# PRECISE SPECTRAL ASYMPTOTICS FOR NONLINEAR STURM-LIOUVILLE PROBLEMS

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**Abstract**

We consider the nonlinear Sturm-Liouville problem

$$-u''(t) + u(t)^p = \lambda u(t), \quad u(t) > 0, \quad t \in I := (0,1), \quad u(0) = u(1) = 0,$$

where $p > 1$ is a constant and $\lambda > 0$ is an eigenvalue parameter. To understand the global structure of the bifurcation diagram in $\mathbf{R}_+ \times L^2(I)$ completely, we establish the *asymptotic expansion* of $\lambda(\alpha)$ (associated with eigenfunction $u_\alpha$ with $\|u_\alpha\|_2 = \alpha$) as $\alpha \to \infty$. We also obtain the corresponding asymptotics of the width of the boundary layer of $u_\alpha$ as $\alpha \to \infty$.

Proposed running head: precise spectral asymptotics

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# 1 Introduction

We consider the following nonlinear Sturm-Liouville problem

$$-u''(t) + u(t)^p = \lambda u(t), \quad t \in I := (0,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(0) = u(1) = 0, \tag{1.3}$$

where $p > 1$ is a constant and $\lambda > 0$ is an eigenvalue parameter. It is known by Berestycki [1] and Fraile *et al.* [7] that for each $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{I})$ with $\|u_\alpha\|_2 = \alpha$. The set $\{(\lambda(\alpha), u_\alpha), \alpha > 0\}$ gives all solutions of (1.1)–(1.3) and is an unbounded curve of class $C^1$ in $\mathbf{R}_+ \times L^2(I)$ emanating from $(\pi^2, 0)$.

The purpose of this paper is to understand the global structure of this bifurcation diagram in $\mathbf{R}_+ \times L^2(I)$ completely. To this end, we establish the *asymptotic expansion* of $\lambda(\alpha)$ as $\alpha \to \infty$. We also establish the corresponding asymptotics of the width of the boundary layer of $u_\alpha$ as $\alpha \to \infty$.

The equation (1.1)–(1.3) has been extensively investigated by many authors in $L^\infty$-framework from a viewpoint of local and global bifurcation theory. We refer to Berestycki [1], Fraile *et al.* [7], Holzmann and Kielhöfer [11], Rabinowitz [12], [13] and the references therein for the works in this direction. On the other hand, since (1.1)–(1.3) is regarded as an eigenvalue problem, it is significant to investigate (1.1)–(1.3) in $L^2$-framework. For the works in this direction, we refer to Bongers *et al.* [2], Chabrowski [3], Chiappinelli [4], [5], [6], Heinz [8], [9], [10], Shibata [14] and the references therein. In particular, Chiappinelli [4], [5] obtained the asymptotic formula for $\lambda(\alpha)$ as $\alpha \to 0$. On the other hand, in Shibata [14], the following asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ has been given: There exists a constant $C > 0$ such that for $\alpha \gg 1$,

$$C^{-1}\alpha^{(p-1)/2} \le \lambda(\alpha) - \alpha^{p-1} \le C\alpha^{(p-1)/2}. \tag{1.4}$$

(1.4) gives the optimal estimate for the second term of $\lambda(\alpha)$ as $\alpha \to \infty$. However, the exact second term was not obtained. Our main aim in this paper is to improve the formula (1.4) to gain a complete picture of the bifurcation diagram in $\mathbf{R}_+ \times L^2(I)$.

Now we state our results. Let $\mathbf{N}_0 := \{0, 1, 2, \cdots\}$.

THEOREM 1. *For any $n \in \mathbf{N}_0$, the following asymptotic formula holds as $\alpha \to \infty$:*

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2} \alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}), \tag{1.5}$$

*where*

$$C_1 = (p+3)\int_I \sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}\,ds \tag{1.6}$$

*and $a_k(p)$ is a polynomial (deg $a_k(p) \le k+1$) which is determined by $a_0, a_1, \cdots, a_{k-1}$.*

For example,

$$a_0(p) = 1, \quad a_1(p) = \frac{(5-p)(9-p)}{24}, \quad a_2(p) = \frac{(3-p)(5-p)(7-p)}{24}.$$

The following theorem gives the asymptotic formula for the boundary layer of $u_\alpha$ as $\alpha \to \infty$.

THEOREM 2. *For any $n \in \mathbf{N}_0$, the following formula holds as $\alpha \to \infty$:*

$$\begin{aligned} u_\alpha'(0)^2 = u_\alpha'(1)^2 \;&=\; \frac{p-1}{p+1}\alpha^{p+1} + C_1\alpha^{(p+3)/2} + \sum_{k=0}^{n} \frac{2A_k(p)}{(p-1)^{k+1}} C_1^{k+2}\alpha^{2+k(1-p)/2} \\ &\quad + o(\alpha^{2+n(1-p)/2}), \end{aligned} \tag{1.7}$$

*where $A_k(p)$ is a polynomial (deg $A_k(p) \le k+1$) which is determined by $a_0, a_1, \cdots, a_{k-1}$.*

For example,

$$A_0(p) = 1, \quad A_1(p) = \frac{(9-p)(13-p)}{48}, \quad A_2(p) = \frac{(5-p)(7-p)(9-p)}{48}.$$

The following theorem gives the relationship between $\|u_\alpha\|_2$ and $\|u_\alpha\|_\infty$ for $\alpha \gg 1$.

THEOREM 3. *For any $n \in \mathbf{N}_0$, the following formula holds as $\alpha \to \infty$:*

$$\|u_\alpha\|_\infty^{p-1} = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2}\alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}). \tag{1.8}$$

As a corollary of Theorems 1–3, we obtain the analogous results for the nonlinear Sturm-Liouville problem

$$-u''(t) + |u(t)|^{p-1}u(t) = \lambda u(t), \quad t \in I := (0,1), \tag{1.9}$$

$$u(0) = u(1) = 0. \tag{1.10}$$

For (1.9)–(1.10), it is known that for each $\alpha > 0$, there exists a unique solution $(\lambda, u) = (\lambda(m,\alpha), u_{m,\alpha}) \in \mathbf{R}_+ \times C^2(\bar{I})$ $(m \in \mathbf{N})$ such that $u_{m,\alpha}$ has exactly $m-1$ interior simple zeros in $I$, $u_{m,\alpha} > 0$ near 0 and $\|u_{m,\alpha}\|_2 = \alpha$ (cf. [1, 7]). Moreover, the set $\{(\lambda(m,\alpha), \pm u_{m,\alpha}), \alpha > 0, m \in \mathbf{N}\}$ gives all solutions of (1.9)–(1.10). $\{(\lambda(m,\alpha), u_{m,\alpha}), \alpha > 0\}$ is called the m-th branch of nodal solutions of (1.9)–(1.10) and is an unbounded curve of class $C^1$ in $\mathbf{R}_+ \times L^2(I)$ emanating from $((m\pi)^2, 0)$. Then clearly, $(\lambda(1,\alpha), u_{1,\alpha}) = (\lambda(\alpha), u_\alpha)$ and it is seen from an easy symmetry argument (cf. [10, p. 313]) that the interior zeroes of $u_{m,\alpha}$ $(m \geq 2)$ are exactly $\{1/m, \cdots, (m-1)/m\}$. Therefore, the restriction of $u_{m,\alpha}$ to $[0, 1/m]$ corresponds to a positive solution (with a different eigenvalue) on $I$ via a dilation. We then obtain the explicit correspondence between $\lambda(\alpha)$ and $\lambda(m,\alpha)$ and obtain the analogous asymptotic formulas for all the branches of (1.9)–(1.10).

