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Introduction

In [10] and [11], E.Rees and E.Thomas studied the divisibility of some Chern numbers of the complex cobordism classes and the homotopy groups of $MU(n)$. The purpose of this paper is to study the symplectic cobordism theory by using their methods.

Let $MSp(n)$ be the Thom space of the universal symplectic vector bundle over the classifying space $BSp(n)$, and $MSp = \{MSp(n), \varepsilon_n\}$ be the Thom spectrum of the symplectic cobordism theory, where $\varepsilon_n : \Sigma^4 MSp(n) \longrightarrow MSp(n+1)$ is the structure map. Let $b_n : MSp(n) \longrightarrow \Omega^{4N}MSp(n+N)$ be the adjoint map of the composition $\varepsilon_{n,N} : \Sigma^{4N}MSp(n) \longrightarrow MSp(n+N)$ of $\Sigma^i \varepsilon_{n+i}$, where $N > 3n > 0$. Converting b_n into a fibering with fiber F_n, we consider the fibering

$$(1) \qquad F_n \longrightarrow MSp(n) \xrightarrow{b_n} \Omega^{4N}MSp(n+N).$$

Then F_n is $(8n-2)$-connected, and we can determine the cohomology groups of F_n in dimensions less than $12n-2$ (see Proposition 2.15).

Let $P_i \in H^{4i}(BSp)$ be the i-th symplectic Pontrjagin class. For a symplectic cobordism class $u \in \pi_{4k}(MSp)$ and a class $P_{i_1} \cdots P_{i_j} \in H^{4i}(BSp)$ with $\sum_{t=1}^{j} i_t = i$, $P_{i_1} \cdots P_{i_j}[u]$ denotes the

Pontrjagin number of u for a class $P_{i_1} \cdots P_{i_j}$.

The one of our purpose is to obtain the divisibility of some Pontrjagin numbers of the symplectic cobordism classes by making use of the cohomology groups of F_n. As a concrete result, we have the following theorem (see Theorem 3.8):

Theorem I. Let $n \geqq 1$. Then

(i) $P_n[u] \equiv 0 \mod 8$ for any $u \in \pi_{4n}(MSp)$.

(ii) $P_1 P_n[u] - ((n+4)/2)P_{n+1}[u] \equiv 0 \mod 24$ for any $u \in \pi_{4n+4}(MSp)$.

The divisibility of Pontrjagin numbers of some symplectic cobordism classes has been studied in [14], [13], [3], [6] to investigate the structure of $\pi_*(MSp)$. For the divisibility (i) of the above theorem, E.E.Floyd [3] has proved it with some restriction by using the alternative method, and some application of the method of Floyd's is considered in [4].

The second purpose of this paper is to study the homotopy groups $\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ by using the fibering (1) and some examples of the symplectic cobordism classes. Our second results are stated as follows (see Corollaries 4.4, 4.5 and Theorems 4.6, 4.7).

Theorem II. (i) Let $m(n)$ be the greatest common measure of $\{(1/8)P_n[u] \mid u \in \pi_{4n}(MSp)\}$. Then the induced homomorphism

$$b_{n*} : \pi_{8n-1}(MSp(n)) \longrightarrow \pi_{4n-1}(MSp)$$

of b_n in (1) is epimorphic and its kernel is a cyclic group of order $4m(n)$ generated by the Whitehead product $[i, i]$ for the homotopy class i of the natural inclusion map $S^{4n} \longrightarrow MSp(n)$.

(ii) If $2\pi_{4n-1}(MSp) = 0$ and n is not a power of 2, then b_{n*} in (i) is split epimorphic, that is,

$$\pi_{8n-1}(MSp(n)) \cong Z_{4m(n)} \oplus \pi_{4n-1}(MSp).$$

(iii) $m(n)$ is a power of 2 for $n \neq 1, 3$, and $m(1)=m(3)=3$.

(iv) $m(n) = 1$ if $n = 2^k+2^\ell-1$ or 2^k+2^ℓ $(k,\ell \geqq 0)$ and $n \neq 1, 3$.

Theorem III. (i) $\pi_{8n+3}(MSp(n))$ $(n \geqq 3)$ has no p-torsion for any odd prime p.

(ii) The homomorphism $b_{n*} : \pi_{8n+3}(MSp(n)) \longrightarrow \pi_{4n+3}(MSp)$ is epimorphic for $n \geqq 1$.

(iii) If $n = 2^k+2^\ell-1$ $(k,\ell \geqq 1)$, then b_{n*} in (ii) is isomorphic, i.e., $\pi_{8n+3}(MSp(n)) \cong \pi_{4n+3}(MSp)$.

We notice that the assumption $2\pi_{4n-1}(MSp) = 0$ in Theorem II (ii) is valid for $n \leqq 8$ by the result of D.M.Segal [12].

This paper is organized as follows. In §1 we summarize the necessary lemmas concerning the iterated cohomology suspension investigated by R.J.Milgram [5]. In §2 we study the cohomology groups of F_n, and in §3 we state the divisibility of some Pontrjagin numbers and prove Theorem I. In §4 we consider the homotopy exact sequence concerning $\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ and state Theorems II and III. In §5 we prepare some symplectic cobordism classes and prove these theorems in §4.

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§1. Preliminaries

In this section, we summarize the necessary lemmas concerning the iterated cohomology suspension studied by R.J.Milgram [5].

