

On the Sustainability of Government Borrowing in a Dynamic General Equilibrium¹

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Abstract

This paper constructs a dynamic general equilibrium model with money in consumers's utility functions, and investigates the equilibrium dynamics of government's debt. The limitation level of the government borrowing for which a dynamic equilibrium and the no Ponzi-Game condition are compatible each other is explicitly derived. The critical level depends on the long-run interest rate, primary balance, money supply, and so on.

Key Words: sustainability of fiscal deficit, no-Ponzi-Game condition, dynamic general equilibrium

JEL Classification Numbers: E6, H6

1 Introduction

Since Nuemeyer and Yano [1995], it has been standard to take careful attention to intertemporal budget constraints of economic agents for monetary dynamic general equilibrium analyses. This sort of feasibility problem for government, which is logically equivalent with repayability of fiscal liability, is a burning issue for the Japanese economy. Thus, many empirical studies, which focus the sustainability of the Japanese government's borrowing, have been conducted. However, to my best knowledge, there is no dynamic general equilibrium analysis highlighting an upper bound of national debt balance that is compatible with the intertemporal budget constraint of the government.

The main purpose of this paper is to theoretically study a limitation level of government's deficit that is compatible with a dynamic general equilibrium and the intertemporal budget constraint of the government. Using a closed economy version of Neumeyer and Yano's model, I explicitly derive debt dynamics and the critical level of the government's borrowing. One may consider that the critical level depends on the long-run interest rate level, primary balance, disposable income level, money supply, and so on. This paper rigorously supports the plausible intuition in a dynamic general equilibrium framework, and explicitly reveals how the limitation level depends on these factors.

Since the Japanese government's fiscal deficit began to spike, especially during the depression of the 1990's, it has been said that the sustainability of the government's borrowing can be a point of serious controversy. With the increasing emphasis on this problem, various empirical studies on this subject have been conducted. However, many existing studies assume that the interest rate is constant over time. This study solves both nominal and real interest rates as endogenous variables. Although theoretical studies on a dynamic general equilibrium treat price variables including interest rates as endogenous variables, equilibrium dynamics of financial liabilities and the critical level of the government's borrowing that is repayable over the time horizon are not investigated. This paper highlights the financial market and the time evolution of fiscal deficit along an equilibrium path.

This paper closely relates the following studies. Neumeyer and Yano [1995] study the interdependence of monetary policy across national borders, by using a money-in-utility model. Many papers of Yano [1984a, 1984b, 1990, 1991, 1998, 1999] investigate turnpike properties and the inefficacy

of temporal policies. Kondo extends Yano's results to monetary dynamic general equilibrium models [2004, 2006a, 2006b]. However, the time evolution of the bond market is not focused in their papers. Hamilton and Flavin note the no Ponzi-Game condition that is equivalent with the intertemporal budget constraint, and empirically study the sustainability problem in their paper [1986]. Empirical tests for the Japanese economy have been conducted, e.g. Fukuda and Teruyama [1994], Doi and Nakasato [1998] and Ihori et al. [2001].

The rest of this paper is organized as follows. In Section 2, I set up the basic model. An equilibrium path is derived in Section 3 (Theorem 1). The most basic case, in which money supply level is constant over time, is taken special care (Corollary 1). The main result of this paper is offered in Section 4, i.e., I explicitly derive the upper bound of the government's borrowing that is compatible with an equilibrium and the intertemporal budget constraint of the government (Theorem 2 and Corollary 2). In Appendix (Section 5), I study an equilibrium bond dynamics and derive the no Ponzie-Game condition, which is equivalent with the intertemporal budget constraint.

2 Model

Think of an economy with the infinite time horizon and with discrete time points $t = -1, 0, 1, \dots$. I call the period between time $t - 1$ and t the period t . There are representative consumer and a government, and they transact a consumption good, money and bond at each period. The government imposes lump-sum fashion tax τ_{ht} that is in consumption good form on every consumer, and spends it g_t , $t = 0, 1, \dots$. The government provides money M_t^g and bond B_t^g to the consumers. Money brings liquidity service to the consumers, and a unit of bond generates interest rate $1 + i_t$ from the period t to $t + 1$.

