Quantum state during and after the False Vacuum Decay

(**Thesis**)

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* T. Tanaka, M. Sasaki and K. Yamamoto,
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* M. Sasaki, T. Tanaka, K. Yamamoto and J. Yokoyama,
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* M. Sasaki, T. Tanaka, K. Yamamoto and J. Yokoyama,
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* K. Yamamoto, to be published in Prog. Theor. Phys. Vol.**91** No.3 (March).

Abstract

The transition of a quantum state associated with the false vacuum decay of a metastable system is investigated based on the WKB wave functional approach. In a covariant manner, we reformulate the WKB wave function for a multidimensional tunneling system with finite degrees of freedom, which describes the quasi-ground state of the system. Then we extend the formalism to the case of a field theory and develop a systematic method to construct the wave functional which determines the quantum state after the false vacuum decay. A clear interpretation of the resulting quantum state is given in the language of the conventional second quantized picture. Using this formalism, we investigate the quantum state during and after the nucleation of an $O(4)$-symmetric bubble. We find that the quantum state inside the nucleated bubble is Lorentz-invariant but very different from the Minkowski vacuum. There exists a family of hypersurfaces on which the energy density is constant as a consequence of the invariance of the state, and the expectation value of the energy momentum tensor behaves like radiation. Then we find the possibility of creating a homogeneous and isotropic open universe through the nucleation of an $O(4)$-symmetric bubble. To extend this investigation to more general cases, we study tunneling phenmenon in simple quantum mechanical systems with emphasis on the interaction between the tunneling mode and that coupled to it. Analysis using the WKB wave function shows that the energy is transferred to the tunneling mode, and implication to a false vacuum decay in the presence of field excitations before the tunneling is discussed.

CONTENTS

1. Introduction

The quantum tunneling is an old problem as the quantum mechanics itself. At present, it is important in many areas such as nuclear physics [1], the scanning tunneling microscopy [2,3], mesoscopic device in solid state physics [4,5], chemical reaction [6,7,8] and biology [6,9]. In the field of cosmology, too, a tunneling process is important because it is believed that there were several epochs at which the universe underwent phase transitions at its early stage. Among such phase transitions, most influential ones are those caused by the decay of a false vacuum, *i.e.*, a first-order phase transition of a metastable vacuum to a more stable state through the tunneling. The particular interest is taken in the false vacuum decay during an inflationary stage of the universe [10], which was recently revived as the extended inflation model [11] and subsequently in several other scenarios [12,13].

The false vacuum decay is a tunneling phenomenon in field theory, whose dynamics was first studied by Voloshin, Kovzarev and Okun [14], and a method to calculate the bubble nucleation rate and to describe the dynamics of a nucleated bubble was given by Coleman [15] and by Callan and Coleman [16], using the Euclidean path integral. Subsequently, a number of efforts were made to study false vacuum decay in various situations [17-28]. One of the most important results is the fact that the decay rate is predominantly given by the path integral around a Euclidean classical solution with $O(4)$-symmetry, called an $O(4)$-symmetric bounce. In particular, in the situation when the effect of gravity can be neglected, it was proved rather generally that the $O(4)$-symmetric bounce solution has the minimum action among the Euclidean classical solutions [17] and has a unique negative mode around it [18]. The $O(4)$-symmetric bounce solution has a bubble-like structure, with the field value approaching to that of the false vacuum at the Euclidean infinity outside the bubble and close to that of the true vacuum inside the bubble. When this solution is analytically continued to the corresponding Lorentzian solution, it describes the motion of a nucleated bubble. Because of the $O(4)$-symmetry of the bounce solution, it has the $O(3,1)$-invariance, *i.e.*, Lorentz invariance, and expands with a hyperbolic trajectory.

However, this description of a false vacuum decay and the subsequent motion of a vacuum bubble is merely the lowest-order WKB picture of the system, as is clear from the fact that it entirely owes to the classical solution of the field equation. What one would expect is that the quantum state with the other infinite degrees of freedom would be significantly affected by the drastic change in a vacuum state and becomes highly non-trivial after the false vacuum decay.

Study of such a higher order effect was initiated by Rubakov [27] and Vachaspati and Vilenkin [28]. They introduced a system consisting of two interacting real scalar fields σ and ϕ in the Minkowski background, and studied the effect on the quantum state of ϕ due to the false vacuum decay of the σ-field. Although their approaches are very interesting, neither is satisfactory. Rubakov studied it using the unjustified method of non-unitary Bogoliubov transformation, and focused on evaluating the number of particles created during the false vacuum decay by defining a particle in terms of the instantaneous Hamiltonian diagonalization. The latter procedure is time- and observer-dependent, and the concept of particle is quite ambiguous in the presence of interaction. In particular, the $O(4)$-symmetry of a bubble is not respected in his method. On the other hand, Vachaspati and Vilenkin discussed the quantum state by solving the functional Schrödinger equation with full respect to the symmetry of the bubble. However, it is not clear if their boundary condition for the wave functional corresponds to the false vacuum state before its decay. Further, their analysis focused on the quantum state outside the bubble, while a matter of more importance is that inside the bubble.

In the present paper, in order to improve the investigation of this problem, we construct different formalism. We start with the Schrödinger equation in a multidimensional tunneling system with finite degrees of freedom, and reformulate the method to construct the WKB wave function [29,30,31,32], which describes the quasi-ground state of a metastable system. Here the quasi-ground state means the lowest energy state sufficiently localized at the meta-stable vacuum minimum. Keeping in mind that gravity should be consistently taken into account ultimately in the cosmological context, we develop the formalism in a covariant manner so that it would be applicable to that case as well. We then extend the result to the field theory, and develop a general framework to find the quasi-ground state wave functional, Ψ, in both the classically forbidden and allowed regions. In the field theory we give an interpretation of Ψ in the second quantized picture, and develop a systematic method to construct the mode function which describes the quantum state after the false vacuum decay. We emphasize that our formalism can give a clear answer to the problem, as we are dealing with the wave function(al) itself and the boundary condition is explicitly taken into account.

This thesis is organized as follows. In §2, we give a derivation of the WKB wave function in a multidimensional tunneling system (multidimensional tunneling wave function)[32]. In §3, we formally extend our formalism to field theory, and develop a systematic method to construct the mode functions which determine the quantum state after the tunneling. Then the result is interpreted in the language of the conventional second quantization picture. For simplicity, we here introduce the tunneling field σ and the other field ϕ coupled with that, and examine the effect on the quantum state of ϕ due to a false vacuum decay of σ-field. In §4, we consider a spatially homogeneous decay of a false vacuum as an example of the field theoretical case, and show that the quantum state after the tunneling generally contains a spectrum of field excitations. The model considered there is much simplified, but contains the essence. The resulting spectrum of the excitation is similar to the thermal spectrum. The same problem was examined by Rubakov, and justification of his peculiar approach is described briefly in Appendix A. In §5, we consider the false vacuum decay associated with $O(4)$-symmetric bubble nucleation [33]. A specific model under the thin-wall approximation allows us an analytic treatment, and we find that the resulting state is Lorentz invariant (*i.e.*, $O(3,1)$-invariant) but is different from the Minkowski vacuum, as expected. We calculate the two-point function and the expectation value of the energy momentum tensor inside the vacuum bubble, with full use of the $O(3,1)$-symmetry: An alternative, more explicit evaluation of the energy momentum tensor by the point-splitting method is also given in Appendix B. Then we find the field excitation in the vacuum bubble from the energy momentum tensor that behaves like radiation at late times. Due to the Lorentz invariance, there exists a family of hypersurfaces over which the energy density is constant. Hence we point out that the nucleating process of the $O(4)$-symmetric vacuum babble can be regarded as the process creating a homogeneous and isotropic open universe filled with radiation, and discuss the possibility to provide a model of our universe [34]. In §6, attempting to generalize these investigations, we study the quantum tunneling from a general excited state in very simple two-dimensional quantum mechanical systems. The final section is devoted to the summary and discussions of the results and the issues that should be solved. In Appendix C, a solution of mathematical reduction formulae is explained, associated with that in §6. We note here that the signature of Lorentzian(Euclidean) metric is taken as $-+++(++++)$.

2. Construction of a Multidimensional Tunneling Wave Function

Let us begin with the construction of the WKB wave function in a multidimensional tunneling system (multidimensional tunneling wave function). Our approach is essentially based on the Gervais and Sakita's formalism, which introduces the concept of the tunneling path, *i.e.*, a classical trajectory in the configuration space, and evaluate the fluctuation of the next order wave function along the tunneling path. We reformulate the multidimensional tunneling wave function in an alternative covariant manner, which will be useful in the future investigation taking the gravity into consideration.

We develop the formalism in the system which has the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}(\phi)\dot{\phi}^\alpha \dot{\phi}^\beta - V(\phi) \qquad (\alpha = 0, \cdots, D), \tag{2.1}$$

where ϕ^α's are the coordinates of the $(D+1)$-dimensional space of dynamical variables (*i.e.*, superspace) and $g_{\alpha\beta}(\phi)$ is the superspace metric. We assume that the signature of the metric is positive definite. The potential $V(\phi)$ is supposed to have a local minimum at $\phi^\alpha = \phi^\alpha_{LM}$. Figure 1 is the case $D = 1$. We call it the local potential minimum or the false(metastable) vacuum minimum throughout this thesis, and use the convention that Greek and Latin indices run from 0 to D and from 1 to D, respectively. The Hamiltonian operator is obtained by replacing the conjugate momentum of the Hamiltonian with the differential operator in the coordinate representation. Though there is ambiguity of the operator ordering in this system, we chose it in such a way that the resulting Hamiltonian takes a covariant form;

$$\hat{H} = -\frac{\hbar^2}{2} g^{\alpha\beta}(\phi)\nabla_\alpha \nabla_\beta + V(\phi), \tag{2.2}$$

where $g^{\alpha\beta}(\phi)$ is the inverse of $g_{\alpha\beta}(\phi)$. Note that this is a Hermite operator, if we define the inner product using the integration associated with the invariant volume element.

Now let us begin with the construction of the wave function. Following the WKB ansatz, the wave function is assumed to have the following form in the classically forbidden region,

$$\Psi = e^{-\frac{1}{\hbar}(W^{(0)}+\hbar W^{(1)}+\cdots)}, \tag{2.3}$$

As the static potential is assumed, we should solve the Schrödinger equation,

$$\hat{H}\Psi = E\Psi. \tag{2.4}$$

We solve this equation order by order with respect to $\hbar$. The equation in the lowest order of $\hbar$ becomes

$$-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha W^{(0)}\nabla_\beta W^{(0)} + V(\phi) = E_0. \tag{2.5}$$

Here E_0 is the zeroth-order part of the energy eigenvalue E. This is the Hamilton-Jacobi equation with potential, $-V$, and energy, $-E_0$. The minus sign appears because we are considering classically forbidden region, and taking the ansatz (2.3). Introducing a parameter τ in terms of

$$\frac{d\phi^\alpha(\tau)}{d\tau} := g^{\alpha\beta}\nabla_\beta W^{(0)}, \tag{2.6}$$

(2.5) reads

$$\frac{d^2\phi^\alpha(\tau)}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\dot{\phi}^\beta\dot{\phi}^\gamma = g^{\alpha\beta}\nabla_\beta V, \tag{2.7}$$

where $\Gamma^\alpha{}_{\beta\gamma}$ is the connection coefficient of the superspace metric $g_{\alpha\beta}$ [35]. This is nothing but the classical equation of motion with an imaginary time. Thus τ is called the Euclidean time.

As we consider the case E_0 is chosen to be $V(\phi^\alpha_{LM})$, there exist solutions of the Euclidean equation of motion which start from the false vacuum minimum and reach the region outside the potential barrier. Among the solutions, there is a solution that gives the minimum Euclidean action, which we call the tunneling solution, $\bar{\phi}^\alpha(\tau)$, and its trajectory the tunneling path or the dominant escape path (hereafter DEP). It is the path of least resistance [29] or the most probable escape path [30]. In our case in which E_0 is equal to $V(\phi^\alpha_{LM})$, the tunneling solution is a half way of the bounce solution [15]. We can set parametrization of the Euclidean time so that the tunneling solution leaves the false vacuum minimum, $\phi^\alpha = \phi^\alpha_{LM}$, at $\tau \to -\infty$, and reaches the turning point at $\tau = 0$, without any loss of generality.

Usually, the tunneling process is described by this tunneling solution. Integrating the equation derived from Eqs.(2.5) and (2.6);

$$\frac{dW^{(0)}}{d\tau} = 2\left(V(\phi) - E_0\right), \tag{2.8}$$

the tunneling rate can be naively evaluated by the ratio of the squared amplitude at the turning point to that at the false vacuum origin as

$$\Gamma \sim \exp\left(2\left[W^{(0)}(-\infty) - W^{(0)}(0)\right]\right). \tag{2.9}$$

Next let us turn to the second order Equation;

$$-g^{\alpha\beta}\nabla_\alpha W^{(0)}\nabla_\beta W^{(1)} + \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^{(0)} = \frac{E_1}{\hbar}. \tag{2.10}$$

Here E_1 is $O(\hbar)$ part of the energy eigenvalue E. If solutions of the Euclidean equation of motion (and also $W^{(0)}(\phi^\alpha)$) are known with a sufficient number of integral constants in the vicinity of the tunneling solution, we obtain a congruence of solutions in the superspace. Then we can introduce a set of new coordinates $\{\lambda^{\bar{\alpha}}\} := \{\tau, \lambda^{\bar{n}}\}$ which have one-to-one correspondence to the original coordinates $\{\phi^\alpha\}$, where $\{\lambda^{\bar{n}}\}$ are the coordinates labeling different orbits of the congruence. Using these new coordinates, we find

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta W^{(0)} = \frac{\partial}{\partial\tau}\log\left[\det\left(\frac{\partial\phi^\alpha}{\partial\lambda^{\bar{\beta}}}\right)\sqrt{g}\right], \tag{2.11}$$

where $\sqrt{g}$ is the determinant of $g_{\alpha\beta}$. Then Eq.(2.10) is integrated as

$$W^{(1)} = \frac{1}{2}\log\left[\det\left(\frac{\partial\phi^\alpha}{\partial\lambda^\beta}\right)\sqrt{g}\right] - \frac{E_1}{\hbar}\tau + \text{constant}. \tag{2.12}$$

Therefore the wave function is, to the second lowest order, formally given by

$$\Psi = \frac{C}{\sqrt{\det\left(\frac{\partial\phi^\alpha}{\partial\lambda^\beta}\right)}}\exp\left[-W^{(0)}(\lambda^{\bar{\alpha}})/\hbar + E_1\tau/\hbar\right]. \tag{2.13}$$

This wave function is a general one and we need to choose a congruence of orbits parametrized by $\lambda^{\bar{n}}$ in the vicinity of the DEP which satisfies an appropriate boundary condition at $\tau \to -\infty$. For this purpose, we first expand the wave function (2.13) around the DEP introducing the orthonormal basis along it, and then we require the thus-expanded wave function to have the correct asymptotic behavior at $\tau \to -\infty$, so that it is correctly matched to the ground state wave function at the local potential minimum.

The first step can be achieved by using a technique similar to the Fermi-Walker transport of a vector and by deriving an equation similar to the geodesic deviation equation [35,36]. Consider a set of orthonormal bases $e^\alpha_{[\mu]}(\tau)$ along the DEP; $g_{\alpha\beta}e^\alpha_{[\mu]}e^\beta_{[\nu]} = \delta_{[\mu][\nu]}$, where $[\mu]$ runs through the range $0, 1, \cdots, D$. For notational convenience we introduce another set of indices $(\mathbf{0}, \mathbf{a})$ to denote $[\mu]$. We choose $e^\alpha_{\mathbf{0}}$ to be the unit vector tangent to the DEP ;

$$e^\alpha_{\mathbf{0}} := \frac{N^\alpha}{N}, \tag{2.14}$$

where

$$\begin{aligned} N^\alpha &:= \frac{d\phi^\alpha}{d\tau} = g^{\alpha\beta}\nabla_\beta W^{(0)}, \\ N^2 &:= N_\alpha N^\alpha = \nabla_\alpha W^{(0)}\nabla^\alpha W^{(0)} = 2(V - E_0). \end{aligned} \tag{2.15}$$

If we define a differential operator $D_F/\partial\tau$ for a vector X^α as

$$\frac{D_F}{\partial\tau}X^\alpha := \frac{D}{\partial\tau}X^\alpha + \frac{N^\alpha}{N^2}X_\beta\frac{D}{\partial\tau}N^\beta - \frac{X_\beta N^\beta}{N^2}\frac{D}{\partial\tau}N^\alpha, \tag{2.16}$$

where $D/\partial\tau = N^\alpha\nabla_\alpha$ is the covariant derivative tangent to the DEP, it is easily seen that $D_F e^\alpha_{\mathbf{0}}/\partial\tau = 0$. Hence we can choose all the basis vectors $e^\alpha_{[\mu]}$ along the

DEP to satisfy

$$\frac{D_F}{\partial \tau} e^{\alpha}_{[\mu]} = 0. \tag{2.17}$$

(The basis vectors thus defined satisfy the relation $e_{\alpha \mathbf{a}} \acute{e}^{\alpha}_{\mathbf{b}} = 0$ along the DEP [30]) Then introduce coordinates around the DEP. At each point q on the DEP, we can find a hypersurface perpendicular to the DEP, $\Sigma(q)$, which is spanned by all possible geodesics tangent to linear combinations of $e^{\alpha}_{\mathbf{a}}$ at q at least in a sufficiently small neighborhood of q (see Fig. 2). Then it is known that there exists an exponential map from the tangent space at q of $\Sigma(q)$ to the hypersurface $\Sigma(q)$ [36], on which we can introduce the Riemann normal coordinates $\eta^{\mathbf{a}}$ with the identification $e^{\alpha}_{\mathbf{a}} \partial/\partial\phi^{\alpha} = \partial/\partial\eta^{\mathbf{a}}$; *i.e.*, the bases $e^{\alpha}_{\mathbf{a}}$ becomes coordinate bases. Hence we have

$$\begin{aligned} \left.\frac{\partial W^{(0)}}{\partial \eta^{\mathbf{a}}}\right|_{\eta^{\mathbf{a}}=0} &= W^{(0)}{}_{;\alpha} e^{\alpha}_{\mathbf{a}} = N_{\alpha} e^{\alpha}_{\mathbf{a}} = 0, \\ \left.\frac{\partial^2 W^{(0)}}{\partial \eta^{\mathbf{a}} \partial \eta^{\mathbf{b}}}\right|_{\eta^{\mathbf{a}}=0} &= W^{(0)}_{;\alpha\beta} e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} =: \Omega_{\mathbf{ab}}, \end{aligned} \tag{2.18}$$

where the semicolon denotes covariant differentiation with respect to the metric $g_{\alpha\beta}(\phi)$. Consequently $W^{(0)}(\lambda^{\bar{a}})$ is expanded as

$$W^{(0)}(\lambda^{\bar{a}}) = W^{(0)}(\tau) + \frac{1}{2}\Omega_{\mathbf{ab}} \eta^{\mathbf{a}} \eta^{\mathbf{b}} + \cdots. \tag{2.19}$$

Now we show how the matrix $\Omega_{\mathbf{ab}}$ is determined in the above expression. First we set

$$z^{\alpha}_{\bar{\mu}} := \frac{\partial \phi^{\alpha}}{\partial \lambda^{\bar{\mu}}}. \tag{2.20}$$

Then a straight-forward calculation yields the following equation for $z^{\alpha}_{\bar{\mu}}$ along the DEP,

$$\begin{aligned} \frac{D}{\partial \tau} z^{\alpha}_{\bar{\mu}} &= N^{\alpha}{}_{;\beta} z^{\beta}_{\bar{\mu}}, \\ \frac{D^2}{\partial \tau^2} z^{\alpha}_{\bar{\mu}} &= V^{;\alpha}{}_{;\beta} z^{\beta}_{\bar{\mu}} - R^{\alpha}{}_{\sigma\beta\gamma} N^{\sigma} N^{\gamma} z^{\beta}_{\bar{\mu}}, \end{aligned} \tag{2.21}$$

where we follow the convention of [36] for the Riemann tensor. The second equation is similar to the geodesic deviation equation, except for the first term on the r.h.s.

This is because the DEP is the solution for the equation of motion in the space with the potential, which makes the DEP to be different from the geodesics. In deriving this equation, we used the Euclidean equation of motion (2.7), which now should read

$$\frac{D}{\partial\tau}\frac{\partial\phi^\alpha}{\partial\tau} - V^{;\alpha} = 0. \tag{2.22}$$

Next we rewrite Eqs.(2.21) in terms of the ordinary partial derivatives along the DEP, we consider the components of $z^\alpha{}_{\bar{n}}$ ($\bar{n} = 1, 2, \cdots, D$) projected in the direction of $e^\alpha_{\mathbf{a}}$;

$$z^{\mathbf{a}}{}_{\bar{n}} := e^{\mathbf{a}}_\alpha z^\alpha{}_{\bar{n}} =: K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)\chi^{\mathbf{b}}{}_{\bar{n}}, \tag{2.23}$$

where $\chi^{\mathbf{b}}{}_{\bar{n}}$ is a τ-independent matrix introduced as a normalization factor of $K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)$. Then it is straightforward to find the equations for $z^{\mathbf{a}}{}_{\bar{n}}$ along the DEP;

$$\frac{d}{d\tau}z^{\mathbf{a}}{}_{\bar{n}} = e^\alpha_{\mathbf{a}} W_{;\alpha\beta} e^\beta_{\mathbf{b}} z^{\mathbf{b}}{}_{\bar{n}} = \Omega_{\mathbf{ab}} z^{\mathbf{b}}{}_{\bar{n}}\,, \tag{2.24}$$

$$\frac{d^2}{d\tau^2}z^{\mathbf{a}}{}_{\bar{n}} = V_{;\mathbf{ab}} z^{\mathbf{b}}{}_{\bar{n}} - 3N^{-2} V_{;\mathbf{a}} V_{;\mathbf{b}} z^{\mathbf{b}}{}_{\bar{n}} - N^2 R_{\mathbf{a0b0}} z^{\mathbf{b}}{}_{\bar{n}}\,, \tag{2.25}$$

where

$$\begin{aligned} V_{;\mathbf{ab}} &:= e^\alpha_{\mathbf{a}} V_{;\alpha\beta}\, e^\beta_{\mathbf{b}}, \\ V_{;\mathbf{a}} &:= e^\alpha_{\mathbf{a}} V_{;\alpha}\,, \\ R_{\mathbf{a0b0}} &:= e^\alpha_{\mathbf{a}} e^\beta_{\mathbf{b}} R_{\alpha\sigma\beta\gamma} N^\sigma N^\gamma. \end{aligned} \tag{2.26}$$

The first term in the r.h.s. of (2.25) is the effect due to the change of the potential curvature along the DEP, the second is the effect how the DEP is bent in the configuration space, and the third is the effect due to the nonflatness of the space. Note also that the matrix $K^{\mathbf{a}}{}_{\mathbf{b}}$ defined in Eq.(2.23) satisfies exactly the same equations as $z^{\mathbf{a}}{}_{\bar{n}}$ does. Using Eq.(2.24), we express $\Omega_{\mathbf{ab}}$ in terms of $K^{\mathbf{a}}{}_{\mathbf{b}}$; multiplying the both sides of the equation by the inverse of $z^{\mathbf{a}}{}_{\bar{n}}$, we find

$$\Omega_{\mathbf{ab}} = \dot{z}_{\mathbf{a}\bar{n}}\left(z^{-1}\right)^{\bar{n}}_{\mathbf{b}} = \dot{K}_{\mathbf{a}}{}^{\mathbf{c}}\left(K^{-1}\right)_{\mathbf{cb}}, \tag{2.27}$$

where the dot denotes τ-differentiation. It is worth noticing that there exist arbitrariness of constant matrix $\chi^{\mathbf{b}}{}_{\bar{n}}$ to determine the matrix $\Omega_{\mathbf{ab}}$.