Let Y be an $(r-1)$-connected CW-complex, and $i : Y \longrightarrow \Omega^k\Sigma^k Y$ be the natural inclusion. Then Milgram [5; Th.1.11] proved that the cofiber $\Omega^k\Sigma^k Y/Y$ of i is homotopy equivalent in dimensions less than $3r-1$ to the space $S^{k-1} \ltimes_T Y \wedge Y$, where $S^{k-1} \ltimes_T Y \wedge Y$ is the quotient space of $S^{k-1} \times (Y \wedge Y)$ by the identification of (x, y_1, y_2) with $(-x, y_2, y_1)$ and $(x, *)$ with the base point.

When $Y = \Omega^k X$ for a $(k+r-1)$-connected CW-complex X, we can consider the evaluation map $e : \Sigma^k\Omega^k X \longrightarrow X$ and the fibering

$$F \xrightarrow{j} \Sigma^k Y \xrightarrow{e} X \quad (Y = \Omega^k X).$$

Then the inclusion $i : Y \longrightarrow \Omega^k\Sigma^k Y$ is a section of the fibering $\Omega^k F \xrightarrow{\Omega^k j} \Omega^k\Sigma^k Y \xrightarrow{\Omega^k e} Y$, and we have the maps $F \xleftarrow{e} \Sigma^k\Omega^k F \xrightarrow{\Sigma^k(q\cdot\Omega^k j)} \Sigma^k(\Omega^k\Sigma^k Y/Y)$, where $q : \Omega^k\Sigma^k Y \longrightarrow \Omega^k\Sigma^k Y/Y$ is the canonical projection. Since these maps are $(k+3r-1)$-equivalent, we have the following lemma (cf. Proof of [5; Cor.4.4]).

Lemma 1.1. In dimensions less than $k+3r-1$, F is homotopy equivalent to $\Sigma^k(S^{k-1} \ltimes_T \Omega^k X \wedge \Omega^k X)$.

Take X to be the Thom space $MSp(n+N)$ of the universal symplectic vector bundle over $BSp(n+N)$. Then we have the fibering

$$(1.2) \qquad F(e) \xrightarrow{\ i\ } \Sigma^{4N}\Omega^{4N}MSp(n+N) \xrightarrow{\ e\ } MSp(n+N).$$

Hereafter we shall take integers n and N to satisfy $N > 3n > 0$. By Lemma 1.1, we have

Corollary 1.3. In dimensions less than $4N+12n-1$, $F(e)$ is homotopy equivalent to $\Sigma^{4N}\Gamma(n,N)$, where we use the notation

$$\Gamma(n,N) = S^{4N-1} \ltimes_T \Omega^{4N}MSp(n+N) \wedge \Omega^{4N}MSp(n+N).$$

Put $\Lambda = Z$ or Z_p (p:prime). By this corollary, we have the isomorphisms

$$H^{i+4N}(F(e);\ \Lambda) = H^i(\Gamma(n,N);\ \Lambda) \quad \text{for} \quad i \leqq 12n-2.$$

Therefore the Serre cohomology exact sequence of (1.2) turns out to be the exact sequence

$$(1.4) \quad \cdots \longrightarrow H^{i-1}(\Gamma(n,N);\ \Lambda) \xrightarrow{\ \tau\ } H^{i+4N}(MSp(n+N);\ \Lambda) \xrightarrow{\ \sigma\ }$$
$$H^i(\Omega^{4N}MSp(n+N);\ \Lambda) \xrightarrow{\ j\ } H^i(\Gamma(n,N);\ \Lambda) \longrightarrow \cdots$$

for $i \leqq 12n-2$, where τ, σ and j are the transgression, the induced homomorphisms e^* and i^* composed with the suspension isomorphisms respectively, and σ is known to be the iterated cohomology suspension.

We shall use the following notations:

(1.5) (i) By a series $R = (r_1, \cdots, r_j)$, we mean that r_i's are non negative integers.

(ii) For a series $R = (r_1, \cdots, r_j)$, we set $|R| = \sum_{i=1}^{j} ir_i$.

(iii) For series $R = (r_1, r_2, \cdots)$ and $S = (s_1, s_2, \cdots)$, $R > S$ means that $r_i = s_i$ $(1 \leqq i \leqq m-1)$ and $r_m > s_m$ for some $m \geqq 1$.

Let $P_i \in H^{4i}(BSp)$ be the universal i-th symplectic Pontrjagin class. Then it is known that $H^*(BSp(n+N)) = Z[\![P_1, \cdots, P_{n+N}]\!]$. We set $P^R = P_1^{r_1} \cdots P_j^{r_j} \in H^{4|R|}(BSp)$ for a series $R = (r_1, \cdots, r_j)$.

Let $U \in H^{4(n+N)}(MSp(n+N))$ be the Thom class of $MSp(n+N)$, and consider the composition

$$V : H^{i-4n}(BSp(n+N)) \xrightarrow[\cong]{U} H^{i+4N}(MSp(n+N)) \xrightarrow{\sigma} H^i(\Omega^{4N}MSp(n+N)),$$

where U is the Thom isomorphism given by $U(x) = Ux$ and σ is the iterated cohomology suspension in (1.4). Here σ is isomorphic for $i \leqq 8n-1$, and $H^*(\Omega^{4N}MSp(n+N))$ for $* \leqq 8n-1$ is the free abelian group with basis $\{V(P^R) \mid |R| < n\}$, where

(1.6) $V(P^R) = \sigma(UP^R) \in H^{4(n+|R|)}(\Omega^{4N}MSp(n+N))$.