The consumer initially holds money M_{-1} and bond B_{-1} at $t = -1$, and gets consumption good endowment y_t for every period $t = 0, 1, \dots$. The stream $\{y_t\}$ is supposed to be bounded over time. The consumer has his own periodwise preference over his consumption $c_t \in \mathbb{R}_+$ and real money balances $m_t (= M_t/p_t) \in \mathbb{R}_+$ that is represented by a utility function $u : \mathbb{R}_+^2 \rightarrow \mathbb{R}$.

The consumer must subject to the flow budget constraints:

$$M_{t+s} + B_{t+s} \leq p_{t+s}(y_{t+s} - \tau_{t+s} - c_{t+s}) + M_{t+s-1} + (1 + i_{t+s-1})B_{t+s-1}, \quad (1)$$

$s = 0, 1, \dots$, and the no Ponzi-Game (NPG) condition:

$$\liminf_{T \rightarrow \infty} \left(\prod_{j=1}^T \frac{1}{1 + r_{t+j-1}} \right) \frac{A_{t+T}}{p_{t+T}} \geq 0, \quad (2)$$

where $1 + r_t = (1 + i_t) p_t / p_{t+1}$ is the real interest rate from period t to $t + 1$ and $A_t = M_{t-1} + (1 + i_{t-1}) B_{t-1}$ is the consumer's financial asset at the beginning of the period t , $t = 0, 1, \dots$. Following Neumeyer and Yano [1995], I integrate the flow budget constraints and the NPG condition at period t into the intertemporal budget constraints. (See (3).)

The consumer's behavior is summarized by the following maximizing problem:

$$\begin{aligned} \{c_t^*, M_t^*, B_t^*\} &\in \arg \max_{\{c_t, M_t, B_t\}} \sum_{s=0}^{\infty} \beta^s u(c_{t+s}, m_{t+s}) \\ \text{s.t.} \quad &\sum_{s=0}^{\infty} \left(\prod_{j=1}^s \frac{1}{1 + r_{t+j-1}} \right) (c_{t+s} + \delta_{t+s} m_{t+s}) \leq w_t, \\ \text{where } w_t &= \frac{A_t}{p_t} + \sum_{s=0}^{\infty} \left(\prod_{j=0}^{s-1} \frac{1}{1 + r_j} \right) (y_{t+s} - \tau_{t+s}), \end{aligned} \quad (3)$$

$t = 0, 1, \dots$, where $\beta \in (0, 1)$ is a discount factor of the future utility and $\delta_t = i_t / (1 + i_t)$ is the opportunity cost of real money holding at the period t . To save notation I define an operator taking the discounted value $\tilde{x}_t = \sum_{s=0}^{\infty} \left(\prod_{j=1}^s \frac{1}{1 + r_{t+j-1}} \right) x_{t+s}$ for given t and $\{x_{t+s}\}_{s=0}^{\infty}$. Using this operator, I can rewrite $w_t = A_t / p_t + \tilde{y}_t - \tilde{\tau}_t$.

The government sets up a stream of its policy variables $\{M_t^g, B_t^g, g_t, \tau_t\}$. The stream must subject to the flow budget constraints

$$M_{t+s}^g + B_{t+s}^g = p_{t+s} (g_{t+s} - \tau_{t+s}) + M_{t+s-1}^g + (1 + i_{t+s-1}) B_{t+s-1}^g, \quad (4)$$

$s = 0, 1, \dots$

A stream of price and allocation of the economy is determined so that all markets are cleared simultaneously:

$$c_t^* + g_t = y_t; \quad M_t^* = M_t^g; \quad B_t^* = B_t^g; \quad \text{for every } t = 0, 1, \dots \quad (5)$$

The prices in the equilibrium will be distinguished with an asterisk, as well. Although, by Walras's law, the bond market clearing condition is redundant for every $t = 0, 1, \dots$, I rather focus on the time evolution of the bond market in this paper.