Corollary 4. *Let $m \in \mathbf{N}$ be fixed. Then for any $n \in \mathbf{N}_0$, the following asymptotic formulas hold as $\alpha \to \infty$:*

$$\lambda(m,\alpha) = \alpha^{p-1} + mC_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}}(mC_1)^{k+2}\alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}), \tag{1.11}$$

$$u'_{m,\alpha}(0)^2 = u'_{m,\alpha}(1)^2 = \frac{p-1}{p+1}\alpha^{p+1} + mC_1\alpha^{(p+3)/2} + \sum_{k=0}^{n} \frac{2A_k(p)}{(p-1)^{k+1}}(mC_1)^{k+2}\alpha^{2+k(1-p)/2} + o(\alpha^{2+n(1-p)/2}), \tag{1.12}$$

$$\|u_{m,\alpha}\|_\infty^{p-1} = \alpha^{p-1} + mC_1\alpha^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}}(mC_1)^{k+2}\alpha^{k(1-p)/2} + o(\alpha^{n(1-p)/2}). \tag{1.13}$$

The remainder of this paper is organized as follows. In Section 2, we establish the second term of the asymptotics of $\lambda(\alpha)$. In Section 3, we establish the third and fourth terms of the asymptotics of $\lambda(\alpha)$. This step is needed to use the mathematical induction in Section 4. In Section 4, we prove Theorems 1–3 by using the mathematical induction and the arguments developed in Sections 2 and 3. We also give the proof of Corollary 4 at the end of Section 4.

## 2 Second term of $\lambda(\alpha)$

We begin with notations and the fundamental properties of $\lambda(\alpha)$ and $u_\alpha$. Let $\|\cdot\|_q$ $(q \ge 1, \infty)$ be the usual $L^q$-norm. $C_k$ $(k = 2, 3, \cdots)$ denotes positive constants independent of $\alpha \gg 1$. Let

$$\lambda_2(\alpha) := \lambda(\alpha) - \alpha^{p-1}, \tag{2.1}$$

$$\gamma(\alpha) := \|u'_\alpha\|_2^2 + \frac{2}{p+1}\|u_\alpha\|_{p+1}^{p+1}, \tag{2.2}$$

$$\gamma_2(\alpha) := \gamma(\alpha) - \frac{2}{p+1}\alpha^{p+1}. \tag{2.3}$$

It is known by Berestycki [1] and Fraile *et al.* [7] that (1.1)–(1.3) has a unique solution $u_\lambda$ for a given $\lambda > \pi^2$,

$$\lim_{\lambda\to\infty} \frac{u_\lambda}{\lambda^{1/(p-1)}} = 1 \tag{2.4}$$

uniformly on compact subsets on $I$. Moreover, the mapping $\lambda \mapsto u_\lambda \in C^2(\bar{I})$ is strictly increasing (i.e., $du_\lambda/d\lambda > 0$ in $I$) and $C^1$ for $\lambda > \pi^2$ (cf. [7, p. 203]). Therefore, we see that $\alpha(\lambda) = \|u_\lambda\|_2$ is $C^1$ and strictly increasing, namely, $d\alpha(\lambda)/\lambda > 0$ for $\lambda > \pi^2$. Therefore, $\lambda(\alpha)$, the inverse of $\alpha(\lambda)$, is also $C^1$ and $d\lambda(\alpha)/d\alpha > 0$ for $\alpha > 0$.

LEMMA 2.1. $\|u'_\alpha\|_2^2 = 2C_1(1 + o(1))\alpha^{(p+3)/2}/(p+3)$ *for* $\alpha \gg 1$.

*Proof.* Since (1.1) is autonomous, we know that $u_\alpha$ satisfies

$$u_\alpha(t) = u_\alpha(1-t), \quad 0 \le t \le 1, \tag{2.5}$$

$$u_\alpha\left(\frac{1}{2}\right) = \max_{0\le t\le 1} u_\alpha(t) = \|u_\alpha\|_\infty, \tag{2.6}$$

$$u'_\alpha(t) > 0, \quad 0 \le t < \frac{1}{2}. \tag{2.7}$$

Since it follows from (1.1) that

$$\frac{d}{dt}\left[\frac{1}{2}u_\alpha'(t)^2 - \frac{1}{p+1}u_\alpha(t)^{p+1} + \frac{1}{2}\lambda u_\alpha(t)^2\right] = 0 \quad \text{for } 0 \le t \le 1,$$

the expression between brackets is constant in $[0,1]$, and taking $t = 0, 1/2$, for $0 \le t \le 1$, we obtain

$$\begin{aligned}\frac{1}{2}u_\alpha'(0)^2 &= \frac{1}{2}u_\alpha'(t)^2 - \frac{1}{p+1}u_\alpha(t)^{p+1} + \frac{1}{2}\lambda u_\alpha(t)^2 \\ &= -\frac{1}{p+1}\|u_\alpha\|_\infty^{p+1} + \frac{1}{2}\lambda\|u_\alpha\|_\infty^2.\end{aligned} \tag{2.8}$$

This along with (2.7) implies that for $0 \le t \le 1/2$

$$u_\alpha'(t) = \sqrt{\lambda(\alpha)(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2) - \frac{2}{p+1}(\|u_\alpha\|_\infty^{p+1} - u_\alpha(t)^{p+1})}. \tag{2.9}$$

Then by (2.5), (2.7), (2.9) and putting $s = u_\alpha(t)/\|u_\alpha\|_\infty$, we obtain

$$\begin{aligned}\|u_\alpha'\|_2^2 &= 2\int_0^{1/2}\sqrt{\lambda(\alpha)(\|u_\alpha\|_\infty^2 - u_\alpha(t)^2) - \frac{2}{p+1}(\|u_\alpha\|_\infty^{p+1} - u_\alpha(t)^{p+1}})u_\alpha'(t)dt \\ &= 2\|u_\alpha\|_\infty\int_0^1\sqrt{\lambda(\alpha)\|u_\alpha\|_\infty^2(1-s^2) - \frac{2}{p+1}\|u_\alpha\|_\infty^{p+1}(1-s^{p+1})}ds.\end{aligned} \tag{2.10}$$

By (1.4) and (2.4), we see that $\|u_\alpha\|_\infty = \lambda(\alpha)^{1/(p-1)}(1+o(1))$ for $\alpha \gg 1$. By this, for $\alpha \gg 1$ and $0 \le s \le 1$, we obtain

$$\frac{2\|u_\alpha\|_\infty}{\lambda(\alpha)^{(p+3)/(2(p-1))}}\sqrt{\lambda(\alpha)\|u_\alpha\|_\infty^2(1-s^2) - \frac{2}{p+1}\|u_\alpha\|_\infty^{p+1}(1-s^{p+1})} \le C_2.$$