The following lemma is an immediate consequence of [5; Prop.3.1] (cf. [11; (2.1)]), where $\langle\theta,\theta'\rangle$ and $e^i \cdot \theta\otimes\theta$ are the notations used in [11].

Lemma 1.7. (i) The cohomology group $H^i(\Gamma(n,N))$ for $i \leqq 12n-2$ is a direct sum of some copies of Z and Z_2. A basis of its free part consists of the following classes:

$\langle V(P^R), V(P^S)\rangle \in H^{8n+4(|R|+|S|)}(\Gamma(n,N))$ for any series R and S with $R > S$ and $|R|+|S| \leqq n-1$, and

$1\cdot V(P^R)\otimes V(P^R) \in H^{8n+8|R|}(\Gamma(n,N))$ for any series R with $2|R| \leqq n-2$.

A basis of its Z_2-summands consists of the following classes:

$e^{2k}\cdot V(P^R)\otimes V(P^R) \in H^{8n+2k+8|R|}(\Gamma(n,N))$ for any integer $k \geqq 1$ and any series R with $2k+8|R| \leqq 4n-2$.

(ii) A basis of $H^i(\Gamma(n,N); Z_2)$ for $i \leqq 12n-2$ consists of the mod 2 reductions of the classes given in (i) and moreover the classes

$e^{2k+1}\cdot V(P^R)\otimes V(P^R) \in H^{8n+2k+1+8|R|}(\Gamma(n,N); Z_2)$ for any integer $k \geqq 0$ and any series R with $2k+8|R| \leqq 4n-4$.

We remark that the classes $u\cdot\theta\otimes\theta$ in [11; (2.1)] do not appear in $H^i(\Gamma(n,N))$ for $i \leqq 12n-2$ since $N > 3n$.

By the above lemma, we have $H^j(\Gamma(n,N)) = 0$ if j is odd and $j < 12n-2$, and the following

Lemma 1.8. (i) The sequence (1.4) for $\Lambda = Z$ and $i \leqq 12n-2$ is short exact:

$$0 \longrightarrow H^{i+4N}(MSp(n+N)) \xrightarrow{\sigma} H^i(\Omega^{4N}MSp(n+N)) \xrightarrow{j} H^i(\Gamma(n,N)) \longrightarrow 0.$$

(ii) $H^i(\Omega^{4N}MSp(n+N)) = 0$ if i is odd and $i \leqq 12n-2$.

For the maps j and τ in (1.4), we have the following lemma by [5; Th.4.6] (cf. [11; (2.10), (2.5)]):

Lemma 1.9. (i) In the integral cohomology groups,

$$j(V(P^R)V(P^S)) = \begin{cases} \langle V(P^R), V(P^S)\rangle & \text{if } R > S, \\ 2(1\cdot V(P^R)\otimes V(P^R)) & \text{if } R = S. \end{cases}$$

(ii) In the mod 2 cohomology groups,

$$\tau(e^{4j-1}\cdot V(P^R)\otimes V(P^R)) = Sq^{4(n+j+|R|)}(UP^R).$$

The next lemma can be proved by a similar argument to E.Rees and E.Thomas [11; 2.4, 2.6, 2.8].

Lemma 1.10. For $i < 3n$, the cohomology group $H^{4i}(\Omega^{4N}MSp(n+N))$ is a free abelian group.

Proof. $H^{4i}(\Omega^{4N}MSp(n+N))$ has no odd torsion by Lemma 1.7. We prove that

(*) $\tau : H^{4i-1}(\Gamma(n,N);\ Z_2) \longrightarrow H^{4N+4i}(MSp(n+N);Z_2)$ is monomorphic.

Then $H^{4i-1}(\Omega^{4N}MSp(n+N);\ Z_2) = 0$ by the exact sequence (1.4), and hence $H^{4i}(\Omega^{4N}MSp(n+N))$ has no 2-torsion by the universal coefficient theorem. Thus we have the lemma.

Now we prove (*). A basis of $H^{4i-1}(\Gamma(n,N);\ Z_2)$ for $i < 3n$ consists of the classes $e^{4i-8(n+|R|)-1}\cdot V(P^R)\otimes V(P^R)$ for any series R with $4i-8(n+|R|)-1 > 0$ by Lemma 1.7, and it holds

$$\tau(e^{4i-8(n+|R|)-1}\cdot V(P^R)\otimes V(P^R)) = Sq^{4(i-n-|R|)}UP^R$$

by Lemma 1.9 (ii). We have $Sq^{4t}(UP^R) = \sum_{0\leqq k\leqq Min(t,|R|)} UP_{t-k}Sq^{4k}P^R$ by the Cartan formula, and $Sq^{4s}P_j = \sum_{0\leqq \ell\leqq s}\binom{j-s+\ell-1}{s}P_{s-\ell}P_{j+\ell}$ by the Wu formula. Using these relations and the condition

$2(n+|R|) < i < 3n$, we have

$$Sq^{4(i-n-|R|)}UP^R = UP_{i-n-|R|}P^R + \sum_S UP^S$$

for some series $S = (s_1, s_2, \cdots)$ with $s_j = 0$ $(j \geqq i-n-|R|)$. By the above relations, we see that τ is monomorphic and we have (*).

q.e.d.

The formulas for the cohomology operations on $H^*(\Gamma(n,N);Z_2)$ are given by Milgram [5; Th.3.7.] (cf. [11;(2.3)]) as follows:

Lemma 1.11. (i) $Sq^{4i}<V(P^R),V(P^S)> = \sum_{0\leqq r<i/2}<Sq^{4r}V(P^R), Sq^{4(i-r)}V(P^S)>$, $Sq^j<V(P^R),V(P^S)> = 0$ if $j \not\equiv 0 \mod 4$.