3 Equilibrium Path

In this section, following Neumeyer and Yano [1995], I explicitly derive the equilibrium path. (The result is summarized as Theorem 1 and Corollary 1.) To this end, I specify the utility function of the representative consumer as

$$u(c, m) = \log c + \gamma \log m. \quad (6)$$

Then, I can obtain the following demand functions

$$c_t^* = \frac{1}{1 + \gamma} (1 - \beta) w_t; \quad \delta_t m_t^* = \frac{\gamma}{1 + \gamma} (1 - \beta) w_t; \quad (7)$$

for $t = 0, 1, \dots$. Note that the parameter γ determines the ratio of expenditure between the real consumption and money in an equilibrium, i.e.

$$\gamma = \frac{\delta_t m_t^*}{c_t^*} = \frac{i_t^*}{1 + i_t^*} \frac{M_t^*}{p_t^* c_t^*} \quad (8)$$

for every $t = 0, 1, \dots$

As was seen in (7), to get the consumption path I need to obtain the dynamics of the real wealth $\{w_t\}$ and the real interest rate. In an equilibrium, the dynamics of the consumer's real wealth are determined as following:

Lemma 1 *It holds that $w_{t+1} = \beta(1 + r_t)w_t$ for every $t = 0, 1, \dots$*

Proof. By the flow budget constraint, it holds that

$$c_t^* + \delta_t m_t^* = y_t - \tau_t + \frac{A_t^*}{p_t} - \frac{1}{1 + r_t} \frac{A_{t+1}^*}{p_{t+1}}.$$

Since (7) holds, the LHS satisfies

$$LHS = (1 - \beta) w_t.$$

Moreover, since $(\tilde{y}_t - \tilde{\tau}_t) - \frac{1}{1+r_t}(\tilde{y}_{t+1} - \tilde{\tau}_{t+1}) = y_t - \tau_t$, by the definition of the wealth constraint, I obtain

$$\begin{aligned} RHS &= y_t - \tau_t + w_t - (\tilde{y}_t - \tilde{\tau}_t) - \frac{1}{1 + r_t} [w_{t+1} - (\tilde{y}_{t+1} - \tilde{\tau}_{t+1})] \\ &= w_t - \frac{1}{1 + r_t} w_{t+1}. \end{aligned}$$

Hence, $w_{t+1} = \beta(1 + r_t) w_t$. ■

The real expenditure to the consumption and the money balance along the equilibrium path are determined by (7) and Lemma 1.

Lemma 2 *It holds that $c_{t+1}^* = \beta(1 + r_t) c_t^*$ and $\delta_{t+1} m_{t+1}^* = \beta(1 + r_t) \delta_t m_t^*$ for every $t = 0, 1, \dots$*

Proof. By (7) and Lemma 1, the following holds

$$c_{t+1}^* = \frac{1 - \beta}{1 + \gamma} \beta(1 + r_t) w_t = \frac{1 - \beta}{1 + \gamma} \beta(1 + r_t) \frac{1 + \gamma}{1 - \beta} c_t^* = \beta(1 + r_t) c_t^*.$$

In much the same way, I can obtain $\delta_{t+1} m_{t+1}^* = \beta(1 + r_t) \delta_t m_t^*$. ■

The real interest rate, which is crucial to the equilibrium path as well as the real wealth, is essentially influenced by the discount factor β and the disposable products in the present model.

Lemma 3 *It holds that*

$$1 + r_t^* = \frac{1}{\beta} \frac{y_{t+1} - g_{t+1}}{y_t - g_t}$$

for every $t = 0, 1, \dots$

Proof. By the market clearing condition (5) and Lemma 2, it holds that

$$\frac{y_{t+1} - g_{t+1}}{y_t - g_t} = \frac{c_{t+1}^*}{c_t^*} = \beta(1 + r_t^*).$$

This ends the proof. ■

The nominal variables (the price level, the opportunity cost of the real money balance and the nominal interest rate) must satisfy the following relationships.