Next we express $\det\left|\partial\phi^\alpha/\partial\lambda^{\bar\beta}\right|$ in terms of $K^{\mathbf{a}}{}_{\mathbf{b}}$. In order to do so, we write down the superspace line element in the coordinates $\{\lambda^{\bar\mu}\} = \{\tau, \lambda^{\bar n}\}$ in two different ways;

$$
\begin{aligned}
ds^2 = g_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial\lambda^{\bar\mu}}\frac{\partial\phi^\beta}{\partial\lambda^{\bar\nu}}d\lambda^{\bar\mu}d\lambda^{\bar\nu} &= \left(e^{\mathbf{0}}_\alpha e_{\mathbf{0}\beta} + e^{\mathbf{a}}_\alpha e_{\mathbf{a}\beta}\right)\frac{\partial\phi^\alpha}{\partial\lambda^{\bar\mu}}\frac{\partial\phi^\beta}{\partial\lambda^{\bar\nu}}d\lambda^{\bar\mu}d\lambda^{\bar\nu} \\
&= N^2 d\tau^2 + \delta_{\mathbf{ab}} z^{\mathbf{a}}{}_{\bar n} z^{\mathbf{b}}{}_{\bar m} d\lambda^{\bar n} d\lambda^{\bar m}.
\end{aligned}
\tag{2.28}
$$

Then equating the volume elements in the two expressions, we find

$$
\begin{aligned}
\sqrt{g}\left|\det\left(\frac{\partial\phi^\alpha}{\partial\lambda^{\bar\mu}}\right)\right| &= N\left|\det z^{\mathbf{a}}{}_{\bar n}\right| \\
&= \sqrt{2\left(V(\bar\phi^\alpha(\tau)) - E_0\right)}\ \left|\det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)\right|\left|\det \chi^{\mathbf{c}}{}_{\bar n}\right|.
\end{aligned}
\tag{2.29}
$$

Substituting Eqs.(2.29) and (2.19) into (2.13), we arrive at a desired expression,

$$
\begin{aligned}
\Psi = &\frac{C\ e^{-W^{(0)}(\tau)/\hbar} e^{E_1\tau/\hbar}}{\left(\sqrt{2\left(V(\bar\phi^\alpha(\tau)) - E_0\right)}\left|\det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)\right|\left|\det \chi^{\mathbf{c}}{}_{\bar n}\right|\right)^{1/2}} \\
&\times \exp\left[-\frac{1}{2\hbar}\eta^{\mathbf{a}}\eta^{\mathbf{b}}\Omega_{\mathbf{ab}}(\tau)\right].
\end{aligned}
\tag{2.30}
$$

Now let us turn to the next step, and consider the matching condition for the wave function. As we are interested in the quantum tunneling decay from the false vacuum state in the local potential minimum, we consider construction of the quasi-ground state wave function, which describes the state of the lowest energy sufficiently localized at the false vacuum minimum. Here, we assume that the system can be well approximated by a collection of harmonic oscillators near the false vacuum minimum and quasi-ground state wave function there can be approximated by the ground-state wave function for this collection of harmonic oscillators. This type of matching for the tunneling wave function is explicitly done in the Refs.[32,37]. Specifically we assume that the potential and the superspace metric have the following asymptotic forms near $\phi^\alpha = \phi^\alpha_{LM}$,

$$
\begin{aligned}
V(\phi^\alpha) - E_0 &\to \frac{1}{2}\left(\omega^2\right)_{\alpha\beta}(\phi^\alpha - \phi^\alpha_{LM})(\phi^\beta - \phi^\beta_{LM}), \\
g_{\alpha\beta} &\to g^{(0)}_{\alpha\beta},
\end{aligned}
\tag{2.31}
$$

respectively. Here $g^{(0)}_{\alpha\beta}$ is a constant positive definite metric and $\omega_{\alpha\beta}$ is assumed to be a positive definite matrix. As we can set $g^{(0)}_{\alpha\beta} = \delta_{\alpha\beta}$ without loss of generality, we do so. The ground-state wave function for this system is

$$\Psi = \left(\det \frac{\omega}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2\hbar}\omega_{\alpha\beta}(\phi^\alpha - \phi^\alpha_{LM})(\phi^\beta - \phi^\beta_{LM})\right], \tag{2.32}$$

which should be matched to the WKB wave function (2.30). From the assumption (2.31), the Euclidean equation of motion (2.22) at $\tau \to -\infty$ takes the following form,

$$\frac{\partial^2}{\partial\tau^2}\phi^\alpha = \left(\omega^2\right)_{\alpha\beta}(\phi^\beta - \phi^\beta_{LM}). \tag{2.33}$$

Hence with the boundary condition that $\phi^\alpha(\tau) \to \phi^\alpha_{LM}$ as $\tau \to -\infty$, the relevant solution which describes a congruence along the DEP is given by

$$\phi^\alpha - \phi^\alpha_{LM} = \left(e^{\omega\tau}\right)^\alpha_{\ \beta} C^\beta, \tag{2.34}$$

where C^β are some constants and are related to the arbitrariness of the reparametrization of τ. Integrating the equation $\partial W^{(0)}/\partial\phi^\alpha = \partial\phi^\alpha/\partial\tau = \omega_{\alpha\beta}(\phi^\beta - \phi^\beta_{LM})$, we get

$$W^{(0)}(\phi^\alpha) = \frac{1}{2}\omega_{\alpha\beta}(\phi^\alpha - \phi^\alpha_{LM})(\phi^\beta - \phi^\beta_{LM}), \tag{2.35}$$

where we have set $W^{(0)} = 0$ at the local potential minimum, $\phi^\alpha = \phi^\alpha_{LM}$. This also implies that $\Omega_{\mathbf{ab}} \to \omega_{\alpha\beta} e^\alpha_{\mathbf{a}} e^\beta_{\mathbf{b}} =: \bar{\omega}_{\mathbf{ab}}$ from (2.18). Then from Eq.(2.27), the asymptotic boundary condition that $K^{\mathbf{a}}_{\ \mathbf{b}}$ should satisfy becomes

$$K^{\mathbf{a}}_{\ \mathbf{b}} = \left(e^{\bar{\omega}\tau}\right)^{\mathbf{a}}_{\ \mathbf{c}}\, \kappa^{\mathbf{c}}_{\ \mathbf{b}}, \tag{2.36}$$

where $\kappa^{\mathbf{c}}_{\ \mathbf{b}}$ is a constant matrix. This condition requires that $K^{\mathbf{a}}_{\ \mathbf{b}}$ decreases exponentially at the local potential minimum. This condition is the necessary condition to obtain the wave function which decreases away from the classical trajectory DEP and to match it with the harmonic oscillator wave function at the local potential minimum. Taking the trace of the equation, $\dot{K}^{\mathbf{a}}_{\mathbf{c}}(K^{-1})^{\mathbf{c}}_{\mathbf{b}} = \bar{\omega}_{\mathbf{ab}}$, we find

$$\frac{1}{\det K}\frac{d}{d\tau}\left(\det K\right) = \mathrm{Tr}\,\bar{\omega}. \tag{2.37}$$

Further, using the facts that

$$\begin{aligned} &e^{\alpha}_{\mathbf{0}}\omega_{\alpha\beta}e^{\beta}_{\mathbf{0}} = \frac{1}{N^2}\omega_{\alpha\beta}\dot{\phi}^{\alpha}\dot{\phi}^{\beta} = \frac{1}{2}\frac{\partial}{\partial\tau}\log N^2, \\ &\mathrm{Tr}\,\bar{\omega} = \mathrm{Tr}\,\omega - e^{\alpha}_{\mathbf{0}}\omega_{\alpha\beta}e^{\beta}_{\mathbf{0}}, \end{aligned} \tag{2.38}$$

we can show that the following equality holds in the asymptotic region,

$$\frac{1}{N\det K}\frac{d}{d\tau}\left(N\det K\right) = \mathrm{Tr}\,\omega. \tag{2.39}$$

Integrating this equation, we get

$$\sqrt{2\big(V(\bar{\phi}^{\alpha}(\tau)) - E_0\big)}\,\big|\det K^{\mathbf{a}}_{\ \mathbf{b}}(\tau)\big| = C'\,e^{\mathrm{Tr}\,\omega\,\tau}, \tag{2.40}$$

where C' comes out of the integration constant, and reflects the arbitrariness to normalize the matrix $K^{\mathbf{a}}_{\mathbf{b}}$. Substituting (2.35) and (2.40) into (2.30), and comparing it with the harmonic oscillator wave function (2.32), we find

$$E_1 = \frac{\hbar}{2}\mathrm{Tr}\,\omega, \qquad \frac{C}{\sqrt{|\det \chi^{\mathbf{c}}_{\ \bar{n}}|}} = \left(\det\frac{\omega}{\pi}\right)^{1/4} C'^{1/2}. \tag{2.41}$$

Thus E_1 is the vacuum fluctuation energy of the false vacuum. Finally we obtain the WKB quasi-ground state wave function to the second lowest order, which is matched to the ground state wave function at the false vacuum minimum, as follows,

$$\begin{aligned} \Psi = &\frac{A\left(\det\omega/\pi\right)^{1/4}}{\Big[2\big(V(\bar{\phi}^{\alpha}(\tau)) - E_0\big)\Big]^{1/4}\sqrt{|\det K^{\mathbf{a}}_{\ \mathbf{b}}(\tau)|}} \\ &\times \exp\Big(-\frac{1}{\hbar}\int_{-\infty}^{\tau} d\tau' 2\big(V(\bar{\phi}^{\alpha}(\tau')) - E_0\big) + \frac{1}{2}\mathrm{Tr}\,\omega\tau\Big) \\ &\times \exp\Big(-\frac{1}{2\hbar}\Omega_{\mathbf{ab}}(\tau)\eta^{\mathbf{a}}\eta^{\mathbf{b}}\Big), \end{aligned} \tag{2.42}$$

where $A = C'^{1/2}$, $\Omega_{\mathbf{ab}}$ is expressed in terms of $K^{\mathbf{a}}_{\ \mathbf{b}}$ by (2.27), and it is determined by solving Eq.(2.25) with the boundary condition (2.36), *i.e.*, the exponentially decreasing solution as $\tau \to -\infty$. If we fix the normalization of matrix $K^{\mathbf{a}}{}_{\mathbf{b}}$, then that of wave function, A, is determined. Though we do not fix it here, and keep the arbitrariness of the normalization of the matrix $K^{\mathbf{a}}{}_{\mathbf{b}}$, for later convenience.

We have found the quasi-ground state wave function in the forbidden region. But we want to know the quantum state of the field after the tunneling. We therefore must obtain the wave function in the region beyond the turning point, *i.e.*, classically allowed region (hereafter, following the conventional terminology, we call the classically forbidden region the Euclidean region and the classically allowed region the Lorentzian region). The construction of the general form of the Lorentzian wave function is not much different from that of the Euclidean wave function. That is, the procedure is to construct classical trajectory of leading WKB order first, and evaluate the second order wave function along the classical trajectory. The essential issue is the matching condition at the turning point at which the WKB approximation breaks down. Nevertheless, in the case that the potential varies slowly in the direction of tunneling (τ-direction) around the turning point, the matching problem reduces to that of one-dimensional quantum system [38]. Note that since the wave functional Ψ is the eigen-function of the quasi-ground state, it is real everywhere. Hence, when it is analytically continued to the Lorentzian region, it consists of the *outgoing* and *incoming* wave parts which are complex conjugate to each other. However, since we are interested in the wave functional which represents tunneling out of the barrier from the false vacuum, we focus on the outgoing part. Then the Lorentzian wave function will simply be given by the analytic continuation of the Euclidean wave function, *i.e.*, replacing the Euclidean time parameter τ by the Lorentzian time t with $\tau \to it$. The matching problem for a general case has not been formulated so far and we hope to come back to this issue in future.

3. Field Theoretical Description

In this section, we consider the application of the formulation developed in the previous section to the field theory. For simplicity, we introduce a field σ which undergoes a false vacuum decay and another field ϕ that is coupled to it, then investigate the effect of the false vacuum decay on the latter field. In the beginning, we construct the tunneling wave functional of this system following the procedure in the previous section, and derive the two point correlation function from it as an example of the observables after the tunneling. Then we show that it is possible to interpret the results in the conventional second quantized picture.

Now we write the Lagrangian of the system as

$$\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_\phi \tag{3.1}$$

where

$$\begin{aligned} \mathcal{L}_\sigma &:= -\int d^3\boldsymbol{x}\left[\frac{1}{2}(\partial_\mu\sigma)^2 + U(\sigma)\right], \\ \mathcal{L}_\phi &:= -\int d^3\boldsymbol{x}\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2(\sigma)\phi^2\right]. \end{aligned} \tag{3.2}$$

Here we have assumed that the potential $U(\sigma)$ has the form as shown in Fig.3, and that the false vacuum ($\sigma = \sigma_-$) decays to the true vacuum ($\sigma = \sigma_+$) through the tunneling effect. As is assumed in the previous section, the origin of $U(\sigma)$ is chosen so that $U(\sigma_-) = 0$ (*i.e.*, the quasi-ground state of the false vacuum has the vanishing energy at the lowest WKB order; $E_0 = 0$). The function $m^2(\sigma)$ describes the interaction between the two fields. The Hamiltonian of the system is given by

$$\begin{aligned} \hat{H} = &\int d^3\boldsymbol{x}\left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\sigma^2} + \frac{1}{2}(\partial_i\sigma)^2 + U(\sigma)\right] \\ &+ \int d^3\boldsymbol{x}\left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2(\sigma)\phi^2\right]. \end{aligned} \tag{3.3}$$

Then the quasi-ground state WKB wave functional of the system satisfies the following functional Schrödinger equation,

$$\hat{H}\Psi = E_1\Psi, \tag{3.4}$$

where E_1 is the correction to the energy in the first WKB order (which in reality diverges in field theory, but we will not go into the problem of regularization and renormalization here).

In this system the tunneling path(DEP) is found by the classical solution of the fields, *i.e.*, for the tunneling field the half way of the bounce solution in the classically forbidden (Euclidean) region, which we denote by $\sigma_0(\boldsymbol{x},\tau)$, and its analytic continuation $\tau \to it(t > 0)$ in the classically allowed (Lorentzian) region, and for the ϕ-field $\phi(\boldsymbol{x}) = 0$. To avoid complexity, we neglecte the fluctuation of tunneling field σ. We only take the one degree of freedom in it, *i.e.*, the tunneling solution σ_0, which is parametrized by one parameter τ, then investigate the quantum fluctuation of the ϕ-field. This is the same approximation as that adopted by Rubakov [27] and Vachaspati and Vilenkin [28].

As the tunneling degree of freedom and the fluctuation degrees of freedom is orthogonal from the beginning, we have the following correspondence, to apply the previous formalism to the field theory,

$$\begin{aligned} \bar{\phi}^{\alpha}(\tau) &\to \sigma_0(\boldsymbol{x},\tau), \\ \eta^{\mathbf{a}} &\to \phi(\boldsymbol{x}). \end{aligned} \tag{3.5}$$

Thus, as far as the fluctuating degrees of freedom are concerned, the extension to the field theory is done by replacing the suffix $\mathbf{a}$ with the spatial coordinates $\boldsymbol{x}$. To find the quasi-ground state wave functional, we have to solve the matrix $K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)$, which we denote in the field theory by $K(\boldsymbol{x},\boldsymbol{y};\tau)$. The equation for it is derived as follows. Interpreting the potential of the system as

$$\begin{aligned} V = &\int d^3\boldsymbol{x}\left[\frac{1}{2}(\partial_i\sigma_0)^2 + U(\sigma_0)\right] \\ &+ \int d^3\boldsymbol{x}\left[\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2(\sigma_0)\phi^2\right], \end{aligned} \tag{3.6}$$

$V_{,\mathbf{a}}$ and $V_{,\mathbf{ab}}$ in (2.25) are read as

$$\begin{aligned} V_{;\mathbf{a}} &= \left.\frac{\delta V}{\delta\phi(\boldsymbol{x})}\right|_{\sigma=\sigma_0,\phi=0} = 0 \\ V_{;\mathbf{ab}} &= \left.\frac{\delta^2 V}{\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{y})}\right|_{\sigma=\sigma_0,\phi=0} = \left[-\Delta_{\boldsymbol{x}} + m^2(\sigma_0(\boldsymbol{x},\tau))\right]\delta(\boldsymbol{x}-\boldsymbol{y}), \end{aligned} \tag{3.7}$$

then Eq.(2.25) now,

$$\left[\frac{\partial^2}{\partial\tau^2} + \Delta_{\boldsymbol{x}} - m^2(\sigma_0(\boldsymbol{x},\tau))\right] K(\boldsymbol{x},\boldsymbol{y};\tau) = 0, \tag{3.8}$$

with the boundary condition that it decreases exponentially at $\tau \to -\infty$. Here instead of directly dealing with $K(\boldsymbol{x},\boldsymbol{y};\tau)$, we express $K(\boldsymbol{x},\boldsymbol{y};\tau)$ in terms of a complete set of functions $g_{\boldsymbol{k}}(\boldsymbol{x},\tau)$ which satisfy the same field equation (3.8) with the same boundary condition, together with a complete set of orthonormal spatial harmonics $Y_{\boldsymbol{k}}(\boldsymbol{y})$;

$$K(\boldsymbol{x},\boldsymbol{y};\tau) = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}}(\boldsymbol{x},\tau) Y_{\boldsymbol{k}}(\boldsymbol{y}). \tag{3.9}$$

The boundary condition of $g_{\boldsymbol{k}}(\boldsymbol{x},\tau)$ at $\tau \to -\infty$ is

$$g_{\boldsymbol{k}}(\boldsymbol{x},\tau) \to \exp(\omega_{\boldsymbol{k}}\tau) Y_{\boldsymbol{k}}(\boldsymbol{x}), \tag{3.10}$$

where $\omega_{\boldsymbol{k}} = \sqrt{k^2 + m_-^2}$, $m_-^2 = \lim_{\tau\to-\infty} m^2(\sigma_0(\boldsymbol{x},\tau))$, and $-k^2$ is the eigenvalue of $Y_{\boldsymbol{k}}$:

$$\left(\Delta_{\boldsymbol{x}} + k^2\right) Y_{\boldsymbol{k}}(\boldsymbol{x}) = 0. \tag{3.11}$$

In what follows, we use $x, y, \cdots$ to denote $(\boldsymbol{x},\tau), (\boldsymbol{y},\tau), \cdots$, for notational simplicity.

According to the result obtained in the previous section, the quasi-ground state wave function in the Euclidean region is written as

$$\begin{aligned} \Psi &= A\, e^{-S_0(\tau)/\hbar - S_1(\tau)} \Phi\left[\phi(\cdot),\tau\right]; \\ \Phi\left[\phi(\cdot),\tau\right] &= \frac{e^{E_1\tau/\hbar}}{\sqrt{\det K(\boldsymbol{x},\boldsymbol{y};\tau)}} \exp\left(-\frac{1}{2\hbar}\int d^3\boldsymbol{x} d^3\boldsymbol{y}\, \phi(\boldsymbol{x})\Omega(\boldsymbol{x},\boldsymbol{y};\tau)\phi(\boldsymbol{y})\right), \end{aligned} \tag{3.12}$$

where A is a normalization factor,

$$S_0(\tau) = W^{(0)}(\tau) = \int_{-\infty}^{\tau} d\tau' \mathcal{L}_E[\sigma_0; \tau'], \quad S_1(\tau) = \frac{1}{4} \log |\mathcal{L}_E[\sigma_0; \tau]| , \tag{3.13}$$
$$\Omega(\boldsymbol{x}, \boldsymbol{y}; \tau) = \int d^3 z \partial_\tau K(\boldsymbol{x}, \boldsymbol{z}; \tau) K^{-1}(\boldsymbol{z}, \boldsymbol{y}; \tau),$$

and the Euclidean Lagrangian $\mathcal{L}_E$ is defined as $\mathcal{L}_E[\sigma; \tau] := -\mathcal{L}_\sigma[\sigma; i\tau]$. Using (3.9), $\Omega(\boldsymbol{x}, \boldsymbol{y}; \tau)$ can be expressed in terms of $g_{\boldsymbol{k}}(x)$ as

$$\Omega(\boldsymbol{x}, \boldsymbol{y}; \tau) = \sum_{\boldsymbol{k}} \dot{g}_{\boldsymbol{k}}(x) g_{\boldsymbol{k}}^{-1}(y), \tag{3.14}$$

where $g_{\boldsymbol{k}}^{-1}(y)$ is the inverse of $g_{\boldsymbol{k}}(x)$ such that

$$\sum_{\boldsymbol{k}} g_{\boldsymbol{k}}^{-1}(x) g_{\boldsymbol{k}}(y) = \delta^3(\boldsymbol{x} - \boldsymbol{y}), \qquad \int d^3 \boldsymbol{x}\, g_{\boldsymbol{k}}^{-1}(x) g_{\boldsymbol{p}}(x) = \delta_{\boldsymbol{k}\boldsymbol{p}} \,. \tag{3.15}$$

Thus, to obtain the Euclidean wave functional, all we need to know are the mode functions $g_{\boldsymbol{k}}(x)$.