(ii) $Sq^{4i}(1\cdot V(P^R)\otimes V(P^R)) = \sum_{0\leqq r<i/2}<Sq^{4r}V(P^R),Sq^{4(i-r)}V(P^R)>$ $+ \sum_{j\geqq 0}\binom{n+|R|-j}{i-2j}e^{4i-8j}\cdot Sq^{4j}V(P^R)\otimes Sq^{4j}V(P^R)$,

$Sq^j(1\cdot V(P^R)\otimes V(P^R)) = 0$ if $j \not\equiv 0 \mod 4$.

(iii) For $k \geqq 1$,

$$Sq^i(e^k\cdot V(P^R)\otimes V(P^R)) = \sum_{j,r\geqq 0}\binom{k}{r}\binom{4(n+|R|-j)}{i-r-8j}e^{k+i-8j}\cdot Sq^{4j}V(P^R)\otimes Sq^{4j}V(P^R).$$

Especially, we have

Corollary 1.12. For $k \geqq 1$,

$$Sq^1(e^k\cdot V(P^R)\otimes V(P^R)) = ke^{k+1}\cdot V(P^R)\otimes V(P^R),$$
$$Sq^2(e^k\cdot V(P^R)\otimes V(P^R)) = \binom{k}{2}e^{k+2}\cdot V(P^R)\otimes V(P^R),$$
$$Sq^4(e^k\cdot V(P^R)\otimes V(P^R)) = (\binom{k}{4}+n+|R|)e^{k+4}\cdot V(P^R)\otimes V(P^R).$$

Let X be a $(k+r-1)$-connected space and $r \geqq 2$. The evaluation map $e_i : \Sigma^i\Omega^i X \longrightarrow X$ is the composition of the evaluation maps $e' : \Sigma^j\Omega^j X \longrightarrow \Sigma^{j-1}\Omega^{j-1}X$ $(i\geqq j\geqq 1)$, and we have the commutative diagram

$$(1.13)\qquad \begin{array}{ccccc} F(e_k) & \longrightarrow & \Sigma^k\Omega^k X & \xrightarrow{e_k} & X \\ \downarrow f_1 & & \downarrow e' & & \| \\ F(e_{k-1}) & \longrightarrow & \Sigma^{k-1}\Omega^{k-1}X & \xrightarrow{e_{k-1}} & X, \end{array}$$

where $F(e_i)$ $(i=k-1,k)$ are the fibers of the respective fiberings and f_1 is the restriction of e' to the fiber. Then, through the identifications of $F(e_i)$ with $\Sigma^i(S^{i-1} \ltimes_T \Omega^i X \wedge \Omega^i X)$ in dimensions less than $i+3r-1$ given in Lemma 1.1, we see that f_1 is identified with the composition of

$$\Sigma^k(S^{k-1} \ltimes_T Y \wedge Y) \xrightarrow{\Sigma^k\tau_1} \Sigma^k\Omega(S^{k-2} \ltimes_T \Sigma Y \wedge \Sigma Y) \xrightarrow{\tilde{e}} \Sigma^{k-1}(S^{k-2} \ltimes_T \Omega^{k-1}X \wedge \Omega^{k-1}X),$$

where $Y = \Omega^k X$ and τ_1 is the natural map $\Omega^k\Sigma^k Y/Y \longrightarrow \Omega(\Omega^{k-1}\Sigma^{k-1}(\Sigma Y)/\Sigma Y)$ (see [5; §2]) with identifications $\Omega^i\Sigma^i W/W \simeq S^{i-1} \ltimes_T W \wedge W$ $(W=Y,\Sigma Y)$ and $\tilde{e}$ is the map induced by the evaluation maps.

In the diagram (1.13), set $X = MSp(n+N)$ $(N>3n+4)$ and $k = 4N, \cdots, 4N-3$ to obtain the commutative diagram

$$(1.14)\qquad \begin{array}{ccccc} F(e_{4N}) & \longrightarrow & \Sigma^{4N}\Omega^{4N}MSp(n+N) & \xrightarrow{e_{4N}} & MSp(n+N) \\ \downarrow f & & \downarrow e'' & & \| \\ F(e_{4N-4}) & \longrightarrow & \Sigma^{4N-4}\Omega^{4N-4}MSp(n+N) & \xrightarrow{e_{4N-4}} & MSp(n+N), \end{array}$$

where $e'' = (e')^4$ and $f = (f_1)^4$. Let $\sigma' : H^{i+4N-4}(MSp(n+N)) \longrightarrow H^i(\Omega^{4N-4}MSp(n+N))$ be the iterated cohomology suspension. Then, by using the identifications of $F(e_{4N})$ with $\Sigma^{4N}\Gamma(n,N)$, $F(e_{4N-4})$ with $\Sigma^{4N-4}\Gamma(n+1,N-1)$ and f_1 with $\tilde{e}(\Sigma^k\tau_1)$ as is stated above, we have the following lemma by [5; Th.3.8] on τ_1:

Lemma 1.15. Set $V'(x) = \sigma'(Ux)$. Then

$f^*(e^k \cdot V'(P^R) \otimes V'(P^R)) = e^{k+4} \cdot V(P^R) \otimes V(P^R)$ for any $k \geqq 0$,

$f^*(\langle V'(P^R), V'(P^S)\rangle) = 0$.