Lemma 4 *It holds that*

$$\frac{p_{t+1}^*}{p_t^*} = \frac{M_{t+1}^g}{M_t^g} \frac{\delta_{t+1}^*}{\delta_t^*} \frac{y_t - g_t}{y_{t+1} - g_{t+1}}$$

for every $t = 0, 1, \dots$

Proof. By Lemma 2 and 3, it holds in an equilibrium that $\frac{\delta_{t+1}^* m_{t+1}^*}{\delta_t^* m_t^*} = \frac{y_{t+1} - g_{t+1}}{y_t - g_t}$. Thus, I can obtain the lemma. ■

Lemma 5 *It holds that*

$$1 + i_t^* = \frac{1}{\beta} \frac{M_{t+1}^g}{M_t^g} \frac{\delta_{t+1}^*}{\delta_t^*}$$

for every $t = 0, 1, \dots$

Proof. By Lemma 3 and 4, it holds that

$$1 + i_t^* = (1 + r_t^*) \frac{p_{t+1}^*}{p_t^*} = \frac{1}{\beta} \frac{y_{t+1} - g_{t+1}}{y_t - g_t} \frac{\delta_{t+1}^*}{\delta_t^*} \frac{M_{t+1}^g}{M_t^g} \frac{y_t - g_t}{y_{t+1} - g_{t+1}} = \frac{1}{\beta} \frac{\delta_{t+1}^*}{\delta_t^*} \frac{M_{t+1}^g}{M_t^g}.$$

■

I denote the growth rate of money supply by $\mu_t = \frac{M_{t+1}^g}{M_t^g} - 1$. The next lemma is a crucial sub-step for proving Lemma 7.

Lemma 6 *For every $t = 1, 2, \dots$, the following hold*

$$(i) \frac{p_t^*}{p_{t+1}^*} = \frac{1}{1 + \mu_t} \frac{m_{t+1}^*}{m_t^*}; \quad (ii) \delta_t^* = 1 - \frac{1}{1 + \mu_t} \frac{1}{1 + r_t^*} \frac{m_{t+1}^*}{m_t^*};$$

$$(iii) \frac{1}{1 + \mu_t} \frac{1}{1 + r_t^*} = \frac{1}{1 + i_t^*} \frac{m_t^*}{m_{t+1}^*}.$$

Proof. (i) Since $\frac{m_{t+1}^*}{m_t^*} = \frac{M_{t+1}^g/p_{t+1}^*}{M_t^g/p_t^*} = (1 + \mu_t) \frac{p_t^*}{p_{t+1}^*}$, the result follows.

(ii) By (i), I can easily demonstrate the desired result:

$$\delta_t^* = \frac{i_t^*}{1 + i_t^*} = 1 - \frac{1}{1 + i_t^*} = 1 - \frac{1}{1 + r_t^*} \frac{p_t^*}{p_{t+1}^*} = 1 - \frac{1}{1 + r_t^*} \frac{1}{1 + \mu_t} \frac{m_{t+1}^*}{m_t^*}.$$

(iii) By the proof of (ii),

$$\frac{1}{1 + r_t^*} \frac{1}{1 + \mu_t} \frac{m_{t+1}^*}{m_t^*} = 1 - \delta_t^* = \frac{1}{1 + i_t^*}.$$

Thus, (iii) holds. ■

Now, I make some assumptions to explicitly derive the real money balance in an equilibrium.

• **Assumption.**

A. $\mu_t = \mu$ for every $t = 0, 1, \dots$

B. $\beta < 1 + \mu$.

C. $\lim_{T \rightarrow \infty} \left(\prod_{j=1}^T \frac{1}{1+i_{t+j-1}^*} \right) = 0$.

All the assumptions are pretty standard in the literature. There may exist no equilibrium if $1 + \mu \leq \beta$. For this point, see Brock [1974]. Assumption C excludes hyperdeflationary paths¹.

Under the assumptions, I obtain the equilibrium level of the real money balance m_t^* .