One important point to note here is that the wave functional Ψ in Eq.(3.12) can be regarded as being composed of two distinct parts; namely, the part $e^{-S_0/\hbar - S_1}$ which describes the background tunneling wave function, and the rest $\Phi[\phi(\cdot), \tau]$ which describes the fluctuation of the ϕ-field. Our wave functional is related to the formalism developed by Rubakov [27]. To solve the Schrödinger equation (3.4), Rubakov put an ansatz that the wave functional is written as a product of the WKB wave function for the tunneling system and the one for the fluctuating system as in Eq.(3.12). With this ansatz, he showed that the wave functional for the fluctuating system $\Phi[\phi(\cdot), \tau]$ satisfies the Euclidean version of the Schrödinger equation with $t \to -i\tau$. In our case, we can also show that $\Phi[\phi(\cdot), \tau]$ satisfies the Euclidean Schrödinger equation as

$$-\frac{\partial}{\partial \tau} \Phi[\phi(\cdot), \tau] = \left(\hat{H}_\phi - E_1 \right) \Phi[\phi(\cdot), \tau],$$

where

$$\hat{H}_\phi := \int d^3\boldsymbol{x} \left[-\frac{\hbar^2}{2} \frac{\delta^2}{\delta\phi^2} + \frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m^2(\sigma_0(\boldsymbol{x},\tau))\phi^2 \right]$$

Thus the Φ-part describes the fluctuation of the ϕ-field. In particular, all the information of the quantum state is contained in the function $\Omega(\boldsymbol{x},\boldsymbol{y};\tau)$. It is also worthwhile to note that the boundary condition of $g_{\boldsymbol{k}}(x)$, Eq.(3.10), correctly reproduces the (quasi-)ground state wave functional before tunneling because $\Omega(\boldsymbol{x},\boldsymbol{y};\tau) \to \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} Y_{\boldsymbol{k}}(\boldsymbol{x}) Y_{\boldsymbol{k}}^{-1}(\boldsymbol{y})$ as $\tau \to -\infty$.

Once we obtain the Euclidean wave functional, the remaining task is to derive the Lorentzian wave functional by matching these two at the turning point $\tau = 0$. This matching procedure can be quite complicated in general. But, as noted in the last of the previous section, when the potential varies slowly in the direction of tunneling (τ-direction), the matching problem reduces to that in the case of one dimensional quantum system [38]. In this case, we can obtain the Lorentzian wave functional, Ψ_L, which represents tunneling out of the barrier form the false vacuum, by the analytic continuation of Ψ with $\tau \to it$. Then introducing a function $v_{\boldsymbol{k}}(x)$ in the Lorentzian region, the complex conjugate of which, $v_{\boldsymbol{k}}^*(x)$, is the analytic continuation of $g_{\boldsymbol{k}}(x)$ with $\tau \to it$, we find

$$\Psi_L = F(it) \exp\left[-\frac{1}{2} \int d^3\boldsymbol{x} d^3\boldsymbol{y} \phi(\boldsymbol{x}) \Omega_L(\boldsymbol{x},\boldsymbol{y};t) \phi(\boldsymbol{y}) \right], \tag{3.16}$$

with

$$\Omega_L(\boldsymbol{x},\boldsymbol{y};t) = -i \sum_{\boldsymbol{k}} \dot{v}_{\boldsymbol{k}}^*(x) v_{\boldsymbol{k}}^{*\,-1}(y), \tag{3.17}$$

where $F(it)$ is a function of t that appears by the procedure of continuation and the dot denotes t-differentiation.

Now we investigate the nature of the quantum state described by Ψ_L. As we are interested in only that of the ϕ-field, let us consider the equal-time two-point correlation function, which is expressed as

$$
\begin{aligned}
G^{(1)}(x,y) &= \frac{\int d\phi(\cdot)\Psi_L^*\{\phi(x),\phi(y)\}\Psi_L}{\int d\phi(\cdot)\Psi_L^*\Psi_L} \\
&= \Big(\Omega_L(\boldsymbol{x},\boldsymbol{y};t)+\Omega_L^*(\boldsymbol{x},\boldsymbol{y};t)\Big)^{-1}+(\boldsymbol{x}\leftrightarrow\boldsymbol{y}), \qquad (3.18)\\
&= \Big(i\sum_k \dot{v}_k(x)v_k^{-1}(y)-i\sum_k \dot{v}_k^*(x)v_k^{*-1}(y)\Big)^{-1}+(\boldsymbol{x}\leftrightarrow\boldsymbol{y}).
\end{aligned}
$$

This expression, as it is, does not give us much information. The reason is that although the functions $v_k(x)$ form a complete set, they are not properly orthonormalized in general. Hence, in order to rewrite Eq.(3.18) in a more comprehensible form, we introduce a set of normal mode functions $u_q(x)$, each of which is a linear combination of $v_k(x)$,

$$u_q(x)=\sum_k {\Lambda_q}^k v_k(x), \qquad (3.19)$$

with $\det {\Lambda_q}^k \neq 0$, and are normalized as

$$-i\int d^3\mathbf{z}\Big(u_q(z)\dot{u}_{q'}^*(z)-\dot{u}_q(z)u_{q'}^*(z)\Big)=\delta_{qq'}. \qquad (3.20)$$

We note that, in principle, these functions can be constructed by Schmidt's orthogonalization procedure. Contracting the both sides of the above normalization condition by the inverse of $u_q(x)$ and $u_{q'}^*(y)$, we find

$$-i\sum_q\Big(\dot{u}_q^*(x)u_q^{*-1}(y)-u_q^{-1}(x)\dot{u}_q(y)\Big)=\sum_q u_q^{-1}(x)u_q^{*-1}(y). \qquad (3.21)$$

Since $\sum_k \dot{v}_k^*(x)v_k^{*-1}(y)=\sum_q \dot{u}_q^*(x){u^*}_q^{-1}(y)$, the equal-time two-point function is expressed in terms of $u_q(x)$ as

$$G^{(1)}(x,y)=\sum_q\Big(u_q(x)u_q^*(y)+u_q^*(x)u_q(y)\Big). \qquad (3.22)$$

This expression coincides with the one for the Heisenberg state $|\Phi\rangle$ defined by

$$\hat{a}_q|\Phi\rangle=0, \qquad \text{for } \forall q, \qquad (3.23)$$

where the field operator $\hat{\phi}(x)$ is expanded as

$$\hat{\phi}(x) = \sum_{\boldsymbol{q}} \left(\hat{a}_{\boldsymbol{q}} u_{\boldsymbol{q}}(x) + \hat{a}_{\boldsymbol{q}}^{\dagger} u_{\boldsymbol{q}}^{*}(x) \right), \tag{3.24}$$

using the annihilation (creation) operator $\hat{a}_{\boldsymbol{q}}$ ($\hat{a}_{\boldsymbol{q}}^{\dagger}$) associated with the mode function $u_{\boldsymbol{q}}(x)$ ($u_{\boldsymbol{q}}^{*}(x)$). Summarizing the results, the quantum state of ϕ-field after the tunneling can be described by a "vacuum" whose positive frequency mode function is given by $u_{\boldsymbol{q}}(x)$, as is done in the second quantization of a field. Note that this mode function $u_{\boldsymbol{q}}(x)$ is generally different from the true positive frequency function after tunneling, say $w_{\boldsymbol{q}}(x)$, if it can be defined. Then $u_{\boldsymbol{q}}(x)$ and $w_{\boldsymbol{q}}(x)$ are related to each other by a non-trivial Bogoliubov transformation. This implies the quantum state after tunneling contains a spectrum of excitations of the field ϕ. We will see this in the following sections considering specific models of false vacuum decay.

4. Homogeneous Decay of a False Vacuum

In this section, we consider a specific example of the tunneling in field theory discussed in the previous section. In particular we consider the case when the decay of a false vacuum occurs homogeneously in the entire universe. The aim is to demonstrate the significance of our formalism and to show how non-trivial the resulting quantum state after tunneling will be, as well as to clarify its relation to the previous work by Rubakov [27]. A false vacuum decay that occurs homogeneously can be realized if we consider a spatially closed universe [21], or it may be regarded as the limiting case of a sufficiently large vacuum bubble compared to the scale of interest.

For simplicity, we choose the potential of the tunneling field as,

$$U(\sigma) = \frac{\lambda}{8}\left(\sigma^2 - \sigma_c^2\right)^2 - \epsilon\frac{\sigma + \sigma_c}{2\sigma_c}, \qquad (\epsilon > 0), \tag{4.1}$$

and assume that ϵ is small enough, so that the true vacuum and the false vacuum are approximately given by $\sigma(\boldsymbol{x}) = \sigma_c$ and $\sigma(\boldsymbol{x}) = -\sigma_c$, respectively, and that the energy difference between these two state is small compared to the height of the barrier, *i.e.*, $U(0) = \lambda\sigma_c^4/8 \gg \epsilon$. This assumption enables us to use the thin-wall approximation as in the inhomogeneous vacuum decay with $O(4)$-symmetric bubble([15],see also next section). We also assume that the background universe is the static spatially closed universe of a radius R_0, neglecting the gravity. The classical solution of the tunneling field in the Euclidean region can be written as

$$\sigma_0(\tau) = \sigma_c \tanh\left(\frac{\sqrt{\lambda}\sigma_c(\tau + T)}{2}\right), \qquad (-\infty < \tau < 0), \tag{4.2}$$

where $T \simeq (1/\sqrt{\lambda}\sigma_c)\log(\lambda\sigma_c^4/\epsilon)$. The classical solution passes through the top of the potential barrier at $\tau = -T$.

As for the coupling between σ-field and ϕ-field, we take the following model,

$$m^2(\sigma) = m_0^2 + \beta_c\sigma, \tag{4.3}$$

here we only consider the case $m^2(\sigma) > 0$ all the time. If we introduce the spherical harmonics on a unit three sphere S_3,

$$\begin{aligned} &[\Delta_{\Omega_3} + j(j+2)]\, Y_{jlm}(\boldsymbol{x}) = 0, \\ &j = 0, 1, \cdots, \quad l = 0, 1, \cdots, j, \quad m = -l, \cdots, l, \end{aligned} \tag{4.4}$$

where Δ_{Ω_3} is the Laplacian operator on the unit three sphere, the advantage of the model of homogeneous decay becomes clear. That is, each mode of ϕ-field decouples with each other, and we find the equation for $g_{\boldsymbol{k}}(x)$

$$\left[\frac{\partial^2}{\partial \tau^2} + k^2 - \left(m_0^2 + \beta_c \sigma_c \tanh\left\{\frac{\sqrt{\lambda}\sigma_c(\tau + T)}{2}\right\}\right)\right] g_{\boldsymbol{k}}(x) = 0, \tag{4.5}$$

with the boundary condition at $\tau \to -\infty$,

$$g_{\boldsymbol{k}}(x) \to e^{\sqrt{k^2 + m_0^2 - \beta_c \sigma_c}\,\tau} Y_{\boldsymbol{k}}(\boldsymbol{x}), \tag{4.6}$$

where we have defined $k^2 = j(j+2)/{R_0}^2$ and denoted $\boldsymbol{k} = (j, l, m)$. Here, to make the problem simpler, we take the limit of thin-wall. That is, taking the limit $T \gg 1/\sqrt{\lambda}\sigma_c$, we approximate the classical solution $\sigma_c(\tau)$ by using the step function. Then Eq.(4.5) can be easily solved to give,

$$g_{\boldsymbol{k}}(x) = \begin{cases} e^{\omega_- \tau} Y_{\boldsymbol{k}}(\boldsymbol{x}) & (-\infty < \tau < -T), \\ \left(A_{\boldsymbol{k}} e^{\omega_+ \tau} + B_{\boldsymbol{k}} e^{-\omega_+ \tau}\right) Y_{\boldsymbol{k}}(\boldsymbol{x}) & (-T < \tau < 0), \end{cases} \tag{4.7}$$

with

$$\begin{aligned} A_{\boldsymbol{k}} &= \frac{\omega_+ + \omega_-}{2\omega_+} e^{(\omega_+ - \omega_-)T}, \\ B_{\boldsymbol{k}} &= \frac{\omega_+ - \omega_-}{2\omega_+} e^{-(\omega_+ + \omega_-)T}, \end{aligned} \tag{4.8}$$

where we have defined $\omega_\pm := \sqrt{k^2 + m_\pm^2}$ and $m_\pm^2 := m_0^2 \pm \beta_c \sigma_c$. Once we obtain a wave function in the Euclidean region, that in the Lorentzian region is obtained by the continuation, $\tau \to it$, as discussed in the previous section. Fortunately, in the present model, we find the following mode function $u_k(x)$ that is properly normalized in the Lorentzian region,

$$\begin{aligned} u_k(x) &= \frac{A_k e^{-i\omega_+ t} + B_k e^{i\omega_+ t}}{\sqrt{2\omega_+(A_k^2 - B_k^2)}} Y_k(x) \\ &= \frac{A_k}{\sqrt{A_k^2 - B_k^2}} w_k(x) + \frac{B_k}{\sqrt{A_k^2 - B_k^2}} w_k^*(x), \end{aligned} \tag{4.9}$$

where $w_k(x)$ is the usual positive frequency mode function,

$$w_k(x) = \frac{e^{-i\omega_+ t}}{\sqrt{2\omega_+}} Y_k(x). \tag{4.10}$$

In this model, since the field after tunneling is a simple massive scalar field, there is no ambiguity in the definition of a particle. This allows us to compare our result with that of Rubakov [27]. The number of created particles is definitely estimated in terms of the Bogoliubov coefficient of $w_k^*(x)$ in the expression (4.9) as follows

$$\begin{aligned} N_k &= \frac{B_k^2}{A_k^2 - B_k^2} \\ &= \frac{(\omega_+ - \omega_-)^2}{(\omega_+ + \omega_-)^2 e^{4\omega_+ T} - (\omega_+ - \omega_-)^2}. \end{aligned} \tag{4.11}$$

In Appendix A, it is shown that this agrees with the result obtained in Rubakov's formalism. Here, we note that the above particle spectrum differs from that in the case of particle creation due to a sudden change of the mass in the real Lorentzian spacetime. The latter would be the case if the false vacuum decay were considered in the classical picture and were assumed to occur suddenly at, say, $t = 0$, in which the number of created particles would be given by N_k of Eq.(4.11) with $T = 0$.

Let us consider some implications of Eq.(4.11). First note that N_k decreases exponentially as the absolute value of T becomes large. In particular, in the limit $\omega_+ \gg \omega_-$ or $\omega_+ \ll \omega_-$, which holds if $m_-^2 \ll m_+^2$ and $k^2 \lesssim m_+^2$, or $m_-^2 \gg m_+^2$ and

$k^2 \lesssim m_-^2$, respectively, N_k takes the same form as the thermal distribution with temperature $1/4T$,

$$N_k \simeq \frac{1}{e^{4\omega_+ T} - 1}. \tag{4.12}$$

However, the behavior in the large momentum limit differs from the thermal spectrum as

$$N_k \simeq \frac{1}{(4k^2/\Delta m^2)^2 \, e^{4\omega_+ T} - 1} \qquad \text{for } k^2 \gg m_\pm^2\,, \tag{4.13}$$

where $\Delta m^2 := m_+^2 - m_-^2$.

To gain a bit more insight into the quantum state after tunneling, let us consider the case when the mass difference between the true and false vacua is small; $|\Delta m^2| \ll m_+^2$. Then, the number of created particles becomes

$$N_k \simeq \left(\frac{\Delta m^2}{4\omega_+^2}\right)^2 e^{-4\omega_+ T}. \tag{4.14}$$

and the energy density due to the created particles of the ϕ-field is given by

$$\begin{aligned} \Delta\mathcal{E} &\simeq \int_0^\infty d^3k \omega_+ N_k \\ &\simeq \Delta m^4 \, e^{-4m_+ T} \int_0^\infty dx \frac{\sqrt{x(x+2x_0)}}{(x+x_0)^2} e^{-4x}, \end{aligned} \tag{4.15}$$

where $x_0 := m_+ T$. Thus, the energy density generated through the tunneling is of $O(\Delta m^4)$ for $x_0 \lesssim 1$, while it becomes negligibly small for $x_0 \gg 1$. Since $1/T$ is related to a certain mass scale M associated with the tunneling field σ, the particle creation is expected to be rather significant for models with $m_+ \lesssim M$. We note that this conclusion qualitatively holds for general values of the masses $m_\pm^2$ as well, though it has been derived by assuming $|\Delta m^2| \ll m_+^2$.

5. False Vacuum Decay with an $O(4)$-Symmetric Bubble

In this section, we consider the case when a false vacuum decays by nucleating an $O(4)$-symmetric bubble in the Minkowski spacetime, assuming the following potential for the tunneling field again

$$U(\sigma) = \frac{\lambda}{8}\left(\sigma^2 - \sigma_c^2\right)^2 - \epsilon\frac{\sigma + \sigma_c}{2\sigma_c}, \qquad (\epsilon > 0), \tag{5.1}$$

with thin-wall approximation $U(0) \gg \epsilon$, for simplicity. Introducing the coordinates in the Euclidean region as

$$\begin{aligned} ds_E^2 &= d\tau^2 + dr^2 + r^2 d\Omega^2 & (-\infty < \tau \leq 0) \\ &= {\xi_E}^2 dT_E^2 + d{\xi_E}^2 + {\xi_E}^2 \cos^2 T_E \, d\Omega^2 & (-\frac{\pi}{2} \leq T_E \leq 0), \end{aligned} \tag{5.2}$$

where $\tau = \xi_E \sin T_E$ and $r = \xi_E \cos T_E$, the tunneling solution of classical field is described by a half way of the bounce solution and is approximately given in the $O(4)$-symmetric form as [15]

$$\sigma_0(\xi_E) = -\sigma_c \tanh\left(\frac{\xi_E - R}{2\Delta R}\right), \tag{5.3}$$

with

$$R = \frac{\sqrt{\lambda}{\sigma_c}^3}{\epsilon}, \qquad \Delta R = \frac{1}{\sqrt{\lambda}\sigma_c}.$$

Since $R/\Delta R = \lambda\sigma_c^4/\epsilon = 8U(0)/\epsilon \gg 1$ by assumption, the solution describes an $O(4)$-symmetric bubble of radius R with very thin wall of thickness ΔR. The name, thin-wall approximation, comes form this fact.

As for the mass term of the ϕ-field, we adopt the following form,

$$m^2(\sigma) = \alpha(\sigma_c^2 - \sigma^2), \tag{5.4}$$

which in the thin-wall limit reduces to

$$m^2(\sigma_0) = m_s\delta(\xi_E - R); \qquad m_s := 4\frac{\alpha\sigma_c}{\sqrt{\lambda}}. \tag{5.5}$$

Thus ϕ-field interacts with σ-field only on the bubble wall, which allows analytic calculations. Note also that because of the thin-wall assumption, $m_s R = 4\alpha\sigma_c^4/\epsilon = 4(\alpha/\lambda)(R/\Delta R) \gg 1$ unless $\alpha \ll \lambda$.