§2. The cohomology groups of F_n

The structure map $\varepsilon_n : \Sigma^4 MSp(n) \longrightarrow MSp(n+1)$ in the Thom spectrum $MSp = \{MSp(n), \varepsilon_n\}$ of the symplectic cobordism theory is defined to be the map induced by the bundle map of $\xi_n \oplus 1$ to ξ_{n+1}, where ξ_i is the universal symplectic vector bundle over $BSp(i)$ and 1 means the trivial symplectic line bundle. Consider the composition $\varepsilon_{n,N} : \Sigma^{4N} MSp(n) \longrightarrow MSp(n+N)$ of $\Sigma^i \varepsilon_{n+i}$, and its adjoint map $b_{n,N} : MSp(n) \longrightarrow \Omega^{4N} MSp(n+N)$. Converting $b_{n,N}$ into a fibering with fiber $F_{n,N}$, we consider the fibering

(2.1) $$F_{n,N} \longrightarrow MSp(n) \xrightarrow{b_{n,N}} \Omega^{4N} MSp(n+N).$$

For any $N' > N \geqq n \geqq 1$, the homotopy groups, the cohomology groups of $F_{n,N}$ and $F_{n,N'}$ are naturally isomorphic in dimensions less than $12n-1$, because ε_{n+N} is $(8n+8N+6)$-equivalent. Therefore, for a positive integer n, we shall take an integer N large enough to satisfy $N > 3n$, and we denote simply by

$$b_n = b_{n,N} \quad \text{and} \quad F_n = F_{n,N}.$$

We remark that F_n is $(8n-2)$-connected.

In this section, we investigate the cohomology groups of F_n in dimensions less than $12n-2$.

Let $I_n \subset H^*(BSp)$ be the ideal generated by $\{P_i \mid i > n\}$, and

(2.2) $UI_n^j \subset H^{4(n+N)+j}(MSp(n+N))$ be the subgroup generated by $\{UP^R \mid P^R \in I_n^j = I_n \cap H^j(BSp)\}$.

Then we have

Lemma 2.3. (i) The composition

$$e(\Sigma^{4N}b_n) : \Sigma^{4N}MSp(n) \longrightarrow \Sigma^{4N}\Omega^{4N}MSp(n+N) \longrightarrow MSp(n+N)$$

is homotopic to $\varepsilon_{n,N}$, where e is the evaluation map in (1.2).

(ii) The following commutative diagram of four short exact sequences holds for $i \leqq 12n-2$:

$$\begin{array}{ccccc}
UI_n^{i-4n} & \xrightarrow{\tilde{\sigma}} & H^{i-1}(F_n) & \xrightarrow{\tilde{j}} & H^i(\Gamma(n,N)) \\
\downarrow \cap & & \downarrow \tau & & \| \\
H^{i+4N}(MSp(n+N)) & \xrightarrow{\sigma} & H^i(\Omega^{4N}MSp(n+N)) & \xrightarrow{j} & H^i(\Gamma(n,N)) \\
\downarrow \varepsilon_{n,N}^* & & \downarrow b_n^* & & \\
H^{i+4N}(\Sigma^{4N}MSp(n)) & \underset{\cong}{\xleftarrow{\Sigma^{4N}}} & H^i(MSp(n)). & &
\end{array}$$

Here the central horizontal sequence is the one in Lemma 1.8 (i), the central vertical sequence is the Serre cohomology exact sequence of the fibering (2.1) where τ denotes its transgression, $\tilde{\sigma}$ is the restriction of σ, and $\tilde{j}$ is the composition $j\tau$.

Proof. (i) is clear by definition.

(ii) The left hand vertical sequence is exact by the definition (2.2) of UI_n^j. By (i), the lower square commutes. Since $\varepsilon_{n,N}^*$ is epimorphic, so is b_n^*, and the central vertical sequence is short exact. Since the central horizontal sequence is short exact as is shown in Lemma 1.8 (i), the upper one is so by the 9 lemma, and these complete the proof. q.e.d.

We remark that the transgression $\tau : H^{i-1}(F_n; \Lambda) \longrightarrow H^i(\Omega^{4N}MSp(n+N); \Lambda)$ is monomorphic for $i \leqq 12n-2$ and for any coefficient group Λ, by the proof of the above lemma.

Lemma 2.4. In $H^{8n+4i}(\Omega^{4N}MSp(n+N))$ for $i < n$, the element $V(P^R)V(P^S) - V(P_nP^RP^S)$ belongs to $\mathrm{Ker}\, b_n^*$ for any series R and S with $|R|+|S| = i$.

Proof. Let $\tilde{U}$ denote the Thom class of $MSp(n)$. Then, by Lemma 2.3, we have

$$b_n^*(V(P^R)V(P^S)) = b_n^*\sigma(UP^R)\cdot b_n^*\sigma(UP^S) = (\Sigma^{4N})^{-1}\varepsilon_{n,N}^*(UP^R)\cdot(\Sigma^{4N})^{-1}\varepsilon_{n,N}^*(UP^S)$$
$$= \tilde{U}P^R\cdot\tilde{U}P^S = \tilde{U}P_nP^RP^S = b_n^*(V(P_nP^RP^S)),$$

and the lemma holds. q.e.d.