Lemma 7 *It holds that*

$$m_t^* = \frac{\gamma(1-\beta)}{1+\gamma} w_t^* \frac{1+\mu}{1+\mu-\beta}$$

for every $t = 0, 1, \dots$

Proof. Take $t = 0, 1, \dots$ arbitrary and fix it momentarily. By using (7) and (ii) of Lemma 6, I can obtain, by induction, that

$$m_t^* - \left(\prod_{j=1}^T \frac{1}{1+\mu_{t+j-1}} \frac{1}{1+r_{t+j-1}} \right) m_{t+T}^* = \gamma \sum_{s=0}^{T-1} \left(\prod_{j=1}^s \frac{1}{1+\mu_{t+j-1}} \frac{1}{1+r_{t+j-1}} \right) c_{t+s}^*$$

for any $T=1, 2, \dots$ In an equilibrium, the limit of the both side of the above expression as $T \rightarrow \infty$ can be obtained as follows. By Assumption A, B and Lemma 2,

$$\begin{aligned} RHS &= \gamma \sum_{s=0}^{T-1} \left(\prod_{j=1}^s \frac{1}{1+\mu_{t+j-1}} \frac{1}{1+r_{t+j-1}^*} \right) \beta^s \left(\prod_{j=1}^s (1+r_{t+j-1}^*) \right) c_t^* \\ &= \gamma c_t^* \sum_{s=0}^{T-1} \left(\frac{\beta}{1+\mu} \right)^s \\ &\longrightarrow \gamma c_t^* \frac{1+\mu}{1+\mu-\beta} \text{ as } T \rightarrow \infty. \end{aligned}$$

¹Brock [1974] proved that hyperdeflationary path may be an equilibrium in money-in-the-utility-function models.

By Assumption C and Lemma 6-(iii), it holds that

$$\begin{aligned}
LHS &= m_t^* - \left(\prod_{j=1}^T \frac{1}{1 + i_{t+j-1}^*} \frac{m_{t+j-1}^*}{m_{t+j}^*} \right) m_{t+T}^* \\
&= m_t^* - m_t^* \left(\prod_{j=1}^T \frac{1}{1 + i_{t+j-1}^*} \right) \\
&\longrightarrow m_t^* \text{ as } T \rightarrow \infty.
\end{aligned}$$

Thus, $m_t^* = \gamma c_t^* \frac{1+\mu}{1+\mu-\beta} = \frac{\gamma(1-\beta)}{1+\gamma} w_t^* \frac{1+\mu}{1+\mu-\beta}$ for every $t = 0, 1, \dots$ ■

As a direct consequence, I get the equilibrium price of the real money.

Lemma 8 *It holds that*

$$\delta_t^* = \frac{1 + \mu - \beta}{1 + \mu}$$

for every $t = 0, 1, \dots$

Proof. Since $\delta_t m_t^* = \frac{\gamma(1-\beta)}{1+\gamma} w_t^*$, it holds, in an equilibrium, that

$$\delta_t^* = \frac{\gamma(1-\beta)}{1+\gamma} w_t^* \frac{1+\gamma}{\gamma(1-\beta) w_t^*} \frac{1}{1+\mu} \frac{1+\mu-\beta}{1+\mu} = \frac{1+\mu-\beta}{1+\mu}.$$

■

I assume that real endowment y_t , the government's consumption g_t and real tax τ_t are constant over the time horizon to completely solve the model.

• **Assumption.**

D. $y_t = y$, $g_t = g$, $\tau_t = \tau$ for every $t = 0, 1, \dots$

E. $y > \tau \geq 0$, $y > g \geq 0$.

Let me note that given policy parameters g , τ , μ , M_{-1}^g , B_{-1}^g , the production level y and the preference parameter γ , all endogenous variables in an equilibrium is determined.

Now, I formally define the terminology “sustainability” in this paper.

Definition 1 Given y and γ , policy parameters g , τ , μ , M_{-1}^g , B_{-1}^g are said to be sustainable if, under the parameter values, an equilibrium satisfies NPG condition

$$\limsup_{T \rightarrow \infty} \left(\prod_{j=1}^T \frac{1}{1 + r_{t+j-1}^*} \right) \frac{D_{t+T}^{g*}}{p_{t+T}^*} \leq 0, \quad (9)$$

where $D_t^{g*} = M_{t-1}^g + (1 + i_{t-1}^*)B_{t-1}^g$ is the government's financial liabilities in an equilibrium.