5.1. Construction of the mode function

Now let us find the mode function $g_{\boldsymbol{k}}(x)$ for the above model. We have to solve the Euclidean field equation,

$$\left[\frac{\partial^2}{\partial\tau^2} + \Delta - m_s\delta(\xi_E - R)\right] g_{\boldsymbol{k}}(x) = 0, \tag{5.6}$$

with the following boundary condition at $\tau \to -\infty$.

$$g_{\boldsymbol{k}}(x) = e^{k\tau}\sqrt{\frac{k}{\pi}} j_l(kr) Y_{lm}(\Omega), \tag{5.7}$$

where $j_l(z)$ is the spherical Bessel function, Y_{lm} is the spherical harmonic function on the unit two sphere, and we have chosen the spatial harmonic function $Y_{\boldsymbol{k}}(\boldsymbol{x})$ to be that in the spherical coordinates for later convenience (the eigenvalue $\boldsymbol{k}$ denotes (k, l, m) in this case). Using the $O(4)$-symmetric coordinates (5.2), we rewrite the Euclidean field equation as

$$\begin{aligned}\Bigg[\frac{1}{\cos^2 T_E}\frac{\partial}{\partial T_E}\left(\cos^2 T_E \frac{\partial}{\partial T_E}\right) + \frac{1}{\xi_E}\frac{\partial}{\partial \xi_E}\left({\xi_E}^3\frac{\partial}{\partial \xi_E}\right)& \\ + \frac{\Delta_\Omega}{\cos^2 T_E} - {\xi_E}^2 m_s \delta(\xi_E - R)\Bigg]\, g_{\boldsymbol{k}}(x) = 0,&\end{aligned} \tag{5.8}$$

where Δ_Ω is the Laplacian operator on a unit two sphere. Then it is apparent that $g_{\boldsymbol{k}}$ can be expressed in the form,

$$g_{\boldsymbol{k}}(x) = \sum_{plm} \chi_{plm}(T_E) F_p(\xi_E) Y_{lm}(\Omega), \tag{5.9}$$

where the functions $\chi_{plm}(T_E)$ and $F_p(\xi_E)$ satisfy

$$\begin{aligned}&\left[\frac{1}{\xi_E}\frac{d}{d\xi_E}\left({\xi_E}^3\frac{d}{d\xi_E}\right) - {\xi_E}^2 m_s\delta(\xi_E - R) + \lambda_p\right] F_p(\xi_E) = 0,\\ &\left[\frac{1}{\cos^2 T_E}\frac{d}{dT_E}\left(\cos^2 T_E\frac{d}{dT_E}\right) - \lambda_p - \frac{l(l+1)}{\cos^2 T_E}\right]\chi_{plm}(T_E) = 0,\end{aligned} \tag{5.10}$$

respectively, with λ_p being the eigenvalue of the function $F_p(\xi_E)$. In the absence of interaction, *i.e.*, $m_s = 0$, one has $F_p^{(0)}(\xi_E) = {\xi_E}^{ip-1}$ with the eigenvalue $\lambda_p = p^2 + 1$, where $-\infty < p < \infty$. In the presence of the δ-function mass, we can construct $F_p(\xi_E)$ by matching the function $F_p^{(0)}(\xi_E)$ outside the bubble wall ($\xi_E > R$) to a linear combination of them inside the wall ($\xi_E < R$). The junction condition is

$$\lim_{\delta\to 0}\left(\left.\frac{dF_p(\xi_E)}{d\xi_E}\right|_{\xi_E=R+\delta} - \left.\frac{dF_p(\xi_E)}{d\xi_E}\right|_{\xi_E=R-\delta}\right) = m_s F_p(\xi_E)\Big|_{\xi_E=R}. \tag{5.11}$$

Then the eigen-function $F_p(\xi_E)$ is given as

$$F_p(\xi_E) = \begin{cases} {\xi_E}^{ip-1} & (R \le \xi_E < \infty), \\ {\xi_E}^{ip-1}\left[1 - \frac{m_s R}{2ip}\left(1 - \left(\frac{R}{\xi_E}\right)^{2ip}\right)\right] & (0 < \xi_E \le R), \end{cases} \tag{5.12}$$

with $\lambda_p = p^2 + 1$. On the other hand, the equation for $\chi_{plm}(T_E)$ has following two independent solutions,

$$\chi_{plm}(T_E) = \frac{1}{\sqrt{\cos T_E}}\left(b_1 P_{ip-1/2}^{-l-1/2}(-\sin T_E) + b_2 Q_{ip-1/2}^{-l-1/2}(-\sin T_E)\right), \tag{5.13}$$

where $P_\mu^\nu(z)$ and $Q_\mu^\nu(z)$ are the associated Legendre functions of the first and second kind, respectively, and b_1 and b_2 are constants to be determined by the boundary condition. Here we follow Ref. [39] as for the definition of the Legendre functions. We can determine the coefficient constant b_1 and b_2, by using a wonderful transformation formula given by Gerlach [40];

$$e^{kT}\sqrt{\frac{k}{\pi}}\, j_l(kr) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, \frac{\Gamma(-ip+l+1)P_{ip-1/2}^{-l-1/2}(-\sin T_E)}{\sqrt{2\cos T_E}}\, {\xi_E}^{ip-1} k^{ip-1/2}, \tag{5.14}$$

for $\xi_E > 0$ and $-\sin T_E > 0$. Since the above expression is just that of $g_k(x)$ outside the wall, one readily finds $b_2 = 0$ and the appropriate form of b_1 as

$$b_1 = \frac{\Gamma(-ip+l+1)k^{ip-1/2}}{\sqrt{2}2\pi}. \tag{5.15}$$

Combining the above results, we obtain

$$g_k(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\Gamma(-ip+l+1)P_{ip-1/2}^{-l-1/2}(-\sin T_E)}{\sqrt{2\cos T_E}} F_p(\xi_E) k^{ip-1/2}\, Y_{lm}(\Omega), \quad (5.16)$$

where $F_p(\xi_E)$ is the one given by Eq.(5.12). Now, using the inverse transformation formula;

$$\frac{\Gamma(ip+l+1)P_{ip-1/2}^{-l-1/2}(-\sin T_E)}{\sqrt{2\cos T_E}} = \int_0^{\infty} \frac{du}{\sqrt{\pi}} u^{ip} e^{u\sin T_E} j_l(u\cos T_E), \quad (5.17)$$

which exists for $-\sin T_E > 0$, we can rewrite Eq.(5.16) as

$$g_k(x) = \sqrt{\frac{k}{\pi}} \left(e^{k\tau} j_l(kr) + \theta(R^2 - {\xi_E}^2) \frac{m_s R}{2} \int_1^{R^2/{\xi_E}^2} du e^{k\tau u} j_l(kru) \right) Y_{lm}(\Omega), \quad (5.18)$$

where we have used an identity, $P_{ip-1/2}^{-l-1/2}(z) = P_{-ip-1/2}^{-l-1/2}(z)$, and the integral representation of the step-function,

$$\theta(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dp \frac{e^{ipz}}{p}. \quad (5.19)$$

The above expression of $g_k(x)$ is the desired formula. We can directly check that it satisfies the Euclidean field equation.

To find the mode function in the Lorentzian region normalized properly, we follow the prescription described below. Consider another set of mode functions $f_k(x)$ besides $g_k(x)$ in the Euclidean region which satisfy the same field equation but with a different boundary condition,

$$f_k(x) \to \exp(-\omega_k \tau) Y_k(x). \quad (5.20)$$

Then we can easily normalize $g_k(x)$ and $f_k(x)$ at $\tau \to -\infty$ to satisfy the relation

$$\int d^3x \Big(f_k(x)\partial_\tau g_{k'}(x) - \partial_\tau f_k(x) g_{k'}(x) \Big) = \delta_{k,k'}. \quad (5.21)$$

Since both $g_k(x)$ and $f_k(x)$ satisfy the same field equation, this relation is conserved in the Euclidean time evolution. Now define a function $\tilde{v}_{\boldsymbol{k}}(x)$ in the Lorentzian region by the analytic continuation of $f_{\boldsymbol{k}}(x)$. If $v^*_{\boldsymbol{k}}(x)$, which is the analytic continuation of $g_{\boldsymbol{k}}(x)$, happens to be complex conjugate to $\tilde{v}_{\boldsymbol{k}}(x)$, modulo a constant factor independent of $\boldsymbol{k}$, then we can easily find the orthonormalized mode functions in the Lorentzian region, because the relation (5.21) continues to hold in the Lorentzian region and gives the proper normalization condition. As is clear from the expression (5.18), if one defines the function $f_{\boldsymbol{k}}(x)$ by replacing τ with $-\tau$ in it, it also satisfies the field equation but with the opposite boundary condition at $\tau \to -\infty$; $f_{\boldsymbol{k}}(x) \to e^{-k\tau} Y_{\boldsymbol{k}}(\boldsymbol{x})$. Then these functions satisfy the normalization condition (5.21), and $f_{\boldsymbol{k}}$ and $g_{\boldsymbol{k}}$ become complex conjugate to each other when analytically continued to the Lorentzian region. Thus the present case corresponds to the special case described above.

Thus the orthonormalized mode function in the Lorentzian region is given by

$$u_{\boldsymbol{k}}(x) = \sqrt{\frac{k}{\pi}} \left(e^{-ikt} j_l(kr) + \theta(R^2 - \xi^2) \frac{m_s R}{2} \int_1^{R^2/\xi^2} du e^{-iktu} j_l(kru) \right) Y_{lm}(\Omega), . \tag{5.22}$$

where $\xi = \sqrt{r^2 - t^2}$. Note that this reduces to the Minkowski positive frequency function outside the bubble, *i.e.*, the quantum state outside the bubble is the trivial Minkowski vacuum as it should be, since an observer there would not know if a bubble is nucleated or not. We also note that $u_{\boldsymbol{k}}(x)$ takes the form of a linear combination of the Minkowski positive frequency functions. Hence one might wonder if the state inside the bubble is really non-trivial or not. However, the essential point is that the coefficients of the linear combination are *spacetime-dependent*, and this fact leads to the non-triviality of the state inside the bubble.

Here, we mention a delicate issue associated with this mode function. In the expression (5.22), one finds that the mode function is well-defined only outside the light cone; $\xi^2 = r^2 - t^2 > 0$, and it becomes singular at $\xi = 0$. This problem is originated from the irregular behavior of the mode function $g_{\boldsymbol{k}}(x)$ at $t = r = 0$. Therefore some regularization is necessary in order to make the analytic continuation possible and to make the expression (5.22) meaningful inside the light cone.

We can avoid this problem by the following way. Namely, we perform the analytic continuation at an infinitesimally small time ε before $\tau = 0$ as shown in Fig.4. With this analytic continuation, the regularity of the mode function is recovered; *i.e.*, it is smoothly continued to the region inside the light cone and the orthonormality condition is properly maintained. Note that in terms of this procedure the transformation formula (5.14) can be continued to the Lorentzian region. Then we may compute any physical quantities in the Lorentzian region with the regularized mode functions and only after the completion of a calculation we take the limit $\varepsilon \to 0$.

As for the ill behavior of $u_{\boldsymbol{k}}(x)$ on the light cone, we do not have a rigorous answer why this happens. However, it is not because of the oversimplification of our model, but because of a breakdown of the WKB approximation at the turning point and the $O(4)$-symmetry of the background. In our case, the "turning point" corresponds to the spatial configuration of σ on the hypersurface $\tau = 0$. However, because of the $O(4)$-symmetry, the configuration at points other than $r = 0$ should not be related to the breakdown of the WKB approximation. In other words, the only fixed point under $O(4)$-transformations is $\xi_E = 0$ and it is the only point which remains on the "turning point" hypersurface for any choice of another observer. In any case, the WKB approximation breaks down at this point (also at the light cone), therefore we should not stick to this problem, which lies beyond the scope of our formalism.

Finally, we mention the fact that in our model in which ϕ interacts with σ only on the bubble wall, it can be shown that our mode functions $u_{\boldsymbol{k}}(x)$ are equivalent to the ones obtained by Vachaspati and Vilenkin [28] in which they used different coordinates to express the mode functions.

5.2. Evaluation of the quantum state inside the bubble

As we have obtained the mode function, which describes the quantum state of the ϕ-field, we now investigate the state after the bubble nucleation carefully. Let us start considering the symmetric two-point function (Hadamard's elementary function). Since we are interested in the quantum state inside the true vacuum bubble, we concentrate on the case in which two points are both in the vacuum bubble. As noted before, our mode function is a spacetime-dependent linear combination of the Minkowski positive frequency function. Therefore we can write

$$\begin{aligned} G^{(1)}(x,x') :=& \left\langle \hat{\phi}(x)\hat{\phi}(x') + \hat{\phi}(x')\hat{\phi}(x) \right\rangle \\ =& \sum_{k} \left(u_k(x)u_k^*(x') + (x \leftrightarrow x') \right) \\ =& D^{(1)}(x-x') \\ &+ \frac{m_s R}{2} \left(\int_1^{R^2/\zeta} D^{(1)}(ux - x')du + \int_1^{R^2/\zeta'} D^{(1)}(x - vx')dv \right) \\ &+ \left(\frac{m_s R}{2} \right)^2 \int_1^{R^2/\zeta} \int_1^{R^2/\zeta'} D^{(1)}(ux - vx')dudv, \end{aligned} \tag{5.23}$$

where $\zeta = x^2 = -t^2 + r^2$, $\zeta' = x'^2 = -t'^2 + r'^2$, and

$$D^{(1)}(x - x') := \frac{1}{2\pi^2} \frac{1}{(x - x')^2}, \tag{5.24}$$

is the Minkowski two-point function for a massless field [41]. Then we find

$$\begin{aligned} &G^{(1)}(x,x') \\ &= \frac{1}{2\pi^2} \left\{ \frac{1}{\epsilon^2} + \frac{m_s R}{2} \left(\int_1^{R^2/\zeta} \frac{du}{\zeta u^2 - 2\eta u + \zeta'} + \int_1^{R^2/\zeta'} \frac{dv}{\zeta' v^2 - 2\eta v + \zeta} \right) \right. \\ &\left. + \left(\frac{m_s R}{2} \right)^2 \int_1^{R^2/\zeta} du \int_1^{R^2/\zeta'} dv \frac{1}{\zeta u^2 - 2\eta uv + \zeta' v^2} \right\}, \end{aligned} \tag{5.23}$$

where $\eta = x \cdot x'$ and $\epsilon^\mu = x^\mu - x'^\mu$.

To obtain more specific information of the quantum state described by the above two-point function, let us consider the coincidence limit of it. In the limit

$\epsilon^\mu \to 0$, the integrals in the above expression can be explicitly performed (see Appendix B). Then regularizing the result by separating out divergent terms, we obtain

$$\begin{aligned}\left\langle \phi(x)^2 \right\rangle_{\text{reg}} =& \frac{1}{2} G^{(1)}_{\text{reg}}(x,x) \\ =& -\frac{1}{4\pi^2}\left\{ \frac{m_s R}{2} \frac{2}{R^2-\xi^2} \right. \\ &+ \left. \left(\frac{m_s R}{2}\right)^2 \frac{1}{\xi^2} \ln\left(1-\frac{\xi^2}{R^2}\right)^2 \right\} + C_1 + C_2 \frac{1}{\xi^2},\end{aligned} \tag{5.25}$$

where C_1 and C_2 are arbitrary constants which come from the divergent terms. One finds $\left\langle \phi^2 \right\rangle_{\text{reg}}$ diverges on the bubble wall ($\xi = R$) and on the light cone ($\xi = 0$) if $C_2 \neq 0$. The former divergence is due to the oversimplification of our model in which ϕ has the δ-function mass at the bubble wall. Hence it will disappear if a more realistic model is considered. However, the divergence on the light cone is real (at least in the sense of the WKB approximation) and unless there is a good reason to put $C_2 = 0$, it cannot be removed in any other models as discussed in the end of previous subsection.

The situation becomes clearer when we consider the expectation value of the energy momentum tensor $\langle T^{\mu\nu} \rangle$, which we now turn to evaluate. A possible reasoning to get rid of the divergence on the light cone is discussed in the end. For the time being, we circumvent the difficulty by focusing on the spacetime region between the light cone and the bubble wall; $0 < \xi^2 < R^2$. In order to obtain a regularized expression for $\langle T^{\mu\nu} \rangle$, it is customary to use the point-splitting method. However, the manipulation is quite involved, while it turns out that the same result can be obtained with a much simpler method. This method takes a full advantage of the fact that the resulting $\langle T^{\mu\nu} \rangle_{\text{reg}}$ should be Lorentz-invariant and it should satisfy the energy momentum conservation law. Hence we present it in the following. For completeness, the point-splitting regularization of $\langle T^{\mu\nu} \rangle$ is described in Appendix B. First we decompose $T_{\mu\nu}$ into the trace and traceless parts;

$$T_{\mu\nu} = S_{\mu\nu} + \eta_{\mu\nu} S, \tag{5.26}$$

with

$$
\begin{aligned}
S_{\mu\nu} &:= \phi_{,\mu}\phi_{,\nu} - \frac{1}{4}\eta_{\mu\nu}(\phi_{,\alpha})^2, \\
S &:= -\frac{1}{4}(\phi_{,\alpha})^2 = -\frac{1}{8}(\phi^2)^{,\alpha}_{\ ,\alpha},
\end{aligned}
\tag{5.27}
$$

where $\eta_{\mu\nu}$ is the Minkowski metric and we have used the field equation; $\phi^{,\alpha}_{\ ,\alpha} = 0$, in the last equality in the expression for S. Note that the only independent component of $S_{\mu\nu}$ is the (ξ, ξ)-component, because of the Lorentz invariance. That is, introducing the coordinates,

$$
ds^2 = -\xi^2 dT_s^2 + d\xi^2 + \xi^2 \cosh^2 T_s \, d\Omega^2, \tag{5.28}
$$

where $T_s = \operatorname{arctanh}(t/r)$ (see Fig.5), the (ξ, ξ)-component is written as

$$
S^\xi_\xi = S_{\xi\xi} = S_{\mu\nu}\left(\frac{\partial x^\mu}{\partial \xi}\right)\left(\frac{\partial x^\nu}{\partial \xi}\right) = S_{\mu\nu}\frac{x^\mu x^\nu}{\xi^2}. \tag{5.29}
$$

Taking this into consideration, we can express $T_{\mu\nu}$ as

$$
T^{\mu\nu} = -\frac{1}{3}S^\xi_\xi\left(\eta^{\mu\nu} - 4\frac{x^\mu x^\nu}{\xi^2}\right) + \eta^{\mu\nu}S, \tag{5.30}
$$

where S^ξ_ξ and S are functions of only ξ. Then the energy momentum conservation law gives

$$
\left(\xi^4 S^\xi_\xi\right)_{,\xi} = -\xi^4 S_{,\xi}. \tag{5.31}
$$

Here we take the expectation value of the energy momentum tensor. As $\langle S \rangle$ can be expressed in terms of $\langle \phi^2 \rangle$ from Eq.(5.27), inserting the explicit form of $\langle \phi(x) \rangle_{\text{reg}}$ in Eq.(5.25), we get

$$
\langle S \rangle = \frac{m_s R}{4\pi^2}\frac{R^2}{(R^2 - \xi^2)^3} - \frac{(m_s R)^2}{16\pi^2}\frac{1}{(R^2 - \xi^2)^2}. \tag{5.32}
$$

Note that it does not depend on regularization constant C_1 and C_2. Then, integrating Eq.(5.31), we have

$$\begin{aligned}\left\langle S_{\xi}^{\xi}\right\rangle =& -\frac{m_s R}{4\pi^2}\frac{\xi^2}{(R^2-\xi^2)^3} \\ &-\frac{(m_s R)^2}{16\pi^2}\left(\frac{2R^2-3\xi^2}{\xi^2(R^2-\xi^2)^2}+\frac{1}{\xi^4}\ln\left[1-\frac{\xi^2}{R^2}\right]\right)+\frac{d_2}{\xi^4},\end{aligned} \tag{5.33}$$

where d_2 is a integration constant. Combining these expressions, from Eq.(5.30), we obtain

$$\begin{aligned}\langle T^{\mu\nu}\rangle =&\frac{m_s R}{4\pi^2}\left[\frac{\xi^2}{3(R^2-\xi^2)^3}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)+\frac{R^2}{(R^2-\xi^2)^3}\eta^{\mu\nu}\right] \\ &+\frac{(m_s R)^2}{16\pi^2}\left[\left\{\frac{2R^2-3\xi^2}{3\xi^2(R^2-\xi^2)^2}+\frac{1}{3\xi^4}\ln\left(1-\frac{\xi^2}{R^2}\right)^2\right\}\right. \\ &\qquad\left.\times\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)-\frac{1}{(R^2-\xi^2)^2}\eta^{\mu\nu}\right] \\ &-\frac{d_2}{3}\frac{1}{\xi^4}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right).\end{aligned} \tag{5.34}$$

We can compare this result with that obtained in terms of the point splitting method(see Eq.(B.23) in Appendix B):

$$\begin{aligned}\langle T^{\mu\nu}\rangle_{\text{reg}} =&\frac{m_s R}{4\pi^2}\left[\frac{\xi^2}{3(R^2-\xi^2)^3}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)+\frac{R^2}{(R^2-\xi^2)^3}\eta^{\mu\nu}\right] \\ &+\frac{(m_s R)^2}{16\pi^2}\left[\left\{\frac{2R^2-3\xi^2}{3\xi^2(R^2-\xi^2)^2}+\frac{1}{3\xi^4}\ln\left(1-\frac{\xi^2}{R^2}\right)^2\right\}\right. \\ &\qquad\left.\times\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)-\frac{1}{(R^2-\xi^2)^2}\eta^{\mu\nu}\right] \\ &+D_1\eta^{\mu\nu}+D_2\frac{1}{\xi^4}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right),\end{aligned} \tag{5.35}$$

where D_1 and D_2 are arbitrary regularization constants. We have the same result, except for the term $D_1\eta_{\mu\nu}$. D_1 is just a cosmological constant which appears in any theory, while D_2 is an integration constant whose appearance is a new feature in the present model. The latter arises because the background has only the

Lorentz invariance but not the Poincare invariance. In fact, this term corresponds to the C_2-term in $\langle\phi^2\rangle_{\text{reg}}$ and diverges badly on the light cone. It should be noted that the D_2-term is traceless and satisfies the conservation law by itself; *i.e.*, it is transverse-traceless.

Since one has no definite theory to predict the value of the cosmological constant, we set $D_1 = 0$ as usual. If we also set $D_2 = 0$, we find the rest of $\langle T^{\mu\nu}\rangle_{\text{reg}}$ is perfectly regular inside the bubble wall, hence it can be extended to the region inside the light cone without any problem. On the other hand, if we were to retain the D_2-term, the conventional WKB picture of false vacuum decay breaks down, because the diverging energy momentum tensor indicates a significant effect of backreaction onto the background field. Although we cannot deny this possibility, however, the WKB approximation of our formalism breaks down at this point as noted before. Therefore we take a conservative stand point and forcus on the part that does not depend on the regularization constatnt, setting $D_2 = 0$, and then put forward the discussions.

5.3. Creation of a homogeneous and isotropic open universe

The above result shows that the resulting quantum state inside the bubble is highly non-trivial, implying the importance of the effect of fluctuating fields during and after the bubble nucleation, if not leading to the breakdown of the WKB picture. Now let us discuss the cosmological implication of this result. The spacetime inside the light cone ($t > r$) is most conveniently expressed in the following coordinates;

$$ds^2 = -dT^2 + T^2 d\chi^2 + T^2 \sinh^2\chi\, d\Omega^2, \tag{5.36}$$

where $T^2 = t^2 - r^2 = -\xi^2$ and $\chi = \text{arctanh}\,(r/t)$ (see Fig.5). It is the hyperbolic time-slicing of the Minkowski spacetime and represents a cosmological model of the universe with vanishing energy density, called the Milne universe. In this region, $\langle T^{\mu\nu}\rangle_{\text{reg}}$ is given by Eq.(5.35) with ξ^2 replaced by $-T^2$. As mentioned before, it is natural to assume $m_s R = 4\alpha\sigma_c^4/\epsilon \gg 1$ under the thin-wall approximation. Then the energy momentum tensor at sufficiently late times $T \gg R$ is given by

$$\langle T^{\mu\nu}\rangle_{\rm reg} \simeq \frac{(m_s R)^2}{16\pi^2}\left[\frac{2}{3T^4}\ln(T^2/R^2)\left(\frac{4x^\mu x^\nu}{T^2}+\eta^{\mu\nu}\right)-\frac{1}{T^4}\eta^{\mu\nu}\right]. \tag{5.37}$$

This implies that the energy density on the $T = const.$ hypersurface is homogeneous and isotropic, and behaves almost like radiation, namely, which decreases as $\rho \propto 1/T^4$ and has the nature of quasi-traceless. This suggests that the bubble nucleation may be interpreted as the creation of a homogeneous and isotropic open universe with radiation, provided that the bubble nucleation rate is exponentially small so that the probability of bubble collision is negligible.