Especially, $(V(P^R))^2 - V(P_n(P^R)^2)$ is contained in $\mathrm{Ker}\, b_n^*$ by the above lemma for $R = S$. On the other hand, its j-image is $2(1\cdot V(P^R)\otimes V(P^R))$ by Lemma 1.9 (i). Therefore, by the commutative diagram in Lemma 2.3 (ii), we see the following:

(2.5) There is an element $w_n(R)$ $(= w_{n,N}(R)) \in I_n^{4n+8|R|}$ such that $(V(P^R))^2 - V(P_n(P^R)^2) + V(w_n(R))$ is divisible by 2 in $H^{8(n+|R|)}(\Omega^{4N}MSp(n+N))$.

On the other hand, we have

Lemma 2.6. There is an element $v_n(R)$ $(= v_{n,N}(R)) \in I_n^{4n+8|R|}$ for any series R with $2|R| < n$ satisfying the following conditions (i)-(ii):

(i) $v_n(R) = P_{n+|R|}P^R + \sum_S m_S P^S$ for some series $S = (s_1, s_2, \cdots)$ with $s_j = 0$ $(j \geqq n+|R|)$ and some integers m_S.

(ii) $Uv_n(R) = Sq^{4(n+|R|)}(UP^R) + UP_n(P^R)^2$ in $H^{8n+4N+8|R|}(MSp(n+N); Z_2)$.

Proof. By the Cartan formula, we have

$$Sq^{4(n+|R|)}(UP^R) = UP_{n+|R|}P^R + \sum_{k=1}^{|R|-1} UP_{n+|R|-k}Sq^k(P^R) + UP_n(P^R)^2.$$

Hence we can take an element $v_n(R) \in I_n^{4n+8|R|}$ such that its mod 2 reduction is $P_{n+|R|}P^R + \sum_{k=1}^{|R|-1} P_{n+|R|-k}Sq^k(P^R)$ and it satisfies (i). q.e.d.

For classes $w_n(R)$ in (2.5) and $v_n(R)$ in Lemma 2.6, we notice that $w_n(R) = v_n(R) = 0$ for $R = 0$ (the 0-series) since $I_n^{4n} = 0$. Furthermore,

Lemma 2.7. $V(w_n(R)) \equiv V(v_n(R))$ mod 2 for any series R with $2|R| \leqq n-3$.

Proof. Consider the following commutative diagram for $j \leqq 12n-9$ and $n \geqq 2$:

$$\begin{array}{ccccc}
H^{j-1}(\Gamma(n,N); Z_2) & \xrightarrow{\tau} & H^{j+4N}(MSp(n+N); Z_2) & \xrightarrow{\sigma} & H^j(\Omega^{4N}MSp(n+N); Z_2) \\
\downarrow f^* & & \| & & \downarrow \sigma'' \\
H^{j-5}(\Gamma(n-1,N+1); Z_2) & \xrightarrow{\tau'} & H^{j+4N}(MSp(n+N); Z_2) & \xrightarrow{\sigma'} & H^{j-4}(\Omega^{4N-4}MSp(n+N); Z_2)
\end{array}$$

Here two exact sequences are the ones in (1.4) for $\Lambda = Z_2$, f^* is the homomorphism in Lemma 1.15, and σ, σ', σ'' are the iterated cohomology suspensions. Let $j = 8(n+|R|)$ for $2|R| \leqq n-3$. Then $(V(P^R))^2 - V(P_n(P^R)^2) + V(w_n(R)) = 0$ in $H^{8(n+|R|)}(\Omega^{4N}MSp(n+N); Z_2)$ by (2.5), and we have $V'(P_n(P^R)^2) + V'(w_n(R)) = 0$ where $V'(x) = \sigma'(Ux)$. Hence there is a class $z \in H^{8(n+|R|)-5}(\Gamma(n-1,N+1); Z_2)$ satisfying $\tau'(z) = UP_n(P^R)^2 + Uw_n(R)$. Since $w_n(R) \in I_n$, we have $z = e^3 \cdot V'(P^R) \otimes V'(P^R) + \sum_{\ell \geqq 1, T} \lambda_{\ell,T} e^{8\ell+3} \cdot V'(P^T) \otimes V'(P^T)$ for some $\lambda_{\ell,T} \in Z_2$, by Lemma 1.9. These two equalities imply

$$UP_n(P^R)^2 + Uw_n(R) = Sq^{4(n+|R|)}(UP^R) + \tau(\sum_{\ell \geqq 1, T} \lambda_{\ell,T} e^{8\ell-1} \cdot V(P^T) \otimes V(P^T))$$

by Lemmas 1.15 and 1.9. Therefore, if we put $X = w_n(R) - v_n(R) \in I_n^{4n+8|R|}$, then the mod 2 reduction of UX equals $\tau(\sum_{\ell \geqq 1, T} \lambda_{\ell,T} e^{8\ell-1} \cdot V(P^T) \otimes V(P^T))$ and the integral class $V(X)$ is divisible by 2. Therefore we have $V(w_n(R)) \equiv V(v_n(R)) \mod 2$ for any series R with $2|R| \leqq n-3$, and this completes the proof. q.e.d.

Lemma 2.8. Assume $N > 7n-4$. Then, for any integer $k \geqq 1$ and any series R with $k+2|R| \leqq n-1$, the element $V(P_{n+k}(P^R)^2) - V(v_{n+k}(R))$ is divisible by 2 in $H^{8n+4k+8|R|}(\Omega^{4N}MSp(n+N))$, where $v_{n+k}(R) = v_{n+k,N-k}(R)$.

