds.

From the equation in $(4.1)_\sigma$ we have

$$|\psi_{rr}(r)| \leq \left(\frac{1}{r} + \frac{\tau}{2} r \right) |\psi_r(r)| + \sigma e^{-r^2/4} e^{\psi(r)} \leq \left(\frac{1}{r} + \frac{\tau}{2} r \right) |\psi_r(r)| + \sigma e^{\psi(0)}.$$

We note here that

$$\left(\frac{1}{r} + \frac{\tau}{2} r \right) \frac{1}{r} \int_0^r \xi e^{-\xi^2/4} d\xi \leq \frac{1}{r^2} \int_0^r \xi d\xi + \frac{\tau}{2} \int_0^\infty \xi e^{-\xi^2/4} d\xi = \frac{1}{2} + \tau. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\left(\frac{1}{r} + \frac{\tau}{2} r \right) |\psi_r(r)| \leq \frac{1 + 2\tau}{2} \sigma e^{\psi(0)}.$$

Therefore we obtain the right hand side of (4.3). This completes the proof of Lemma 4.1. □

To prove Proposition 4.1 we consider the initial value problem

$$(4.7)_s \qquad \begin{cases} w_{rr} + \left(\dfrac{1}{r} + \dfrac{\tau}{2}r\right) w_r + e^{-r^2/4}e^w = 0, & r > 0, \\ w_r(0) = 0 \quad \text{and} \quad w(0) = s, \end{cases}$$

where $s \in \mathbb{R}$. We denote by $w(r; s)$ the solution of the problem $(4.7)_s$. We easily see that $w(r; s)$ and $w_r(r; s)$ satisfy, respectively,

$$(4.8) \qquad w(r; s) = s - \int_0^r \frac{1}{\xi} e^{-\tau\xi^2/4} \left(\int_0^\xi \eta e^{(\tau-1)\eta^2/4} e^{w(\eta; s)} d\eta \right) d\xi$$

and

$$(4.9) \qquad w_r(r; s) = -\frac{1}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{w(\xi; s)} d\xi.$$

Define $I(\tau)$ as

$$I(\tau) = \int_0^\infty \frac{1}{\xi} e^{-\tau\xi^2/4} \left(\int_0^\xi \eta e^{(\tau-1)\eta^2/4} d\eta \right) d\xi.$$

From [21, Lemma 1] it follows that $I(\tau) = (\log \tau)/(\tau - 1)$ if $\tau \neq 1$, $I(\tau) = 1$ if $\tau = 1$. We easily obtain $w_r(r; s) < 0$ for $r > 0$ and $w(r; s) \geq s - e^s I(\tau)$ for $r \geq 0$. (See [21, Lemma 2].) Then $\lim_{r\to\infty} w(r; s)$ exists and is a finite value. Put $t(s) = \lim_{r\to\infty} w(r; s)$.

Lemma 4.2. *For $s \in \mathbb{R}$, let $\psi(r; s) = w(r; s) - t(s)$. Then $\psi(r; s)$ is a solution to $(4.1)_\sigma$ with $\sigma = e^{t(s)}$. Conversely, let $\psi(r)$ be a solution of $(4.1)_\sigma$. Then, for some $s \in \mathbb{R}$, $\psi(r) = \psi(r; s)$ and $\sigma = e^{t(s)}$.*

Proof. It is clear that $\psi(r; s)$ is a solution to $(4.1)_\sigma$ with $\sigma = e^{t(s)}$. Conversely, let $\psi(r)$ be a solution of $(4.1)_\sigma$, and let $w(r) = \psi(r) + \log \sigma$. Then $w(r)$ satisfies $(4.7)_s$ with $s = \psi(0) + \log \sigma$. By the uniqueness we obtain $w(r) = w(r; s)$ with $s = \psi(0) + \log \sigma$. We have $\lim_{r\to\infty} w(r; s) = \lim_{r\to\infty} w(r) = \log \sigma$. Then $t(s) = \log \sigma$, that is, $\sigma = e^{t(s)}$. Hence we obtain $\psi(r) = w(r) - \log \sigma = w(r; s) - t(s)$, which implies $\psi(r) = \psi(r; s)$. □

From [21, (ii) of Lemma 5] it follows that, for $s_1, s_2 \in \mathbb{R}$,

$$(4.10) \qquad \sup_{r \geq 0} |w(r; s_1) - w(r; s_2)| \leq C_1 |s_1 - s_2|,$$

where $C_1 = \exp(e^m I(\tau))$ and $m = \max\{s_1, s_2\}$. Moreover we have the following:

Lemma 4.3. *Let $s_1, s_2 \in \mathbb{R}$, and let $m = \max\{s_1, s_2\}$. Then we have*
(i) $\sup_{r\geq 0} |w_r(r; s_1) - w_r(r; s_2)| \leq C_2 |s_1 - s_2|$, *where $C_2 = \pi^{1/2} e^m C_1$;*

(ii) $\sup_{r\geq 0}|w_{rr}(r;s_1)-w_{rr}(r;s_2)|\leq C_3|s_1-s_2|$, *where* $C_3=(3+2\tau)e^m C_1/2$.

Proof. From (4.9) we have

$$|w_r(r;s_1)-w_r(r;s_2)|\leq \frac{1}{r}e^{-\tau r^2/4}\int_0^r \xi e^{(\tau-1)\xi^2/4}|e^{w(\xi;s_1)}-e^{w(\xi;s_2)}|d\xi.$$

Note that $|e^{w(t;s_1)}-e^{w(t;s_2)}|\leq e^m|w(t;s_1)-w(t;s_2)|$ with $m=\max\{s_1,s_2\}$. Then from (4.10) we have $|e^{w(t;s_1)}-e^{w(t;s_2)}|\leq C_1e^m|s_1-s_2|$. Then it follows that

$$|w_r(r;s_1)-w_r(r;s_2)|\leq C_1e^m|s_1-s_2|\left(\frac{1}{r}\int_0^r \xi e^{-\xi^2/4}d\xi\right). \tag{4.11}$$

From $(1/r)\int_0^r \xi e^{-\xi^2/4}d\xi\leq \int_0^\infty e^{-\xi^2/4}d\xi=\pi^{1/2}$, we obtain (i).

From $(4.7)_s$ we see that $w_{rr}(r;s)=-(1/r+\tau r/2)w_r(r;s)-e^{-r^2/4}e^{w(r;s)}$. Then we have

$$|w_{rr}(r;s_1)-w_{rr}(r;s_2)|\leq\left(\frac{1}{r}+\frac{\tau}{2}r\right)|w_r(r;s_1)-w(r;s_2)|+e^m|w(r;s_1)-w(r;s_2)|.$$

Then from (4.11) and (4.6) we obtain

$$\left(\frac{1}{r}+\frac{\tau}{2}r\right)|w_r(r;s_1)-w_r(r;s_2)|\leq\frac{1+2\tau}{2}C_1e^m|s_1-s_2|.$$

Therefore we obtain (ii). □

Lemma 4.4. *Let* $s_1,\ s_2\in\mathbb{R}$, *and let* $m=\max\{s_1,s_2\}$. *Then we have*
(i) $|t(s_1)-t(s_2)|\leq C_1|s_1-s_2|$, *where* $C_1=\exp(e^mI(\tau))$;
(ii) $\lim_{s\to-\infty}(s-t(s))=0$;
(iii) $\sup_{s\in\mathbb{R}}t(s)\leq-\log I(\tau)$.

