

Universal BER Performance Ordering of MIMO Systems over Flat Channels

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Abstract—We present universal BER performance ordering for different antenna sizes in Multiple-Input Multiple-Output (MIMO) wireless systems with zero-forcing (ZF) equalization, which hold for all SNR. We first show that when the number of transmit antennas is fixed, BER of each symbol degrades with a decrease in the number of receive antennas even if the received SNR is kept constant. Then, we prove that when the number of receive antennas is fixed, the average BER improves with a decrease in the number of transmit antennas, which shows the tradeoff between BER and bandwidth efficiency in MIMO with ZF equalization. Furthermore, discussions on the relation of BER with the same change in the number of receive and transmit antennas show that there is no universal order in BER for any particular channel. These highlight the advantage and the limit of MIMO with ZF equalization.

Index Terms—BER, MIMO, Equalization

I. INTRODUCTION

MULTIPLE-Input Multiple-Output (or the so-called MIMO) system, which employs multiple antennas at both ends of the receiver and transmitter terminals, has been the subject of intensive research efforts in the past decade with potential application in future high speed wireless communications network. This is motivated by the benefits of 1) diversity gain, which can be achieved by averaging over multiple path gains to combat fading, to improve bit-error rate (BER); 2) the fading-induced spatial multiplexing gain, which makes use of the degrees of freedom in communication system by transmitting independent symbol streams in parallel through spatial channels, to improve capacity and/or BER (see e.g., [1]–[7] and references therein).

It has been shown that for linear detectors, the diversity order of MIMO transmissions with N_t transmit and N_r receive antennas over i.i.d. Rayleigh channels is $N_r - N_t + 1$ at full multiplexing [8]. The diversity order is usually measured by the slope of the BER curve at high SNR. From this we can infer that the diversity order is improved by increasing the number N_r of receive antennas, whereas the diversity order is degraded by increasing the number N_t of transmit antennas (which also contributes to multiplexing gain). In [1], gains induced by different schemes of MIMO systems were analytically and numerically compared. For a fix number of

receive antennas, numerical simulations show a loss in signal-to-noise ratio (SNR) with an increase in the number of transmit antennas but no analytical explanation for this phenomenon is given. On the other hand, the exact expressions for the symbol error-rate (SER) of MIMO with minimum mean squared error (MMSE) equalization is rigorously derived in [9], while an approximate BER expression of MIMO with zero-forcing (ZF) equalization is derived in [10]. However, these analysis are heavily dependent on the specific channel probability density function (pdf). They require integration over a given channel pdf, without which no general conclusion can be made.

Indepth theoretical study of MIMO systems which includes Vertical Bell Laboratories Layered Space-Time (V-BLAST), has also been reported in [11] which focuses on the tradeoff between the multiplexing and the diversity gain based on an approximate outage probability expression that is satisfied only asymptotically at high SNR. Diversity-multiplexing tradeoff with regard to group detection for MIMO at high SNR has been done in [12]. The insights glimpsed from these analysis are important and beneficial. However, the common shortcoming of these works is that they are approximations or bounds in the high/low SNR regimes which may be obsolete at practical range of SNR. We also bring attention to the fact that diversity gain at high SNR is not synonymous with BER or diversity gain at a particular value of SNR. Furthermore, diversity gain achieved for Rayleigh channels may not be achieved for other types of channels.

In this paper, we develop a novel approach to analyze the error-rate performance in MIMO system with linear equalizations that is not limited to the SNR extremes but apply for all range of SNR. In particular, we focus on the impact of antenna size on the BER performance. As suggested from the diversity order at high SNR, increasing the number of receive antennas should enhance the BER performance since the receive SNR increases, while decreasing the number of transmit antennas should do the same, because the symbols transmitted from other antennas can be regarded as interferences. However, it is not obvious that these still hold after linear equalization which stimulates the need for our theoretical analysis. Especially for the former case, under the condition that the receive SNR is kept constant, i.e., without power gain/loss due to the increase/decrease in the number of receive antennas, it will be interesting to analyze how the BER will be affected by the change in the number of receive antennas. We explicitly show that when the number of transmit antennas is fixed, the BER degrades with a decrease in the number of receive antennas, even if the receive SNR is fixed. This receive diversity loss or BER loss is due to the inherent convexity property of BER

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functions. Then, we analytically prove that for a fix number of receive antennas, the BER improves with a decrease in the number of transmit antennas, which implies that there exists a pure tradeoff between BER and bandwidth efficiency in MIMO system employing ZF equalization.

Albeit we do not evaluate how much gains there actually are, which require the knowledge of the channel coefficients or the associated channel pdf, our results are universal in the sense that performance ordering with the number of transmit antennas and the number of receive antennas holds true at all SNR, irrespective of channel pdf. Last but not least, for completeness, we also show that the BER is not ordered for any particular channel when both the number of transmit antennas and the number of receive antennas simultaneously change. Simulations to corroborate our theoretical analysis are presented.

II. PRELIMINARIES AND SYSTEM MODEL

We consider a MIMO transmission with N_t transmit and N_r receive antennas ($N_r \geq N_t$) over flat channels. Let us define ρ/N_t as the transmit power at each transmit antenna for the $N_r \times N_t$ MIMO system. At the receiver, the N_r received samples, $\mathbf{x} = [x_1, \dots, x_{N_r}]^T$, is expressed as

$$\mathbf{x} = \sqrt{\frac{\rho}{N_t}} \mathbf{H} \mathbf{s} + \mathbf{w}, \quad (1)$$

where the $N_t \times 1$ combined data vector \mathbf{s} having i.i.d. entries with unit variance, the $N_r \times 1$ vector \mathbf{w} of zero mean circular complex additive white Gaussian noise (AWGN) entries with unit variance and the $N_r \times N_t$ channel matrix \mathbf{H} are respectively given by

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{N_t} \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_{N_r} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} h_{11} & \dots & h_{1N_t} \\ \vdots & \ddots & \vdots \\ h_{N_r 1} & \dots & h_{N_r N_t} \end{bmatrix}, \quad (2)$$

such that h_{mn} denotes the path gain from transmit antenna n ($n \in [1, N_t]$) to receive antenna m ($m \in [1, N_r]$). The path gains are assumed to be perfectly known at the receiver but unknown at the transmitter.

Let the m th row (which corresponds to the m th receive antenna) of the channel matrix \mathbf{H} be \mathbf{h}_m for $m \in [1, N_r]$, and the n th column (which corresponds to the n th transmit antenna) of the channel matrix \mathbf{H} be $\tilde{\mathbf{h}}_n$ for $n \in [1, N_t]$. Consequently, we can also express \mathbf{H} as

$$\mathbf{H} = [\mathbf{h}_1, \dots, \mathbf{h}_{N_r}]^T = [\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_{N_t}]. \quad (3)$$

The signal-to-noise ratio (SNR) at receive antenna m is found to be $\rho \|\mathbf{h}_m\|^2 / N_t$, where $\|\cdot\|$ is the 2-norm of a vector, while the total receive power of the symbol transmitted from antenna n , i.e., the sum of power for symbol s_n at all receive antennas, is $\rho \|\tilde{\mathbf{h}}_n\|^2 / N_t$.

MIMO system offers the benefits of diversity gain to improve bit-error rate (BER) and/or the fading-induced spatial multiplexing gain to increase channel capacity. Mathematical capacity analysis reveals that the channel capacity scales with the minimum of the number of transmit antennas and the number of receive antennas [3], while the analysis of the diversity gain, which is fully achieved by non-linear Maximum

Likelihood (ML) equalization, shows a tradeoff between the number of transmit antennas and diversity advantage [11]. In this paper, we consider more practical linear equalizations and analyze their performance with respect to antenna size.