By this, (2.10) and Lebesgue's convergence theorem, we obtain

$$\lim_{\alpha\to\infty}\frac{\|u_\alpha'\|_2^2}{\lambda(\alpha)^{(p+3)/(2(p-1))}} = 2\int_0^1\sqrt{\frac{p-1}{p+1} - s^2 + \frac{2}{p+1}s^{p+1}}ds = \frac{2C_1}{p+3}.$$

This along with (1.4) implies our assertion. Thus the proof is complete. ∎

LEMMA 2.2. *$d\gamma_2(\alpha)/d\alpha = 2\alpha\lambda_2(\alpha)$ for all $\alpha > 0$.*

*Proof.* By (1.1) and (2.2), we obtain

$$\begin{aligned}\frac{d\gamma(\alpha)}{d\alpha} &= 2\int_I u_\alpha'(t)\frac{du_\alpha'(t)}{d\alpha}dt + 2\int_I u_\alpha(t)^p\frac{du_\alpha(t)}{d\alpha}dt \\ &= 2\int_I\{-u_\alpha''(t) + u_\alpha(t)^p\}\frac{du_\alpha(t)}{d\alpha}dt \\ &= 2\lambda(\alpha)\int_I u_\alpha(t)\frac{du_\alpha(t)}{d\alpha}dt = \lambda(\alpha)\frac{d}{d\alpha}\int_I u_\alpha(t)^2dt \\ &= 2\alpha\lambda(\alpha).\end{aligned} \tag{2.11}$$

By this, (2.1) and (2.3), we obtain our assertion. ∎

LEMMA 2.3. *$\lambda_2(\alpha) = C_1(1+o(1))\alpha^{(p-1)/2}$ for $\alpha \gg 1$.*

*Proof.* Multiply $u_\alpha$ by (1.1). Then integration by parts yields

$$\|u_\alpha'\|_2^2 + \|u_\alpha\|_{p+1}^{p+1} = \lambda(\alpha)\alpha^2. \tag{2.12}$$

By this and (2.2), we obtain

$$\gamma(\alpha) - \frac{2}{p+1}\lambda(\alpha)\alpha^2 = \frac{p-1}{p+1}\|u_\alpha'\|_2^2.$$

By substituting (2.1) and (2.3) for this, we obtain

$$\gamma_2(\alpha) - \frac{2}{p+1}\lambda_2(\alpha)\alpha^2 = \frac{p-1}{p+1}\|u_\alpha'\|_2^2. \tag{2.13}$$

Then by this and Lemma 2.2, we find that $\gamma_2$ satisfies the differential equation

$$\gamma_2'(\alpha) - \frac{p+1}{\alpha}\gamma_2(\alpha) = h(\alpha) := -\frac{(p-1)\|u_\alpha'\|_2^2}{\alpha}. \tag{2.14}$$

We see from Lemma 2.1 that $|h(s)|/s^{p+1} \le C_3 s^{-(p+1)/2}$ for $s \gg 1$. This implies that for $\alpha \gg 1$

$$\int_\alpha^\infty \frac{|h(s)|}{s^{p+1}}ds \le \frac{2}{p-1}C_3\alpha^{(1-p)/2}. \tag{2.15}$$

Then we see that the solution $\gamma_2$ of (2.14) is represented as

$$\gamma_2(\alpha) = I_1(\alpha) + I_2(\alpha) := \alpha^{p+1}\int_\alpha^\infty \frac{-h(s)}{s^{p+1}}ds + C_4\alpha^{p+1}. \tag{2.16}$$

We see from (2.15) that $I_1(\alpha) = o(\alpha^{p+1})$ for $\alpha \gg 1$. By (1.4) and (2.4), we obtain $\|u_\alpha\|_{p+1}^{p+1} = (1+o(1))\alpha^{p+1}$ for $\alpha \gg 1$. Therefore, by (2.2) and Lemma 2.1, for $\alpha \gg 1$, we obtain

$$\gamma(\alpha) = \frac{2}{p+1}\alpha^{p+1} + o(\alpha^{p+1}). \tag{2.17}$$

By this and (2.3), we see that $\gamma_2(\alpha) = o(\alpha^{p+1})$. Therefore, we find that $C_4 = 0$, and consequently, by (2.16), we obtain

$$\gamma_2(\alpha) = \alpha^{p+1}\int_\alpha^\infty \frac{-h(s)}{s^{p+1}}ds. \tag{2.18}$$

Then by this, Lemma 2.1 and l'Hopital's rule, we obtain

$$\lim_{\alpha\to\infty} \frac{\int_\alpha^\infty -h(s)/s^{p+1}ds}{\alpha^{(1-p)/2}} = \lim_{\alpha\to\infty} \frac{2\|u'_\alpha\|_2^2}{\alpha^{(p+3)/2}} = \frac{4}{p+3}C_1.$$

This along with (2.18) implies that for $\alpha \gg 1$

$$\gamma_2(\alpha) = \frac{4C_1}{p+3}\alpha^{(p+3)/2} + o(\alpha^{(p+3)/2}). \tag{2.19}$$

Now, by (2.13) and Lemma 2.1, for $\alpha \gg 1$, we obtain

$$\lambda_2(\alpha)\alpha^2 = \frac{p+1}{2}\gamma_2(\alpha) - \frac{p-1}{2}\|u'_\alpha\|_2^2 = (1+o(1))C_1\alpha^{(p+3)/2}. \tag{2.20}$$

This implies our conclusion. Thus the proof is complete. ∎

# 3 The third and fourth terms of $\lambda(\alpha)$

Taking (2.1), (2.3), (2.19) and Lemma 2.3 into account, we put

$$\lambda_3(\alpha) := \lambda(\alpha) - \alpha^{p-1} - C_1\alpha^{(p-1)/2}, \tag{3.1}$$

$$\gamma_3(\alpha) := \gamma(\alpha) - \frac{2}{p+1}\alpha^{p+1} - \frac{4C_1}{p+3}\alpha^{(p+3)/2}. \tag{3.2}$$

Then by (2.11), (3.1) and (3.2), we have

$$\frac{d\gamma_3(\alpha)}{d\alpha} = 2\alpha\lambda_3(\alpha). \tag{3.3}$$

The following estimate for $\|u_\alpha\|_\infty$ enables us to repeat the arguments in the previous section.