Proof. Letting $r\to\infty$ in (4.10), we have (i). Since $w(r;s)<s$ for $r>0$, it follows from (4.8) that

$$0<s-w(r;s)\leq e^s\int_0^r\frac{1}{\xi}e^{-\tau\xi^2/4}\left(\int_0^\xi \eta e^{(\tau-1)\eta^2/4}d\eta\right)d\xi.$$

Letting $r\to\infty$ we have $0<s-t(s)\leq e^sI(\tau)$ for $s\in\mathbb{R}$. This implies that (ii) holds.

Since $w(r;s)$ is decreasing in $r>0$, it follows form (4.9) that

$$w_r(r;s)\leq-\frac{1}{r}e^{w(r;s)}e^{-\tau r^2/4}\int_0^r\xi e^{(\tau-1)\xi^2/4}d\xi.$$

Then we obtain

$$\frac{d}{dr}\left(-e^{-w(r;s)}\right)\leq-\frac{1}{r}e^{-\tau r^2/4}\int_0^r\xi e^{(\tau-1)\xi^2/4}d\xi.$$

Integrating the above on $[0,\infty)$ we have $e^{-t(s)}-e^{-s}\geq I(\tau)$ or $e^{-t(s)}\geq I(\tau)$. This implies that (iii) holds. □

Proof of Proposition 4.1. By Lemma 4.2 we have $\mathcal{C} = \{(\sigma(s), \psi(\cdot; s)) : s \in \mathbb{R}\}$, where $\sigma(s) = e^{t(s)}$ and $\psi(r; s) = w(r; s) - t(s)$. We see that $w(\cdot; s) \in C^2[0, \infty)$ and $t(s) \in \mathbb{R}$ are continuous for $s \in \mathbb{R}$ by Lemma 4.3 and (i) of Lemma 4.4, respectively. Thus (i) holds.

By (ii) of Lemma 4.4 we have $\sigma(s) = e^{t(s)} \to 0$ and $\psi(0; s) = s - t(s) \to 0$ as $s \to -\infty$. Then, by Lemma 4.1 we conclude that $\psi(\cdot; s) \to 0$ in $C^2[0, \infty)$ as $s \to -\infty$. Thus (ii) holds.

From Lemma 4.1 we have $\|\psi(\cdot; s)\|_{L^\infty[0,\infty)} = \psi(0; s)$. From (iii) of Lemma 4.4 we have $\lim_{s\to\infty} \psi(0; s) = \lim_{s\to\infty}(s - t(s)) \geq \lim_{s\to\infty}(s + \log I(\tau)) = \infty$. Thus (iii) holds. This completes the proof of Proposition 4.1. □

5. Blow-up analysis to self-similar solutions

This section is concerned with the case (iii) of Theorem 2. We study the asymptotic behavior of sequences $\{(\phi_k, \psi_k)\} \subset \mathcal{S}$ satisfying $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \to \infty$ as $k \to \infty$. We show the following:

Proposition 5.1. *Let $(\phi_k, \psi_k) \in \mathcal{S}$, and let $\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)}$. Assume that*

$$\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \to \infty \quad \text{as } k \to \infty. \tag{5.1}$$

and that $\{\lambda_k\}$ is bounded. Then there exists a subsequence, which we call again (ψ_k, ϕ_k) and λ_k, satisfying $\lambda_k \to 8\pi$ as $k \to \infty$ and

$$\phi_k(x)dx \rightharpoonup 8\pi\delta_0(dx) \quad \text{as } k \to \infty \tag{5.2}$$

in the sense of measure, where $\delta_0(dx)$ is Dirac's delta function with the support in origin.

In order to prove Proposition 5.1 we make use of Theorems A and Theorem 4 in Section 1. We also need the following result by Brezis and Merle [2].

Theorem B [2]. *Assume $\{u_k\}$ is a sequence of solutions of* (1.11) *such that*

$$\|V_k\|_{L^\infty(\Omega)} \leq C, \quad \|u_k^+\|_{L^1(\Omega)} \leq C, \quad \text{and} \quad \int_\Omega V_k e^{u_k} dx < 4\pi,$$

for some constant $C > 0$, where $u^+ = \max\{u, 0\}$. Then $\{u_k^+\}$ is bounded in $L^\infty_{\rm loc}(\Omega)$.

Now we prepare several lemmas.

Lemma 5.1. *Assume that $f \in C(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Let $w \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ be a solution of*

$$-\Delta w - \frac{\tau}{2} x \cdot \nabla w = f \qquad \text{for } x \in \mathbb{R}^2. \tag{5.3}$$

Then we have $\|w\|_{L^1(\mathbb{R}^2)} + \|\nabla w\|_{L^1(\mathbb{R}^2)} \le C\|f\|_{L^1(\mathbb{R}^2)}$ *for some positive constant* C.

Proof. Define W and F respectively as

$$W(x,t) = w\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad F(x,t) = \frac{1}{t}f\left(\frac{x}{\sqrt{t}}\right).$$

Then W and F satisfy

$$\|W(\cdot,t)\|_{L^1(\mathbb{R}^2)} = t\|w\|_{L^1(\mathbb{R}^2)} \quad \text{and} \quad \|F(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} \tag{5.4}$$

for $t > 0$. Furthermore, from (5.3) we have $\tau W_t = \Delta W + F$ in $\mathbb{R}^2 \times (0,\infty)$. Since $W \to 0$ in $L^1(\mathbb{R}^2)$ as $t \to 0$ from (5.4), we obtain

$$W(x,t) = \frac{1}{\tau}\int_0^t e^{((t-s)/\tau)\Delta}F(\cdot,s)ds.$$

Then it follows from (5.4) that

$$t\|w\|_{L^1(\mathbb{R}^2)} = \|W(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le \frac{1}{\tau}\int_0^t \|F(\cdot,s)\|_{L^1(\mathbb{R}^2)}ds \le \frac{t}{\tau}\|f\|_{L^1(\mathbb{R}^2)}.$$

Therefore we obtain $\|w\|_{L^1(\mathbb{R}^2)} \le \tau^{-1}\|f\|_{L^1(\mathbb{R}^2)}$.