For transition to our analysis in the proceeding sections, let us review linear equalizations for MIMO systems. The output of a zero-forcing (ZF) equalizer is obtained by multiplying $\mathbf{G} = \sqrt{\frac{\rho}{N_t}} (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H$ to \mathbf{x} , which gives us $\hat{\mathbf{s}} = \mathbf{s} + \mathbf{G} \mathbf{w}$, where $(\cdot)^H$ stands for complex conjugate transposition. To enable ZF equalization, we require that:

Assumption 1: The channel matrix is tall and has column full rank.

The covariance of $\mathbf{G} \mathbf{w}$ is given by $(\frac{\rho}{N_t} \mathbf{H}^H \mathbf{H})^{-1}$. Let us define

$$\mathbf{R}_{N_r, N_t} = \mathbf{H}^H \mathbf{H} = \sum_{m=1}^{N_r} \mathbf{h}_m^H \mathbf{h}_m, \quad (4)$$

and denote the n th diagonal entry of $\mathbf{R}_{N_r, N_t}^{-1}$ as $\lambda_{N_r, N_t, n}$. Then, it follows from $\hat{\mathbf{s}} = \mathbf{s} + \mathbf{G} \mathbf{w}$ that the (post-processing) receive SNR of symbol s_n after ZF equalization is expressed as

$$\text{SNR}_{N_r, N_t, n} = \frac{\rho}{N_t} \frac{1}{\lambda_{N_r, N_t, n}}, \quad \text{for } n \in [1, N_t]. \quad (5)$$

On the other hand, the MMSE equalizer is given by $\mathbf{G} = \sqrt{\frac{\rho}{N_t}} \mathbf{H}^H (\frac{\rho}{N_t} \mathbf{H}^H \mathbf{H} + \mathbf{I})^{-1}$. Based on $\hat{\mathbf{s}} = \mathbf{G} \mathbf{x}$, we define the n th entry of the equalized output as $\hat{s}_n = p_n s_n + v_n$, where v_n is the effective noise contaminating symbol s_n . Then, we can show that the covariance of the effective noise meets $E\{|v_n|^2\} = p_n(1 - p_n)$. The receive signal-to-interference noise ratio (SINR) of symbol n after MMSE equalization is then expressed as $\text{SINR}_{N_r, N_t, n} = \frac{\rho}{N_t} \frac{1}{\xi_n} - 1$, where ξ_n is the n th diagonal entry of $[\mathbf{H}^H \mathbf{H} + \frac{N_t}{\rho} \mathbf{I}]^{-1}$. Similar SINR expressions are also found for block transmission with linear equalization [13] and MIMO-OFDM with linear processing [14].

We remark that SNRs or SINRs are fundamental parameters of system performances. If a symbol-by-symbol detection is employed, the BER or symbol-error rate (SER) function can usually be described by SNR or SINR. Suppose that we draw symbols from a fixed digital modulation with finite constellation. For the constellation, let us denote $f(\cdot)$ as a function in SNR or SINR to describe the bit-error probability of the transmitted symbols. It is obvious that $f(\cdot)$ is a decreasing function in SNR or SINR. Take for example, the symbol-by-symbol hard detection of QPSK constellation and ZF equalization. Then, the BER of symbol s_n for $N_r \times N_t$ system is expressed as $\text{BER}_{N_r, N_t, n} = f(\text{SNR}_{N_r, N_t, n}) = Q(\sqrt{\text{SNR}_{N_r, N_t, n}})$, where $Q(x)$ denotes the Gaussian-Q function $Q(x) \equiv (1/\sqrt{2\pi}) \int_x^\infty e^{-t^2/2} dt$.

For ZF equalization, the BER of symbol s_n decreases with an increase in $\text{SNR}_{N_r, N_t, n}$, which is inversely proportional to $\lambda_{N_r, N_t, n}$. To investigate the BER performance of the MIMO system, we need to investigate the properties of $\lambda_{N_r, N_t, n}$. From the properties, we would like to study how the antenna number affects the BER performance. Unlike diversity analysis, e.g., [1], [3], we do not specify any pdf of the channel gains. Our BER analysis also holds at practical range of SNR unlike [8], [11], [12].

In the sequel, we focus our attention on ZF equalization. The same results for SNR of ZF equalization can be developed for SINR of MMSE equalization. However, since the effective noises of MMSE equalization are in general non-Gaussian and depend on the channel structure, e.g., the number of transmit and receive antennas, we cannot describe the BER function of MMSE equalized symbols by one function. If BER of any antenna size can be approximated as one function of SINR, then the discussion on BER in the rest of the paper will also hold for BER with MMSE equalization.

III. DECREASING THE NUMBER OF RECEIVE ANTENNAS

Now, let us study the BER performance of MIMO system when we decrease the number of receive antennas, while fixing the number of transmit antennas. As the number of receive antennas decreases, the total receive power of a transmitted symbol decreases. Thus, it may be obvious that the BER performance degrades due to the power loss. However, under the condition that the total receive power of each symbol is kept constant even if the number of receive antennas decreases, it is not clear if the same conclusion can also be made. We investigate how the BER performance is affected by the number of receive antennas when the total receive power of each symbol is fixed.

Let us assume that $N_r - 1 \geq N_t$. We fix the number of transmit antennas at N_t and decrease the number N_r of receive antennas by one. When receive antenna μ is removed from the $N_r \times N_t$ system, the corresponding channel matrix is denoted as $\mathbf{H}^{(\mu)}$, which is assumed to have column full rank. The $(N_r - 1) \times N_t$ channel matrix $\mathbf{H}^{(\mu)}$ yields the $N_t \times N_t$ matrix $\mathbf{R}_{N_r-1, N_t}^{(\mu)}$, corresponding to (4), expressed as $\mathbf{R}_{N_r-1, N_t}^{(\mu)} = \mathbf{H}^{(\mu)H} \mathbf{H}^{(\mu)} = \sum_{m=1, m \neq \mu}^{N_r} \mathbf{h}_m^H \mathbf{h}_m$. It is easy to see that the matrices \mathbf{R}_{N_r, N_t} and $\mathbf{R}_{N_r-1, N_t}^{(\mu)}$ are related as

$$\sum_{\mu=1}^{N_r} \mathbf{R}_{N_r-1, N_t}^{(\mu)} = (N_r - 1) \mathbf{R}_{N_r, N_t}. \quad (6)$$

It follows that $\text{SNR}_{N_r, N_t, n}$ in (5) can be expressed as

$$\begin{aligned} \text{SNR}_{N_r, N_t, n} &= \frac{\rho}{N_t} \frac{1}{[\mathbf{R}_{N_r, N_t}^{-1}]_{n, n}} \\ &= \frac{\rho}{N_t} \frac{1}{(N_r - 1) [(\sum_{\mu=1}^{N_r} \mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]_{n, n}}, \end{aligned} \quad (7)$$

where $[\cdot]_{m, n}$ denotes the (m, n) th entry of a matrix.

To compare the $N_r \times N_t$ system with the $(N_r - 1) \times N_t$ system, it is reasonable to uniformly remove one antenna among N_r antennas, i.e., the selection of any one receive antenna has the same probability $1/N_r$. If receive antenna μ is removed from the $N_r \times N_t$ system, then the total receive power of symbol s_n reduces to $\rho \sum_{m=1, m \neq \mu}^{N_r} |h_{mn}|^2 / N_t$. Thus, for $(N_r - 1) \times N_t$ system, the average total receive power of symbol s_n with respect to random receive antenna dropping is given by

$$\frac{1}{N_r} \sum_{\mu=1}^{N_r} \left(\sum_{m=1, m \neq \mu}^{N_r} \rho \frac{|h_{mn}|^2}{N_t} \right) = \left(\frac{N_r - 1}{N_r} \right) \rho \frac{\|\tilde{\mathbf{h}}_n\|^2}{N_t}. \quad (8)$$

To ensure that the average total receive power of each symbol remains constant even when the number of receive antennas reduce by one, we increase the transmit power of the $(N_r - 1) \times N_t$ system by a factor of $\frac{N_r}{N_r - 1}$, i.e., we replace ρ in (8) by $\frac{N_r}{N_r - 1} \rho$. Subsequently, for this $(N_r - 1) \times N_t$ system, the receive SNR at receive antenna m increases to $\frac{N_r}{N_r - 1} \frac{\rho \|\mathbf{h}_m\|^2}{N_t}$ and hence the average total receive power of the $(N_r - 1) \times N_t$ systems becomes equal to the total receive power of the $N_r \times N_t$ system. Thus, the effects of power loss due to the reduction of the number of receive antennas disappears on the average.