The definition of “sustainability” is theoretically appropriate because under the flow budget constraints (4), NPG condition (9) is equivalent with the intertemporal budget constraint

$$\sum_{s=0}^{\infty} \left(\prod_{j=0}^{s-1} \frac{1}{1 + r_{t+j}^*} \right) (\delta_{t+s}^* m_{t+s}^{g*} + \tau_{t+s} - g_{t+s}) \geq \frac{D_t^{g*}}{p_t^*}. \quad (10)$$

If one accept the validity of (10) as a budget constraint of the government, he must also admit that of (9). Indeed, the definition has been widely adopted in various studies, e.g. Hamilton and Flavin [1986].

I can stipulate an equilibrium path, given p_0^* .

Lemma 9 An equilibrium path can be expressed as below

$$\begin{aligned} (i) \quad & 1 + r_t^* = \frac{1}{\beta}; & (ii) \quad & 1 + i_t^* = \frac{1 + \mu}{\beta}; & (iii) \quad & p_t^* = (1 + \mu)^t p_0^*; \\ (iv) \quad & w_t^* = \frac{A_0}{p_0^*} + \frac{y - \tau}{1 - \beta}; & (v) \quad & c_t^* = \frac{1 - \beta}{1 + \gamma} \frac{A_0}{p_0^*} + \frac{y - \tau}{1 + \gamma}; \\ (vi) \quad & m_t^* = \frac{\gamma}{1 + \gamma} \frac{1 + \mu}{1 + \mu - \beta} \left[(1 - \beta) \frac{A_0}{p_0^*} + y - \tau \right]; \\ (vii) \quad & M_t^* = (1 + \mu)^{t+1} M_{-1}^g = \frac{\gamma}{1 + \gamma} \frac{(1 + \mu)^{t+1}}{1 + \mu - \beta} [(1 - \beta) A_0 + p_0^*(y - \tau)]; \end{aligned}$$

for every $t = 0, 1, \dots$

Proof. (i) I can easily ascertain it by Lemma 3.

(ii) follows from Lemma 5.

(iii) This is a direct corollary of Lemma 4 and 5.

(iv) holds from the definition of the initial wealth, (i) and Lemma 1.

Since proofs of (v) – (vii) are quite easy, I omit them here. ■

By Assumption B, if $p_0^* > 0$, all endogenous variables are also positive.

I can solve p_0^* , by using the definition of w_0 $\left(= \frac{A_0}{p_0^*} + \frac{y-\tau}{1-\beta}\right)$ and $\delta_0 m_0^* = \frac{\gamma(1-\beta)}{1+\gamma} w_0$:

$$\begin{aligned} p_0^* &= \frac{(1+\gamma)(1+\mu-\beta)M_{-1}^g - \gamma(1-\beta)D_0^g}{\gamma(y-\tau)} \\ &= \frac{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g}{\beta\gamma(y-\tau)}, \end{aligned} \quad (11)$$

where I consider $1+i_{-1} = (1+\mu)/\beta$. Thus,

$$\frac{A_0}{p_0^*} = \frac{\beta\gamma M_{-1}^g + \gamma(1+\mu)B_{-1}^g}{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g}(y-\tau). \quad (12)$$

By employing Lemma 9, (11) and (12), I obtain the equilibrium path explicitly.