Next we show $\|\nabla w\|_{L^1(\mathbb{R}^2)} \le C\|f\|_{L^1(\mathbb{R}^2)}$. By the L^p-L^q estimates (2.5) with $p = q = 1$ we have

$$\|\nabla e^{((t-s)/\tau)\Delta}F(\cdot,s)\|_{L^1(\mathbb{R}^2)} \le C(t-s)^{-1/2}\|F(\cdot,s)\|_{L^1(\mathbb{R}^2)} = C(t-s)^{-1/2}\|f\|_{L^1(\mathbb{R}^2)}.$$

Then we obtain

$$\|\nabla W(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le \frac{1}{\tau}\int_0^t \|\nabla e^{((t-s)/\tau)\Delta}F(\cdot,s)\|_{L^1(\mathbb{R}^2)}ds \le Ct^{1/2}\|f\|_{L^1(\mathbb{R}^2)}.$$

By the definition of W it follows that $\|\nabla W(\cdot,t)\|_{L^1(\mathbb{R}^2)} = t^{1/2}\|\nabla w\|_{L^1(\mathbb{R}^2)}$. Therefore we conclude that $\|\nabla w\|_{L^1(\mathbb{R}^2)} \le C\|f\|_{L^1(\mathbb{R}^2)}$. This completes the proof of Lemma 5.1. □

Let $(\phi_k, \psi_k) \in \mathcal{S}$, and let $\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)}$. Then (λ_k, ψ_k) solves (1.10), that is,

$$\Delta\psi_k + \frac{\tau}{2}x\cdot\nabla\psi_k + \lambda_k e^{-|x|^2/4}e^{\psi_k}\Big/\int_{\mathbb{R}^2} e^{-|y|^2/4}e^{\psi_k(y)}dy = 0 \qquad \text{for } x \in \mathbb{R}^2. \tag{5.5}$$

From Theorem 3 we have $\psi_k \in L^1(\mathbb{R}^2)$, $\psi_k = \psi_k(r)$, $r = |x|$, and $\partial\psi_k/\partial r < 0$ for $r > 0$. Assume that (5.1) holds. Then $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} = \psi_k(0) \to \infty$ as $k \to \infty$. We always use B_r to denote a ball of radius r centered at origin, that is, $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$.

Lemma 5.2. (i) *We have* $\|\psi_k\|_{L^1(\mathbb{R}^2)} + \|\nabla\psi_k\|_{L^1(\mathbb{R}^2)} = O(1)$ *as* $k \to \infty$.
(ii) *For all* $r > 0$ *we have* $\sup_k \|\psi_k\|_{L^\infty(\mathbb{R}^2\setminus B_r)} < \infty$.

Proof. (i) Put

$$f_k(x) = \lambda_k e^{-|x|^2/4} e^{\psi_k(x)} \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy .$$

Then $f_k \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. We have $\psi_k \in L^1(\mathbb{R}^2)$ and

$$-\Delta\psi_k - \frac{\mathcal{T}}{2} x \cdot \nabla\psi_k = f_k \qquad \text{for } x \in \mathbb{R}^2.$$

By Lemma 5.1 we obtain $\|\psi_k\|_{L^1(\mathbb{R}^2)} + \|\nabla\psi_k\|_{L^1(\mathbb{R}^2)} \le C\|f_k\|_{L^1(\mathbb{R}^2)}$ for some constant $C > 0$. Since $\|f_k\|_{L^1(\mathbb{R}^2)} = \lambda_k = O(1)$ as $k \to \infty$, the assertion of (i) holds.

(ii) Assume to the contrary that $\sup_k \|\psi_k\|_{L^\infty(\mathbb{R}^2 \setminus B_{r_0})} = \infty$ for some $r_0 > 0$. Since $\psi_k(r)$ is decreasing in $r > 0$, there exists a subsequence, which we call again $\{\psi_k\}$, such that $\inf_{y \in B_{r_0}} \psi_k(y) \to \infty$ as $k \to \infty$. Then $\|\psi_k\|_{L^1(\mathbb{R}^2)} \to \infty$ as $k \to \infty$, which contradicts the assertion (i). □

Take $R > 0$. Let g_k be a unique solution of the problem

$$-\Delta g_k = \frac{\mathcal{T}}{2} x \cdot \nabla\psi_k \quad \text{in } B_R, \qquad g_k = 0 \quad \text{on } \partial B_R.$$

Lemma 5.3. *We have* $\|g_k\|_{L^\infty(B_R)} = O(1)$ *and* $\|\nabla g_k\|_{L^\infty(B_R)} = O(1)$ *as* $k \to \infty$.

Proof. We have $g_k = g_k(r)$, $r = |x|$, since $\psi_k = \psi_k(r)$. We see that $g_k(r)$ satisfies

$$-(r g_k')' = \frac{\mathcal{T}}{2} r^2 \psi_k, \quad 0 < r < R, \qquad g_k'(0) = g_k(R) = 0,$$

where $' = d/dr$. We will show that

$$\text{(5.6)} \qquad \|g_k\|_{L^\infty[0,R]} = O(1), \quad \|g_k'\|_{L^\infty[0,R]} = O(1) \quad \text{as } k \to \infty.$$

By integrating the equation above, we obtain

$$-r g_k'(r) = \frac{\mathcal{T}}{2} \int_0^r s^2 \psi_k'(s) ds.$$

Then it follows that

$$|g_k'(r)| \le \frac{\mathcal{T}}{2r} \int_0^r s^2 |\psi_k'(s)| ds \le \frac{\mathcal{T}}{2} \int_0^r s|\psi_k'(s)| ds \quad \text{for } 0 \le r \le R.$$

Thus we obtain

$$\text{(5.7)} \qquad \|g_k'\|_{L^\infty[0,R]} \le \frac{\mathcal{T}}{2} \int_0^R s|\psi_k'(s)| ds.$$

We note that $\int_r^R g_k'(s) ds = g_k(R) - g_k(r) = -g_k(r)$. Then

$$|g_k(r)| \le \int_0^R |g_k'(s)| ds \le R\|g_k'\|_{L^\infty[0,R]} \quad \text{for } 0 \le r \le R.$$

From (5.7) we obtain

$$\|g_k\|_{L^\infty[0,R]} \le \frac{\tau R}{2} \int_0^R s|\psi_k'(s)|ds. \tag{5.8}$$

By (i) of Lemma 5.2 we have

$$2\pi \int_0^R s|\psi_k'(s)|ds = \|\nabla\psi_k\|_{L^1(B_R)} \le \|\nabla\psi_k\|_{L^1(\mathbb{R}^2)} = O(1) \quad \text{as } k \to \infty.$$

From (5.7) and (5.8) we obtain (5.6). This completes the proof of Lemma 5.3. □

Now define v_k as

$$v_k(x) = \psi_k(x) - g_k(x) - \log\left(\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy\right). \tag{5.9}$$

It follows from (5.5) that

$$-\Delta v_k = -\Delta\psi_k - \frac{\tau}{2} x \cdot \nabla\psi_k = \lambda_k e^{-|x|^2/4} e^{g_k} e^{v_k} \qquad \text{for } x \in B_R. \tag{5.10}$$

Then we have

$$-\Delta v_k = V_k(x) e^{v_k} \qquad \text{in } B_R, \tag{5.11}$$

where $V_k(x) = \lambda_k e^{-|x|^2/4} e^{g_k}$. Since $\{\lambda_k\}$ is bounded and by Lemma 5.3, we have $0 \le V_k(x) \le C_0$ and $\|\nabla V_k\|_{L^\infty(B_R)} \le C_1$ for some constants C_0 and C_1. Since v_k is radial symmetry and satisfies $-\Delta v_k \ge 0$ in B_R, $v_k(r)$ is nonincreasing in $r \in (0, R)$ by the maximum principle.