LEMMA 3.1. *For* $\alpha \gg 1$

$$(\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{1/(p-1)} \le \|u_\alpha\|_\infty < \lambda(\alpha)^{1/(p-1)}. \tag{3.4}$$

*Proof.* Since the second inequality is known by Berestycki [1], we have only to prove the first inequality. We put $v_\alpha(t) := \lambda(\alpha)^{-1/(p-1)}u_\alpha(t+1/2)$ and $w_\alpha := 1 - v_\alpha$. By (1.1) and (2.6), we see that $w_\alpha$ satisfies:

$$\begin{aligned} w''_\alpha(t) &= \lambda(\alpha)(1 - w_\alpha(t) - (1-w_\alpha(t))^p), \quad t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ w_\alpha(0) &= 1 - \|v_\alpha\|_\infty, \\ w'_\alpha(0) &= 0. \end{aligned}$$

Since $0 < v_\alpha < 1$ in $I$ by the second inequality of (3.4), we have $0 < w_\alpha < 1$ in $(-1/2, 1/2)$. By (2.4), we see that $v_\alpha \to 1$ and $w_\alpha \to 0$ uniformly on $I_\delta := [-\delta, \delta]$ as $\alpha \to \infty$, where $0 < \delta \ll 1$ is a fixed constant. Therefore, for a fixed constant $0 < \epsilon \ll p-1$, we obtain by Taylor expansion that for $\alpha \gg 1$

$$\begin{aligned} \lambda(\alpha)(p-1-\epsilon)w_\alpha &< w_\alpha''(t) < \lambda(\alpha)(p-1+\epsilon)w_\alpha, \quad t \in I_\delta, \\ w_\alpha(0) &= 1 - \|v_\alpha\|_\infty, \\ w_\alpha'(0) &= 0. \end{aligned}$$

Since $W_\pm(\alpha, t) := (1/2)(1 - \|v_\alpha\|_\infty)(e^{\sqrt{(p-1\pm\epsilon)\lambda(\alpha)}t} + e^{-\sqrt{(p-1\pm\epsilon)\lambda(\alpha)}t})$ satisfy

$$\begin{aligned} W_\pm''(\alpha, t) &= \lambda(\alpha)(p-1\pm\epsilon)W_\pm(\alpha, t), \quad t \in I_\delta, \\ W_\pm(\alpha, 0) &= 1 - \|v_\alpha\|_\infty, \\ W_\pm'(0) &= 0, \end{aligned}$$

we easily see that $W_-(\alpha, t) \le w_\alpha(t) \le W_+(\alpha, t)$ for $t \in I_\delta$ and $\alpha \gg 1$. This implies that as $\alpha \to \infty$

$$\frac{1}{2}(1 - \|v_\alpha\|_\infty)e^{\sqrt{(p-1-\epsilon)\lambda(\alpha)}\delta} \le W_-(\alpha, \delta) \le w_\alpha(\delta) \to 0.$$

This yields $1 - C_6 e^{-\delta\sqrt{(p-1-\epsilon)\lambda(\alpha)}} \le \|v_\alpha\|_\infty$. Hence, there exists a constant $C_5 > 0$ such that for $\alpha \gg 1$

$$\begin{aligned} (\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{1/(p-1)} &\le \lambda(\alpha)^{1/(p-1)}(1 - C_6 e^{-\delta\sqrt{(p-1-\epsilon)\lambda(\alpha)}}) \\ &\le \lambda(\alpha)^{1/(p-1)}\|v_\alpha\|_\infty = \|u_\alpha\|_\infty. \end{aligned}$$

Thus the proof is complete. ∎

Next, we study the asymptotics of $\lambda_3(\alpha)$. To this end, we prove the following lemma.

LEMMA 3.2. *For $\alpha \gg 1$*

$$(p+3)\gamma_3(\alpha) - 4\lambda_3(\alpha)\alpha^2 = \eta_0(\alpha)\alpha^2, \tag{3.5}$$

*where $\eta_0(\alpha) \to C_1^2$ as $\alpha \to \infty$.*

*Proof.* For a fixed $\alpha > 0$, it is easy to see that $L(\alpha, t) := \lambda(\alpha)t^2/2 - t^{p+1}/(p+1)$ is strictly increasing for $0 \le t \le \lambda(\alpha)^{1/(p-1)}$. Therefore, by Lemma 3.1, we obtain

$$L(\alpha, (\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{1/(p-1)}) \le L(\alpha, \|u_\alpha\|_\infty) \le L(\alpha, \lambda(\alpha)^{1/(p-1)}). \tag{3.6}$$

First, we study the asymptotics of $L(\alpha, \|u_\alpha\|_\infty)$. By (2.2) and (2.12), we have

$$\|u'_\alpha\|_2^2 = \frac{p+1}{p-1}\left(\gamma(\alpha) - \frac{2}{p+1}\lambda(\alpha)\alpha^2\right), \tag{3.7}$$

$$\|u_\alpha\|_{p+1}^{p+1} = \frac{p+1}{p-1}(\lambda(\alpha)\alpha^2 - \gamma(\alpha)). \tag{3.8}$$

Integrate (2.8) over $I$. Then by (3.7) and (3.8), we obtain

$$\begin{aligned} \frac{1}{2}u'_\alpha(0)^2 &= \frac{1}{2}\|u'_\alpha\|_2^2 - \frac{1}{p+1}\|u_\alpha\|_{p+1}^{p+1} + \frac{1}{2}\lambda(\alpha)\alpha^2 \\ &= \frac{p+3}{2(p-1)}\gamma(\alpha) + \frac{p-5}{2(p-1)}\lambda(\alpha)\alpha^2. \end{aligned} \tag{3.9}$$

By substituting (3.1) and (3.2) for (3.9), we see from (2.8) that

$$\begin{aligned} L(\alpha, \|u_\alpha\|_\infty) &= \frac{1}{2}u'_\alpha(0)^2 \\ &= \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \frac{p+3}{2(p-1)}\gamma_3(\alpha) \\ &\quad + \frac{p-5}{2(p-1)}\lambda_3(\alpha)\alpha^2. \end{aligned} \tag{3.10}$$

Secondly, we study the asymptotics of $L(\alpha, \lambda(\alpha)^{1/(p-1)})$. By (3.1) and Taylor expansion, for $\alpha \gg 1$, we obtain

$$\begin{aligned} L(\alpha, \lambda(\alpha)^{1/(p-1)}) &= \frac{(p-1)}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)} \\ &= \frac{p-1}{2(p+1)}(\alpha^{p-1} + C_1\alpha^{(p-1)/2} + \lambda_3(\alpha))^{(p+1)/(p-1)} \\ &= \frac{p-1}{2(p+1)}\alpha^{p+1}(1 + C_1\alpha^{(1-p)/2} + \lambda_3(\alpha)\alpha^{1-p})^{(p+1)/(p-1)} \\ &= \frac{p-1}{2(p+1)}\alpha^{p+1}\left\{1 + \frac{p+1}{p-1}(C_1\alpha^{(1-p)/2} + \lambda_3(\alpha)\alpha^{1-p})\right. \\ &+ \left.\frac{p+1}{(p-1)^2}(C_1\alpha^{(1-p)/2} + \lambda_3(\alpha)\alpha^{1-p})^2 + o((C_1\alpha^{(1-p)/2} + \lambda_3(\alpha)\alpha^{1-p})^2)\right\}. \end{aligned} \tag{3.11}$$

Note that $\lambda_3(\alpha) = o(\alpha^{(p-1)/2})$ by (3.1) and Lemma 2.3. By using this and (3.11), we obtain

$$\begin{aligned} L(\alpha, \lambda(\alpha)^{1/(p-1)}) &= \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \frac{1}{2}\lambda_3(\alpha)\alpha^2 \\ &\quad + \frac{1}{2(p-1)}C_1^2\alpha^2 + o(\alpha^2). \end{aligned} \tag{3.12}$$