Let us define the symbol SNR of symbol s_n after ZF equalization when receive antenna μ is removed as $\text{SNR}_{N_r-1, N_t, n}^{(\mu)}$ for $n \in [1, N_t]$. Then, similar to (5), the symbol SNR of symbol s_n for the $(N_r - 1) \times N_t$ system is expressed as

$$\text{SNR}_{N_r-1, N_t, n}^{(\mu)} = \frac{N_r}{N_r - 1} \frac{\rho}{N_t} \frac{1}{[(\mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]_{n, n}}. \quad (9)$$

To compare the $(N_r - 1) \times N_t$ system with the original $N_r \times N_t$ system, we utilize the following lemma (see Appendix I for a proof):

Lemma 1: For a given channel matrix, if $\mathbf{H}^{(\mu)}$ has column full rank for $\mu \in [1, N_r]$, then for $n \in [1, N_t]$,

$$\frac{1}{[(\sum_{\mu=1}^{N_r} \mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]_{n, n}} \geq \sum_{\mu=1}^{N_r} \frac{1}{[(\mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]_{n, n}}. \quad (10)$$

From (7) and Lemma 1, we obtain

$$\text{SNR}_{N_r, N_t, n} \geq \frac{1}{N_r} \sum_{\mu=1}^{N_r} \text{SNR}_{N_r-1, N_t, n}^{(\mu)}, \quad (11)$$

where $\frac{1}{N_r} \sum_{\mu=1}^{N_r} \text{SNR}_{N_r-1, N_t, n}^{(\mu)}$ denotes the average symbol SNR of symbol s_n when one receive antenna is randomly dropped. This shows that the average SNR of symbol s_n degrades when we randomly remove one receive antenna even if the average total receive symbol power remains constant.

We denote the BER of symbol s_n for $(N_r - 1) \times N_t$ system when receive antenna μ is removed as $\text{BER}_{N_r-1, N_t, n}^{(\mu)} = f(\text{SNR}_{N_r-1, N_t, n}^{(\mu)})$. Then, its BER averaged with respect to random receive antenna dropping is simply

$$\text{BER}'_{N_r-1, N_t, n} = \frac{1}{N_r} \sum_{\mu=1}^{N_r} \text{BER}_{N_r-1, N_t, n}^{(\mu)}, \quad (12)$$

where we utilize the superscript ' to denote the average value with respect to random antenna dropping. Although from (11), $\text{SNR}_{N_r, N_t, n} \geq \frac{1}{N_r} \sum_{\mu=1}^{N_r} \text{SNR}_{N_r-1, N_t, n}^{(\mu)}$, this does not necessarily imply that $\text{BER}_{N_r, N_t, n}$ is lower than the BER averaged with respect to random receive antenna dropping, i.e., $\text{BER}'_{N_r-1, N_t, n}$. To show this, we require that

Assumption 2: $f(\cdot)$ is a convex function in SNR.

This assumption is reasonable. For example, the Gaussian-Q function $Q(\sqrt{x})$ is convex in $x \geq 0$. The BER functions of most digital modulations are expressed (at least approximately) as a Gaussian-Q function or a linear combination of Gaussian-Q functions. For such a digital modulation, the BER function is invariably convex in all SNR.

Coupled with Assumption 2, since $f(\cdot)$ is a decreasing function in SNR, we have

$$\begin{aligned} f(\text{SNR}_{N_r, N_t, n}) &\leq f\left(\frac{1}{N_r} \sum_{\mu=1}^{N_r} \text{SNR}_{N_r-1, N_t, n}^{(\mu)}\right) \\ &\leq \frac{1}{N_r} \sum_{\mu=1}^{N_r} f(\text{SNR}_{N_r-1, N_t, n}^{(\mu)}), \end{aligned} \quad (13)$$

for $n \in [1, N_t]$. This reveals that removing one receive antenna randomly degrades the average BER of each symbol even if we increase the transmit power to keep the average total receive symbol power of $(N_r - 1) \times N_t$ system equal to the total receive symbol power of the original $N_r \times N_t$ system. We summarize this result in the following theorem:

Theorem 1: Suppose ZF equalization in an $N_r \times N_t$ MIMO transmission over a fixed static channel, where $N_r - 1 \geq N_t$. We randomly remove one receive antenna but increase the transmit power by a factor of $N_r/(N_r - 1)$. If the channel matrices satisfy Assumption 1 and the BER function is convex, then for all SNR, we have

$$\text{BER}_{N_r, N_t, n} \leq \text{BER}'_{N_r-1, N_t, n}. \quad (14)$$

Theorem 1 clearly states the BER gain of a symbol in MIMO transmission over a *fixed static* channel obtained by simply increasing the number of receive antennas. Receive diversity is said to be acquired. Remember that the effect of power loss is eliminated. It has already been shown in [8] that at high SNR, the diversity order of $N_r \times N_t$ systems for linear detection over i.i.d. Rayleigh distributed channels is $N_r - N_t + 1$ at full multiplexing. This implies that BER gain/receive diversity gain is obtained by increasing N_r . Unlike [8], we embraced a more pragmatic approach where no approximation is made and no fading is assumed. Theorem 1 can be applied to all digital modulations satisfying Assumption 2, regardless of the underlying channel pdf. Importantly, it states a universal and deterministic characteristics of the BER performance of MIMO systems that is contributed in large part by the *convexity* property of the BER function. For a given channel environment and at all SNR, if a receive antenna is randomly dropped, the average BER performance deteriorates. To know how much the actual deterioration is, one has to evaluate using the exact channel coefficients. Indeed, the BER averaged with respect to random receive antenna dropping depends on the number of receive antennas and *a fortiori* deteriorates as the number of receive antennas is lessened. This highlights the advantage/disadvantage of MIMO system upon increasing/decreasing the number of receive antennas.

So far, we have not specified any channel pdf. To gain more insights, let us denote the channel pdf of channel \mathbf{H} as $P(\mathbf{H})$ and of $\mathbf{H}^{(\mu)}$ as $P(\mathbf{H}^{(\mu)})$. To see the BER of symbol s_n averaged over random channels, we consider the following channel characteristics:

Assumption 3: $P(\mathbf{H}^{(1)}) = P(\mathbf{H}^{(2)}) = \dots = P(\mathbf{H}^{(N_r)})$.

This means that when any one row is removed from the $N_r \times N_t$ channel matrix, the resultant $(N_r - 1) \times N_t$ channel matrix has the same pdf. Clearly, if the entries of \mathbf{H} are i.i.d., then the assumption holds true. However, it should be remarked that a more general class of channels which includes

for example, non i.i.d. channels having correlation between channel gains, also meets the assumption.