Lemma 5.4. *There exists a subsequence, which we call again $\{v_k\}$, such that $v_k(0) \to \infty$ and $v_k(x) \to -\infty$ uniformly on compact subset of $B_R \setminus \{0\}$ as $k \to \infty$. Moreover,*

$$\int_{B_R} V_k e^{v_k} dx \to 8\pi \quad \text{as } k \to \infty \tag{5.12}$$

and

$$\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \to \infty \quad \text{as } k \to \infty. \tag{5.13}$$

Proof. We see that

$$\int_{B_R} e^{v_k(y)} dy \le e^{\|g_k\|_{L^\infty(B_R)}} \int_{B_R} e^{\psi_k(y)} dy \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4 + \psi_k(y)} dy \le C$$

for some constant $C > 0$. Hence, by applying Theorem A, there exists a subsequence (still denoted by $\{v_k\}$) satisfying one of the alternatives (i), (ii), and (iii) in Theorem A.

Assume that the first alternative (i) holds. Since $\{v_k\}$ and $\{g_k\}$ are bounded in $L^\infty_{\text{loc}}(B_R)$ and $\psi_k(0) \to \infty$ as $k \to \infty$, it follows from (5.9) that

$$\log\left(\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y) dy}\right) = \psi_k(0) - g_k(0) - v_k(0) \to \infty \quad \text{as } k \to \infty.$$

Let $y_0 \in B_R \setminus \{0\}$. Then from (5.9) we have $\psi_k(y_0) \to \infty$ as $k \to \infty$. This contradicts (ii) of Lemma 5.2.

Assume that the second alternative (ii) holds. Since $v_k(r)$ is nonincreasing in r, we have $v_k \to -\infty$ uniformly on B_R. Then

$$\int_{B_R} e^{v_k} dx \to 0 \quad \text{as } k \to \infty. \tag{5.14}$$

Put

$$w_k = \psi_k - g_k \quad \text{and} \quad W_k(x) = V_k(x) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy.$$

Then we have $-\Delta w_k = W_k e^{w_k}$ in B_R. Because $\psi_k \geq 0$, we have

$$W_k(x) \leq V_k(x) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} dy \leq C$$

for some constant $C > 0$. We find that $\|w_k\|_{L^1(B_R)} \leq \|\psi_k\|_{L^1(B_R)} + \|g_k\|_{L^1(B_R)} = O(1)$ as $k \to \infty$ by Lemmas 5.2 and 5.3. It follows from (5.14) that

$$\int_{B_R} W_k(y) e^{w_k(y)} dy = \int_{B_R} V_k(y) e^{v_k(y)} dy \leq C_0 \int_{B_R} e^{v_k(y)} dy \to 0 \quad \text{as } k \to \infty.$$

Hence, by applying Theorem B we obtain $\|w_k^+\|_{L^\infty(B_r)} = O(1)$ as $k \to \infty$. This contradicts $w_k(0) = \psi_k(0) - g_k(0) \to \infty$ as $k \to \infty$.

Therefore, the third alternative (iii) must hold. By (ii) of Lemma 5.2 we have the blow-up set $\mathcal{B} = \{0\}$. Then $v_k(0) \to \infty$ and $v_k(x) \to -\infty$ uniformly on compact subset of $B_R \setminus \{0\}$. Moreover

$$\int_{B_R} V_k e^{v_k} dx \to \alpha \quad \text{as } k \to \infty \tag{5.15}$$

for some $\alpha \geq 4\pi$. Since v_k is radial symmetry, we have $\max_{\partial B_R} v_k - \min_{\partial B_R} v_k = 0$. By applying Theorem 4, we obtain $\alpha = 8\pi$ in (5.15).

Let $x_0 \in B_R \setminus \{0\}$. From $v_k(x_0) \to -\infty$ as $k \to \infty$ we have

$$\log\left(\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy\right) = \psi_k(x_0) - g_k(x_0) - v_k(x_0) \to \infty \quad \text{as } k \to \infty,$$

which implies that (5.13) holds. □

Proof of Proposition 5.1. Let $\{v_k\}$ be a subsequence obtained in Lemma 5.4. First we verify that, for all $r > 0$,

$$\int_{\mathbb{R}^2 \setminus B_r} V_k e^{v_k} dy \to 0 \quad \text{as } k \to \infty. \tag{5.16}$$

From (ii) of Lemma 5.2 there exists a constant $M = M(r) > 0$ such that $|\psi_k(x)| \leq M$ for $|x| \geq r$. Since

$$\int_{\mathbb{R}^2\setminus B_r} V_k(y)e^{v_k(y)}dy = \frac{\lambda_k \int_{\mathbb{R}^2\setminus B_r} e^{-|y|^2/4}e^{\psi_k(y)}dy}{\int_{\mathbb{R}^2} e^{-|y|^2/4}e^{\psi_k(y)}dy} \leq \frac{\lambda_k e^M \int_{\mathbb{R}^2\setminus B_r} e^{-|y|^2/4}dy}{\int_{\mathbb{R}^2} e^{-|y|^2/4}e^{\psi_k(y)}dy},$$

it follows from (5.13) that (5.16) holds.

From (5.10), (5.11), and the second equation of (1.4) we have

$$V_k e^{v_k} = -\Delta v_k = -\Delta\psi_k - \frac{\tau}{2}x \cdot \nabla\psi_k = \phi_k.$$

From (5.12) and (5.16) we have

$$\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} V_k e^{v_k}dy = \int_{B_R} V_k e^{v_k}dy + \int_{\mathbb{R}^2\setminus B_R} V_k e^{v_k}dy \to 8\pi \quad \text{as } k \to \infty.$$

Thus $\lambda_k \to 8\pi$ as $k \to \infty$. Since $\{\phi_k\}$ is bounded in $L^1(\mathbb{R}^2)$, we may extract a subsequence, which we call again $\{\phi_k\}$, such that ϕ_k converges in the sense of measures on $\mathbb{R}^2$ to some nonnegative bounded measure μ, i.e.,

$$\int_{\mathbb{R}^2} \phi_k(x)\eta dx \to \int_{\mathbb{R}^2} \eta d\mu$$

for every $\eta \in C(\mathbb{R}^2)$ with compact support. From (5.16) we have $\int_{\mathbb{R}^2\setminus B_r} \phi_k(x)dx \to 0$ as $k \to \infty$ for every $r > 0$. Then $\phi_k \to 0$ in $L^1_{\rm loc}(\mathbb{R}^2 \setminus \{0\})$ and hence μ is supported on $\{0\}$. Thus we obtain $d\mu = \alpha\delta_0(dx)$ with $\alpha = 8\pi$, which implies that (5.2) holds. This completes the proof of Proposition 5.1. □

6. L^1-norms of self-similar solutions

This section is concerned with the case (iv) of Theorem 2 and we investigate the upper bounds of $\|\phi\|_{L^1(\mathbb{R}^2)}$ for $(\phi, \psi) \in \mathcal{S}$.