Finally, we study the asymptotics of $L(\alpha, (\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{1/(p-1)})$. By (1.4) and Taylor expansion, for $\alpha \gg 1$, we have

$$\begin{aligned} &L(\alpha, (\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{1/(p-1)}) \\ &= (\lambda(\alpha) - e^{-C_5\sqrt{\lambda(\alpha)}})^{2/(p-1)} \left( \frac{p-1}{2(p+1)}\lambda(\alpha) + \frac{e^{-C_5\sqrt{\lambda(\alpha)}}}{p+1} \right) \\ &= \lambda(\alpha)^{2/(p-1)} \left(1 - \frac{e^{-C_5\sqrt{\lambda(\alpha)}}}{\lambda(\alpha)}\right)^{2/(p-1)} \left( \frac{p-1}{2(p+1)}\lambda(\alpha) + \frac{e^{-C_5\sqrt{\lambda(\alpha)}}}{p+1} \right) \\ &= \lambda(\alpha)^{2/(p-1)} \left(1 - \frac{2e^{-C_5\sqrt{\lambda(\alpha)}}}{(p-1)\lambda(\alpha)} + \frac{(3-p)e^{-2C_5\sqrt{\lambda(\alpha)}}}{(p-1)^2\lambda(\alpha)^2} + o(\lambda(\alpha)^{-2}e^{-2C_5\sqrt{\lambda(\alpha)}})\right) \\ &\times \left( \frac{p-1}{2(p+1)}\lambda(\alpha) + \frac{e^{-C_5\sqrt{\lambda(\alpha)}}}{p+1} \right) \\ &= \frac{p-1}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)} - \frac{1}{2(p-1)}(1+o(1))e^{-2C_5\sqrt{\lambda(\alpha)}}\lambda(\alpha)^{(3-p)/(p-1)} \\ &= \frac{p-1}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)} - O(e^{-2C_7\alpha^{(p-1)/2}}\alpha^{3-p}) \\ &= \frac{p-1}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)} - o(\alpha^2). \end{aligned} \tag{3.13}$$

Therefore, by (3.10), (3.12) and (3.13), we obtain (3.5). Thus the proof is complete. ∎

LEMMA 3.3. *$\lambda_3(\alpha) = (1 + o(1))C_1^2/(p-1)$ as $\alpha \to \infty$.*

*Proof.* By (3.3) and (3.5), we obtain

$$\gamma_3'(\alpha) - \frac{p+3}{2\alpha}\gamma_3(\alpha) = -\frac{\eta_0(\alpha)\alpha}{2}. \tag{3.14}$$

By solving this equation, we see that

$$\gamma_3(\alpha) = \alpha^{(p+3)/2}\int_\alpha^\infty \frac{\eta_0(s)}{2s^{(p+1)/2}}ds + C_8\alpha^{(p+3)/2}. \tag{3.15}$$

$\gamma_3(\alpha) = o(\alpha^{(p+3)/2})$ by (2.19) and (3.2). Moreover, the first term of the right hand side of (3.15) is also $o(\alpha^{(p+3)/2})$. Therefore, we see that $C_8 = 0$. Moreover, by l'Hopital's rule, we have

$$\lim_{\alpha\to\infty} \frac{\int_\alpha^\infty \eta_0(s)/(2s^{(p+1)/2})ds}{\alpha^{(1-p)/2}} = \lim_{\alpha\to\infty} \frac{\eta_0(\alpha)}{p-1} = \frac{1}{p-1}C_1^2.$$

This along with (3.15) implies

$$\gamma_3(\alpha) = \frac{1}{p-1}C_1^2(1+o(1))\alpha^2. \tag{3.16}$$

Then by (3.5) and (3.16), we obtain

$$\lambda_3(\alpha)\alpha^2 = \frac{p+3}{4}\gamma_3(\alpha) - \frac{1}{4}(1+o(1))C_1^2\alpha^2 = \frac{1}{p-1}C_1^2(1+o(1))\alpha^2. \tag{3.17}$$

Thus the proof is complete. ▮

Taking Lemma 3.3 and (3.16) into account, we put

$$\lambda_4(\alpha) := \lambda_3(\alpha) - \frac{1}{p-1}C_1^2, \tag{3.18}$$

$$\gamma_4(\alpha) := \gamma_3(\alpha) - \frac{1}{p-1}C_1^2\alpha^2. \tag{3.19}$$

Then by (3.3), (3.18) and (3.19), we obtain

$$\frac{d\gamma_4(\alpha)}{d\alpha} = 2\alpha\lambda_4(\alpha). \tag{3.20}$$

LEMMA 3.4. $\lambda_4(\alpha) = (5-p)(9-p)(1+o(1))C_1^3\alpha^{(1-p)/2}/(24(p-1)^2)$ *as* $\alpha \to \infty$.

*Proof.* By (3.10), (3.18) and (3.19), we obtain

$$\begin{aligned} L(\alpha, \|u_\alpha\|_\infty) &= \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \frac{1}{p-1}C_1^2\alpha^2 \\ &\quad + \frac{p+3}{2(p-1)}\gamma_4(\alpha) + \frac{p-5}{2(p-1)}\lambda_4(\alpha)\alpha^2. \end{aligned} \tag{3.21}$$

By the same argument as that to obtain (3.12), by (3.1), (3.18) and Taylor expansion, for $\alpha \gg 1$, we obtain

$$L(\alpha, \lambda(\alpha)^{1/(p-1)}) = \frac{p-1}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)}$$

$$
\begin{aligned}
&= \frac{p-1}{2(p+1)}\left(\alpha^{p-1}+C_1\alpha^{(p-1)/2}+\frac{1}{p-1}C_1^2+\lambda_4(\alpha)\right)^{(p+1)/(p-1)} \\
&= \frac{p-1}{2(p+1)}\alpha^{p+1}+\frac{1}{2}C_1\alpha^{(p+3)/2}+\frac{1}{p-1}C_1^2\alpha^2+\frac{1}{2}\lambda_4(\alpha)\alpha^2 \qquad (3.22)\\
&\quad+\frac{9-p}{6(p-1)^2}C_1^3\alpha^{(5-p)/2}+o(\alpha^{(5-p)/2}).
\end{aligned}
$$

Therefore, by (3.6), (3.13), (3.21) and (3.22), we obtain

$$(p+3)\gamma_4(\alpha)-4\lambda_4(\alpha)\alpha^2=\frac{9-p}{3(p-1)}(1+o(1))C_1^3\alpha^{(5-p)/2}. \tag{3.23}$$

This along with (3.20) implies

$$\gamma_4'(\alpha)-\frac{p+3}{2\alpha}\gamma_4(\alpha)=\eta_1(\alpha)\alpha^{(3-p)/2}, \tag{3.24}$$

where $\eta_1(\alpha)\to(p-9)C_1^3/(6(p-1))$ as $\alpha\to\infty$. Then by the same calculation as that to obtain (3.15), we obtain

$$\gamma_4(\alpha)=\alpha^{(p+3)/2}\int_\alpha^\infty\frac{-\eta_1(s)}{s^p}ds=\frac{9-p}{6(p-1)^2}(1+o(1))C_1^3\alpha^{(5-p)/2}. \tag{3.25}$$

Then by (3.23) and (3.25), we obtain

$$
\begin{aligned}
\lambda_4(\alpha)\alpha^2 &= \frac{p+3}{4}\gamma_4(\alpha)-\frac{9-p}{12(p-1)}(1+o(1))C_1^3\alpha^{(5-p)/2} \qquad (3.26)\\
&= \frac{(5-p)(9-p)}{24(p-1)^2}(1+o(1))C_1^3\alpha^{(5-p)/2}.
\end{aligned}
$$

This implies our assertion. ∎

# 4 Proof of Theorems

In this section, for $i,k,\mathrm{m}\in\mathbf{N}_0$, the notation $(4.\mathrm{m})_{i=k}$ means "(4.m) for the case $i=k$". By using the arguments in Section 3, we prove Theorem 1 by showing the following Proposition 4.1.