Proof. Consider the iterated cohomology suspension

$$\sigma : H^{i+4k}(\Omega^{4(N-k)}MSp(n+N)) \longrightarrow H^i(\Omega^{4N}MSp(n+N)).$$

Then we see that

$$\sigma((V'(P^R))^2 - V'(P_{n+k}(P^R)^2) + V'(v_{n+k}(R)) = -V(P_{n+k}(P^R)^2) + V(v_{n+k}(R)).$$

Thus the lemma holds by (2.5) and Lemma 2.7. q.e.d.

Now, consider the short exact sequence

$$0 \longrightarrow H^{i-1}(F_n) \xrightarrow{\tau} H^i(\Omega^{4N}MSp(n+N)) \xrightarrow{b_n^*} H^i(MSp(n)) \longrightarrow 0$$

in Lemma 2.3 (ii). Then, by Lemma 2.4, (2.5) and Lemma 2.8, we can define the following classes $a(R,S)$, $b(R)$, $c(2i,R)$ and $d(R)$ in $H^*(F_n)$ as follows:

(2.9) $a(R,S) \in H^{8n+4(|R|+|S|)-1}(F_n)$ for series R and S with $R > S$ and $|R|+|S| \leqq n-1$ satisfying

$$\tau(a(R,S)) = V(P^R)V(P^S) - V(P_nP^RP^S).$$

(2.10) $b(R) \in H^{8n+8|R|}(F_n)$ for a series R with $2|R| \leqq n-1$ satisfying

$$\tau(b(R)) = (1/2)\{(V(P^R))^2 - V(P_n(P^R)^2) + V(w_n(R))\},$$

where $w_n(R)$ is a class in (2.5).

(2.11) $c(4k,R) \in H^{8n+4k-1+8|R|}(F_n)$ for an integer $k \geqq 1$ and a series R with $k+2|R| \leqq n-1$ satisfying

$$\tau(c(4k,R)) = (1/2)\{V(P_{n+k}(P^R)^2) - V(v_{n+k}(R))\},$$

where $N > 7n-4$ and $v_{n+k}(R)$ is a class in Lemma 2.8.

(2.12) $c(4k+2,R) \in H^{8n+4k+1+8|R|}(F_n)$ for an integer $k \geqq 0$ and a series R with $k+2|R| \leqq n-1$ satisfying

$$\tilde{j}(c(4k+2,R)) = e^{4k+2}\cdot V(P^R)\otimes V(P^R),$$

where $\tilde{j} = j\tau : H^{i-1}(F_n) \longrightarrow H^i(\Gamma(n,N))$ in Lemma 2.3 (ii) is isomorphic if $i \equiv 1 \mod 4$.

(2.13) $d(R) \in H^{4n+4|R|}(F_n)$ for a series $R = (r_1, r_2, \cdots)$ with $|R| \leqq 2n-1$, $r_t=1$ for some $t \geqq n+1$ and $r_j=0$ $(j \geqq (n+|R|+1)/2)$ satisfying

$\tau(d(R)) = V(P^R)$.

For the epimorphism $\tilde{j} : H^{i-1}(F_n) \longrightarrow H^i(\Gamma(n,N))$ in Lemma 2.3 (ii), we have

Lemma 2.14. (i) $\tilde{j}(a(R,S)) = \langle V(P^R), V(P^S)\rangle$, $\tilde{j}(b(R)) = 1\cdot V(P^R)\otimes V(P^R)$ and $\tilde{j}(c(4k,R)) = e^{4k}\cdot V(P^R)\otimes V(P^R)$ for $k \geqq 1$.

(ii) The set of $d(R)$ in (2.13) and $2c(4k,T)$ of $c(4k,T)$ in (2.11) forms a basis of $\operatorname{Ker}\tilde{j}$.

Proof. (i) By Lemma 1.9, (2.9) and (2.10), we have

$$\tilde{j}(a(R,S)) = j(V(P^R)V(P^S) - V(P_nP^RP^S)) = \langle V(P^R), V(P^S)\rangle,$$

$$\tilde{j}(b(R)) = j((1/2)\{(V(P^R))^2 - V(P_n(P^R)^2) + V(w_n(R))\}) = 1\cdot V(P^R)\otimes V(P^R).$$

By the definition of f in (1.14), we have the commutative diagram

$$\begin{array}{ccc}
H^{i+4k}(\Omega^{4(N-k)}MSp(n+N)) & \xrightarrow{\ j\ } & H^{i+4k}(\Gamma(n+k,N-k)) \\
\downarrow \sigma & & \downarrow (f^*)^k \\
H^i(\Omega^{4N}MSp(n+N)) & \xrightarrow{\ j\ } & H^i(\Gamma(n,N))
\end{array}$$

for $i \leqq 12n-2$, where $N > 7n-4$, j's are the homomorphisms in (1.4) and σ is the iterated cohomology suspension. By this diagram, Lemmas 1.9, 1.15 and (2.11), we have

$$\begin{aligned}
\tilde{j}(c(4k,R)) &= j((1/2)\{V(P_{n+k}(P^R)^2) - V(v_{n+k}(R))\}) \\
&= -j\sigma((1/2)\{(V(P^R))^2 - V(P_{n+k}(P^R)^2) + V(v_{n+k}(R))\}) \\
&= -(f^*)^k(1\cdot V(P^R)\otimes V(P^R)) = e^{4k}\cdot V(P^R)\otimes V(P^R).
\end{aligned}$$

(ii) By the definition of $v_n(R)$ in Lemma 2.6, the set $\{UP^R \mid R = (r_1, r_2, \cdots)$ with $|R| \leqq 2n-1$, $r_t = 1$ for some $t \geqq n+1$ and

$r_j = 0$ $(j \geqq (n+|R|+1)/2)\}$ $\cup$ $\{UP_{n+k}(P^T)^2 - Uv_{n+k}(T) \mid k \geqq 1,$ T with $k+2|T| \leqq n-1\}$ forms a basis of UI_n^* for $* \leqq 8n-2$. Hence, by the definition of $d(R)$ and $c(4k,T)$ and $\tilde{j}(c(4k,T)) = e^{4k} \cdot V(P^T) \otimes V(P^T)$ in (i), we have the desired result. q.e.d.