Under Assumption 3, we have for $\mu \in [1, N_r]$,

$$\int \text{BER}_{N_r-1, N_t, n}^{(\mu)} P(\mathbf{H}^{(\mu)}) d\mathbf{H}^{(\mu)} \equiv \overline{\text{BER}}_{N_r-1, N_t, n}, \quad (15)$$

where $\overline{\text{BER}}_{N_r-1, N_t, n}$ is the BER of symbol s_n averaged over random $(N_r - 1) \times N_t$ channels. The over-bar is used to denote the average over random channels to differentiate with the average over antenna dropping. Utilizing (14) of Theorem 1, straightforward manipulation yields

$$\int \text{BER}_{N_r, N_t, n} P(\mathbf{H}) d\mathbf{H} \leq \int \text{BER}'_{N_r-1, N_t, n} P(\mathbf{H}) d\mathbf{H}. \quad (16)$$

It follows from (12) and (15) that the R.H.S of (16) is equivalent to $\overline{\text{BER}}_{N_r-1, N_t, n}$.

On the other hand, if we denote the BER of symbol s_n of $N_r \times N_t$ system averaged over random $N_r \times N_t$ channels as $\overline{\text{BER}}_{N_r, N_t, n}$, then L.H.S. of (16) $\int \text{BER}_{N_r, N_t, n} P(\mathbf{H}) d\mathbf{H} = \overline{\text{BER}}_{N_r, N_t, n}$. Since the equality in (10) holds only for some special channels, we can conclude that:

Theorem 2: Suppose an $N_r \times N_t$ MIMO transmission with ZF equalization, where $N_r - 1 \geq N_t$. Then, under Assumptions 1–3 and for a fix number of transmit antennas, the BER of symbol s_n averaged over random channels is a decreasing function in the number of receive antennas for all SNR such that

$$\overline{\text{BER}}_{N_r, N_t, n} < \overline{\text{BER}}_{N_r-1, N_t, n}. \quad (17)$$

In addition to degrading the BER of each symbol averaged over random receive antenna dropping (as proven in Theorem 1), Theorem 2 states that decreasing the number of receive antennas also degrades the BER of each symbol averaged over random channels (or equivalently, increasing the number of receive antennas improves the average BER performance). We stress that the BER gain attributed to an increase in the number of receive antennas comes from the convexity of the BER function, irrespective of channel pdf and SNR. The implication is that receive diversity is always available for any channel pdf and at any value of SNR.

To further emphasize the importance of the convexity property, let us suppose that the BER function is concave (which is of course impossible in practice). Then, all the inequality signs in the equations are reversed. In this case, all the results derived so far will also be reversed, and we get $\overline{\text{BER}}_{N_r, N_t, n} > \overline{\text{BER}}_{N_r-1, N_t, n}$, i.e., BER gain can only be achieved with a decrease in the number of receive antennas.

IV. DECREASING THE NUMBER OF TRANSMIT ANTENNAS

In this section, we consider the BER for a fix number N_r of receive antennas when the number N_t of transmit antennas is reduced by one, assuming that $2 \leq N_t \leq N_r$. For comparison between $N_r \times N_t$ system and $N_r \times (N_t - 1)$ system, as in the previous section, we uniformly remove one transmit antenna among N_t transmit antennas, i.e., the selection of any one transmit antenna has the same probability $1/N_t$.

It is often the case that the total transmit power of all transmit antennas is kept constant for different number of transmit antennas. But here, we fix the transmit power of each

transmit antenna to be ρ/N_t . This implies that the sum of transmit power reduces from ρ to $\rho(N_t-1)/N_t$, if one transmit antenna is removed. In this case, the total receive power from a transmit antenna remains constant, while the average receive power at each receive antenna of $N_r \times (N_t - 1)$ system with respect to transmit antenna dropping is $(N_t - 1)/N_t$ of the receive power at each receive antenna of the original $N_r \times N_t$ system. Note that even if the sum of transmit power is kept constant, our subsequent analysis still hold.

When transmit antenna ν is dropped from the $N_r \times N_t$ system, the corresponding channel matrix is denoted as $\mathbf{H}^{[\nu]}$. Let us define an $(N_t - 1) \times (N_t - 1)$ matrix $\mathbf{R}_{N_r, N_t-1}^{[\nu]} = \mathbf{H}^{[\nu]H} \mathbf{H}^{[\nu]}$. We also define the n th diagonal entry of $(\mathbf{R}_{N_r, N_t-1}^{[\nu]})^{-1}$ as $\lambda_{N_r, N_t-1, n}^{[\nu]}$ for $n \in [1, \nu - 1]$ and the $(n - 1)$ st diagonal entry of $(\mathbf{R}_{N_r, N_t-1}^{[\nu]})^{-1}$ as $\lambda_{N_r, N_t-1, n}^{[\nu]}$ for $n \in [\nu + 1, N_t]$ so that $\lambda_{N_r, N_t-1, n}^{[\nu]}$ corresponds to the SNR of symbol s_n after equalization.

The following lemma is fundamental to our BER analysis (see Appendix II for a proof):

Lemma 2: For $n \in [1, N_t]$ and $n \neq \nu$,

$$\lambda_{N_r, N_t-1, n}^{[\nu]} \leq \lambda_{N_r, N_t, n}, \quad (18)$$

where equality holds if and only if $\tilde{\mathbf{h}}_\mu$ is orthogonal to all $\tilde{\mathbf{h}}_n$ for $n \neq \mu$.

The SNR of symbol s_n for the $N_r \times (N_t - 1)$ system without transmit antenna ν is expressed as

$$\text{SNR}_{N_r, N_t-1, n}^{[\nu]} = \frac{\rho}{N_t} \frac{1}{\lambda_{N_r, N_t-1, n}^{[\nu]}}. \quad (19)$$

We have from (5), and direct application of Lemma 2 that for $n \neq \nu$,

$$\begin{aligned} \text{SNR}_{N_r, N_t-1, n}^{[\nu]} &= \frac{\rho}{N_t} \frac{1}{\lambda_{N_r, N_t-1, n}^{[\nu]}} \\ &\geq \frac{\rho}{N_t} \frac{1}{\lambda_{N_r, N_t, n}} = \text{SNR}_{N_r, N_t, n}. \end{aligned} \quad (20)$$

Notice that in the case where the sum of transmit power is kept constant, then $\text{SNR}_{N_r, N_t-1, n}^{[\nu]} = \frac{\rho}{N_t-1} \frac{1}{\lambda_{N_r, N_t-1, n}^{[\nu]}} > \frac{\rho}{N_t} \frac{1}{\lambda_{N_r, N_t-1, n}^{[\nu]}} \geq \text{SNR}_{N_r, N_t, n}$, and the relation in (20) is similarly obtained except that the equality sign is removed. Hence, removing one transmit antenna (reducing the bandwidth or spectral efficiency) improves the SNR of each symbol transmitted from the remaining antennas and hence its BER, i.e., if we denote the BER of symbol s_n of $N_r \times (N_t - 1)$ system as $\text{BER}_{N_r, N_t-1, n}^{[\nu]} = f(\text{SNR}_{N_r, N_t-1, n}^{[\nu]})$, then $\text{BER}_{N_r, N_t-1, n}^{[\nu]} \leq \text{BER}_{N_r, N_t, n}$, for $n \neq \nu$.

Intuitively, this result may be quite reasonable, since in the original $N_r \times N_t$ system symbol s_ν can be considered as an interference to symbol s_n and the effect of symbol s_ν is absent if symbol s_ν is not transmitted. For i.i.d. Rayleigh channels at high SNR, the diversity order of $N_r \times N_t$ systems for linear detection is $N_r - N_t + 1$ and hence reducing N_t increases diversity order [8]. However, diversity gain at high SNR does not equate to BER improvement at all SNR. Therefore, our result is not self-evident. From (20), we can find a fundamental tradeoff between bandwidth efficiency and

SNR, or equivalently, BER performance in ZF equalization that holds at *any value of SNR*. If one increases the number of transmit antennas, then bandwidth efficiency or multiplexing gain is enhanced but the BER of each symbol is degraded.