PROPOSITION 4.1. *There exist polynomials $\{B_i(p)\}_{i\geq 0}$ of $p$ (deg $B_i(p)\leq i$) such that for any $n\in\mathbf{N}_0$, the following asymptotic formulas (4.1) and (4.2) hold as $\alpha\to\infty$:*

$$\lambda(\alpha) = \alpha^{p-1}+C_1\alpha^{(p-1)/2}+\sum_{j=0}^{n}\frac{a_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{j(1-p)/2}+o(\alpha^{n(1-p)/2}), \tag{4.1}$$

$$\gamma(\alpha) = \frac{2}{p+1}\alpha^{p+1} + \frac{4}{p+1}C_1\alpha^{(p+3)/2} + \sum_{j=0}^{n}\frac{b_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{2+j(1-p)/2} + o(\alpha^{2+n(1-p)/2}), \tag{4.2}$$

*where*

$$a_i(p) := \frac{4+i(1-p)}{2(i+1)}B_i(p) \quad (0 \le i \le n), \tag{4.3}$$

$$b_i(p) := \frac{2}{i+1}B_i(p) \quad (0 \le i \le n). \tag{4.4}$$

For example, we know from Lemmas 3.3 and 3.4 that

$$B_0(p) = \frac{1}{2}, B_1(p) = \frac{9-p}{6}, a_0(p) = 1, a_1(p) = \frac{(5-p)(9-p)}{24}, b_0(p) = 1, b_1(p) = \frac{9-p}{6}.$$

To prove Proposition 4.1, we need the following Lemma 4.2, which is the extension of (3.22). For $1 \le i \le k$, we put

$$\lambda_{i+3}(\alpha) := \lambda(\alpha) - \alpha^{p-1} - C_1\alpha^{(p-1)/2} - \sum_{j=0}^{i-1}\frac{a_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{j(1-p)/2}, \tag{4.5}$$

$$\gamma_{i+3}(\alpha) := \gamma(\alpha) - \frac{2}{p+1}\alpha^{p+1} - \frac{4}{p+3}C_1\alpha^{(p+3)/2} - \sum_{j=0}^{i-1}\frac{b_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{2+j(1-p)/2}. \tag{4.6}$$

LEMMA 4.2. *Let $k \in \mathbf{N}$ be fixed. Assume that $(4.1)_{n=k-1}$–$(4.4)_{n=k-1}$ are valid. Then there exists a polynomial $B_k(p)$ of $p$ (deg $B_k(p) \le k$) determined by $a_0(p), a_1(p), \cdots, a_{k-1}(p)$ such that the following asymptotics holds as $\alpha \to \infty$:*

$$L(\alpha, \lambda(\alpha)^{1/(p-1)}) = \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \frac{1}{2}\lambda_{k+3}(\alpha)\alpha^2 + \sum_{j=0}^{k-1}\frac{A_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{2+j(1-p)/2} + \frac{B_k(p)}{(p-1)^{k+1}}C_1^{k+2}\alpha^{2+k(1-p)/2} + o(\alpha^{2+k(1-p)/2}), \tag{4.7}$$

*where*

$$A_i(p) = \frac{8+5i-ip}{4(i+1)}B_i(p) \quad (0 \le i \le k-1). \tag{4.8}$$

*Proof.* We prove the assertion by the mathematical induction with respect to $k \in \mathbf{N}$.

*The case* $k = 1$. $(4.7)_{k=1}$ follows from (3.22) with $A_0(p) = 1, B_0(p) = 1/2$ and $B_1(p) = (9-p)/6$. (In this case, we do not have to assume $(4.1)_{n=0}$–$(4.4)_{n=0}$, since they follow from Lemma 3.3 and (3.16).) Thus the proof of the case $k = 1$ is complete.

*The case* $k \geq 2$. Assume that $(4.1)_{n=k-1}$–$(4.4)_{n=k-1}$ hold. Clearly, $(4.1)_{n=k-1}$–$(4.4)_{n=k-1}$ imply $(4.1)_{n=k-2}$–$(4.4)_{n=k-2}$. Therefore, by the induction assumption, we have (4.7), in which $k$ is replaced by $k-1$. The proof of (4.7) is divided into three steps.

*Step 1.* We define $A_{k-1}(p)$ by $(4.8)_{i=k-1}$. Then we obtain by $(4.3)_{0\leq i\leq k-1}$ and $(4.4)_{0\leq i\leq k-1}$ that for $0 \leq i \leq k-1$

$$A_i(p) = \frac{1}{2}a_i(p) + B_i(p) = \frac{p-5}{2(p-1)}a_i(p) + \frac{p+3}{2(p-1)}b_i(p) = \frac{8+5i-ip}{4(i+1)}B_i(p). \tag{4.9}$$

This implies $(4.8)_{0\leq i\leq k-1}$. By $(4.1)_{n=k-1}$ and $(4.5)_{i=k,k-1}$, for $\alpha \gg 1$, we obtain

$$\lambda_{k+3}(\alpha) = \lambda_{k+2}(\alpha) - \frac{a_{k-1}(p)}{(p-1)^k}C_1^{k+1}\alpha^{(k-1)(1-p)/2} = o(\alpha^{(k-1)(1-p)/2}). \tag{4.10}$$

Substitute (4.10) for (4.7), in which $k$ is replaced by $k-1$. Then by $(4.9)_{i=k-1}$, we obtain

$$\begin{aligned} L(\alpha, \lambda(\alpha)^{1/(p-1)}) &= \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \frac{1}{2}\lambda_{k+3}(\alpha)\alpha^2 \\ &\quad + \sum_{j=0}^{k-1}\frac{A_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{2+j(1-p)/2} + r_k(\alpha), \end{aligned} \tag{4.11}$$

where $r_k(\alpha) = o(\alpha^{2+(k-1)(1-p)/2})$. Therefore, to prove (4.7), it is sufficient to show that

$$r_k(\alpha) = \frac{B_k(p)}{(p-1)^{k+1}}C_1^{k+2}\alpha^{2+k(1-p)/2} + o(\alpha^{2+k(1-p)/2}). \tag{4.12}$$

*Step 2.* To derive (4.12), we calculate $L(\alpha, \lambda(\alpha)^{1/(p-1)})$. We put

$$g_k(x) := 1 + h_k(x) := 1 + C_1x + \sum_{j=0}^{k-1}\frac{a_j(p)}{(p-1)^{j+1}}C_1^{j+2}x^{j+2}.$$