Theorem 1 *An equilibrium path is given as below*

$$\begin{aligned} (i) \quad 1+r_t^* &= \frac{1}{\beta}; & (ii) \quad 1+i_t^* &= \frac{1+\mu}{\beta}; & (iii) \quad \delta_t^* &= \frac{1+\mu-\beta}{1+\mu}; \\ (iv) \quad M_t^* &= (1+\mu)^{t+1} M_{-1}^g; \\ (v) \quad p_t^* &= (1+\mu)^t \frac{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g}{\beta\gamma(y-\tau)}; \\ (vi) \quad w_t^* &= \frac{\beta(1+\gamma)(1+\mu-\beta)M_{-1}^g}{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g} \frac{y-\tau}{1-\beta}; \\ (vii) \quad c_t^* &= \frac{\beta(1+\mu-\beta)M_{-1}^g}{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g} (y-\tau); \\ (viii) \quad m_t^* &= \frac{\gamma\beta(1+\mu)M_{-1}^g}{\beta(1+\mu-\beta+\gamma\mu)M_{-1}^g - \gamma(1-\beta)(1+\mu)B_{-1}^g} (y-\tau); \end{aligned}$$

for every $t = 0, 1, 2, \dots$

Next, I focus the most basic case $\mu = 0$, and obtain the following.

Corollary 1 *Under the constant money supply regime, an equilibrium path is given as below*

$$\begin{aligned}
(i) \quad 1 + r_t^* &= \frac{1}{\beta}; & (ii) \quad 1 + i_t^* &= \frac{1}{\beta}; & (iii) \quad \delta_t^* &= 1 - \beta; & (iv) \quad M_t^* &= M_{-1}^g; \\
(v) \quad p_t^* &= \frac{1 - \beta}{\beta} \frac{\beta M_{-1}^g - \gamma B_{-1}^g}{\gamma(y - \tau)}; & (vi) \quad w_t^* &= \frac{\beta(1 + \gamma) M_{-1}^g (y - \tau)}{\beta M_{-1}^g - \gamma B_{-1}^g (1 - \beta)}; \\
(vii) \quad c_t^* &= \frac{\beta M_{-1}^g}{\beta M_{-1}^g - \gamma B_{-1}^g} (y - \tau); & (viii) \quad m_t^* &= \frac{\beta}{1 - \beta} \frac{\gamma M_{-1}^g}{\beta M_{-1}^g - \gamma B_{-1}^g} (y - \tau);
\end{aligned}$$

for every $t = 0, 1, 2, \dots$

4 NPG Condition

In this section, I provide the main results of this paper, i.e. the critical level of the government's fiscal deficit that can be compatible with a dynamic general equilibrium and the intertemporal budget constraint of the government. Further, for my analyses to be meaningful, it is necessary that all endogenous variables are guaranteed to be positive. I reveal the conditions that are required on the parameters in this model.

I can find that, by (11), the condition $p_0^* > 0$ can be transformed as follows,

$$\begin{aligned}
p_0^* > 0 &\iff \beta(1 + \mu - \beta + \gamma\mu) M_{-1}^g - \gamma(1 - \beta)(1 + \mu) B_{-1}^g > 0 & (13) \\
&\iff B_{-1}^g < \frac{\beta(1 + \mu - \beta + \gamma\mu)}{\gamma(1 - \beta)(1 + \mu)} M_{-1}^g.
\end{aligned}$$

Furthermore, by the examination in Appendix, I obtain

$$NPG \iff B_{-1}^g \leq \frac{\beta(1 + \mu - \beta + \gamma\mu)(\tau - g) + \mu\gamma(y - \tau)}{\gamma(1 - \beta)(1 + \mu)(y - g)} M_{-1}^g. \quad (14)$$

The nonnegativity condition (13) and the NPG condition (14) provide a set of conditions on the parameters of the model to guarantee the validity of my analyses. Fortunately, under Assumption B and E, if the NPG condition (14) is satisfied, (13) is also met. Hence, the condition required on the parameters is simply summarized as (14). I have obtained the following theorem.