Now let us consider the average BER of $N_r \times N_t$ MIMO system in one transmitted block, i.e., the BER averaged over the N_t symbols, which is defined as

$$\text{BER}_{N_r, N_t} = \frac{1}{N_t} \sum_{n=1}^{N_t} f(\text{SNR}_{N_r, N_t, n}). \quad (21)$$

To differentiate this with the BER of each symbol, we call this *block BER*. The block BER is introduced to compare systems having different number of symbols in one transmitted block. The block BER of $N_r \times (N_t - 1)$ system without transmit antenna ν is defined as

$$\text{BER}_{N_r, N_t-1}^{[\nu]} = \frac{1}{N_t - 1} \sum_{n=1, n \neq \nu}^{N_t} f(\text{SNR}_{N_r, N_t-1, n}^{[\nu]}). \quad (22)$$

Let us define the block BER of $N_r \times (N_t - 1)$ system averaged with respect to random transmit antenna dropping as shown in (23). The difference $\text{BER}_{N_r, N_t} - \text{BER}_{N_r, N_t-1}^{[\nu]}$ is given by (24). Since $f(\cdot)$ is a decreasing function in SNR, one finds from (20) that the argument in the brackets of (24) is greater than or equal to 0, which leads to:

Theorem 3: Suppose ZF equalization of an $N_r \times N_t$ MIMO system over a fixed static channel. We randomly remove one transmit antenna and denote the block BER of $N_r \times (N_t - 1)$ systems averaged with respect to random transmit antenna dropping as $\text{BER}_{N_r, N_t-1}^{[\nu]}$. If the channel matrices satisfy Assumption 1, then for all SNR, we have

$$\text{BER}_{N_r, N_t-1}^{[\nu]} \leq \text{BER}_{N_r, N_t}. \quad (25)$$

Theorem 3 shows that for a fix number of receive antennas, if one transmit antenna is randomly removed with probability $1/N_t$, the block BER of $N_r \times (N_t - 1)$ system averaged with respect to random transmit antenna dropping is smaller than the block BER of the original $N_r \times N_t$ system. Since the equality sign only holds for some special channels, we can say that on the average, reducing the number of transmit antennas improves the block BER performance, which is in sharp contrast to Theorem 1. We can find a pure tradeoff between bandwidth efficiency and the block BER performance in ZF equalization. It is worth emphasizing that Theorem 3 holds in any given channel environment satisfying Assumption 1 regardless of the underlying channel pdf and is valid for all SNR.

To get further insights, we assume that

Assumption 4: $P(\mathbf{H}^{[1]}) = P(\mathbf{H}^{[2]}) = \dots = P(\mathbf{H}^{[N_t]})$, where $P(\mathbf{H}^{[\nu]})$ denotes the channel pdf of $\mathbf{H}^{[\nu]}$.

We integrate both sides of (25) with respect to $P(\mathbf{H})$. Then the R.H.S. gives $\overline{\text{BER}}_{N_r, N_t}$. Under Assumption 4, the L.H.S becomes the block BER averaged over random $N_r \times (N_t - 1)$ channels as $\overline{\text{BER}}_{N_r, N_t-1} = \int \text{BER}_{N_r, N_t-1}^{[\nu]} P(\mathbf{H}^{[\nu]}) d\mathbf{H}^{[\nu]} = \int \text{BER}_{N_r, N_t-1}^{[\nu]} P(\mathbf{H}) d\mathbf{H}$. Since the equality sign in (25) only holds in some special channels, we can conclude that:

Theorem 4: Suppose $N_r \times N_t$ MIMO transmission with ZF equalization under Assumptions 1 and 4. Then, for a fix

$$\text{BER}'_{N_r, N_t-1} \equiv \frac{1}{N_t} \sum_{\nu=1}^{N_t} \text{BER}_{N_r, N_t-1}^{[\nu]} = \frac{1}{N_t} \sum_{\nu=1}^{N_t} \left(\frac{1}{N_t-1} \sum_{n=1, n \neq \nu}^{N_t} f(\text{SNR}_{N_r, N_t-1, n}^{[\nu]}) \right). \quad (23)$$

$$\frac{1}{N_t} \sum_{n=1}^{N_t} \left[f(\text{SNR}_{N_r, N_t, n}) - \left(\frac{1}{N_t-1} \sum_{\nu=1, \nu \neq n}^{N_t} f(\text{SNR}_{N_r, N_t-1, n}^{[\nu]}) \right) \right]. \quad (24)$$

number of receive antennas, the block BER averaged over random channels is an increasing function in the number of transmit antennas for all SNR such that

$$\overline{\text{BER}}_{N_r, N_t-1} < \overline{\text{BER}}_{N_r, N_t}. \quad (26)$$

Repeatedly using Theorem 4, one finds an ordered performance with respect to the number of transmit antennas: $\overline{\text{BER}}_{N_r, 1} < \overline{\text{BER}}_{N_r, 2} < \dots < \overline{\text{BER}}_{N_r, N_t}$.

Theorem 4 indicates that for all SNR, increasing the number of transmit antennas, i.e., the bandwidth efficiency, degrades the block BER performance averaged over random channels (or equivalently, decreasing the number of transmit antennas improves the block BER). Theorem 4 is completely different from Theorem 2. While the BER of each symbol is a decreasing function in the number of receive antennas (as shown in Theorem 2), the block BER is an increasing function in the number of transmit antennas. This affirms the tradeoff between BER and bandwidth efficiency in MIMO with ZF equalization, which was also found in ML equalization for i.i.d. channels at high SNR [11]. Unlike [11], any characteristics of random channels are not specified except for Assumption 4, and (26) holds for any value of SNR.

V. DECREASING BOTH THE NUMBER OF TRANSMIT AND RECEIVE ANTENNAS

When we randomly remove one receive antenna from $N_r \times N_t$ system, we can define the block BER averaged with respect to random receive antenna dropping as BER'_{N_r-1, N_t} . It follows direct from Theorem 1 that

$$\text{BER}_{N_r, N_t} \leq \text{BER}'_{N_r-1, N_t}, \quad (27)$$

provided $N_r - 1 \geq N_t$. Thus, not surprisingly, for all SNR, random removal of one receive antenna degrades the block BER as well. To recap from the previous sections, we have verified that decreasing the number of receive antennas has a negative impact on BER performance, while decreasing the number of transmit antennas positive. However, the impact of a simultaneous decrease in both the number of receive antennas and the number of transmit antennas on the BER in MIMO has yet to be clarified. In this section, we show by using a simple example of QPSK signaling that there is no ordered BER for the same change in the number of receive antennas and the number of transmit antennas.

Even when the number of receive antennas and the number of transmit antennas simultaneously decrease by one, i.e., $N_r \rightarrow N_r - 1, N_t \rightarrow N_t - 1$, we keep the total receive SNR in all the receive antennas unchanged. By dropping receive

TABLE I
SNR AND BER FOR CASE 1 AND CASE 2

Case 1			Case 2		
(μ, ν)	$\text{SNR}_{1,1,1}^{(\mu, \nu)}$	$\text{BER}_{1,1,1}^{(\mu, \nu)}$	(μ, ν)	$\text{SNR}_{1,1,1}^{(\mu, \nu)}$	$\text{BER}_{1,1,1}^{(\mu, \nu)}$
(1, 1)	9.7435	0.0009	(1, 1)	3.8439	0.0250
(1, 2)	9.7783	0.0009	(1, 2)	32.9487	0.0000
(2, 1)	23.4698	0.0000	(2, 1)	4.8307	0.0140
(2, 2)	3.6504	0.0280	(2, 2)	6.0063	0.0071
n	$\text{SNR}_{2,2,n}$	$\text{BER}_{2,2,n}$	n	$\text{SNR}_{2,2,n}$	$\text{BER}_{2,2,n}$
1	7.7513	0.0027	1	3.4056	0.0325
2	5.5796	0.0091	2	11.5624	0.0003

antenna μ and transmit antenna ν , the corresponding input-output relation of the $(N_r - 1) \times (N_t - 1)$ system is expressed as

$$\mathbf{y}^{(\mu)} = \sqrt{\frac{N_r \rho}{(N_r - 1)N_t}} \mathbf{H}^{(\mu, \nu)} \mathbf{s}^{(\nu)} + \mathbf{w}^{(\mu)}, \quad (28)$$

where $\mathbf{y}^{(i)}$, $\mathbf{s}^{(i)}$, and $\mathbf{w}^{(i)}$ are obtained by removing the i th entry of \mathbf{y} , \mathbf{s} , and \mathbf{w} , and $\mathbf{H}^{(\mu, \nu)}$ by removing the μ th row and the ν th column of \mathbf{H} . Like in Section III, the transmit power of $(N_r - 1) \times (N_t - 1)$ system increases by a factor $\frac{N_r}{N_r - 1}$.