Here let us discuss the cosmological generation of entropy. Suppose that ΔM is the mass scale corresponding to the potential difference between true and false vacua; $\mathcal{E} = (\Delta M)^4$, and that the true vacuum has vanishing vacuum energy. Then the false vacuum is in the de Sitter phase with the Hubble parameter $H^2 = 8\pi G\mathcal{E}/3 \simeq (\Delta M)^4/m_{pl}^2$, where m_{pl} is the Planck mass and G is the gravitational constant. Within this de Sitter phase, there appears a true vacuum bubble, inside of which is a homogeneous and isotropic open universe with radiation energy density $\rho_{ini} \simeq (m_s R)^2/R^4$ with the initial scale factor (*i.e.*, the curvature radius) $a(t_{ini}) \simeq R$. Now, since the bubble radius cannot be greater than the Hubble radius; $R \lesssim H^{-1}$ [19], and the initial energy density is presumably smaller than the false vacuum energy density; $\rho_{ini} \lesssim \mathcal{E}$, we have

$$\mathcal{E} \gtrsim \rho_{ini} \simeq \frac{m_s^2}{R^2} \gtrsim m_s^2 H^2 \simeq \frac{m_s^2}{m_{pl}^2}\mathcal{E}. \tag{5.38}$$

Hence $R \approx H^{-1}$ for $m_s \simeq m_{pl}$. In this optimal case, the total entropy within the initial curvature radius is

$$S_{tot} \simeq (\rho_{ini} R^4)^{3/4} \simeq (m_s R)^{3/2} \simeq \left(\frac{m_{pl}}{\Delta M}\right)^3 \gg 1 \quad \text{for } \Delta M \ll m_{pl}. \tag{5.39}$$

Thus inside the bubble is a homogeneous isotropic open universe with high entropy. Does the created open universe describe our universe? Unfortunately, no. The created universe is a curvature-dominated one from the beginning in the present case, because $\rho_{ini}/m_{pl}^2 \lesssim H^2 \lesssim R^{-2}$ and the radiation energy density decreases as $a(t)^{-4}$ while the curvature term as $a(t)^{-2}$. Therefore, this model is not a good one for our universe. Nevertheless, it is very interesting that the isotropy and homogeneity of thus nucleated universe is guaranteed by the $O(3,1)$ symmetry of the resultant quantum state and that the process is completely causal since the created universe is inside the light cone. Because we have investigated the quantum fluctuation on the fixed flat Minkowski background, the above discussion is unsatisfactory. The investigation taking the gravity into consideration is necessary to provide a definite answer.

6. Toward generalization of the Initial Condition

So far, we have investigated the tunneling associated with the ground state in the false vacuum. Here we consider a tunneling that is not in the ground state in the false vacuum , and discuss the false vacuum decay in the presence of excited particles before the tunneling [42,43,44]. As a first step for this purpose, we consider a simple quantum mechanical tunneling of two dimensions with a potential, shown in Fig.6. Then we examine the quantum state of a particle penetrating the potential barrier from the local potential minimum on the left hand side of the figure to the classically allowed region on the right hand side. This is just the case $D = 1$ in §2. Introducing coordinates y and η instead of ϕ^0 and ϕ^1 as shown there, we can regard y as the tunneling degree of freedom and η as that coupled to the tunneling sector.

6.1. Wave Function

To specify the system to be considered, we write the Lagrangian as,

$$\mathcal{L} = \mathcal{L}_y + \mathcal{L}_\eta, \tag{6.1}$$

with

$$\begin{aligned} \mathcal{L}_y &:= \frac{1}{2}\dot{y}^2 - U(y), \\ \mathcal{L}_\eta &:= \frac{1}{2}\dot{\eta}^2 - \frac{1}{2}m^2(y)\eta^2, \end{aligned} \tag{6.2}$$

where $m^2(y)$ describes the coupling of y and η. Fig.6 shows the potential of the total system: $V(y,\eta) := U(y) + m^2(y)\eta^2/2$, where the local potential minimum is located at $(y,\eta) = (y_{LM}, 0)$.

Here we derive the wave function that describes the tunneling from the excited state as the following [30]. To find the tunneling wave function, we should solve the Schrödinger equation

$$\{\hat{H}_y + \hat{H}_\eta\}\Psi(y,\eta) = E\Psi(y,\eta), \tag{6.3}$$

with

$$\begin{aligned}\hat{H}_y &:= -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2} + U(y),\\ \hat{H}_\eta &:= -\frac{\hbar^2}{2}\frac{\partial^2}{\partial \eta^2} + \frac{1}{2}m^2(y)\eta^2.\end{aligned} \tag{6.4}$$

As it is clear that the tunneling path lies on the y-axis, we take the following ansatz for the wave function

$$\Psi(y,\eta) = e^{-S_0(y)/\hbar}\Theta(y,\eta). \tag{6.5}$$

Keeping in mind that y and η are the variables of the order $\hbar^0$ and $\hbar^{1/2}$, respectively, because y traces the classical trajectory and η describes the fluctuation around it, we have the following equations from the Schrödinger equation in the order of $\hbar^0$ and $\hbar$, respectively,

$$\begin{aligned}&\frac{1}{2}\left(\frac{dS_0}{dy}\right)^2 - U(y) = -E_0,\\ &\frac{\hbar}{2}\frac{d^2S_0}{dy^2}\Theta(y,\eta) + \hbar\frac{dS_0}{dy}\frac{\partial\Theta(y,\eta)}{\partial y} + \hat{H}_\eta\Theta(y,\eta) = E_1\Theta(y,\eta).\end{aligned} \tag{6.6}$$

Here we have set $E = E_0 + E_1$, with $E_0 = O(\hbar^0)$ being classical part and $E_1 = O(\hbar)$ due to quantum fluctuations. The first equation in (6.6) is the Hamilton-Jacobi equation of the lowest WKB order. Setting the relation

$$\frac{dS_0}{dy} = \frac{d}{d\tau}y(\tau), \tag{6.7}$$

it leads to the classical equation of motion with the imaginary time,

$$\frac{d^2y(\tau)}{d\tau^2} - \frac{\partial U}{\partial y} = 0. \tag{6.8}$$

Combining these equations, we get

$$S_0(y(\tau)) = \int^{\tau} 2\left(U[y(\tau')] - E_0\right) d\tau'. \tag{6.9}$$

The second equation of (6.6) determines the wave function at the next WKB order. If we define

$$\Theta(y(\tau), \eta) =: \frac{\Phi(\eta, \tau)}{\left[2\left(U(y(\tau)) - E_0\right)\right]^{1/4}}, \tag{6.10}$$

we obtain the equation for $\Phi(\eta, \tau)$ as

$$\hat{L}\Phi(\eta, \tau) = E_1 \Phi(\eta, \tau), \tag{6.11}$$

where

$$\hat{L} := \hbar\frac{\partial}{\partial\tau} + \hat{H}_\eta = \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{2}m^2(y(\tau))\eta^2. \tag{6.12}$$

Apparently one may regard this equation as the Schrödinger Equation for η with the Euclidean time [27]. As is expected from the results in §2, we can write a solution as

$$\Phi(\eta, \tau) = \frac{e^{E_1\tau/\hbar}}{[g(\tau)]^{1/2}} \exp\left(-\frac{1}{2\hbar}\Omega(\tau)\eta^2\right), \tag{6.13}$$

with

$$\Omega(\tau) := \frac{\dot{g}(\tau)}{g(\tau)}, \tag{6.14}$$

where the dot denotes τ-differentiation, and function $g(\tau)$ follows

$$\left[\frac{d^2}{d\tau^2} - m^2(y(\tau))\right] g(\tau) = 0. \tag{6.15}$$

The boundary condition for $g(\tau)$ is determined from the matching condition of the wave function near the local minimum. Then, combining (6.5) with (6.10), we obtain a wave function in the forbidden region :

$$\Psi(y(\tau), \eta) = \frac{A\ e^{-S_0(\tau)/\hbar}}{\left[2\left(U(y(\tau)) - E_0\right)\right]^{1/4}} \Phi(\eta, \tau), \tag{6.16}$$

where a normalization constant A is attached.

The matching condition is discussed in the same way as that in §2. Around the local potential minimum $(y, \eta) = (y_{LM}, 0)$, one can approximate the potential as

$$V(y,\eta) \simeq \frac{1}{2}\varpi^2(y - y_{LM})^2 + \frac{1}{2}\omega^2\eta^2, \tag{6.17}$$

where the tunneling wave function is approximately given by

$$\Psi(y,\eta) = \left[\frac{\varpi\omega}{\pi^2}\right]^{1/4} \exp\left[-\frac{1}{2\hbar}\varpi(y - y_{LM})^2 - \frac{1}{2\hbar}\omega\eta^2\right]. \tag{6.18}$$

To match (6.16) and this wave function, we should set

$$E_1 = \frac{\hbar}{2}(\varpi + \omega), \tag{6.19}$$

and impose the boundary condition for $g(\tau)$ as

$$g(\tau) \to c_1 e^{\omega\tau}, \tag{6.20}$$

where c_1 is a nomalization constant. As noted before, information of the quantum state of the subsystem η is described by the Gaussian factor $\Omega(\tau)$ in Eq.(6.13), and the boundary condition for $g(\tau)$ ensures the exponential decrease of the wave function away from the tunneling path. Note also that $\Omega(\tau)$ does not depend on the normalization of $g(\tau)$ as is clear from (6.14). If we fix the normalization constant c_1, the matching condition determines the normalization constant of the the wave function A.

Using the discussion of Gervais and Sakita [30], it is easy to extend this wave function to that includes an excited state of η before tunneling. First, define the operator,

$$\tilde{A}^\dagger := e^{\omega\tau}\left\{f(\tau)\frac{\partial}{\partial\eta} + \dot{f}(\tau)\eta\right\}, \tag{6.21}$$

where we have introduced the function $f(\tau)$ along the tunneling path, which satisfies the same equation as $g(\tau)$, but behaves as

$$f(\tau) \to c_2 e^{-\omega\tau}, \tag{6.22}$$

at $\tau \to -\infty$(near the local potential minimum), where c_2 is some constant. Then we can observe the following commutation relation by a straightforward computation,

$$[\,\hat{L},\ \tilde{A}^\dagger\,] = \omega \tilde{A}^\dagger. \tag{6.23}$$

This relation implies that $\tilde{A}^\dagger\ \Phi(\eta,\tau)$ is a solution of the Eq.(6.11) with E_1 replaced by $E_1+\omega$, or generally $\{\tilde{A}^\dagger\}^n\Phi(\eta,\tau)$ is also a solution with E_1 replaced by $E_1+n\omega$. Thus we have the following WKB solution of the Schrödinger equation

$$\Psi(\tau,\eta) = A\frac{e^{-S_0(\tau)/\hbar}}{[2\left(U(y(\tau)) - E_0\right)]}\{\tilde{A}^\dagger\}^n\Phi(\eta,\tau). \tag{6.24}$$

The above consideration is suggestive of the roll of $\tilde{A}^\dagger$ as the creation operator. In fact, $\tilde{A}^\dagger$ has the following asymptotic form at $\tau \to -\infty$

$$\tilde{A}^\dagger \simeq c_2\left(\frac{\partial}{\partial\eta} - \omega\eta\right), \tag{6.25}$$

which is proportional to the creation operator of the harmonic oscillator. This implies that the solution (6.24) is matched to the harmonic oscillator wave function of the n-th excited state in the η-direction at the local potential minimum

$$\Psi_n = \left[\frac{\varpi}{\pi}\right]^{1/4} e^{-\varpi(y-y_{LM})^2/2}\left[\frac{\omega^{1/2}}{\pi^{1/2}2^{2n}n!}\right]^{1/2} e^{-\omega\eta^2/2}H_n(\sqrt{\omega}\eta), \tag{6.26}$$

where $H_n(z)$ is the Hermite Polynomial. Once we obtain the wave function in the forbidden region, the remaining task is to find the out-going part of the wave function in the allowed region beyond the turning point. We will work out this procedure for simple specific models in the following.

6.2. Simple Model(1)

First let us consider the following potential with

$$U(y) = \begin{cases} \frac{1}{8}Q^2\varpi^4\left(y^2 - \frac{1}{Q^2\varpi^2}\right)^2, & y \leq \frac{1}{Q\varpi}, \\ 0 \qquad , & y \geq \frac{1}{Q\varpi}, \end{cases} \tag{6.27}$$

and

$$m^2(y) = \nu^2\left(1 + \alpha_c Q^2\varpi^2 y^2 + \beta_c Q\varpi y\right). \tag{6.28}$$

A sketch of the total potential $V(y,\eta) = U(y) + m^2(y)\eta^2/2$ is shown in Fig.7. The local minimum is located at $(y,\eta) = (-y_c, 0)$, where we have defined $y_c := 1/Q\varpi$. ϖ^2 is the curvature of the local potential minimum in the direction of y, $1/Q^2$ is the height of the potential barrier. The curvature of the potential along η-direction depends on y, and is given by $\nu^2(1+\alpha_c \mp \beta_c)$ at $y = \mp y_c$, which we denote by

$$\omega^2 = \nu^2(1+\alpha_c-\beta_c), \qquad \upsilon^2 = \nu^2(1+\alpha_c+\beta_c). \tag{6.29}$$

Here we only consider the case $\omega^2, \upsilon^2 > 0$. The WKB approximation is valid if $1/Q^2 \gg \varpi$, ω and υ. Then the solution of classical trajectory (6.8) is easily obtained as

$$y(\tau) = y_c \tanh\left(\frac{\varpi\tau}{2}\right), \tag{6.30}$$

and we have

$$\left[\frac{d^2}{d\tau^2} - \nu^2\left(1 + \alpha_c \tanh^2\frac{\varpi\tau}{2} + \beta_c \tanh\frac{\varpi\tau}{2}\right)\right]g(\tau) = 0. \tag{6.31}$$

The solution that decreases exponentially, *i.e.*, $g(\tau) \rightarrow e^{\omega\tau}$ (we set $c_1 = 1$ in

Eq.(6.20)) at $\tau \to -\infty$ is found analytically [27],

$$g(\tau) = e^{\omega\tau}(1+e^{\varpi\tau})^{\kappa} F(\alpha, \beta, \delta; -e^{\varpi\tau}), \tag{6.32}$$

where

$$\begin{aligned} \alpha &:= \frac{\omega}{\varpi} + \frac{v}{\varpi} + \kappa, & \delta &:= 1 + 2\frac{\omega}{\varpi}, \\ \beta &:= \frac{\omega}{\varpi} - \frac{v}{\varpi} + \kappa, & \kappa &:= \frac{1}{2} + \sqrt{\frac{1}{4} + 4\alpha_c \frac{\nu^2}{\varpi^2}}. \end{aligned} \tag{6.33}$$

$F(\alpha, \beta, \delta; -e^{\varpi\tau})$ denotes Gauss' hypergeometric function. On the other hand, the other solution that behaves as $f(\tau) \to e^{-\omega\tau}$ at $\tau \to -\infty$ (we set $c_2 = 1$ in Eqs.(6.22)) is also found to be

$$f(\tau) = e^{-\omega\tau}(1+e^{\varpi\tau})^{\kappa} F(\alpha', \beta', \delta'; -e^{\varpi\tau}), \tag{6.34}$$

where

$$\begin{aligned} \alpha' &= -\frac{\omega}{\varpi} - \frac{v}{\varpi} + \kappa, & \delta' &= 1 - 2\frac{\omega}{\varpi}, \\ \beta' &= -\frac{\omega}{\varpi} + \frac{v}{\varpi} + \kappa. \end{aligned} \tag{6.35}$$

Applying the results in the previous subsection, we can write the wave function in the forbidden region as

$$\Psi_n = N_n \frac{e^{-S_0(\tau)}}{[2U(y(\tau))]^{1/4}} \left\{ e^{\omega\tau} \left(f(\tau)\frac{\partial}{\partial\eta} + \dot{f}(\tau)\eta \right) \right\}^n \frac{e^{(\omega+\varpi)\tau/2}}{\sqrt{g(\tau)}} \exp\left(-\frac{\Omega(\tau)\eta^2}{2} \right), \tag{6.36}$$

with

$$S_0(\tau) = \int_{-\infty}^{\tau} 2U(y(\tau'))d\tau', \tag{6.37}$$

Here N_n is a normalization constant and we have chosen E_0 to be equal to $U(-y_c) = 0$. This wave function should be matched to the n-th excited state of the harmonic oscillator wave function near the local potential minimum. Let us check this by considering the asymptotic form of this wave function. Taking the limit of $\tau \to -\infty$, from the Eq.(6.30), we have $y \simeq y_c(-1+2e^{\varpi\tau})$, which leads to $2U \simeq (4/Q^2)e^{2\varpi\tau}$ and $S_0 \simeq \varpi(y+y_c)^2/2$. With these expressions, Eq.(6.36) has the asymptotic form

$$\Psi_n \simeq N_n \left[\frac{Q}{2}\right]^{1/2} e^{-\varpi(y+y_c)^2/2} \left\{\frac{\partial}{\partial\eta} - \omega\eta\right\}^n e^{-\omega\eta^2/2}. \tag{6.38}$$

Thus this is properly matched to the harmonic oscillator wave function excited with respect to η-direction, because the operator $\partial/\partial\eta - \omega\eta$ is proportional to the creation operator. Comparing this with the normalized wave function of the harmonic oscillator, Eq.(6.26), we determine the the normalization constant as

$$N_n = \left[\frac{\omega\varpi 2^2}{\pi^2 Q^2}\right]^{1/4} \frac{1}{[2^n n! \omega^n]^{1/2}}. \tag{6.39}$$

Now let us rewrite the expression (6.36) as follows. Since the explicit n-multiple operation of the creation operator in (6.36) gives rise a polynomial of η of at most n-th order, we can expand it in terms of the Hermite Polynomials of up to n-th order. We find (see Appendix C)

$$\Psi_n(y(\tau),\eta) = \sum_{k=0}^{n} \tilde{\Psi}_{nk}(\tau,\eta), \tag{6.40}$$

where

$$\begin{aligned}\tilde{\Psi}_{nk}(\tau,\eta) =& N_n D_{n,k} \frac{e^{-S_0(\tau)}}{[2U]^{1/4}} \frac{e^{\omega(n+1/2)\tau+\varpi\tau/2}}{\sqrt{g}} \left(\frac{\omega}{\sqrt{\Omega g}}\right)^n \\ &\times \left(1-\frac{\Omega}{\omega}fg\right)^{(n-k)/2} H_k(\sqrt{\Omega}\eta)\ \exp\left(-\frac{1}{2}\Omega\eta^2\right),\end{aligned} \tag{6.41}$$

with $D_{n,k}$ for even n given by

$$D_{n,k} = \begin{cases} \dfrac{(n-1)!!}{(k-1)!!}\, 2^{(n-k)/2}\ {}_{n/2}C_{k/2}, & k:\ \text{even}, \\ 0\quad , & k:\ \text{odd}, \end{cases} \tag{6.42}$$

and $D_{n,k}$ for odd n by

$$D_{n,k} = \begin{cases} 0\quad , & k:\ \text{even}, \\ -\dfrac{n!!}{k!!}\, 2^{(n-k)/2}\ {}_{(n-1)/2}C_{(k-1)/2}, & k:\ \text{odd}. \end{cases} \tag{6.43}$$

Here we take $(-1)!! = 1$. Thus the wave function can be written by the summation of the mode $\tilde{\Psi}_{nk}$. As we set $c_1 = c_2 = 1$ in Eqs.(6.20) and (6.22), we have $f(\tau)g(\tau) = 1$ and $\Omega(\tau) = \omega$ in the limit $\tau \to -\infty$, when only the $k = n$ mode survives. This is just the n-th excited state. However, for $y(\tau)$ away from the potential minimum, $f(\tau)g(\tau)$ deviates from unity to generate other k-modes. Note also that odd(even) k-modes never appear for even(odd) n.

In fact at this stage, we can understand the rough behavior of the wave function continued into the allowed region. To show the asymptotic form of $\tilde{\Psi}_{nk}$ near the nucleation point, we use the asymptotic formulas of the functions $g(\tau)$ and $f(\tau)$ at τ large enough,

$$
\begin{aligned}
g(\tau) &\simeq \frac{\Gamma(2\upsilon/\varpi)\Gamma(1+2\omega/\varpi)}{\Gamma(\omega/\varpi+\upsilon/\varpi+\kappa)\Gamma(1+\omega/\varpi+\upsilon/\varpi-\kappa)}\, e^{\upsilon\tau} + O(e^{-\upsilon\tau}) \\
&=: g_c\, e^{\upsilon\tau} + O(e^{-\upsilon\tau}), \\
f(\tau) &\simeq \frac{\Gamma(2\upsilon/\varpi)\Gamma(1-2\omega/\varpi)}{\Gamma(-\omega/\varpi+\upsilon/\varpi+\kappa)\Gamma(1-\omega/\varpi+\upsilon/\varpi-\kappa)}\, e^{\upsilon\tau} + O(e^{-\upsilon\tau}) \\
&=: f_c\, e^{\upsilon\tau} + O(e^{-\upsilon\tau}).
\end{aligned}
\tag{6.44}
$$

Assuming ϖ, ω and υ are quantities of the same order of magnitude, g_c and f_c are of order unity. Since the exponential factor, $e^{\upsilon\tau}$, is large near the nucleation point, the product, $f(\tau)g(\tau)$, becomes large there. Therefore in the expression (6.41), the smaller k is, the larger is the amplitude of $\tilde{\Psi}_{nk}$. As the index k represents the number of oscillations in the η-direction, this means that the mode with smaller number of oscillations has larger amplitude. It is expected that for such a mode the initial oscillating energy of η-direction is transformed into the tunneling degree of freedom. We now investigate this point carefully carrying out the continuation of the wave function into the classically allowed region.