Proposition 6.1. *Let $(\phi, \psi) \in \mathcal{S}$. Then*

$$\|\phi\|_{L^1(\mathbb{R}^2)} \leq \max\left\{\frac{4}{3}\pi^3, \frac{4}{3}\pi^3\tau^2\right\}.$$

Moreover, if $0 < \tau \leq 1/2$ then $\|\phi\|_{L^1(\mathbb{R}^2)} < 8\pi$.

We prove Proposition 6.1, following the idea of Biler[1]. By Theorem 1 the solution $(\phi, \psi) \in \mathcal{S}$ must be radially symmetric about the origin. Define Φ and Ψ, respectively, as

$$\Phi(s) = \frac{1}{2}\int_0^s \phi(\sqrt{t})dt \quad \text{and} \quad \Psi(s) = \frac{1}{2}\int_0^s \psi(\sqrt{t})dt.$$

First we show the following:

Lemma 6.1. *We have* $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s\to\infty} \Phi(s)$. *Moreover,* (Φ, Ψ) *solves*

$$\begin{cases} \Phi'' + \dfrac{1}{4}\Phi' - 2\Phi'\Psi'' = 0 \\ 4s\Psi'' + \tau s\Psi' - \tau\Psi + \Phi = 0 \end{cases} \tag{6.1}$$

for $s > 0$, *where* $' = d/ds$.

Proof. We see that

$$\int_{\mathbb{R}^2} \phi(|y|)dy = 2\pi \int_0^\infty r\phi(r)dr = 2\pi \left(\frac{1}{2}\int_0^\infty \phi(\sqrt{t})dt\right),$$

which implies $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s\to\infty} \Phi(s)$.

Define u and v as

$$u(r,t) = \frac{1}{t}\phi\left(\frac{r}{\sqrt{t}}\right) \quad \text{and} \quad v(r,t) = \psi\left(\frac{r}{\sqrt{t}}\right),$$

respectively. Put U and V as

$$U(r,t) = \int_0^r su(s,t)ds \quad \text{and} \quad V(r,t) = \int_0^r sv(s,t)ds.$$

Then, by the change of variables, we obtain

$$U(r,t) = \frac{1}{2}\int_0^{r^2/t} \phi(\sqrt{s})ds \quad \text{and} \quad V(r,t) = \frac{t}{2}\int_0^{r^2/t} \psi(\sqrt{s})ds.$$

By the definition of Ψ and Φ we have

$$U(r,t) = \Phi\left(\frac{r^2}{t}\right) \quad \text{and} \quad V(r,t) = t\Psi\left(\frac{r^2}{t}\right). \tag{6.2}$$

Now we verify that (U, V) satisfies

$$\begin{cases} U_t = r\left(r^{-1}U_r\right)_r - U_r\left(r^{-1}V_r\right)_r \\ \tau V_t = r\left(r^{-1}V_r\right)_r + U \end{cases} \tag{6.3}$$

for $(r,t) \in [0,\infty) \times (0,\infty)$. Since (u,v) solves (1.1), we see that

$$ru_t = (ru_r)_r - ru_rv_r - u(rv_r)_r \quad \text{and} \quad \tau rv_t = (rv_r)_r + ru.$$

Then we obtain

$$\int_0^r su_t(s,t)ds = ru_r - ruv_r \quad \text{and} \quad \tau\int_0^r sv_t(s,t)ds = rv_r + \int_0^r su(s,t)ds.$$

Thus we obtain (6.3). By virtue of (6.2) we have (6.1). □

Lemma 6.2. *We have*

$$-s\Psi''(s) = \frac{1}{4}e^{-\tau s/4}\int_0^s e^{\tau t/4}\Phi'(t)dt > 0 \quad \textit{for } s > 0. \tag{6.4}$$

Proof. Put $W(s) = -4s\Psi''(s)$. From the second equation of (6.1) we have

$$\Phi' = (-4s\Psi'')' - \tau s\Psi'' = W' + \frac{\tau}{4}W.$$

Since $s\Psi''(s) = \sqrt{s}\psi'(\sqrt{s})/4$, we have $W(0) = \lim_{s\to 0} W(s) = 0$. Then we obtain

$$W(s) = e^{-\tau s/4}\int_0^s e^{\tau t/4}\Phi'(t)dt.$$

Since $\Phi'(s) = \phi(\sqrt{s})/2 > 0$, we obtain the assertion. □

Lemma 6.3. *We have* $s\Psi''(s) \to 0$ *as* $s \to \infty$ *and, for* $s > 0$,

$$0 < \Psi(s) - s\Psi'(s) \le \begin{cases} \dfrac{\tau}{4}\displaystyle\int_0^s \frac{t}{e^{\tau t/4}-1}dt < s & \textit{if } 0 < \tau \le 1, \\ \dfrac{\tau}{4}\displaystyle\int_0^s \frac{t}{e^{t/4}-1}dt & \textit{if } \tau > 1. \end{cases}$$

Proof. From the first equation of (6.1) and (6.4) we have

$$\Phi'' + \frac{1}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi'\int_0^s e^{\tau t/4}\Phi'(t)dt = 0.$$

We note that $\Phi'(s) = \phi(\sqrt{s})/2 > 0$. Then, for the case $0 < \tau \le 1$, we have

$$\Phi'' + \frac{\tau}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi'\int_0^s e^{\tau t/4}\Phi'(t)dt \le 0,$$

that is,

$$\left(e^{\tau s/4}\Phi'\right)' + \frac{1}{2s}\Phi'\int_0^s e^{\tau t/4}\Phi'(t)dt \le 0. \tag{6.5}$$

For the case $\tau > 1$ we have

$$\Phi'' + \frac{1}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi'\int_0^s e^{t/4}\Phi'(t)dt \le 0,$$

that is,

$$\left(e^{s/4}\Phi'\right)' + \frac{1}{2s}e^{(1-\tau)s/4}\Phi'\int_0^s e^{t/4}\Phi'(t)dt \le 0. \tag{6.6}$$

First we consider the case $0 < \tau \leq 1$. Define Z as

$$Z(s) = \int_0^s e^{\tau t/4}\Phi'(t)dt.$$

From (6.5) we have

$$sZ'' + \frac{1}{2}e^{-\tau s/4}Z'Z \leq 0. \tag{6.7}$$

By integrating the above on $[0, s]$ we obtain

$$sZ' - Z + \frac{1}{4}e^{-\tau s/4}Z^2 + \frac{\tau}{16}\int_0^s e^{-\tau t/4}Z^2(t)dt \leq 0.$$

Then we have $sZ' - Z + e^{-\tau s/4}Z^2/4 \leq 0$. Dividing the inequality by Z^2 it follows that $(s/Z)' \geq e^{-\tau s/4}/4$. Therefore we obtain

$$Z(s) \leq \frac{\tau s}{1 - e^{-\tau s/4}}. \tag{6.8}$$

From (6.4) we have $-s\Psi'' = e^{-\tau s/4}Z(s)/4 > 0$. Then

$$0 < -s\Psi''(s) \leq \frac{\tau s}{4(e^{\tau s/4} - 1)} < 1 \quad \text{for } s > 0.$$

This implies $s\Psi''(s) \to 0$ as $s \to \infty$. By integrating the above on $[0, s]$ we obtain the assertion.