Because it is difficult to compare the block BER between an $N_r \times N_t$ system and an $(N_r - 1) \times (N_t - 1)$ system analytically, we consider a simple example by comparing an 1×1 system with an 2×2 system. When receive antenna μ and transmit antenna ν are removed, for the $(N_r - 1) \times (N_t - 1)$ system, we denote $\text{SNR}_{N_r-1, N_t-1, n}^{(\mu, \nu)}$ and $\text{BER}_{N_r-1, N_t-1, n}^{(\mu, \nu)}$ as the SNR and BER of the symbol from the n th antenna respectively. Note that from an 2×2 system, there are 4 possible combinations of removing both the transmit and receive antenna such that only one entry in $\{h_{11}, h_{12}, h_{21}, h_{22}\}$ remains in the 1×1 system. This means that there are 4 possible $\text{SNR}_{1,1,1}^{(\mu, \nu)}$ and $\text{BER}_{1,1,1}^{(\mu, \nu)}$. We denote the (block) BER averaged with respect to random transmit and receive antenna dropping as $\text{BER}'_{1,1} = \frac{1}{4} \sum_{\mu=1}^2 \sum_{\nu=1}^2 \text{BER}_{1,1,1}^{(\mu, \nu)}$, while the block BER of the 2×2 system as $\text{BER}'_{2,2} = \frac{1}{2} (\text{BER}_{2,2,1} + \text{BER}_{2,2,2})$. These block BERs are used for comparison instead of the individual BERs. We also let $\rho = 10\text{dB}$.

Case 1: Let $h_{11} = 0.4962 - 0.3448i$, $h_{12} = -0.8680 - 0.4737i$, $h_{21} = 1.3170 + 0.7827i$ and $h_{22} = 0.9482 + 0.2744i$. Then, the values of $\{\text{SNR}_{1,1,1}^{(\mu, \nu)}\}_{(\mu, \nu)=(1,1)}^{(2,2)}$, $\{\text{BER}_{1,1,1}^{(\mu, \nu)}\}_{(\mu, \nu)=(1,1)}^{(2,2)}$, $\{\text{SNR}_{2,2,n}\}_{n=1}^2$ and $\{\text{BER}_{2,2,n}\}_{n=1}^2$ are respectively calculated and listed in Table I. From Table I, we obtain $\text{BER}'_{1,1} = 0.0075 > \text{BER}'_{2,2} = 0.0059$. On the other hand, consider another channel where

TABLE II
COEFFICIENTS OF THE FIX CHANNEL

h_{11}	$0.4879 - 0.0014i$	h_{31}	$0.5767 - 0.1108i$
h_{12}	$0.0886 + 0.2034i$	h_{32}	$0.8409 - 0.0266i$
h_{13}	$-0.1320 + 0.5132i$	h_{33}	$-0.0964 + 0.0805i$
h_{14}	$-0.0676 - 0.5885i$	h_{34}	$0.5051 + 1.1481i$
h_{21}	$-0.2379 - 0.5502i$	h_{41}	$-0.3059 - 1.1778i$
h_{22}	$-0.8107 + 0.8421i$	h_{42}	$0.2314 + 0.1235i$
h_{23}	$-0.4160 + 1.5438i$	h_{43}	$0.7543 + 0.0419i$
h_{24}	$0.2082 - 0.9448i$	h_{44}	$-0.4892 + 0.6067i$

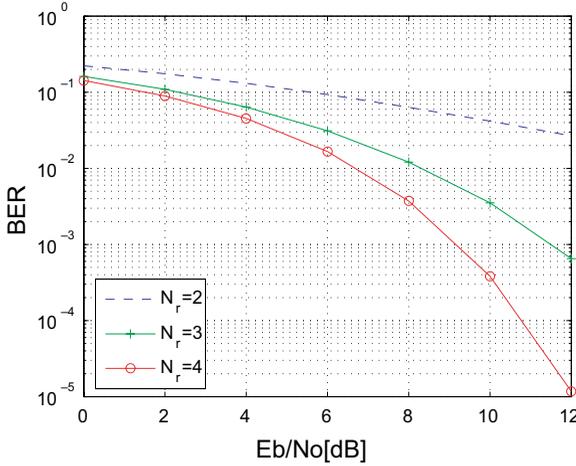


Fig. 1. BER with respect to random receive antenna dropping for a fix $N_t = 2$ and varying N_r over a fix channel.

Case 2: Let $h_{11} = -0.5534 - 0.5426i$, $h_{12} = -0.6816 - 1.6823i$, $h_{21} = -0.0758 - 0.6909i$ and $h_{22} = -0.5927 + 0.1820i$.

Then, like in case 1, the values of $\{\text{SNR}_{1,1,1}^{(\mu,\nu)}\}_{(\mu,\nu)=(1,1)}^{(2,2)}$, $\{\text{BER}_{1,1,1}^{(\mu,\nu)}\}_{(\mu,\nu)=(1,1)}^{(2,2)}$, $\{\text{SNR}_{2,2,n}^2\}_{n=1}^2$ and $\{\text{BER}_{2,2,n}^2\}_{n=1}^2$ are similarly obtained and listed in Table I. From Table I, we have $\text{BER}'_{1,1} = 0.0115 < \text{BER}'_{2,2} = 0.0164$.

These two contrasting cases demonstrate that there is no universal order for the block BER of ZF equalization for the same change in the number of receive and transmit antennas in an MIMO system for any particular channel. Although the MIMO BER is not ordered for a particular channel, the block BER averaged over a large number of random channels may be ordered. To see this, one has to resort to the property of the channel pdf.

VI. NUMERICAL SIMULATIONS

To validate our theoretical findings, we test the MIMO system with ZF equalization for different antenna sizes. The information symbols are drawn from a QPSK constellation. In our simulations, we always keep the average total receive power of each symbol the same as in our theoretical analysis. We plot the BER with respect to E_b/N_0 , where at each E_b/N_0 , the average receive power of each symbol is kept constant regardless of the antenna configuration.

In our first two simulations, we send transmitted symbols over a fix channel with coefficients given in Table II. Fig. 1 illustrates the result for a fix $N_t = 2$ and N_r varying from

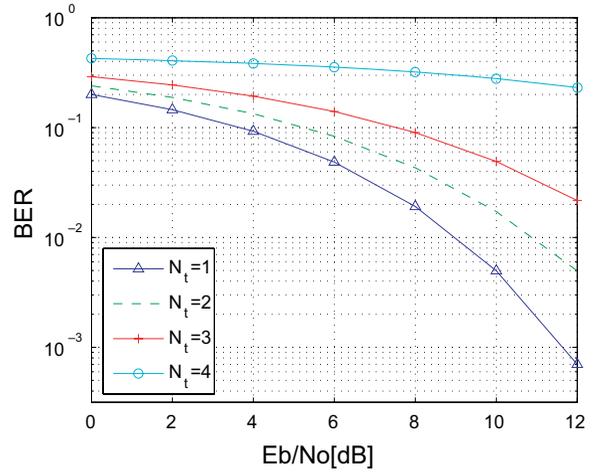


Fig. 2. BER with respect to random transmit antenna dropping for a fix $N_r = 4$ and varying N_t over a fix channel.