Furthermore, let $g_k(k+2, x)$ denote the Taylor expansion of $g_k(x)^{(p+1)/(p-1)}$ of $(k+2)$-th order, which is denoted by

$$g_k(k+2, x) = 1 + c_1(p)x + c_2(p)x^2 + \cdots + c_{k+2}(p)x^{k+2}.$$

Then by $(4.5)_{i=k}$ and Taylor expansion, we have

$$
\begin{aligned}
L(\alpha, \lambda(\alpha)^{1/(p-1)}) &= \frac{p-1}{2(p+1)}\lambda(\alpha)^{(p+1)/(p-1)} \qquad (4.13)\\
&= \frac{p-1}{2(p+1)}\alpha^{p+1}(1 + h_k(\alpha^{(1-p)/2}) + \lambda_{k+3}(\alpha)\alpha^{1-p})^{(p+1)/(p-1)}\\
&= \frac{p-1}{2(p+1)}\alpha^{p+1}\left\{1 + \frac{p+1}{p-1}\left(h_k(\alpha^{(1-p)/2}) + \lambda_{k+3}(\alpha)\alpha^{1-p}\right)\right.\\
&\quad + \sum_{j=2}^{k+2}\frac{(p+1)2(3-p)\cdots(j-(j-2)p)}{j!(p-1)^j}\left(h_k(\alpha^{(1-p)/2}) + \lambda_{k+3}(\alpha)\alpha^{1-p}\right)^j\\
&\quad \left. + o(\alpha^{(k+2)(1-p)/2})\right\}.
\end{aligned}
$$

Let $2 \le j \le k+2$ be fixed. We denote by $\{z_{l,j}(\alpha)\}_l$ the terms of the expansion of $(h_k(\alpha^{(1-p)/2}) + \lambda_{k+3}(\alpha)\alpha^{1-p})^j$ which contain $\lambda_{k+3}(\alpha)$. Then by (4.10), for $\alpha \gg 1$, we obtain

$$
|z_{l,j}(\alpha)| \le C\alpha^{(j-1)(1-p)/2}\cdot(\lambda_{k+3}(\alpha)\alpha^{1-p}) = o(\alpha^{(k+j)(1-p)/2}) = o(\alpha^{(k+2)(1-p)/2}).
$$

Then by this and (4.13), we obtain

$$
\begin{aligned}
L(\alpha, \lambda(\alpha)^{1/(p-1)}) &= \frac{p-1}{2(p+1)}\alpha^{p+1} \qquad (4.14)\\
&\times\left\{1 + \frac{p+1}{p-1}h_k(\alpha^{(1-p)/2}) + \sum_{j=2}^{k+2}\frac{(p+1)2(3-p)\cdots(j-(j-2)p)}{j!(p-1)^j}h_k(\alpha^{(1-p)/2})^j\right.\\
&\quad \left. + o(\alpha^{(k+2)(1-p)/2})\right\} + \frac{1}{2}\lambda_{k+3}(\alpha)\alpha^2\\
&= \frac{p-1}{2(p+1)}\alpha^{p+1}\{1 + c_1(p)\alpha^{(1-p)/2} + c_2(p)\alpha^{2(1-p)/2} + \cdots + c_{k+2}(p)\alpha^{(k+2)(1-p)/2}\}\\
&\quad + \frac{1}{2}\lambda_{k+3}(\alpha)\alpha^2 + o(\alpha^{2+k(1-p)/2}).
\end{aligned}
$$

Then by noting $\alpha^{p+1}\cdot\alpha^{(k+2)(1-p)/2} = \alpha^{2+k(1-p)/2}$, we see from (4.11) and (4.14) that

$$
r_k(\alpha) = \frac{p-1}{2(p+1)}c_{k+2}(p)\alpha^{2+k(1-p)/2} + o(\alpha^{2+k(1-p)/2}). \qquad (4.15)
$$

*Step 3.* $(k+2)!c_{k+2}(p)$ is given by the $(k+2)$-th derivative of $g_k(x)^{(p+1)/(p-1)}$ at $x=0$. We recall that the $n$-th derivative of a composite function $z(x) = Z(y)$ and $y = \psi(x)$ is

$$
\frac{d^n}{dx^n}z(x) = \sum\frac{n!}{(\beta_1)!(\beta_2)!\cdots(\beta_h)!}\frac{d^mZ}{dy^m}\left(\frac{y'}{1!}\right)^{\beta_1}\left(\frac{y''}{2!}\right)^{\beta_2}\left(\frac{y'''}{3!}\right)^{\beta_3}\cdots\left(\frac{y^{(h)}}{h!}\right)^{\beta_h}
$$

Here, the symbol $\sum$ indicates summation over all solutions in non negative integers of the equation $\beta_1 + 2\beta_2 + \cdots + h\beta_h = n$ and $m = \beta_1 + \beta_2 + \cdots + \beta_h$. By using this formula for $n = k+2$, we obtain

$$\begin{aligned} c_{k+2}(p) &= \frac{1}{(k+2)!}\frac{d^{k+2}\left(g_k(x)^{(p+1)/(p-1)}\right)}{dx^{k+2}}|_{x=0} \\ &= \sum \frac{1}{(\beta_1)!(\beta_2)!\cdots(\beta_h)!}\left(\frac{p+1}{p-1}\right)\left(\frac{2}{p-1}\right)\left(\frac{3-p}{p-1}\right)\left(\frac{m-(m-2)p}{p-1}\right) \\ &\quad \times (C_1)^{\beta_1}\left(\frac{a_0(p)C_1^2}{p-1}\right)^{\beta_2}\cdots\left(\frac{a_{h-2}(p)C_1^h}{(p-1)^{h-1}}\right)^{\beta_h}. \end{aligned}$$

Then we first find that the exponent of $(p-1)$ in the denominator of $c_{k+2}(p)$ is $m + \beta_2 + 2\beta_3 + \cdots + (h-1)\beta_h = k+2$. Secondly, $c_{k+2}(p)$ contains $C_1^{\beta_1+2\beta_2+\cdots+h\beta_h} = C_1^{k+2}$. Thirdly, since deg $B_i(p) \le i$ for $0 \le i \le k-1$, we know from $(4.3)_{0\le i\le k-1}$ that deg $a_i(p) \le i+1$ for $0 \le i \le k-1$. Therefore, we see that the degree of the numerator of $c_{k+2}(p)$ is at most $m - 1 + \beta_2 + 2\beta_3 + (h-1)\beta_h = k+1$. Finally, since the numerator of $c_{k+2}(p)$ contains the term $(p+1)$, we see that

$$c_{k+2}(p) = \frac{(p+1)C_1^{k+2}\tilde{c}_{k+2}(p)}{(p-1)^{k+2}}, \tag{4.16}$$

where $\tilde{c}_{k+2}(p)$ is a polynomial of $p$ with deg $\tilde{c}_{k+2}(p) \le k$. Then by (4.12), (4.15) and (4.16), we obtain $B_k = \tilde{c}_{k+2}/2$. Now (4.7) follows from (4.11) and (4.12). ∎

Now we prove Proportion 4.1.

*Proof of Proposition 4.1.* We prove (4.1)–(4.4) by mathematical induction with respect to $n \in \mathbf{N}_0$.