Next we consider the case $\tau > 1$. Define Z as

$$Z(s) = \int_0^s e^{t/4}\Phi'(t)dt.$$

Then from (6.6) we have (6.7). By the similar argument above we obtain (6.8). We see that

$$e^{-\tau s/4}\int_0^s e^{\tau t/4}\Phi'(t)dt = \int_0^s e^{-\tau(s-t)/4}\Phi'(t)dt \leq \int_0^s e^{-(s-t)/4}\Phi'(t)dt = e^{-s/4}Z(s).$$

Then from (6.4) and (6.8) we have

$$0 < -s\Psi''(s) \leq \frac{1}{4}e^{-s/\tau}Z(s) \leq \frac{\tau s}{4(e^{s/4} - e^{(1-\tau)s/4})} \leq \frac{\tau s}{4(e^{s/4} - 1)}.$$

Therefore $s\Psi''(s) \to 0$ as $s \to \infty$. By integrating the above we obtain the assertion. □

Proof of Proposition 6.1. First we consider the case $0 < \tau \leq 1$. From the second equation of (6.1) we have $\Phi(s) = -4s\Psi''(s) + \tau\left(\Psi(s) - s\Psi'(s)\right)$. From Lemma 6.3 we obtain

$$\lim_{s\to\infty}\Phi(s) = \lim_{s\to\infty}\tau\left(\Psi(s) - s\Psi'(s)\right) \leq \frac{\tau^2}{4}\int_0^\infty \frac{s}{e^{\tau s/4} - 1}ds.$$

By the change of variable $z = \tau s/4$ it follows that

$$\lim_{s\to\infty} \Phi(s) \le 4\int_0^\infty \frac{z}{e^z - 1}dz = \frac{2}{3}\pi^2.$$

Since $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s\to\infty} \Phi(s)$ from Lemma 6.1, we obtain the assertion.

Next we consider the case $\tau > 1$. By the similar argument we obtain

$$\lim_{s\to\infty} \Phi(s) \le \frac{\tau^2}{4}\int_0^\infty \frac{s}{e^{s/4}-1}ds = 4\tau^2 \int_0^\infty \frac{z}{e^z-1}dz = \frac{2}{3}\pi^2\tau^2,$$

which implies the assertion.

Finally we consider the case $0 < \tau \le 1/2$. The change of variables

$$t = (\log s)/2, \quad k(t) = \Phi(s), \quad \ell(t) = 2s\Phi'(s), \quad m(t) = \Psi(s), \quad n(t) = 2s\Psi'(s)$$

transforms (6.1) into

$$\begin{cases} \dot{k} = \ell, \qquad \dot{m} = n, \\ \dot{\ell} = \left(2 - k + \tau m - \frac{\tau n}{2} - \frac{e^{2t}}{2}\right)\ell, \\ \dot{n} = 2n + e^{2t}\left(\frac{\tau n}{2} + \tau m - k\right), \end{cases}$$

where $\dot{} = d/dt$. Hence we have

$$\frac{d}{dt}\left((k-2)^2 + 2\ell\right) = 2\ell\left(\tau m - \frac{\tau n}{2} - \frac{e^{2t}}{2}\right) = 4s\Phi'(s)\left(\tau(\Psi(s) - s\Psi'(s)) - \frac{s}{2}\right) \le 0$$

by Lemma 6.3. Then $(k(t)-2)^2 + 2\ell(t)$ is decreasing for $t > -\infty$. We note that $\lim_{t\to-\infty} k(t) = \Phi(0) = 0$ and $\lim_{t\to-\infty} \ell(t) = \lim_{s\to 0} 2s\Phi'(s) = \lim_{s\to 0} s\phi(\sqrt{s}) = 0$. Then we have

$$(k(t)-2)^2 + 2\ell(t) < 4 \quad \text{for } t > -\infty.$$

Since $\ell(t) = 2s\Phi'(s) = s\phi(\sqrt{s}) > 0$ and $\lim_{t\to\infty}((k(t)-2)^2 + 2\ell(t)) < 4$, we obtain $\lim_{t\to\infty} k(t) < 4$. Thus $\lim_{s\to\infty} \Phi(s) < 4$, which implies $\|\phi\|_{L^1(\mathbb{R}^2)} < 8\pi$. □

7. Proof of Theorem 2

By Theorem 3 it is shown that $(\phi, \psi) \in \mathcal{S}$ if and only if $\psi = \psi(r)$, $r = |y|$, solves $(4.1)_\sigma$ for some $\sigma > 0$ and ϕ is given by (1.7). By Proposition 4.1 the set $\mathcal{C}$ defined by (4.2) is written by one parameter families $(\sigma(s), \psi(r; s))$ on $s \in \mathbb{R}$. Let

$$\phi(r; s) = \sigma(s)e^{-r^2/4}e^{\psi(r;s)}. \tag{7.1}$$

Then $\mathcal{S}$ is written by one parameter families $(\phi(r,s), \psi(r,s))$ on $s \in \mathbb{R}$. From (i) and (ii) of Proposition 4.1 and (7.1) we have $s \mapsto (\phi(\cdot; s), \psi(\cdot; s)) \in C^2[0,\infty) \times C^2[0,\infty)$ is continuous and $(\phi(\cdot; s), \psi(\cdot; s)) \to (0,0)$ in $C^2[0,\infty) \times C^2[0,\infty)$ as $s \to -\infty$. We see that

$$\lambda(s) = 2\pi \int_0^\infty r\phi(r; s)dr. \tag{7.2}$$

Then $\lambda(s)$ is continuous and satisfies $\lambda(s) \to 0$ as $s \to -\infty$. Hence, (i) and (ii) holds. By Proposition 6.1 we obtain (iv).

We have $\|\psi(\cdot,s)\|_{L^\infty[0,\infty)} = \psi(0,s) \to \infty$ as $s \to \infty$ from (iii) of Proposition 4.1. Let $\{s_k\}$ be a sequence satisfying $s_k \to \infty$ as $k \to \infty$. We note that $\{\lambda_k\}$ is bounded by Proposition 6.1. By applying Proposition 5.1, there exists a subsequence (still denoted by $\{s_k\}$) such that $\lambda(s_k) \to 8\pi$ and $\phi_k(|x|, s_k)dx \rightharpoonup 8\pi\delta_0(dx)$ as $k \to \infty$. Therefore, (iii) holds. This completes the proof of Theorem 2. □

Appendix A. Existence of solutions to (1.8) with (1.9)

The following theorem refines the previous results [20, Theorem 1], [21, Theorems 1 and 2], and [22, Theorem 1.1].