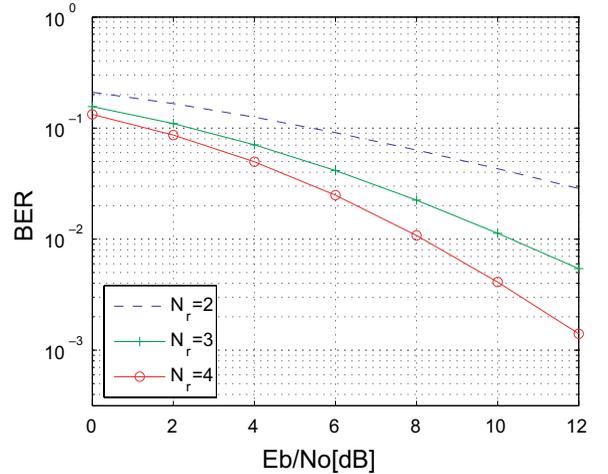


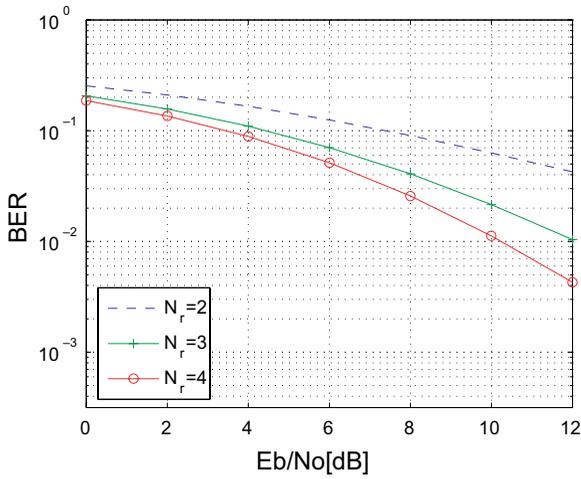
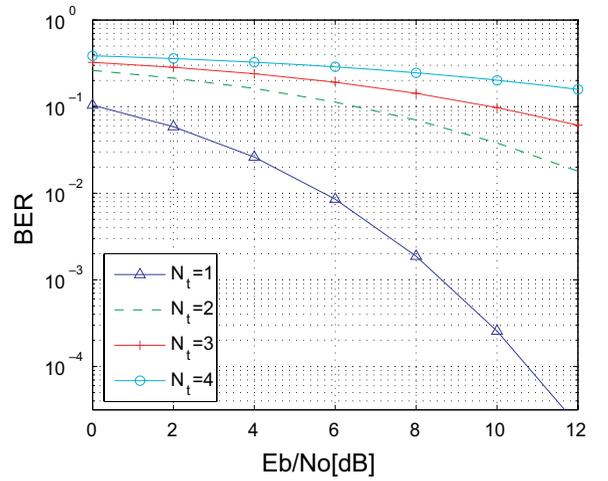
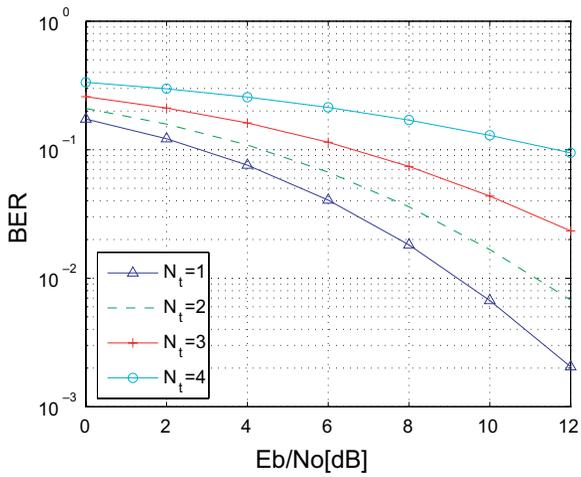
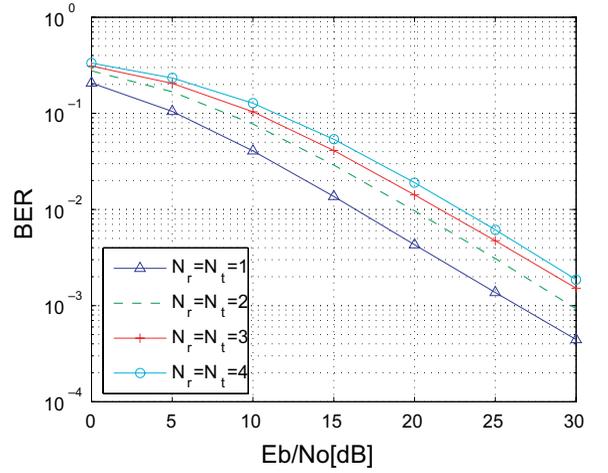
Fig. 3. BER for a fix $N_t = 2$ and varying N_r over random Rayleigh channels.

4 to 2 for ZF equalization for the fix channel. We observe that the block BER averaged with respect to random receive antenna dropping degrades with a decrease in N_r . This result holds not just for this fix channel but for any other channels we tested, which confirms Theorem 1 or (27).

Next, we set $N_r = 4$ and decrease N_t from 4 to 1 for the fix channel. The simulation results are shown in Fig. 2. As the number of transmit antennas is reduced, the block BER averaged with respect to random transmit antenna dropping improves which validates Theorem 3.

In our subsequent simulations, we average the results over 10^5 Rayleigh channels that compose of zero mean Gaussian taps with unit variance, and over 10^5 Rice channels with Rice factor 2. The Rice channels are generated like in [15]. For a fix $N_t = 2$ and N_r varying from 4 to 2 for ZF equalization, Fig. 3 and Fig. 4 depict the results for Rayleigh channels and for Rice channels respectively. From both figures, the block BER averaged over random channels degrades with a decrease in N_r . This is a direct corollary of Theorem 2 since it holds for all symbols and for any channel under Assumption 3 at all range of SNR.

Fig. 5 and Fig. 6 show the results for a fix $N_r = 4$ and

Fig. 4. BER for a fix $N_t = 2$ and varying N_r over random Rice channels.Fig. 6. BER for a fix $N_r = 4$ and varying N_t over random Rice channels.Fig. 5. BER for a fix $N_r = 4$ and varying N_t over random Rayleigh channels.Fig. 7. BER for varying $N_t = N_r$ over random Rayleigh channels.

N_t varying from 4 to 1 for ZF equalization for Rayleigh channels and for Rice channels, respectively. The simulation results confirm Theorem 4 which holds for all SNR, as the block BER averaged over random channels indeed improves significantly with a decrease in N_t (or equivalently, decrease in bandwidth efficiency). This shows unequivocally that there is a tradeoff between the BER performance and bandwidth efficiency.

On the other hand, it is interesting to see what will happen to the block BER of ZF equalization averaged over random channels if we decrease both the number of transmit antennas and the number of receive antennas simultaneously. As mentioned earlier, we know from our theoretical analysis in Sect. III and Sect. IV as well as Fig. 3, Fig. 4, Fig. 5 and Fig. 6 that; 1) the block BER averaged over random channels degrades with a decrease in N_r ; while 2) the block BER averaged over random channels improves with a decrease in N_t . Thus, there appears to be an offset between 1) and 2) which indicates that the trend for the block BER performance averaged over random channels is unpredictable with a decrease in both N_r and N_t . Our discussion in Section V also verifies that for a particular

channel, there is no ordered block BER averaged with respect to antenna dropping when there is a change in both N_r and N_t . Hence, to see how the block BER averaged over random channels changes when both N_r and N_t are decreased, we perform simulations for $N_t = N_r \in [1, 4]$. The results for Rayleigh and for Rice channels are plotted in Fig. 7 and Fig. 8 respectively.