Near the nucleation point, taking τ large enough, we have $y(\tau) \simeq y_c(1-2e^{-\varpi\tau})$, and it leads to $2U \simeq (4/Q^2)e^{-2\varpi\tau}$ and $S_0 \simeq (2/3Q^2\varpi) - \varpi(y-y_c)^2/2$. Using these asymptotic forms and (6.44), $\tilde{\Psi}_{nk}$ becomes

$$
\tilde{\Psi}_{nk} \simeq \tilde{C}_{nk} \left(\frac{2y_c}{-y+y_c}\right)^{\nu(k)+1} \exp\left[\frac{1}{2}\varpi(y-y_c)^2\right] H_k(\sqrt{\upsilon}\eta) \exp\left(-\frac{1}{2}\upsilon\eta^2\right),
\tag{6.45}
$$

where

$$\begin{aligned}
\nu(k) &:= \frac{1}{\varpi}\left\{\omega\left(n+\frac{1}{2}\right)-v\left(k+\frac{1}{2}\right)\right\}, \\
\tilde{C}_{nk} &:= N_n D_{n,k} e^{-2/3Q^2\varpi}\left(\frac{Q}{2}\right)^{1/2}\omega^{n/2}\left(\frac{\omega}{v}\right)^{k/2} g_c^{-(n+k+1)/2}(-f_c)^{(n-k)/2}.
\end{aligned} \tag{6.46}$$

On the other hand, approximating the potential around the turning point as $V(y,\eta) \simeq \varpi^2(y-y_c)^2/2+v^2\eta^2/2$, we can separate the variables in the wave function as

$$\Psi(y,\eta) = \psi(y)\varphi(\eta), \tag{6.47}$$

then the Schrödinger equation becomes

$$\begin{aligned}
\left[-\frac{1}{2}\frac{d^2}{dy^2}+\frac{1}{2}\varpi^2(y-y_c)^2\right]\psi(y) &= E_y\psi(y), \\
\left[-\frac{1}{2}\frac{d^2}{d\eta^2}+\frac{1}{2}v^2\eta^2\right]\varphi(\eta) &= E_\eta\varphi(\eta).
\end{aligned} \tag{6.48}$$

These equations can be reduced to the form,

$$\left[\frac{d^2}{dx^2}+\left(\mu+\frac{1}{2}-\frac{x^2}{4}\right)\right]\phi(x) = 0, \tag{6.49}$$

whose solution is called a parabolic cylinder function. We write the two independent solutions as $U(-\mu-1/2,x)$ and $V(-\mu-1/2,x)$ [45]. The decaying mode solution $U(-\mu-1/2,x)$ gives the harmonic oscillator wave function,

$$U(-\mu-1/2,x) = D_\mu(x) \sim 2^{-\mu/2}H_\mu(x/\sqrt{2})\ e^{-x^2/4}, \tag{6.50}$$

where $D_\mu(x)$ is the Whittaker's notation of the parabolic cylinder function [39].

The definition of $D_\mu(x)$ is given by

$$D_\mu(x) := 2^{\mu/2}\, e^{-x^2/4} \left\{ \frac{\Gamma(1/2)}{\Gamma((1-\mu)/2)}\, {}_1F_1\left(-\frac{\mu}{2}, \frac{1}{2}, \frac{x^2}{2}\right) + \frac{x}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-\mu/2)}\, {}_1F_1\left(\frac{1-\mu}{2}, \frac{3}{2}, \frac{x^2}{2}\right) \right\}, \tag{6.51}$$

with the Kummer's function ${}_1F_1(a, b, x)$,

$${}_1F_1(a, b, x) := \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{x^n}{n!}. \tag{6.52}$$

While $V(-\mu - 1/2, x)$ corresponds to the growing mode,

$$\begin{aligned} V(-\mu - 1/2, x) &= \frac{\Gamma(-\mu)}{\pi}\Big(-\sin \pi(\mu + 1/2) D_\mu(x) + D_\mu(-x)\Big) \\ &\simeq \frac{e^{x^2/4}}{x^{\mu+1}} \qquad (x \gg 1). \end{aligned} \tag{6.53}$$

As it is clear, the η-sector of the wave function $\varphi(\eta)$ should be matched to the decaying mode solution and the y-sector $\psi(y)$ to the the growing mode solution. Hence we find the matched wave function around the turning point,

$$\begin{aligned} \tilde{\Psi}_{nk}(y, \eta) \simeq \tilde{C}_{nk} &\left(\frac{2\sqrt{2}}{Q\sqrt{\varpi}}\right)^{\nu(k)+1} \left(\frac{\pi}{2}\right)^{1/2} V\left(-\nu(k) - 1/2, -\sqrt{2\varpi}(y - y_c)\right) \\ &\times H_k(\sqrt{\upsilon}\eta)\ \exp\left(-\frac{1}{2}\upsilon\eta^2\right), \end{aligned} \tag{6.54}$$

with

$$\begin{aligned} E_\eta &= \upsilon\left(k + \frac{1}{2}\right) =: E_\eta(k), \\ E_y &= \frac{\varpi}{2} + \omega\left(n + \frac{1}{2}\right) - \upsilon\left(k + \frac{1}{2}\right) =: E_y(k). \end{aligned} \tag{6.55}$$

Using the asymptotic formula of the parabolic cylinder function [45], we can extract out-going mode of the wave function in the allowed region beyond the turning point as

$$\tilde{\Psi}_{nk}^{\rm out} = \tilde{C}_{nk} \left(\frac{2}{Q\sqrt{\varpi}}\right)^{\nu(k)+1} \frac{\sqrt{\pi}\; e^{i\pi(\nu(k)+1)/2}}{2\Gamma(\nu(k)/2+1)}$$
$$\times\; \exp\left[i\sqrt{2E_y(k)}(y-y_c)\right] H_k(\sqrt{\upsilon}\eta)\exp\left(-\frac{1}{2}\upsilon\eta^2\right).$$

Surely it represents the state oscillating in the η-direction specified by the index k, and moving in the y-direction with the energy $E_y(k)$, which is the difference between the initial total energy and the oscillation energy $E_\eta(k)$. The superposition of these k-modes is a state of the system. But we must note the relative amplitude of each k-mode. We have assumed that the condition $1/Q\sqrt{\varpi} \gg 1$ is satisfied so that the WKB approximation is applicable. Therefore the mode of the least number of k, *i.e.*,$k = 0$ for even n and $k = 1$ for odd n, dominates the state.

6.3. Simple Model(2)

We discuss another example which allows analytic treatment. We consider the case when there is a small alternation of the potential in the previous section adding the term

$$\Delta U(y) = \begin{cases} -\epsilon Q\varpi\left(y+\dfrac{1}{Q\varpi}\right), & y \leq \dfrac{1}{Q\varpi}, \\ 0 \qquad\qquad , & y \geq \dfrac{1}{Q\varpi}. \end{cases} \tag{6.56}$$

In this case the potential becomes asymmetric, and Eq.(6.8) has the bounce solution. If the correction is small, namely, $Q^2\epsilon \ll 1$, we can use the thin-wall approximation. Taking the thin-wall limit, we find the following equation for $g(\tau)$ and $f(\tau)$ from Eq.(6.31)

$$\left[\frac{d^2}{d\tau^2} - \left(\omega^2 - \varrho\delta(\tau+T) + 2\nu^2\beta_c\theta(\tau+T)\right)\right] g(\tau) = 0, \tag{6.57}$$

with the delta-function $\delta(z)$ and the step-function $\theta(z)$. Here we have defined $\varrho := 4\alpha_c\nu^2/\omega$, and set the parametrization of τ so that the nucleation point corresponds to $\tau = 0$ by using the arbitrariness of its reparametrization. In this case, the classical solution passes through the top of the potential barrier at $\tau = -T$, where $T \simeq (1/\varpi)\log(1/Q^2\epsilon)$. Then the solution is easily found as follows by considering the junction condition at $\tau = -T$,

$$g(\tau) = \begin{cases} e^{\omega\tau} \quad , & (-\infty < \tau \le -T), \\ A\ e^{\upsilon\tau} + B\ e^{-\upsilon\tau}, & (-T \le \tau < 0), \end{cases}$$

and

$$f(\tau) = \begin{cases} e^{-\omega\tau} \quad , & (-\infty < \tau \le -T), \\ C\ e^{\upsilon\tau} + D\ e^{-\upsilon\tau}, & (-T \le \tau < 0), \end{cases}$$

where

$$A = \frac{\upsilon + \omega - \varrho}{2\upsilon} e^{(\upsilon-\omega)T}, \qquad B = \frac{\upsilon - \omega + \varrho}{2\upsilon} e^{-(\upsilon+\omega)T},$$
$$C = \frac{\upsilon - \omega - \varrho}{2\upsilon} e^{(\upsilon+\omega)T}, \qquad D = \frac{\upsilon + \omega + \varrho}{2\upsilon} e^{-(\upsilon-\omega)T}.$$

Inserting these solutions into Eq.(6.36), we get the wave function in the forbidden region. Since we have assumed that $Q^2\epsilon \ll 1$, we get $e^{\upsilon T} \gg 1$, if $\varpi \sim \omega \sim \upsilon$. This leads to $|A| \gg |B|$ and $|C| \gg |D|$, except for some special values of ϖ, ω and υ. Then the functions $g(\tau)$ and $f(\tau)$ are well approximated by only the first terms in their expressions near the turning point $\tau \simeq 0$. In this case we can write $\tilde{\Psi}_{nk}(\tau, \eta)$ as

$$\begin{aligned} \tilde{\Psi}_{nk} \simeq & \tilde{M}_{nk} e^{\omega T(n+1/2)} e^{-\upsilon T(k+1/2)} \\ & \times \frac{e^{-S_0(\tau)}}{[2U]^{1/4}} \exp\left[E_y(k)\tau\right] H_k(\sqrt{\upsilon}\eta) \exp\left(-\frac{1}{2}\upsilon\eta^2\right), \end{aligned} \tag{6.58}$$

where

$$\tilde{M}_{nk} := N_n D_{n,k} \omega^{n/2} \left(\frac{\omega}{\upsilon}\right)^{k/2} \left(\frac{-\upsilon + \omega + \varrho}{2\upsilon}\right)^{(n-k)/2} \left(\frac{\upsilon + \omega - \varrho}{2\upsilon}\right)^{-(n+k+1)/2}. \tag{6.59}$$

Recall that in this model the classical solution becomes bounce, which is different from that in the previous section. Since we have chosen the parametrization of the classical trajectory so as to arrive at the nucleation point at $\tau = 0$, then we can extract out the out-going mode of the wave function in the allowed region by replacing τ by it $(t > 0)$. By this procedure, we have

$$\begin{aligned}\tilde{\Psi}^{\rm out}_{nk} &= \tilde{M}_{nk} e^{-2/3Q^2\varpi} e^{\omega T(n+1/2)} e^{-\upsilon T(k+1/2)} \\ &\times \frac{e^{iS_L(t)}}{[2U]^{1/4}} e^{iE_y(k)t} H_k(\sqrt{\upsilon}\eta) \exp\left(-\frac{1}{2}\upsilon\eta^2\right),\end{aligned} \tag{6.60}$$

with

$$S_L(t) = \int_0^t dt' \mathcal{L}_y(y(t')). \tag{6.61}$$

Now consider the classical Hamilton-Jacobi equation with energy E,

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = E. \tag{6.62}$$

The solution $S(q(t), E)$, which is found by setting $\partial S/\partial q = dq/dt$, is related to another solution with the energy $E + \Delta E$ as

$$\begin{aligned}S(q, E + \Delta E) &\simeq S(q, E) + \Delta E \frac{\partial S}{\partial E} \\ &= S(q, E) + \Delta E\ t + (\text{constant}).\end{aligned} \tag{6.63}$$

Thus $\tilde{\Psi}^{\rm out}_{nk}$ can be interpreted as the state with additional energy $\Delta E = E_y(k) = \varpi/2 + \omega(n + 1/2) - \upsilon(k + 1/2)$ transferred into the tunneling degree of freedom. Because of the condition $e^{\upsilon T} \gg 1$, once again we find the large suppression factor $e^{-\upsilon Tk}$ in the above expression of $\tilde{\Psi}^{\rm out}_{nk}$. Hence the mode of least number of k dominates in the allowed region, which is the same as the case discussed in the previous model.

Summarizing this section, we have investigated the quantum mechanical tunneling in the two dimensional system where the tunneling degree of freedom is coupled to the other excited oscillator, constructing the tunneling wave function explicitly. From the consideration of the above specific models, we obtained the following result. If the condition $e^{\upsilon T} \gg 1$ is satisfied, that is, if the 'duration' for tunneling T becomes longer than the period of the coupled oscillator $2\pi/v$, it is expected that the state after tunneling is dominated by the one in which the initial excitation energy in the oscillator is transferred as much as possible to the tunneling degree of freedom and used to excite the motion in that direction, irrespective of the initial excitation. This result is interpreted as follows. Since it becomes easier to escape away from the local potential minimum by getting the more energy in the direction of tunneling. Therefore such a mode dominantly contributes to the tunneling.

This result is applied to the field theory directly, when we consider a field coupled to another tunneling field which undergoes the decay of a false vacuum homogeneously in the entire universe, as discussed in §4. Because the spatial harmonics expansion separates the system mode by mode, and the problem essentially reduces to that in this section. Then the number of excitation n in the η-direction corresponds to the number of particle of the field coupled to tunneling field before the false vacuum decay. Our result is consistent with those of Rubakov [27] and of Kandrup [42], which report the particle annihilation during the false vacuum decay. In contrast with their approach, we have constructed the wave function explicitly under the tunneling boundary condition, therefore the prescription was clear, and the information in the classically allowed region was obtained in our formalism.

7. Summary and Discussions

In this thesis, the whole attention is paid for the aspect of quantum state of a field during and after the false vacuum decay through the tunneling effect. First we shall summarize our analysis. We started from constructing the quasi-ground state wave function which describes the tunneling in a metastable system with finite degrees of freedom. It is based on the WKB approximation introducing a classical tunneling path in the configuration space, which is developed by Gervais and Sakita [30]. We gave the alternative construction of it in the covariant manner in §2. This covariant formalism will be useful when gravity is taken into account. Extension to the field theory was done formally, then we obtained the wave functional which describes the quantum state of a field during and after the false vacuum decay. For definiteness, we introduced the fluctuating field ϕ coupled to the other tunneling field σ that undergoes a false vacuum decay, and focused on quantum state of the ϕ-field on the tunneling background field. We showed that the resultant quantum state of ϕ-field can be interpreted in the language of conventional second quantized picture, and gave the method to construct appropriate mode functions.

This is done as follows. First we find the Euclidean classical tunneling solution $\sigma_0(\boldsymbol{x}, \tau)$. Then we solve the linearized field equation for ϕ in the background of classical tunneling solution with the condition that the field vanishes exponentially as the Euclidean time τ goes to $-\infty$, and construct a set of Euclidean mode functions $g_{\boldsymbol{k}}(x)$. The Lorentzian mode functions $v_{\boldsymbol{k}}(x)$, which describe the quantum state after tunneling, are obtained by the analytic continuation of $g_{\boldsymbol{k}}(x)$ with $\tau \to it$ and by taking their complex conjugates. As these Lorentzian mode functions are not in general orthonormalized, if necessary, construct a new set of orthonormalized mode functions $u_{\boldsymbol{k}}(x)$ by a suitable linear transformation of the original ones. The resulting quantum state after tunneling is most conveniently described in the Heisenberg picture. That is, if we represent the field operator as $\hat{\phi}(x) = \sum_{\boldsymbol{k}} \left(\hat{a}_{\boldsymbol{k}} u_{\boldsymbol{k}}(x) + \hat{a}^{\dagger}_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(x) \right)$, the state is identical to the "*vacuum*" state annihilated by the operator $\hat{a}_{\boldsymbol{k}}$, but it is not true vacuum state in general. This formalism was applied for the specific examples of the false vacuum decay in §4 and §5.

The false vacuum decay that occurs homogeneously in a closed universe was considered in §4, to demonstrate how non-trivial the resulting quantum state can be after the decay. The resulting spectrum of excitation has some similarity with a thermal spectrum with its temperature given by a Euclidean duration of tunneling that is related to a certain mass scale associated with the tunneling field. Here, the high momentum distribution is more suppressed. As a result, the generated total energy density is determined not by the mass scale of the tunneling field but by the difference of before and after the tunneling. In this model, there exists an asymptotic region and the concept of a particle is definite, then we showed that our approach gives the same results as that of Rubakov [27].

The false vacuum decay with an $O(4)$-symmetric bubble was analyzed in §5, introducing a simple model of the coupling between ϕ and σ, in which the mass term of the ϕ-field is non-vanishing only at the bubble wall. We explicitly constructed the mode functions which describe the quantum state after the false vacuum decay, and found that the constructed mode function were singular on the light cone. We argued, however, that the appearance of the singularity is inevitable for any model with an $O(4)$-symmetric bubble under the WKB approximation. We then presented a method to avoid the singularity during calculations of physical quantities and evaluated the coincidence limit of the two-point function as well as the expectation value of the energy momentum tensor. We found that both of them, even after the usual regularization of divergent terms, became singular on the light cone. However, this singularity depends entirely on the choice of a regularization constant, and we argued that it should be removed in order to retain the presumed consistency of the WKB approximation. The resulting regularized expectation value of the energy momentum tensor has been shown to be perfectly regular everywhere inside the bubble wall. We found that there existed a family of hypersurfaces (the hyperbolic time-slicing) over which the energy density was constant, as a consequence of the Lorentz invariance of the state, and that the expectation value of the energy momentum tensor behaves like radiation.

Hence we pointed out that the bubble nucleation process can be interpreted as creation of a homogeneous and isotropic open universe with high entropy. Unfortunately, however, the created universe in this model is a curvature dominated one, then it is not a realistic model of our universe. Nevertheless, we emphasize that there exists another mechanism, instead of inflation, which gives rise to a homogeneous and isotropic open universe with high entropy by a quantum coherence, and that it may give rise alternative solution to the horizon and flatness problem[♯].

This certainly provides us with a motivation for investigation of a more general, and realistic model. In particular, because we investigated the problem on the fixed flat background, investigation taking gravity into consideration, which is essential in cosmology, is necessary. The first step would be to carry out a similar analysis in a non-trivial curved background spacetime.

Another important issue is the fluctuation of the σ-field itself. Because of the difficulty of zero-mode whose appearance is expected there, this problem has never been considered strictly so far.

In §6, we attempted to generalize our investigation on the initial state, *i.e.*, the false vacuum decay from an excited state in the metastable vacuum [42,43,44]. As a first step for this problem, we considered the quantum mechanical tunneling in very simple systems with only two degrees of freedom. We regarded each degree of freedom as the tunneling one and the other one coupled to it, which was set to be in an excited state before the tunneling. We showed a very interesting phenomenon, that is, the state after the tunneling is dominated by the one in which the initial excitation energy of the oscillator is transferred as much as possible to the tunneling sector to excite the motion in its direction, irrespective of the initial excitation.

Application of this consideration to the field theory was discussed. When a false vacuum decays homogeneously as considered in §4, the problem essentially reduces to that of two degrees of freedom, because the spatial harmonics expansion separates the system mode by mode. Our result implies that the particles annihilate during the false vacuum decay, which is the same result with the previous works of Rubakov [27] and Kandrup [42]. Due to the independence of the initial

♯ In this connection, we mention that Linde [46] discussed the possibility of creating a non-flat (but long-lived) homogeneous and isotropic universe in the context of the self-reproducing universe scenario, provided that the creation probability is exponentially suppressed.

state, they conclude that the spectrum of the number of created particles after the tunneling can be always "thermal", in the model of a homogeneous decay, irrespective of the initial state (even if the field contains many particles initially). It will be interesting to consider the generalization on the initial state in the case when a false vacuum decays nucleating a vacuum bubble.

Finally, in connection with this, we comment on a good point of our formalism. The prescription for the tunneling phenomenon using the Euclidean path integral is excellent method to evaluate a tunneling rate from the ground state, but it does not suit to investigate a tunneling from the excited state. Our formalism which explicitly constructs the wave function can treat this problem, and will be useful when we consider effects of an excitation on tunneling.

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APPENDIX A

In this Appendix, we show that our estimate of the particle creation given in §4 agrees with the result obtained in Rubakov's approach [27]. We omit the details and show only the resulting expression for the particle number in his approach. In Rubakov's formalism, the number of created particles is given by

$$N_{\boldsymbol{k}} = \left(\frac{D^2}{1-D^2}\right)_{\boldsymbol{k}\boldsymbol{k}}, \tag{A.1}$$

where D is a matrix given in Eq.(3.19) of [27]. In the case of spatially homogeneous decay of false vacuum, each $\boldsymbol{k}$-mode decouples and the matrix D becomes diagonal. Hence we can treat each mode separately. Since our mode function $g_{\boldsymbol{k}}$ corresponds to Rubakov's mode function g_α defined in Eq.(3.8) of [27], we find the diagonal component of D with the wavenumber $\boldsymbol{k}$ is expressed in terms of $g_{\boldsymbol{k}}$ as

$$D_{\boldsymbol{k}} = -\left.\frac{\dot{g}_{\boldsymbol{k}} - \omega_+ g_{\boldsymbol{k}}}{\dot{g}_{\boldsymbol{k}} + \omega_+ g_{\boldsymbol{k}}}\right|_{\tau=0} = \frac{\mathcal{W} - \omega_+}{\mathcal{W} + \omega_+}, \tag{A.2}$$

where $\mathcal{W} := \partial_\tau g_{\boldsymbol{k}}/g_{\boldsymbol{k}}\big|_{\tau=0}$. For the model of §4, we have

$$\mathcal{W} = \frac{A_{\boldsymbol{k}} - B_{\boldsymbol{k}}}{A_{\boldsymbol{k}} + B_{\boldsymbol{k}}}\omega_+ .$$

Hence the number spectrum of created particles in Rubakov's formalism, Eq.(A.1), is calculated to be

$$N_{\boldsymbol{k}} = \frac{(\mathcal{W} - \omega_+)^2}{4\mathcal{W}\omega_+} = \frac{B_{\boldsymbol{k}}^2}{A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2}. \tag{A.3}$$

This is exactly in agreement with the result given in Eq.(4.11).

APPENDIX B

In this Appendix, we evaluate the expectation value of the energy momentum tensor explicitly by regularizing the divergence in terms of the point-splitting method. That is, we operate the following derivative operator on $G^{(1)}(x,x')$ to form a bitensor,

$$O_{\mu\nu}\left[G^{(1)}(x,x')\right] := \frac{1}{4}\left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\right) \times \left[\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta} + \frac{\partial}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}\right] G^{(1)}(x,x'), \tag{B.1}$$

and take the coincidence limit, $x'^\mu \to x^\mu$.

We begin by separating the terms linear and quadratic in m_s in the integral form of the two-point function, Eq.(5.23). For notational simplicity, we normalize the unit by setting $R = 1$ in the following. Then,

$$G^{(1)}(x,x') = \frac{1}{2\pi^2}\left\{\frac{1}{\epsilon^2} + \frac{m_s}{2}\mathcal{G}_1 + \left(\frac{m_s}{2}\right)^2\mathcal{G}_2\right\}, \tag{B.2}$$

where

$$\mathcal{G}_1 := \int_1^{1/\zeta} \frac{du}{\zeta u^2 - 2\eta u + \zeta'} + \int_1^{1/\zeta'} \frac{dv}{\zeta' v^2 - 2\eta v + \zeta}, \tag{B.3a}$$

$$\mathcal{G}_2 := \int_1^{1/\zeta} du \int_1^{1/\zeta'} dv \frac{1}{\zeta u^2 - 2\eta uv + \zeta' v^2}. \tag{B.3b}$$

The first divergent $1/\epsilon^2$ term in the above is simply the Minkowskian contribution, which will give rise to a cosmological constant in $\langle T^{\mu\nu}\rangle_{reg}$. Hence we focus on the $\mathcal{G}_1$ and $\mathcal{G}_2$ terms. We will encounter various forms of divergences also in $\mathcal{G}_1$ and $\mathcal{G}_2$, but since the two-point function is manifestly Lorentz-invariant, these divergences are also formally Lorentz-invariant. Hence, we concentrate on finite terms in them. Possible forms of $\langle T^{\mu\nu}\rangle_{reg}$ which may arise from the divergent terms will be discussed in the end.