Theorem A.1. *For any $\tau > 0$ there exists $\sigma^* > 0$ such that*
(i) *if $\sigma > \sigma^*$, then* (1.8) *with* (1.9) *has no solution;*
(ii) *if $\sigma = \sigma^*$, then* (1.8) *with* (1.9) *has at least one solution;*
(iii) *if $0 < \sigma < \sigma^*$, then* (1.8) *with* (1.9) *has at least two distinct solutions $\underline{\psi}_\sigma$, $\overline{\psi}_\sigma$ satisfying $\lim_{\sigma\to 0} \underline{\psi}_\sigma(0) = 0$ and $\lim_{\sigma\to 0} \overline{\psi}_\sigma(0) = \infty$.*

Proof. By Theorem 1 the problem (1.8) with (1.9) is reduced to the problem $(4.1)_\sigma$. By Proposition 4.1 the set $\mathcal{C}$ defined by (4.2) is written by one parameter families $(\sigma(s), \psi(r;s))$ on $s \in \mathbb{R}$. From (7.1) and (7.2) we find that

$$\sigma(s) = \lambda(s) \Big/ \left(2\pi \int_0^\infty r e^{-r^2/4} e^{\psi(r;s)} dr\right) = \lambda(s) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(|y|;s)} dy$$

From (5.13) in Lemma 5.4 we have

$$\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(|y|;s)} dy \to \infty \quad \text{as } s \to \infty.$$

Then $\sigma(s) \to 0$ as $s \to \infty$. Therefore, from (ii) of Proposition 4.1, $\sigma(s)$ satisfies

$$\lim_{s\to\pm\infty} \sigma(s) = 0.$$

Let $\sigma^* = \sup_{s\in\mathbb{R}} \sigma(s)$. Then there exists $s^* \in \mathbb{R}$ such that $\sigma^* = \sigma(s^*)$.

By Proposition 4.1 it is shown that $(4.1)_\sigma$ has a solution if and only if $\sigma = \sigma(s)$ for some $s \in \mathbb{R}$. Therefore, $(4.1)_\sigma$ has no solution, if $\sigma > \sigma^*$, and $(4.1)_\sigma$ has at least one solution, if $\sigma = \sigma^*$. If $\sigma \in (0, \sigma^*)$, by the mean value theorem, there exists s_1, $s_2 \in \mathbb{R}$, $s_1 < s^* < s_2$ such that $\sigma = \sigma(s_1) = \sigma(s_2)$. Then $(4.1)_\sigma$ has at least two solutions $\psi_{\sigma(s_1)}$ and $\psi_{\sigma(s_2)}$. We note that $\lim_{s\to-\infty} \psi_{\sigma(s)}(0) = 0$ and $\lim_{s\to\infty} \psi_{\sigma(s)}(0) = \infty$ by (ii) and (iii) of Proposition 4.1. Since $\lim_{s\to\pm\infty} \sigma(s) = 0$, we can choose solutions $\overline{\psi}_\sigma$ and $\underline{\psi}_\sigma$ satisfying $\lim_{\sigma\to 0} \underline{\psi}_\sigma(0) = 0$ and $\lim_{\sigma\to 0} \overline{\psi}_\sigma(0) = \infty$. This completes the proof of Theorem A.1. □

Appendix B. Proof of Theorem 4

Define $h_k \in C^2(\Omega) \cap C(\overline{\Omega})$ by

$$\Delta h_k = 0 \quad \text{in } \Omega \quad \text{and} \quad h_k = u_k \quad \text{on } \partial\Omega.$$

We may assume that $\{0\} \in \Omega$ with no loss of generality.

Lemma B.1. *Let $r > 0$ satisfying $\overline{B}_r \subset \Omega$. Then $\|\nabla h_k\|_{L^\infty(B_r)} = O(1)$ as $k \to \infty$.*

Proof. By the maximum principle, we have $\max_{\overline{\Omega}} h_k - \min_{\overline{\Omega}} h_k \le \max_{\partial\Omega} h_k - \min_{\partial\Omega} h_k$. Then from (1.15) we obtain

$$\max_{\overline{\Omega}} h_k - \min_{\overline{\Omega}} h_k \le C_2$$

with a positive constant C_2. Let $\widetilde{h}_k(x) = h_k(x) - \min_{\overline{\Omega}} h_k$. Then $\widetilde{h}_k$ satisfies

$$\Delta \widetilde{h}_k = 0 \quad \text{in } \Omega, \qquad 0 \le \widetilde{h}_k \le C_2.$$

Since $\partial \widetilde{h}_k / \partial x_i$, $i = 1, 2$, is harmonic, by the mean value theorem and Gauss-Green Theorem, we obtain

$$\frac{\partial \widetilde{h}_k}{\partial x_i} = \frac{1}{\pi r^2} \int_{B_r} \frac{\partial \widetilde{h}_k}{\partial x_i} dx = \frac{1}{\pi r^2} \int_{\partial B_r} \widetilde{h}_k n_i ds$$

for $i = 1, 2$, where $n = (n_1, n_2)$ is the outer normal unit vector on ∂B_r. Then it follows that

$$\left| \frac{\partial \widetilde{h}_k}{\partial x_i} \right| \le \frac{1}{\pi r^2} \int_{\partial B_r} |\widetilde{h}_k| ds \le \frac{2C_1}{r}, \quad i = 1, 2.$$

Since $|\nabla h_k| = |\nabla \widetilde{h}_k|$, we conclude that $\|\nabla h_k\|_{L^\infty(B_r)} = O(1)$ as $k \to \infty$. □

Let $w_k(x) = u_k(x) - h_k(x)$ in Ω. Then

$$-\Delta w_k = W_k(x) e^{w_k} \quad \text{in } \Omega, \qquad w_k = 0 \quad \text{on } \partial\Omega,$$

where $W_k(x) = e^{h_k(x)} V_k(x)$. Let $G(x, y)$ be the Green's function of $-\Delta$ in Ω with respect to the zero boundary conditions:

$$-\Delta_x G(x, y) = \delta_y, \quad x \in \Omega, \qquad G(x, y) = 0, \quad x \in \partial\Omega.$$

Then we have

$$\nabla w_k(x) = \int_\Omega \nabla_x G(x, y) W_k(y) e^{w_k(y)} dy, \quad x \in \Omega. \tag{B.1}$$

Put $z_k(x) = W_k(x) e^{w_k(x)}$.

Lemma B.2. *For $\psi \in C_0^2(\Omega)$ we have*

$$-\int_\Omega (\Delta\psi) z_k dx = \int_\Omega \left(\nabla(\log W_k)\cdot\nabla\psi\right) z_k dx + \frac{1}{2}\iint_{\Omega\times\Omega} \rho(x,y) z_k(x) z_k(y) dx dy, \tag{B.2}$$

where $\rho(x,y) = \nabla_x G(x,y)\cdot\nabla\psi(x) + \nabla_y G(x,y)\cdot\nabla\psi(y)$.