Theorem 2 *The limitation level of the government's borrowing that is sustainable and compatible with a dynamic general equilibrium is given by (14).*

Next, I will concentrate on the most basic case $\mu = 0$. In this case, the condition required on the parameters (14) can be simply represented by

$$B_{-1}^g \leq \frac{\beta \tau - g}{\gamma y - g} M_{-1}^g. \quad (15)$$

By (8), I can transform the RHS of (15) into

$$\text{RHS of (15)} = \frac{1}{i^*} \frac{\tau - g}{y - g} p^* c^*. \quad (16)$$

Corollary 2 *In the case of $\mu = 0$, the critical level of the government's borrowing that is sustainable and compatible with a dynamic general equilibrium is given by (15), and the upper bound in an equilibrium can be rewritten as (16).*

The main results, Theorem 2 and Corollary 2, can be interpreted as follows; (i) Given the fundamentals of the economy $(y, \gamma, M_{-1}^g, B_{-1}^g)$, the government must set up its expenditure plan g and tax policy τ to satisfy (14) or (15). (ii) Given y, γ , the future plan for expenditure g , imposition of tax τ and current national debt balance B_{-1}^g , the government must increase its money supply immediately till the level that satisfies (14) or (15).

5 Appendix — Derivation of the NPG Condition

In Appendix, I derive the NPG condition (14). To this aim, I derive the equilibrium dynamics of the bond path at the outset. By the flow budget constraint of the government, the equilibrium bond dynamics must follow the following difference equation:

$$B_t^g = \frac{1 + \mu}{\beta} B_{t-1}^g + (1 + \mu)^t [p_0^*(g - \tau) - \mu M_{-1}^g], \quad (17)$$

for every $t = 0, 1, \dots$, where the initial level of the nominal interest rate $1 + i_{-1}$ is supposed to be $(1 + \mu) / \beta$. The solution to the difference equation

(17) is given by

$$B_t^g = \left(\frac{1+\mu}{\beta}\right)^{t+1} B_{-1}^g + (1+\mu)^t [p_0^*(g-\tau) - \mu M_{-1}^g] \sum_{s=0}^t \beta^{-s}, \quad (18)$$

for every $t = 0, 1, \dots$

By (18), I can easily obtain the equilibrium dynamics of the nominal debt:

$$\begin{aligned} D_t^g &= M_{t-1}^g + (1+i_{t-1}) B_{t-1}^g \\ &= (1+\mu)^t M_{-1}^g \\ &\quad + \frac{1+\mu}{\beta} \left[\left(\frac{1+\mu}{\beta}\right)^t B_{-1}^g + (1+\mu)^{t-1} [p_0^*(g-\tau) - \mu M_{-1}^g] \sum_{s=0}^{t-1} \beta^{-s} \right] \\ &= (1+\mu)^t M_{-1}^g + \left(\frac{1+\mu}{\beta}\right)^{t+1} B_{-1}^g + (1+\mu)^t [p_0^*(g-\tau) - \mu M_{-1}^g] \sum_{s=1}^t \beta^{-s} \end{aligned}$$

for $t = 0, 1, \dots$. Thus, the dynamics of the government's real debt can be explicitly described as

$$\begin{aligned} \frac{D_t^g}{p_t^*} &= \frac{D_t^g}{(1+\mu)^t p_0^*} \\ &= \frac{M_{-1}^g}{p_0^*} + \left(\frac{1}{\beta}\right)^t \frac{1+\mu}{\beta} \frac{B_{-1}^g}{p_0^*} + \left[g - \tau - \mu \frac{M_{-1}^g}{p_0^*} \right] \sum_{s=1}^t \beta^{-s}. \end{aligned} \quad (19)$$

Using the fact that $1+r_t = \beta^{-1}$ for every $t = 0, 1, \dots$, I find that the NPG for $t = 0$ can be rewritten as

$$\begin{aligned} NPG &\iff \lim_{t \rightarrow \infty} \beta^t \frac{D_t^g}{p_t^*} \leq 0 \\ &\iff \frac{1+\mu}{\beta} \frac{B_{-1}^g}{p_0^*} + \left[g - \tau - \mu \frac{M_{-1}^g}{p_0^*} \right] \frac{1}{1-\beta} \leq 0. \end{aligned} \quad (20)$$

By using (11), I obtain (14):

$$NPG \iff \frac{B_{-1}^g}{M_{-1}^g} \leq \frac{\beta(1+\mu-\beta+\gamma\mu)(\tau-g) + \mu\gamma(y-\tau)}{\gamma(1-\beta)(1+\mu)(y-g)}.$$

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