*The case* $n = 0$. By (3.1), (3.2), Lemma 3.3 and (3.16), we see that $(4.1)_{n=0}$–$(4.4)_{n=0}$ are valid with $a_0(p) = b_0(p) = 1, B_0(p) = 1/2$. Thus the proof of the case $n = 0$ is complete.

*The case* $n = k$. Assume that $(4.1)_{n=k-1}$– $(4.4)_{n=k-1}$ are valid. Then it follows from $(4.3)_{0\le i\le k-1}$ and $(4.4)_{0\le i\le k-1}$ that

$$a_i(p) = \frac{4 + i(1-p)}{4}b_i(p) \quad (0 \le i \le k-1).$$

By this, (2.11), $(4.5)_{i=k}$ and $(4.6)_{i=k}$, we obtain

$$\frac{d\gamma_{k+3}(\alpha)}{d\alpha} = 2\alpha\lambda_{k+3}(\alpha). \tag{4.17}$$

Substitute $(4.5)_{i=k}$ and $(4.6)_{i=k}$ for (3.9). Then by (3.10) and $(4.9)_{0\le i\le k-1}$, we obtain

$$\begin{aligned} L(\alpha, \|u_\alpha\|_\infty) &= \frac{p-1}{2(p+1)}\alpha^{p+1} + \frac{1}{2}C_1\alpha^{(p+3)/2} + \sum_{j=0}^{k-1}\frac{A_j(p)}{(p-1)^{j+1}}C_1^{j+2}\alpha^{2+j(1-p)/2} \\ &\quad + \frac{p+3}{2(p-1)}\gamma_{k+3}(\alpha) + \frac{p-5}{2(p-1)}\lambda_{k+3}(\alpha)\alpha^2. \end{aligned} \tag{4.18}$$

By this, (3.6), (3.13) and (4.7), we obtain

$$\frac{p+3}{2(p-1)}\gamma_{k+3}(\alpha) - \frac{4}{2(p-1)}\lambda_{k+3}(\alpha)\alpha^2 = \frac{B_k(p)}{(p-1)^{k+1}}(1+o(1))C_1^{k+2}\alpha^{2+k(1-p)/2}. \tag{4.19}$$

Then by (4.17) and (4.19), we obtain

$$\gamma_{k+3}'(\alpha) - \frac{p+3}{2\alpha}\gamma_{k+3}(\alpha) = \eta_k(\alpha)\alpha^{1+k(1-p)/2}, \tag{4.20}$$

where $\eta_k(\alpha) \to -B_k(p)C_1^{k+2}/(p-1)^k$ as $\alpha \to \infty$. By $(4.2)_{n=k-1}$ and $(4.6)_{i=k}$, we see that $\gamma_{k+3}(\alpha) = o(\alpha^{2+(k-1)(1-p)/2}) = o(\alpha^{(p+3)/2})$ for $\alpha \gg 1$. Therefore, by solving (4.20), we obtain

$$\gamma_{k+3}(\alpha) = \alpha^{(p+3)/2}\int_\alpha^\infty -\eta_k(s)s^{(k(1-p)-(1+p))/2}ds. \tag{4.21}$$

Then by l'Hopital's rule, we obtain

$$\lim_{\alpha\to\infty}\frac{\int_\alpha^\infty -\eta_k(s)s^{(k(1-p)-(1+p))/2}ds}{\alpha^{(k+1)(1-p)/2}} = \frac{2B_k(p)}{(k+1)(p-1)^{k+1}}C_1^{k+2}. \tag{4.22}$$

This along with (4.21) implies

$$\gamma_{k+3}(\alpha) = \frac{2B_k(p)}{(k+1)(p-1)^{k+1}}C_1^{k+2}(1+o(1))\alpha^{2+k(1-p))/2}. \tag{4.23}$$

By putting $b_k(p) = 2B_k(p)/(k+1)$, we obtain $(4.4)_{n=k}$. Then we obtain $(4.2)_{n=k}$ by $(4.4)_{n=k}$, $(4.6)_{i=k}$ and (4.23). Now, by (4.19) and (4.23), we obtain

$$\begin{aligned} \lambda_{k+3}(\alpha)\alpha^2 &= \frac{p+3}{4}\gamma_{k+3}(\alpha) - \frac{B_k(p)}{2(p-1)^k}(1+o(1))C_1^{k+2}\alpha^{2+k(1-p)/2} \\ &= \frac{(4+k(1-p))B_k(p)}{2(k+1)(p-1)^{k+1}}(1+o(1))C_1^{k+2}\alpha^{2+k(1-p)/2}. \end{aligned} \tag{4.24}$$

By putting $a_k(p) = (4 + k(1-p))B_k(p)/(2(k+1))$, we obtain $(4.3)_{n=k}$. Then we obtain $(4.1)_{n=k}$ by $(4.3)_{n=k}$, $(4.5)_{i=k}$ and (4.24). Thus the proof is complete. ∎

Now we get Theorem 1 from Proposition 4.1. Theorem 2 is a direct consequence of (2.5), (3.6), (3.10), (3.13), $(4.7)_{k=n+1}$ and $(4.24)_{k=n+1}$. Finally, Theorem 3 is a consequence of Lemma 3.1 and Theorem 1. Thus the proofs of Theorems 1–3 are complete. ∎

We conclude this section by the proof of Corollary 4. We note that $u_{m,\alpha}$ satisfies

$$-u''_{m,\alpha}(t) + u^p_{m,\alpha}(t) = \lambda(m,\alpha)u_{m,\alpha}(t), \quad t \in (0, 1/m), \tag{4.25}$$

$$u_{m,\alpha}(t) > 0, \quad t \in (0, 1/m), \tag{4.26}$$

$$u_{m,\alpha}(0) = u_{m,\alpha}(1/m) = 0. \tag{4.27}$$

We put $s = mt, \beta := m^{-2/(p-1)}\alpha$ and $w_{m,\beta}(s) = m^{-2/(p-1)}u_{m,\alpha}(t)$. Then $\|w_{m,\beta}\|_2 = \beta$ and $(\lambda(m,\alpha)/m^2, w_{m,\beta})$ satisfies (1.1)–(1.3). Then by Theorem 1, for $\alpha \gg 1$, we obtain

$$\begin{aligned}\frac{\lambda(m,\alpha)}{m^2} &= \beta^{p-1} + C_1\beta^{(p-1)/2} + \sum_{k=0}^{n} \frac{a_k(p)}{(p-1)^{k+1}} C_1^{k+2}\beta^{k(1-p)/2} \\ &\quad + o(\beta^{n(1-p)/2}).\end{aligned} \tag{4.28}$$

This along with the definition of $\beta$ implies (1.11). Next, by noting

$$w'_{m,\beta}(0) = m^{-(p+1)/(p-1)}u'_{m,\alpha}(0), \tag{4.29}$$

$$\|w_{m,\beta}\|_\infty = m^{-2/(p-1)}\|u_{m,\alpha}\|_\infty, \tag{4.30}$$

we easily obtain (1.12) by Theorem 2 and (4.29). Finally, (1.13) follows from Theorem 3 and (4.30). ∎

## Acknowledgments

The author should like to thank the referee for his helpful comments that improved the manuscript, and in particular, for suggesting that Corollary 4 should be included in the revised version of the manuscript. This research has been supported by Japan Society for the Promotion of Science.

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