In order to carry out the integrals in Eqs.(B.3), one needs to specify the relative magnitudes among $\zeta(=x^2)$, $\zeta'(=x'^2)$ and $\eta(=x\cdot x')$. In what follows, we assume

$$\begin{aligned} \epsilon^2 &= \left(x'-x\right)^2 > 0, \quad X^2 := \left(\frac{x'+x}{2}\right)^2 > 0, \\ D &:= X^2\epsilon^2 - (X\cdot\epsilon)^2 = \zeta\zeta' - \eta^2 > 0. \end{aligned} \tag{B.4}$$

However, it can be shown that the resulting expression for $\langle T^{\mu\nu}\rangle_{reg}$ is independent of the above choice.

First, let us consider the linear term, Eq.(B.3a). We find

$$\mathcal{G}_l = \frac{1}{\sqrt{D}} \arctan\left(\frac{\zeta y - \eta}{\sqrt{D}}\right)\Bigg|_{y=1}^{y=1/\zeta} + (\zeta \to \zeta'). \tag{B.5}$$

Then using the addition theorems,

$$\begin{aligned} &\arctan\alpha + \arctan\beta = \arctan\frac{\alpha+\beta}{1-\alpha\beta} \quad (|\arctan\alpha + \arctan\beta| < \pi/2), \\ &\arctan\alpha + \arctan\frac{1}{\alpha} = \frac{\pi}{2} \quad (\alpha > 0), \end{aligned}$$

it reduces to

$$\begin{aligned} \mathcal{G}_1 &= -\frac{1}{\sqrt{D}}\left(\arctan\frac{\sqrt{D}}{\eta} + \arctan\frac{\sqrt{D}}{1-\eta} - \pi\right) \\ &= \frac{\pi}{\sqrt{D}} - \frac{1+\eta}{\eta(1-\eta)} + \frac{(1+\eta)^3 - 6\eta}{3\eta^3(1-\eta)^3}D + O(D^2). \end{aligned} \tag{B.6}$$

Now, disregarding the first divergent term, $\pi/\sqrt{D}$, and operating $O_{\mu\nu}$ on the rest of terms, we obtain

$$\lim_{x'\to x} O_{\mu\nu}\left[\mathcal{G}_1\right] = \left(\frac{1}{6\xi^4} + \frac{\xi^2}{3(1-\xi^2)^3}\right)\left(\eta_{\mu\nu} - 4\frac{x_\mu x_\nu}{\xi^2}\right) + \frac{1}{(1-\xi^2)^3}\eta_{\mu\nu}, \tag{B.7}$$

One can easily check that the above contribution to $\langle T^{\mu\nu}\rangle_{reg}$ satisfies the energy momentum conservation law by itself, as it should.

Next we turn to the quadratic term, Eq.(B.3b). It can be transformed as

$$
\begin{aligned}
\mathcal{G}_2 =& \frac{1}{\sqrt{D}} \int_1^{1/\zeta} \frac{du}{u} \left(\arctan \frac{1-\eta u}{u\sqrt{D}} - \arctan \frac{\zeta' - \eta u}{u\sqrt{D}} \right) \\
=& \frac{1}{\sqrt{D}} \int_1^{\zeta} \frac{dz}{z} \left(\arctan \frac{\zeta' z - \eta}{\sqrt{D}} - \arctan \frac{z-\eta}{\sqrt{D}} \right) \\
=& \frac{\ln \zeta}{\sqrt{D}} \left(\arctan \frac{\zeta\zeta' - \eta}{\sqrt{D}} - \arctan \frac{\zeta - \eta}{\sqrt{D}} \right) \\
& - \left(\int_{\zeta'}^{\zeta\zeta'} dz \frac{\ln(z/\zeta')}{(z-\eta)^2 + D} - \int_1^{\zeta} dz \frac{\ln z}{(z-\eta)^2 + D} \right) \\
=& \frac{1}{\sqrt{D}} \left(\ln(\zeta\zeta') \arctan \frac{\zeta\zeta' - \eta}{\sqrt{D}} - \ln(\zeta) \arctan \frac{\zeta - \eta}{\sqrt{D}} - \ln(\zeta') \arctan \frac{\zeta' - \eta}{\sqrt{D}} \right) \\
& - \left(\int_{\zeta'}^{\zeta\zeta'} dz - \int_1^{\zeta} dz \right) \frac{\ln z}{(z-\eta)^2 + D} \\
=& \frac{1}{\sqrt{D}} \left(\ln(\zeta\zeta') \arctan \frac{\zeta\zeta' - \eta}{\sqrt{D}} - \ln(\zeta) \arctan \frac{\zeta - \eta}{\sqrt{D}} - \ln(\zeta') \arctan \frac{\zeta' - \eta}{\sqrt{D}} \right) \\
& - F(\zeta\zeta') + F(\zeta') + F(\zeta) - F(1),
\end{aligned}
\tag{B.8}
$$

where $F(z)$ is an indefinite integral given by

$$
\begin{aligned}
F(z) :=& \int^{z} dz' \ln z' \frac{\sqrt{D}}{(z-\eta)^2 + D} \\
=& \int^{z-\eta} dz' \left(\ln \eta + \ln \left(1 + \frac{z'}{\eta} \right) \right) \frac{1}{z'^2 + D} \\
=& \frac{\ln \eta}{\sqrt{D}} \arctan \frac{z-\eta}{\sqrt{D}} + \int^{z-\eta} dz' \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{z'}{\eta} \right)^n \frac{1}{z'^2 + D} .
\end{aligned}
\tag{B.9}
$$

The final integral term in the last line of the above equation can be further transformed as

$$
\begin{aligned}
&\int^{z-\eta} dz' \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{z'}{\eta}\right)^n \frac{1}{z'^2+D} \\
&= -\sum_{m=1}^{\infty} \frac{1}{2m} \int^{z-\eta} \left(\frac{z'}{\eta}\right)^{2m} \frac{dz'}{z'^2+D} + \sum_{m=0}^{\infty} \frac{1}{2m+1} \int^{z-\eta} \left(\frac{z'}{\eta}\right)^{2m+1} \frac{dz'}{z'^2+D} \\
&= -\sum_{m=1}^{\infty} \frac{1}{2m} \frac{1}{\eta^{2m}} \left\{ \sum_{r=0}^{m-1} \frac{(-D)^r (z-\eta)^{2m-2r-1}}{2m-2r-1} + \frac{(-D)^m}{\sqrt{D}} \arctan \frac{z-\eta}{\sqrt{D}} \right\} \\
&\quad + \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \frac{1}{\eta^{2m+1}} \left\{ \sum_{r=0}^{m-1} \frac{(-D)^r (z-\eta)^{2m-2r}}{2m-2r} \right. \\
&\qquad \left. + \frac{(-D)^m}{2} \ln\left((z-\eta)^2 + D\right) \right\} \\
&= \frac{1}{2\sqrt{D}} \left\{ \ln\left(\frac{\eta^2+D}{\eta^2}\right) \arctan \frac{z-\eta}{\sqrt{D}} + \ln\left((z-\eta)^2+D\right) \arctan \frac{\sqrt{D}}{\eta} \right\} \\
&\quad + H(z),
\end{aligned}
\tag{B.10}
$$

where we have introduced the function $H(z)$ defined by

$$
\begin{aligned}
H(z) := &\sum_{m=0}^{\infty} \frac{1}{(2m+1)} \frac{1}{\eta^{2m+1}} \left\{ \sum_{r=0}^{m-1} \frac{(-D)^r (z-\eta)^{2m-2r}}{2m-2r} \right\} \\
&- \sum_{m=1}^{\infty} \frac{1}{2m} \frac{1}{\eta^{2m}} \left\{ \sum_{r=0}^{m-1} \frac{(-D)^r (z-\eta)^{2m-2r-1}}{2m-2r-1} \right\}.
\end{aligned}
\tag{B.11}
$$

Since we only need to know terms up to $O(\epsilon^2)$ in $\mathcal{G}_2$, so in $H(z)$. Then up to this order we have

$$
\begin{aligned}
H(z) =& \frac{1}{\eta}\left(1+\frac{z}{\eta-z}\ln\frac{z}{\eta}\right) \\
&-\frac{D}{3\eta^3}\left[\left(\frac{\eta}{\eta-z}\right)^2+\frac{1}{2}\left(\frac{\eta}{\eta-z}\right)+\frac{1}{3}+\left\{\left(\frac{\eta}{\eta-z}\right)^3-1\right\}\ln\frac{z}{\eta}\right] \\
&+O(D^2),
\end{aligned}
\tag{B.12}
$$

where we have used the following series formulas,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)} &= \left(1+\frac{1-x}{x}\ln(1-x)\right), \\
\sum_{n=1}^{\infty}\frac{x^n}{n(n+3)} &= \frac{1}{3}\left\{\frac{1}{x^2}+\frac{1}{2x}+\frac{1}{3}+\left(\frac{1}{x^3}-1\right)\ln(1-x)\right\}.
\end{aligned}
\tag{B.13}
$$

Inserting Eq.(B.10) in Eq.(B.9), we find

$$
\begin{aligned}
F(z) =& \frac{1}{2}\left\{\frac{\ln(\eta^2+D)}{\sqrt{D}}\arctan\frac{z-\eta}{\sqrt{D}}+\frac{\ln\left((z-\eta)^2+D\right)}{\sqrt{D}}\arctan\frac{\sqrt{D}}{\eta}\right\} \\
&+H(z).
\end{aligned}
\tag{B.14}
$$

with $H(z)$ given by Eq.(B.12).

Substituting $z=\zeta\zeta'$, ζ, ζ, and 1 into the above expression for $F(z)$, inserting them in the last line of Eq.(B.8), and using the equalities,

$$
\begin{aligned}
\ln\left((\zeta\zeta'-\eta)^2+D\right) &= \ln\zeta\zeta' + \ln\left((1-\eta)^2+D\right), \\
\ln\left((\zeta'-\eta)^2+D\right) &= \ln\zeta + \ln\epsilon^2, \\
\ln\left((\zeta-\eta)^2+D\right) &= \ln\zeta' + \ln\epsilon^2,
\end{aligned}
$$

we finally find

$$
\begin{aligned}
\mathcal{G}_2 =& \mathcal{G}_{2,sing} + \frac{1}{2}\ln(\zeta\zeta')\frac{1}{\sqrt{D}}\left\{\arctan\frac{\zeta\zeta'-\eta}{\sqrt{D}} - \arctan\frac{1-\eta}{\sqrt{D}}\right\} \\
& - \ln\left((1-\eta)^2 + D\right)\frac{1}{\sqrt{D}}\arctan\frac{\sqrt{D}}{\eta} \\
& + H(\zeta) + H(\zeta') - H(\zeta\zeta') - H(1) \\
=& \mathcal{G}_{2,sing} - \frac{2}{\eta} - \frac{1}{\eta}\ln(1-\eta)^2 - \frac{\epsilon^2}{2\eta^2} - \frac{(\epsilon\cdot X)^2}{3\eta^3} \\
& + \left\{\frac{4\eta^2-16\eta+11}{9\eta^3(1-\eta)^2} + \frac{1}{3\eta^3}\ln(1-\eta)^2\right\}D + O(\epsilon^3),
\end{aligned} \tag{B.15}
$$

where $\mathcal{G}_{2,sing}$ is the singular part of $\mathcal{G}_2$;

$$
\begin{aligned}
\mathcal{G}_{2,sing} =& \frac{1}{\sqrt{D}}\arctan\frac{\sqrt{D}}{\eta}\ln\epsilon^2 \\
& - \frac{1}{2}\ln(\zeta/\zeta')\frac{1}{\sqrt{D}}\left\{\arctan\frac{\zeta'-\eta}{\sqrt{D}} - \arctan\frac{\zeta-\eta}{\sqrt{D}}\right\}.
\end{aligned} \tag{B.16}
$$

As in the case of $\mathcal{G}_1$, disregarding the singular part and operating $O_{\mu\nu}$ on the finite terms of $\mathcal{G}_2$, we obtain

$$
\begin{aligned}
\lim_{x'\to x} O_{\mu\nu}\left[\mathcal{G}_2\right] =& -\frac{1}{2(1-\xi^2)^2}\eta_{\mu\nu} \\
& + \left[\frac{5}{18\xi^4} + \frac{2-3\xi^2}{6\xi^2(1-\xi^2)^2} + \frac{1}{6\xi^4}\ln(1-\xi^2)^2\right]\left(\eta_{\mu\nu} - 4\frac{x_\mu x_\nu}{\xi^2}\right),
\end{aligned} \tag{B.17}
$$

which also satisfies the energy momentum conservation law by itself.

Now we discuss the possible contributions to $\langle T^{\mu\nu}\rangle_{reg}$ which may arise from the singular terms in $\mathcal{G}_1$ and $\mathcal{G}_2$. First of all, they must be in the Lorentz-invariant form. Moreover, since both of the regular parts of $\langle T^{\mu\nu}\rangle$, Eqs.(B.7) and (B.17), satisfy the conservation law, those from the singular terms, after regularization, should also satisfy the conservation law by themselves. Hence, let us consider an effective action of the form,

$$S_{reg} = \int \sqrt{-g} d^4x L(\zeta); \quad \zeta = \xi^2 = g_{\mu\nu} x^\mu x^\nu, \tag{B.18}$$

where $g_{\mu\nu}$ is the metric which should be set to $\eta_{\mu\nu}$ after taking the variation. Note that S_{reg} has the Lorentz invariance but does not have the coordinate invariance for general $g_{\mu\nu}$. Now, taking the variation with respect to $g_{\mu\nu}$ and evaluating the result at $g_{\mu\nu} = \eta_{\mu\nu}$, we obtain

$$\Delta T^{\mu\nu}_{reg} := 2\frac{\delta S_{reg}}{\delta g_{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 2\frac{dL}{d\zeta} x^\mu x^\nu + \eta^{\mu\nu} L. \tag{B.19}$$

Then requiring $\Delta T^{\mu\nu}_{reg,\nu} = 0$, we find

$$\zeta \frac{d^2 L}{d\zeta^2} + 3\frac{dL}{d\zeta} = 0, \tag{B.20}$$

which is easily integrated to give

$$L = D_1 + D_2 \frac{1}{\zeta^2} = D_1 + D_2 \frac{1}{\xi^4}, \tag{B.21}$$

where D_1 and D_2 are arbitrary constants. Consequently we have

$$\Delta T^{\mu\nu}_{reg} = D_1 \eta^{\mu\nu} + D_2 \frac{1}{\xi^4}\left(\eta_{\mu\nu} - 4\frac{x_\mu x_\nu}{\xi^2}\right). \tag{B.22}$$

This is the contribution from the regularization of the divergent terms. Comparing it with Eqs.(B.7) and (B.17), we find that there are terms of the same form as the D_2-term in the latter. Hence we absorb them in the D_2-term. Then, recovering the prefactors for the expressions (B.7) and (B.17) (see Eq.(B.2)), we finally obtain

$$
\begin{aligned}
\langle T^{\mu\nu}\rangle_{reg} =& \frac{m_s}{4\pi^2}\left[\frac{\xi^2}{3(1-\xi^2)^3}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)+\frac{1}{(1-\xi^2)^3}\eta^{\mu\nu}\right] \\
&+\frac{m_s^2}{16\pi^2}\left[\left(\frac{2-3\xi^2}{3\xi^2(1-\xi^2)^2}+\frac{1}{3\xi^4}\ln(1-\xi^2)^2\right)\right. \\
&\qquad \left.\times\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right)-\frac{1}{(1-\xi^2)^2}\eta^{\mu\nu}\right] \\
&+D_1\eta^{\mu\nu}+D_2\frac{1}{\xi^4}\left(\eta^{\mu\nu}-\frac{4x^\mu x^\nu}{\xi^2}\right).
\end{aligned}
\tag{B.23}
$$

APPENDIX C

Here we consider the expansion of the following expression in terms of the Hermite polynomials,

$$\left\{f(\tau)\frac{\partial}{\partial\eta} + \dot{f}(\tau)\eta\right\}^n \exp\left(-\frac{\Omega(\tau)\eta^2}{2}\right). \tag{C.1}$$

First we set

$$\left\{f\frac{\partial}{\partial\eta} + \dot{f}\eta\right\}^n \exp\left(-\frac{\Omega\eta^2}{2}\right) =: \sum_{k=0}^{n} Z_{n,k} H_k(\sqrt{\Omega}\eta) \exp\left(-\frac{\Omega\eta^2}{2}\right). \tag{C.2}$$

Then using the relation $\dot{g}f - \dot{f}g = 2\omega$, and the recursion relation,

$$\begin{aligned}\left\{f\frac{\partial}{\partial\eta} + \dot{f}\eta\right\}^2 &\sum_{k=0}^{n} Z_{n,k} H_k(\sqrt{\Omega}\eta) \exp\left(-\frac{\Omega\eta^2}{2}\right) \\ &= \sum_{k=0}^{n+2} Z_{n+2,k} H_k(\sqrt{\Omega}\eta) \exp\left(-\frac{\Omega\eta^2}{2}\right),\end{aligned} \tag{C.3}$$

we obtain for even n,

$$\begin{aligned}
&Z_{n+2,n+2} = \mathcal{F}^2 Z_{n,n} && (n = 0, 2, 4, \cdots) \\
&Z_{n+2,n} = (4n+2)\mathcal{B}\mathcal{F}^2 Z_{n,n} + \mathcal{F}^2 Z_{n,n-2} && (n = 2, 4, \cdots) \\
&Z_{n+2,k} = 4(k+2)(k+1)\mathcal{B}^2\mathcal{F}^2 Z_{n,k+2} + (4k+2)\mathcal{B}\mathcal{F}^2 Z_{n,k} + \mathcal{F}^2 Z_{n,k-2} && \\
& && (n = 4, 6, \cdots, k = n-2, n-4, \cdots, 2) \\
&Z_{n+2,0} = 8\mathcal{B}^2\mathcal{F}^2 Z_{n,2} + 2\mathcal{B}\mathcal{F}^2 Z_{n,0} && (n = 2, 4, \cdots) \\
&Z_{0,0} = 1, \qquad Z_{2,0} = 2\mathcal{B}\mathcal{F}^2,
\end{aligned}$$

and for odd n,

$$\begin{aligned}
&Z_{n+2,n+2} = \mathcal{F}^2 Z_{n,n} && (n = 1, 3, 5, \cdots) \\
&Z_{n+2,n} = (4n+2)\mathcal{B}\mathcal{F}^2 Z_{n,n} + \mathcal{F}^2 Z_{n,n-2} && (n = 3, 5, \cdots) \\
&Z_{n+2,k} = 4(k+2)(k+1)\mathcal{B}^2\mathcal{F}^2 Z_{n,k+2} + (4k+2)\mathcal{B}\mathcal{F}^2 Z_{n,k} + \mathcal{F}^2 Z_{n,k-2} && \\
& && (n = 3, 5, \cdots, k = n-2, n-4, \cdots, 3) \\
&Z_{n+2,1} = 24\mathcal{B}^2\mathcal{F}^2 Z_{n,3} + 6\mathcal{B}\mathcal{F}^2 Z_{n,1} && (n = 3, 5, \cdots) \\
&Z_{1,1} = -1, \qquad Z_{3,1} = -6\mathcal{B}\mathcal{F}^3,
\end{aligned}$$

where we have defined

$$\mathcal{F} := \frac{\omega}{\sqrt{\Omega g}}, \qquad \mathcal{B} := \left(1 - \frac{\Omega}{\omega} f g\right). \tag{C.4}$$

If we set

$$Z_{n,k} =: \mathcal{F}^n \mathcal{B}^{(n-k)/2} D_{n,k}, \tag{C.5}$$

the above reduction formulas are rewritten as, for even n,

$$\begin{aligned}
&D_{n+2,n+2} = D_{n,n} && (n = 0, 2, 4, \cdots)\\
&D_{n+2,n} = (4n+2)D_{n,n} + D_{n,n-2} && (n = 2, 4, \cdots)\\
&D_{n+2,k} = 4(k+2)(k+1)D_{n,k+2} + (4k+2)D_{n,k} + D_{n,k-2} &&\\
& && (n = 4, 6, \cdots, k = n-2, n-4, \cdots, 2)\\
&D_{n+2,0} = 8D_{n,2} + 2D_{n,0} && (n = 2, 4, \cdots)\\
&D_{0,0} = 1, \qquad D_{2,0} = 2, &&
\end{aligned}$$

and for odd n,

$$\begin{aligned}
&D_{n+2,n+2} = D_{n,n} && (n = 1, 3, 5, \cdots)\\
&D_{n+2,n} = (4n+2)D_{n,n} + D_{n,n-2} && (n = 3, 5, \cdots)\\
&D_{n+2,k} = 4(k+2)(k+1)D_{n,k+2} + (4k+2)D_{n,k} + D_{n,k-2} &&\\
& && (n = 3, 5, \cdots, k = n-2, n-4, \cdots, 3)\\
&D_{n+2,1} = 24D_{n,3} + 6D_{n,1} && (n = 3, 5, \cdots)\\
&D_{1,1} = -1, \qquad D_{3,1} = -6. &&
\end{aligned}$$

In principle, we can obtain a solution with arbitrary numbers of n and k by following these reduction formulas. We obtained the solution numerically by the aid of computer machine, and arranged the results carefully, then we found the general form of the solution for even n,

$$D_{n,k} = \begin{cases} \dfrac{(n-1)!!}{(k-1)!!} 2^{(n-k)/2} \; {}_{n/2}C_{k/2}, & k: \text{ even}, \\ 0 \qquad\qquad\qquad , & k: \text{ odd}, \end{cases} \tag{C.6}$$

and for odd n,

$$D_{n,k} = \begin{cases} 0 \quad , & k: \text{ even}, \\ -\dfrac{n!!}{k!!}\, 2^{(n-k)/2} \; {}_{(n-1)/2}C_{(k-1)/2}, & k: \text{ odd}. \end{cases} \tag{C.7}$$

Here ${}_lC_m$ stands for $l!/m!(l-m)!$. This solution satisfies the above reduction formula properly.