From Fig. 7 for Rayleigh channels, the decrease in the block BER averaged over random channels when $N_t = N_r = 2$ is reduced to $N_t = N_r = 1$ is evident but for $N_r = N_t \geq 2$, the change in the block BER for any decrease in both N_t and N_r is small. For ZF, the diversity order is unaffected by the same change in the antenna number on both sides of the link. Some multiplexing gain is obtained when both the number of transmit and receive antennas decreases. Fig. 7 and Fig. 8 show that for ZF, the block BER averaged over random Rayleigh or Rice channels seem to be ordered such that it improves with a decrease in both N_t and N_r .

Furthermore, in order to investigate this phenomenon, we take statistics over a large number of Rayleigh channels to see how the block BER of $(N_r - 1) \times (N_t - 1)$ system fares as compared to the block BER of $N_r \times N_t$ system. With

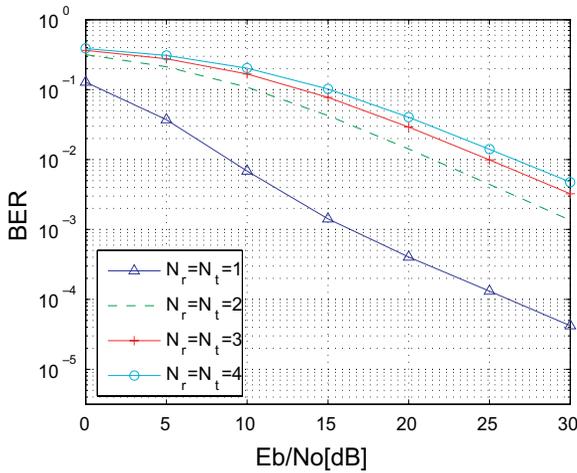


Fig. 8. BER for varying $N_t = N_r$ over random Rice channels.

TABLE III
 $P(\text{BER}_{N-1} < \text{BER}_N)$ FOR VARIOUS N

SNR = 10dB		SNR = 25dB	
N	ZF	N	ZF
2	0.7776	2	0.8131
3	0.6633	3	0.7004
4	0.6074	4	0.6341

the number of transmit antennas and the number of receive antennas set the same as $N_r = N_t = N$, the probability $P(\text{BER}_{N-1} < \text{BER}_N)$ that the BER for N receive (transmit) antennas, denoted as BER_N , is higher than the BER for $N - 1$ receive (transmit) antennas, denoted as BER_{N-1} , is also obtained over 10^5 channel realizations at SNR = 10dB and 25dB. The results are summarized in Table III. Albeit a universal order of the block BER over a static channel is not true for ZF equalization, Table III illustrates that when averaged over random channels, there is a biasness towards block BER improvement with a decreasing antenna size (or block BER degradation with an increasing antenna size) since the probabilities are all greater than 0.5. This suggests that the average effect of BER improvement due to a decrease in the number of transmit antennas is greater than the average effect of BER degradation due to a decrease in the number of receive antennas. This accounts for the ordered performance in Fig. 7.

VII. CONCLUSIONS

We have demonstrated theoretically that for ZF equalization, under the condition of a fix total receive power and a fix number of transmit antennas, the BER averaged over random receive antenna dropping and the BER averaged over random channels degrade with a decrease in the number of receive antennas. For a fix number of receive antennas, we have also proven that a decrease in the number of transmit antennas translates into an amelioration in both the block BER averaged over random transmit antenna dropping and the block BER averaged over random channels, which reveals the tradeoff between BER and bandwidth efficiency. These analytical

results are universal that hold true for all SNR and for any i.i.d. channels. In addition, for the same decrease in both the number of receive and transmit antennas, the block BER of ZF equalization is not strictly ordered for a particular channel but the block BER averaged over random channels may be ordered.

APPENDIX I

Suppose an $N \times 1$ complex vector $\mathbf{y} \in \mathbf{C}^N$ and $N \times N$ positive matrices \mathbf{A}_1 and \mathbf{A}_2 . In [16, pp. 453], for non-zero vector $\mathbf{y} \in \mathbf{C}^N$ and positive definite matrices $\mathbf{A}_1, \mathbf{A}_2$, it is shown that $1/[\mathbf{y}^H(\mathbf{A}_1 + \mathbf{A}_2)^{-1}\mathbf{y}] \geq 1/[\mathbf{y}^H\mathbf{A}_1^{-1}\mathbf{y}] + 1/[\mathbf{y}^H\mathbf{A}_2^{-1}\mathbf{y}]$. It follows that for positive definite matrices \mathbf{A}_i with $i \in [1, K]$, $1/[\mathbf{y}^H(\mathbf{A}_1 + \dots + \mathbf{A}_K)^{-1}\mathbf{y}] \geq \sum_{i=1}^K 1/[\mathbf{y}^H\mathbf{A}_i^{-1}\mathbf{y}]$. Then, since $\mathbf{R}_{N_r-1, N_t}^{(\mu)}$ for $\mu \in [1, N_r]$ is a positive definite matrix, we have that

$$\frac{1}{\mathbf{y}^H[(\sum_{\mu=1}^{N_r} \mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]\mathbf{y}} \geq \sum_{\mu=1}^{N_r} \frac{1}{\mathbf{y}^H[(\mathbf{R}_{N_r-1, N_t}^{(\mu)})^{-1}]\mathbf{y}}. \quad (29)$$

Let \mathbf{y}_n be a vector with only one non-zero entry of 1 in the n th position and zero entries in the other positions. By substituting \mathbf{y} as \mathbf{y}_n in (29), the denominators in all the terms in (29) become the n th diagonal entry of the respective inverse matrices which leads to Lemma 1.

APPENDIX II

Since the ν th column of the channel matrix (or the index of the ν th transmit antenna) can always be rearranged to be in the last (N_t th) column, we just need to consider the case when the N_t th transmit antenna is removed. We prove Lemma 2 as follows: First, we partition \mathbf{R}_{N_r, N_t} into four submatrix components as

$$\mathbf{R}_{N_r, N_t} = \begin{bmatrix} \mathbf{R}_{N_r, N_t-1}^{[N_t]} & \mathbf{c} \\ \mathbf{c}^H & d \end{bmatrix}, \quad (30)$$

where $\mathbf{c} = [\sum_{m=1}^{N_r} h_{m1}^* h_{m, N_t}, \dots, \sum_{m=1}^{N_r} h_{m, N_t-1}^* h_{m, N_t}]^T$ and $d = \sum_{m=1}^{N_r} |h_{m, N_t}|^2$. Then, by applying the well-known matrix inversion lemma [17] to the inverse of (30), we obtain

$$\mathbf{R}_{N_r, N_t}^{-1} = \begin{bmatrix} (\mathbf{R}_{N_r, N_t-1}^{[N_t]})^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} -(\mathbf{R}_{N_r, N_t-1}^{[N_t]})^{-1}\mathbf{c} \\ \mathbf{1} \end{bmatrix} \Delta^{-1} \begin{bmatrix} -\mathbf{c}^H(\mathbf{R}_{N_r, N_t-1}^{[N_t]})^{-1} & \mathbf{1} \end{bmatrix}, \quad (31)$$

where $\Delta = d - \mathbf{c}^H(\mathbf{R}_{N_r, N_t-1}^{[N_t]})^{-1}\mathbf{c}$. Just as $\mathbf{R}_{N_r, N_t-1}^{[N_t]}$ is positive definite, Δ is also positive definite. Comparing the diagonal entries of both sides of (31), we can get (18). The equality in (18) holds only if $\mathbf{c} = \mathbf{0}$, i.e., \mathbf{h}_{N_t} is orthogonal to all \mathbf{h}_n for $n \neq N_t$, which completes the proof of Lemma 2.

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