Proof. We see that

$$\nabla z_k = (\nabla W_k) e^{w_k} + W_k e^{w_k}\nabla w_k = z_k \nabla(\log W_k) + z_k \nabla w_k.$$

Then, for $\psi \in C_0^2(\Omega)$, we obtain

$$-\int_\Omega (\Delta\psi) z_k dx = \int_\Omega \left(\nabla(\log W_k)\cdot\nabla\psi\right) z_k dx + \int_\Omega \left(\nabla w_k \cdot \nabla\psi\right) z_k dx. \tag{B.3}$$

From (B.1) and Fubini's Theorem, we find that

$$\int_\Omega \left(\nabla w_k(x)\cdot\nabla\psi(x)\right) z_k(x) dx = \iint_{\Omega\times\Omega} \left(\nabla_x G(x,y)\cdot\nabla\psi(x)\right) z_k(x) z_k(y) dx dy. \tag{B.4}$$

By changing the role of x and y in (B.4) we obtain

$$\int_\Omega \left(\nabla w_k(y)\cdot\nabla\psi(y)\right) z_k(y) dy = \iint_{\Omega\times\Omega} \left(\nabla_y G(x,y)\cdot\nabla\psi(y)\right) z_k(x) z_k(y) dx dy.$$

Hence, we obtain

$$\int_\Omega \left(\nabla w_k \cdot\nabla\psi\right) z_k dx = \frac{1}{2}\iint_{\Omega\times\Omega} \rho(x,y) z_k(x) z_k(y) dx dy.$$

From (B.3) we obtain (B.2). □

Without loss of generality, we may assume that the blowup set $\mathcal{B}$ contains $\{0\}$, and that there exists a $R > 0$ satisfying $\{x : 0 < |x| < R\} \cap \mathcal{B} = \emptyset$. Therefore, $\{u_k\}$ satisfies

$$\max_{\overline{B}_R} u_k \to \infty \quad \text{and} \quad \max_{\overline{B}_R \setminus B_r} u_k \to -\infty \quad \text{as } k \to \infty \tag{B.5}$$

for all $r \in (0, R)$. Moreover,

$$V_k e^{u_k} dx \rightharpoonup \alpha\delta_0(dx) \tag{B.6}$$

on B_R in the sense of measure for some $\alpha \geq 4\pi$.

Lemma B.3. *There exist constants $r_0 \in (0, R)$ and $a > 0$ such that $V_k(x) \geq a$ for $x \in B_{r_0}$.*

Proof. First we show $\liminf_{k\to\infty} V_k(0) > 0$. Assume to the contrary that

$$\liminf_{k\to\infty} V_k(0) = 0.$$

From (1.12) and (1.14), by taking a subsequence in $\{V_k\}$ (still denoted by $\{V_k\}$), there exists $V_0 \in C(\Omega)$ such that $V_k \to V_0$ in $C(\overline{B_R})$ and $V_0(0) = 0$.

Let $x_k \in B_R$, $u_k(x_k) = \max_{x\in\overline{B}_R} u_k(x)$. It follows from (B.5) that

$$x_k \to 0 \quad \text{and} \quad u_k(x_k) \to \infty. \tag{B.7}$$

Let $\delta_k = e^{-u_k(x_k)/2}$. It follows from (B.7) that $\delta_k \to 0$. For $|x| \le R/(2\delta_k)$, we consider the sequence of functions $v_k(x) = u_k(\delta_k x + x_k) + 2\log\delta_k$. Then v_k satisfies

$$-\Delta v_k(x) = V_k(\delta_k x + x_k)e^{v_k(x)} \quad \text{for } x \in B_{R/(2\delta_k)}.$$

Moreover, we have $v_k(0) = 0$, $v_k(x) \le 0$ in $B_{R/(2\delta_k)}$, and

$$\int_{B_{R/(2\delta_k)}} e^{v_k(x)}dx \le \int_{B_R} e^{u_k(x)}dx \le C$$

for some positive constant C.

For each $r > 0$ the sequence $\{v_k\}$ is well defined in B_r for k large enough. It follows from Theorem A that only alternative (i) may occur, hence $\{v_k\}$ is bounded in $L^\infty_{\rm loc}(B_r)$ and, by standard elliptic estimates, also in $C^{2,\alpha}_{\rm loc}(B_r)$, $0 < \alpha < 1$. Therefore, a subsequence in $\{v_k\}$ converges in $C^2_{\rm loc}(B_r)$. We may do the same arguments for a sequence $r_k \to \infty$, and pass to a diagonal subsequence (which we will still denote as $\{v_k\}$) converging in $C^2_{\rm loc}(\mathbb{R}^2)$ to v which satisfies $-\Delta v = V_0(0)e^v$ in $\mathbb{R}^2$. Moreover, $v(0) = 0$, $v \le 0$ in $\mathbb{R}^2$, and

$$\int_{\mathbb{R}^2} e^v dx \le C. \tag{B.8}$$

Since $V_0(0) = 0$, v is harmonic in $\mathbb{R}^2$. Then v is a constant. This contradicts (B.8). Thus we conclude that $\liminf_{k\to\infty} V_k(0) > 0$.

From (1.14) there exists constants $r_0 \in (0, R)$ and $a > 0$ satisfying $V_k(x) \ge a$ for $x \in B_{r_0}$. □

Proof of Theorem 4. We will show that $\alpha = 8\pi$ in (B.6). Take $\phi \in C^2_0(B_R)$ so that $0 \le \phi \le 1$ and $\phi \equiv 1$ for $x \in B_{r_0}$, where r_0 is a constant in Lemma B.3. Let $\psi(x) = |x|^2\phi(x)$. Then we have $\psi \in C^2_0(B_R)$. Moreover, it follows that $\Delta\psi(x) = 4$ and $\nabla\psi(x) = 2x$ for $x \in B_{r_0}$.

We recall that $W_k(x) = e^{h_k(x)}V_k(x)$. Then we have

$$\nabla(\log W_k) = \frac{\nabla W_k}{W_k} = \nabla h_k + \frac{\nabla V_k}{V_k}.$$

From Lemmas B.1 and B.3 and (1.12) we obtain $|\nabla \log W_k(x)| \le C$ for $x \in B_{r_0}$ with some constant C. Then we have

$$|\nabla\psi(x)\cdot\nabla(\log W_k(x))| \le 2C|x| \quad \text{for } x \in B_{r_0}. \tag{B.9}$$

We see that $G(x,y) = -(1/2\pi)\log|x-y| + K(x,y)$, where $K(x,y)$ is a smooth function on $\overline{\Omega} \times \Omega$. Then $\rho(x,y)$ defined in Lemma B.2 satisfies

$$\rho(x,y) = -\frac{1}{\pi} + 2x \cdot \nabla_x K(x,y) + 2y \cdot \nabla_y K(x,y) \quad \text{for } x \in B_{r_0}. \tag{B.10}$$

We see that $z_k(x) = W_k(x)e^{w_k(x)} = V_k(x)e^{v_k(x)}$. From (B.6) we have $z_k(x)dx \to \alpha\delta_0(dx)$ on B_R in the sense of measure. Furthermore, we have

$$z_k(x)z_k(y)dxdy \to \alpha^2\delta_{x=0}(dx) \otimes \delta_{y=0}(dy) = \alpha^2\delta_{(x,y)=(0,0)}(dxdy)$$

on B_R in the sense of measure. Letting $k \to \infty$ in (B.2), from (B.9) and (B.10), we have $-4\pi\alpha = -\alpha^2/(2\pi)$. From $\alpha \geq 4\pi$, we obtain $\alpha = 8\pi$. This completes the proof of Theorem 4. □

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