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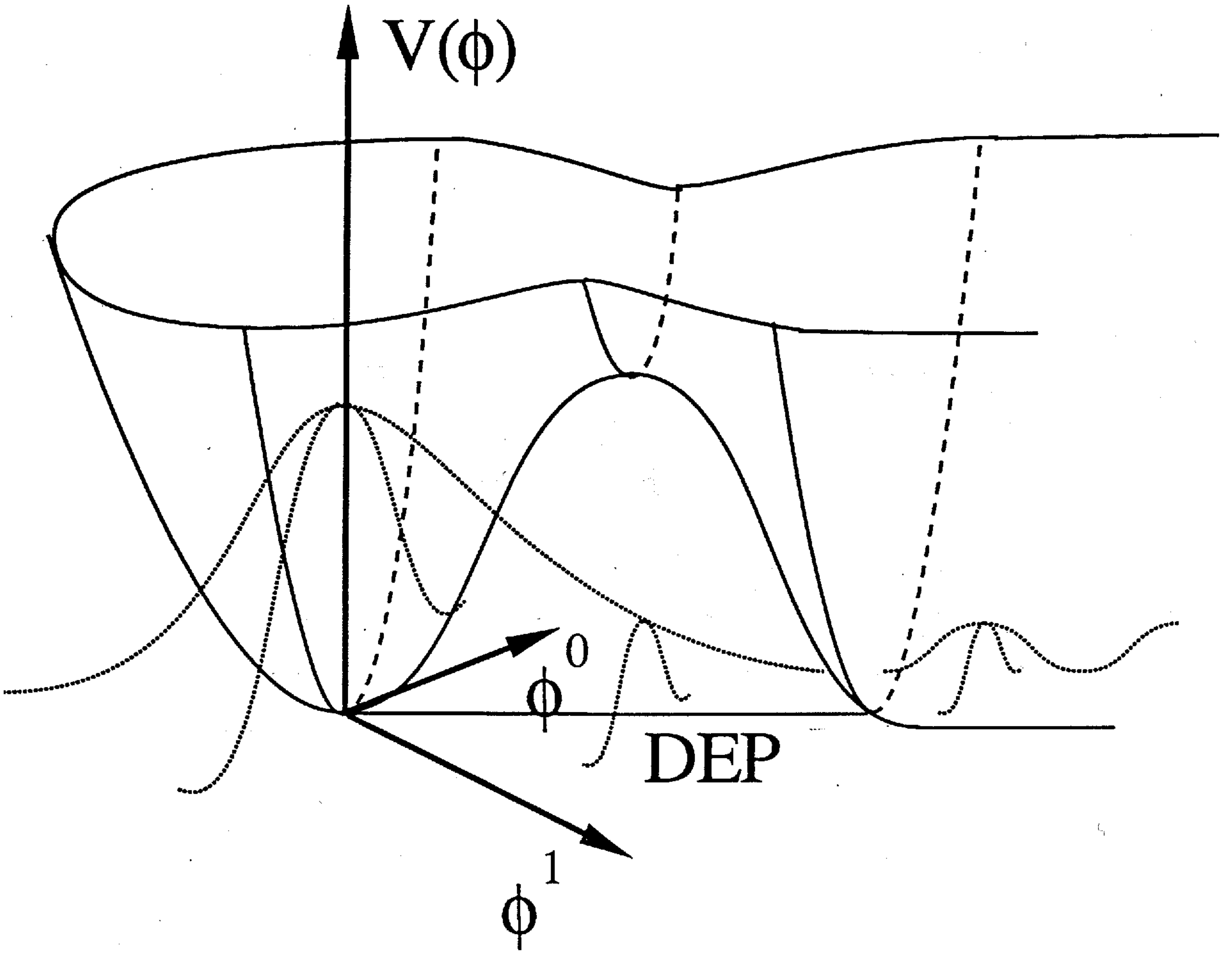

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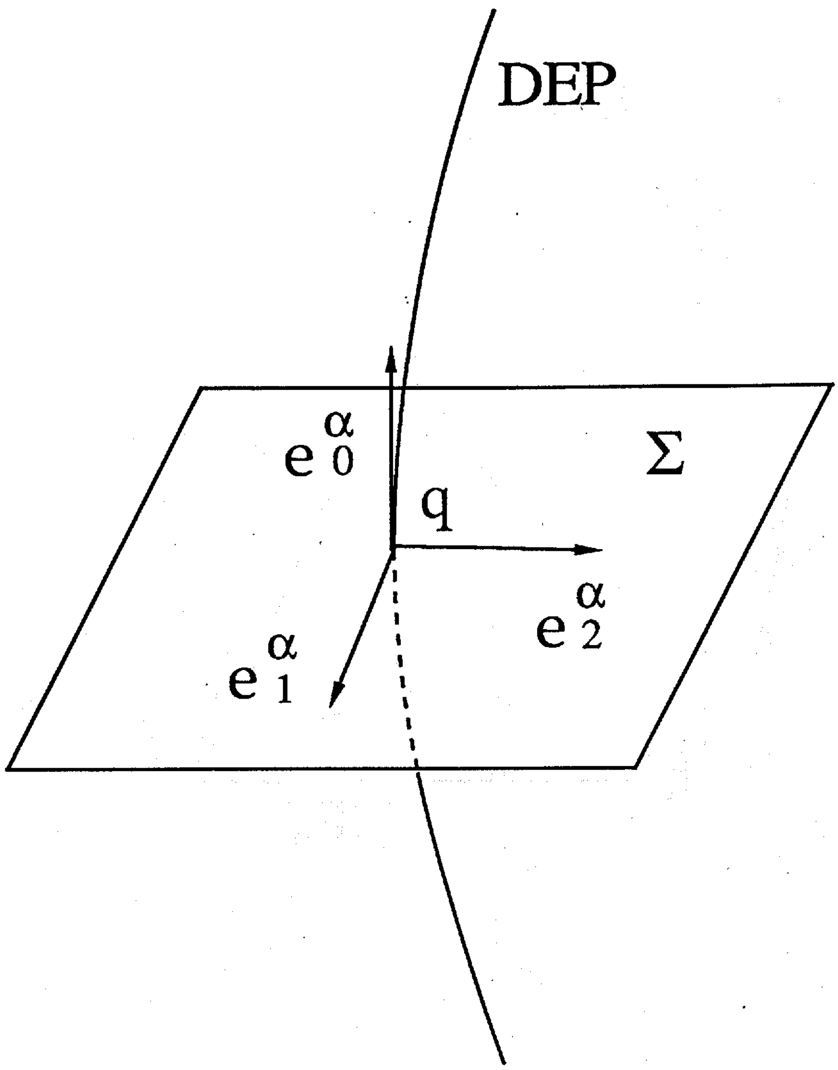

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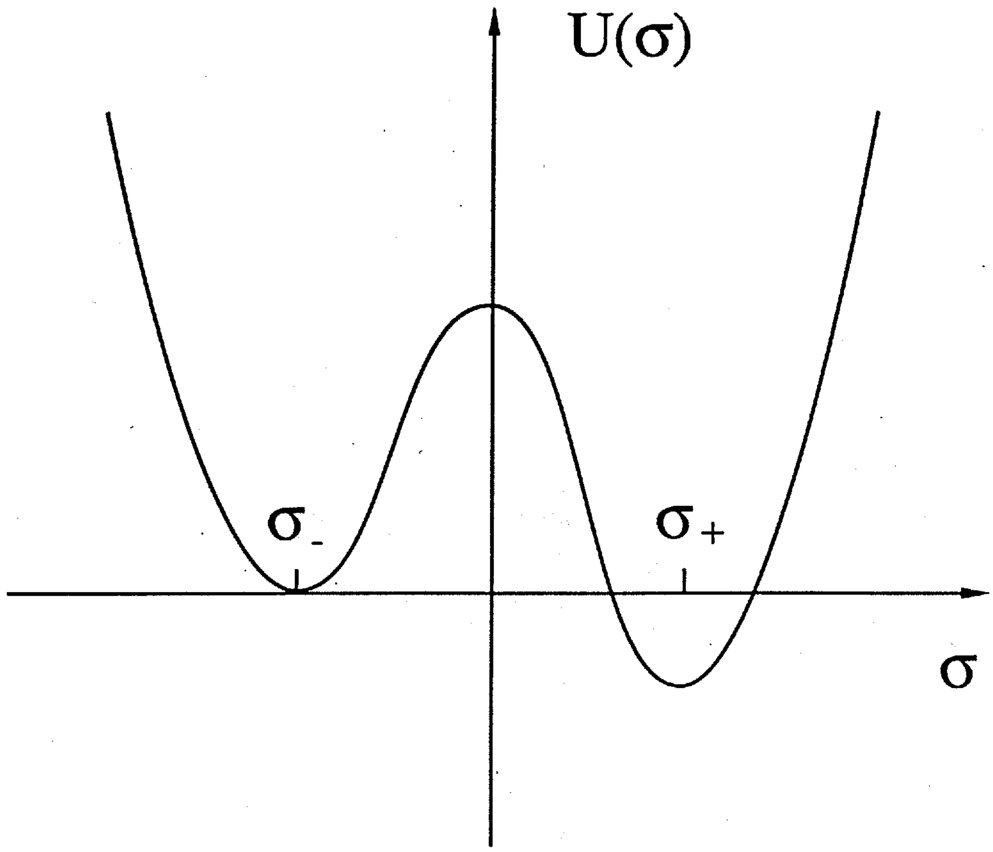

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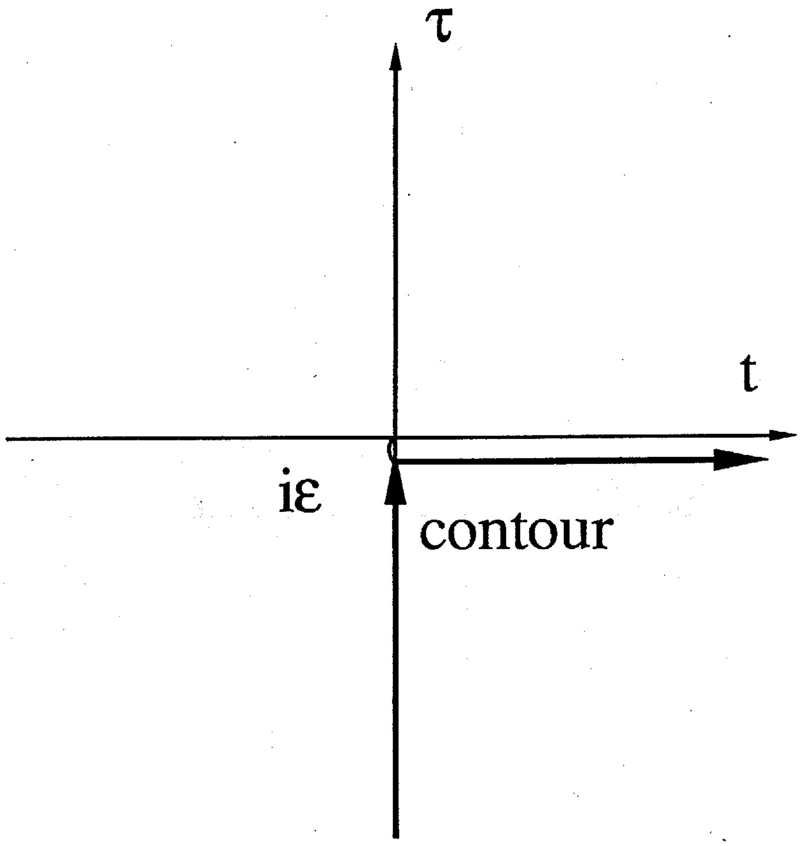

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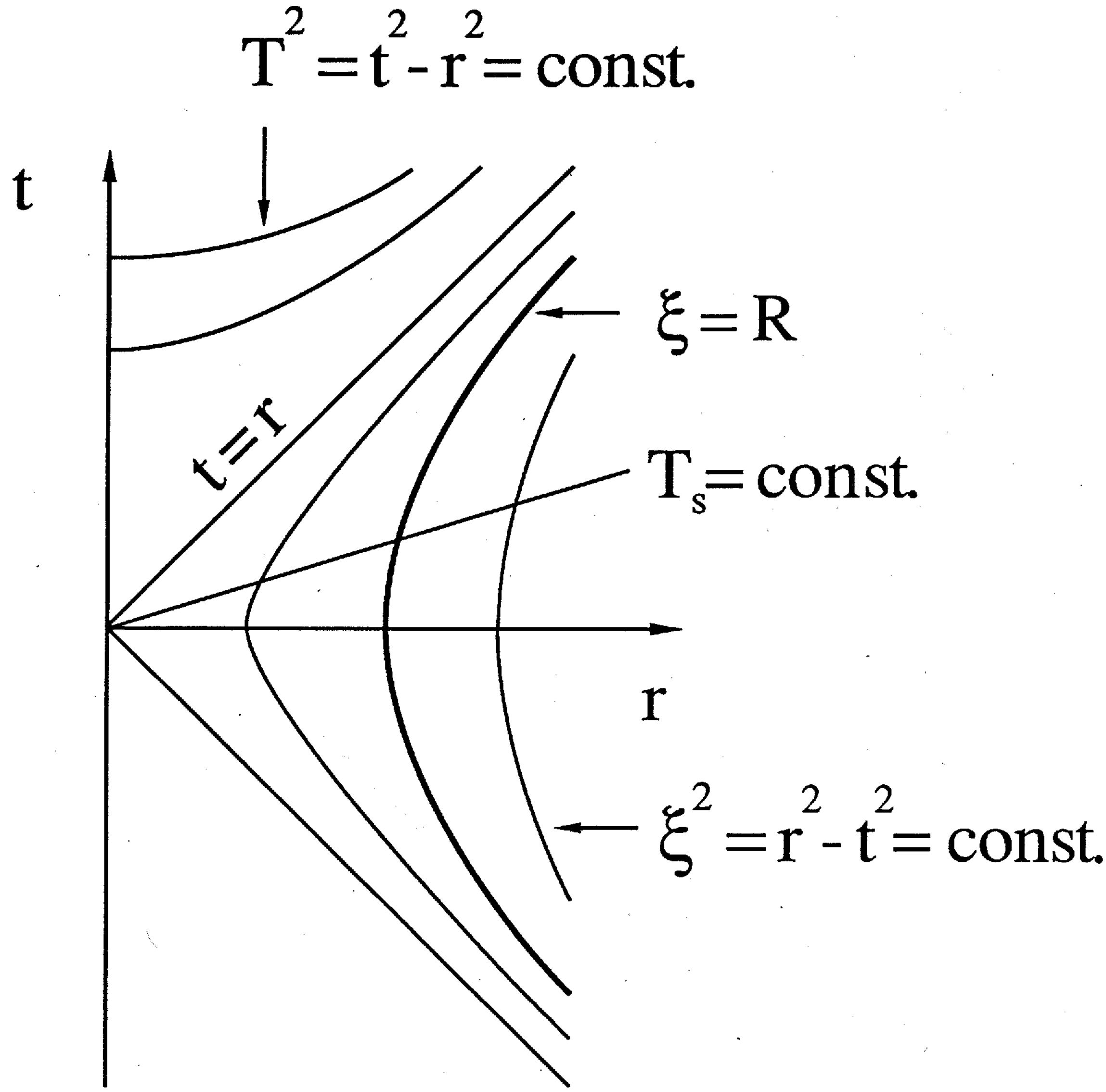

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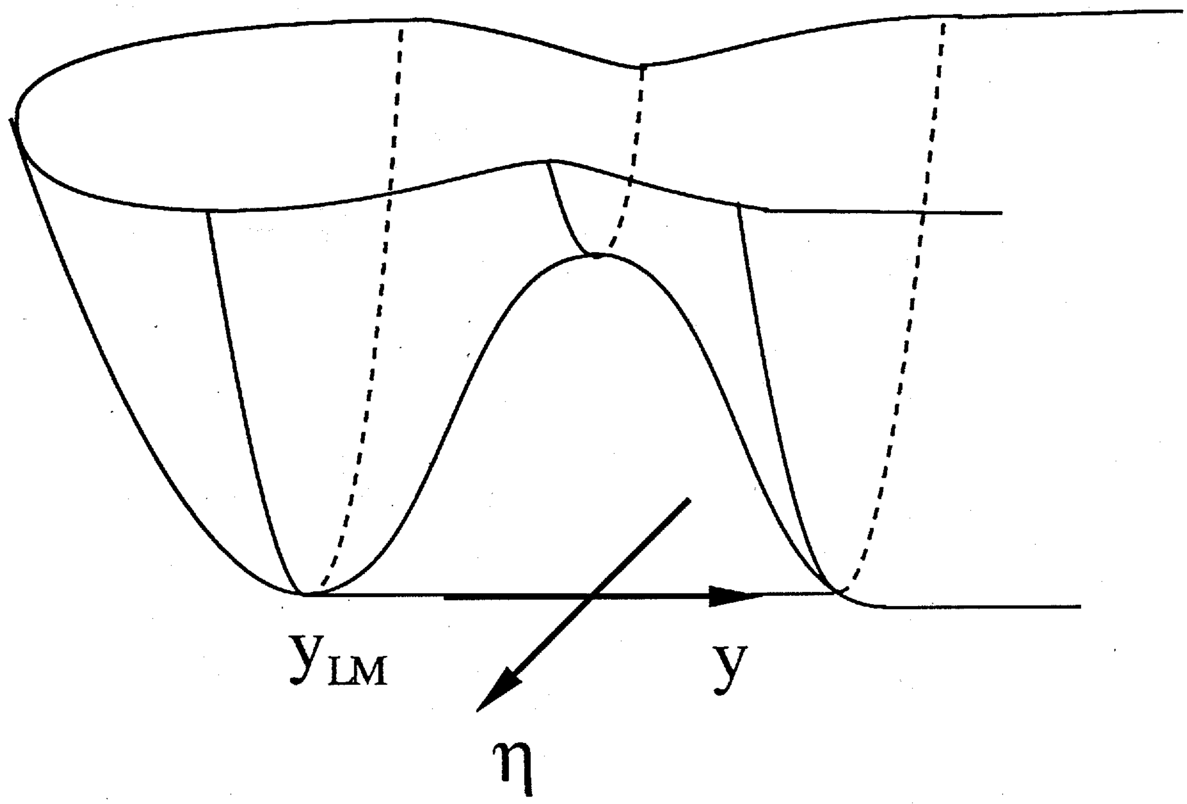

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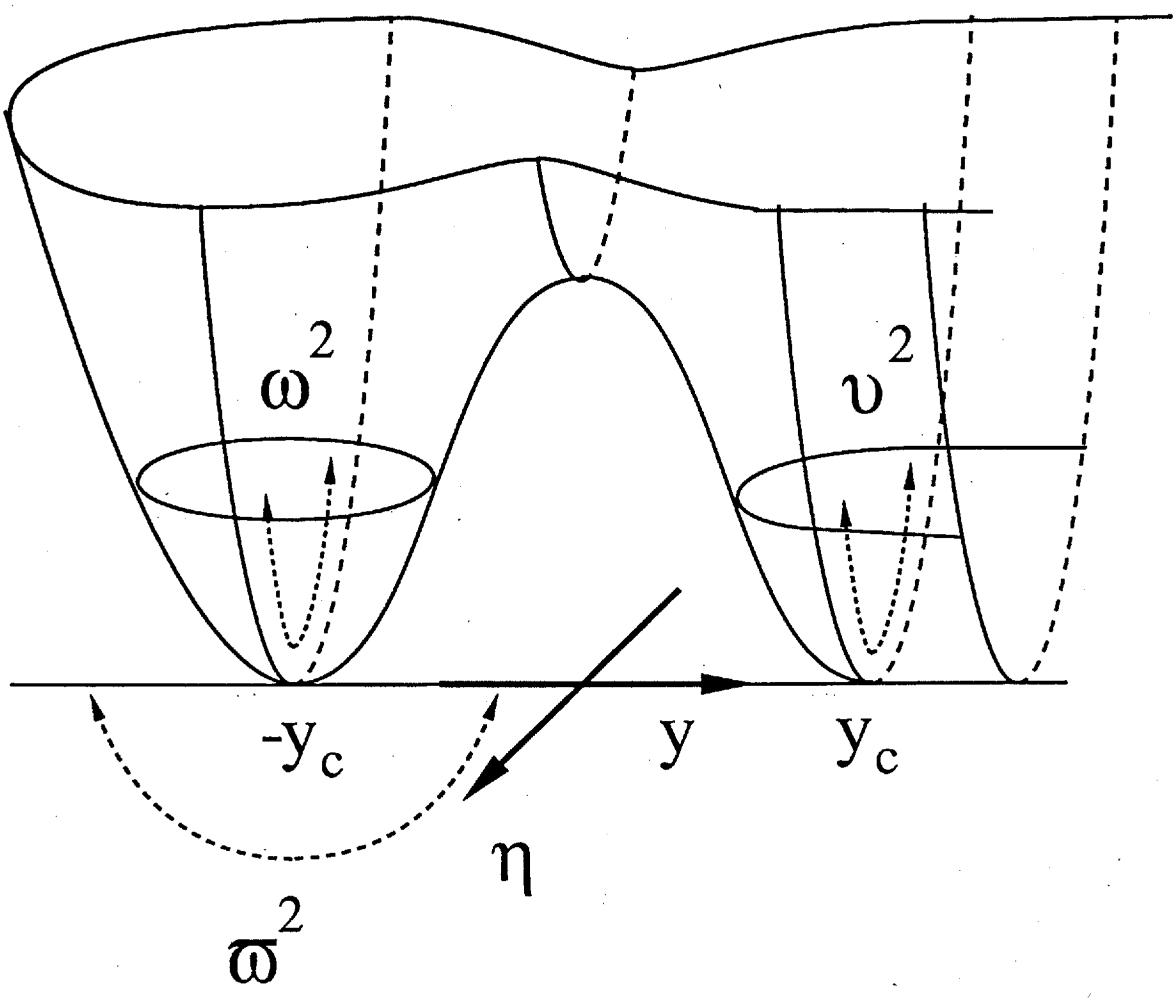

Fig.7