Now, by using the upper short exact sequence in Lemma 2.3 (ii) and Lemmas 1.7 and 2.14, we see immediately the following

Proposition 2.15. Let $j \leqq n-1$. Then

(i) $H^{8n+4j-1}(F_n)$ is the free abelian group with the basis consisting of the following classes:

$a(R,S)$ in (2.9) with $|R|+|S| = j$, $b(R)$ in (2.10) with $2|R| = j$, $c(4k,R)$ in (2.11) with $k+|R| = j$, $d(R)$ in (2.13) with $|R| = j+n$.

(ii) $H^{8n+4j+1}(F_n)$ is isomorphic to a direct sum of some copies of Z_2 with the basis consisting of $c(4k+2,R)$ in (2.12) with $k+|R| = j$.

(iii) $H^{8n+4j}(F_n) = H^{8n+4j+2}(F_n) = 0$.

For the mod 2 cohomology of F_n, we can define the class

(2.16) $c(4k+1,R) \in H^{8n+4k+8|R|}(F_n;\ Z_2)$ for an integer $k \geqq 0$ and a series R with $k+2|R| \leqq n-1$ satisfying

$\tilde{j}(c(4k+1,R)) = e^{4k+1} \cdot V(P^R) \otimes V(P^R)$,

where $\tilde{j} : H^{i-1}(F_n;\ Z_2) \longrightarrow H^i(\Gamma(n,N);\ Z_2)$ is isomorphic if $i \equiv 1 \mod 4$.

By the same way as the above proposition, we have

Lemma 2.17. The set of mod 2 reductions of $a(R,S)$, $b(R)$,

$c(2k,R)$, $d(R)$ in (2.9-13) and $c(4k+1,R)$ in (2.16) forms a basis of $H^i(F_n; Z_2)$ for $i \leqq 12n-3$.

We can study the cohomology operations on $H^*(F_n; Z_p)$ for $* \leqq 12n-3$. When p is an odd prime, the operation P^i on $H^*(F_n; Z_p)$ for $* \leqq 12n-3$ is completely determined by Proposition 2.15 and (2.9-13), because we can compute $\tau(P^i x) = P^i \tau(x)$ for any $x \in H^*(F_n; Z_p)$ and τ is monomorphic. Consider the operation Sq^i on $H^*(F_n; Z_2)$ for $* \leqq 12n-3$. Then we can determine $Sq^i x$ for $x = a(R,S)$, $d(R)$, because we can compute $\tau(Sq^i x) = Sq^i \tau(x)$ by (2.9) and (2.13) and τ is monomorphic. For $x = b(R)$, $c(k,R)$, we can compute $\tilde{j}(Sq^i x) = Sq^i \tilde{j}(x)$ by Lemmas 1.11, 2.14 and (2.12), (2.16). Since $\tilde{j} : H^{i-1}(F_n; Z_2) \longrightarrow H^i(\Gamma(n,N); Z_2)$ is monomorphic if $i \not\equiv 0 \mod 4$ and $i \leqq 12n-2$, $Sq^i b(R)$ for $i \not\equiv 0 \mod 4$ and $Sq^i c(k,R)$ for $i+k \not\equiv 0 \mod 4$ can be determined. $Sq^{4i} b(R)$ and $Sq^j c(k,R)$ for $j+k \equiv 0 \mod 4$ can be determined up to linear combinations of $d(T)$.

Consequently we have

Lemma 2.18. (i) $Sq^i a(R,S) = Sq^i b(R) = Sq^i c(4k,R) = Sq^i d(R) = 0$ if $i \not\equiv 0 \mod 4$.

(ii) $Sq^1 c(4k+1,R) = c(4k+2,R)$, $Sq^2 c(4k+2,R) = c(4k+4,R) + X$, where X is a linear combination of $d(T)$.

In the case $R = 0$ (the 0-series), we have

Lemma 2.19. (i) $Sq^4 b(0) = a((1),0) + n\, c(4,0)$, $P^1 b(0) = -(a((1),0) + (n+1)c(4,0))$ for $p = 3$.

(ii) $Sq^i c(4k+1,0) = c(4k+2,0)$ if $i = 1$, 0 if $i = 2$, $(n+k)c(4k+5,0)$ if $i = 4$,

$Sq^i c(4k+2,0) = 0$ if $i = 1$, $c(4k+4,0)$ if $i = 2$, $(n+k)c(4k+6,0)$ if $i = 4$,

$Sq^i c(4k,0) = 0$ if $i = 1$ and 2, $(n+k)c(4k+4,0) + \tilde{\sigma}(UP_1P_{n+k})$ if $i = 4$,

where $\tilde{\sigma}$ is the homomorphism in Lemma 2.3 (ii).

Proof. First, we prove the formula for $P^1b(0)$. When $p = 3$, it holds $P^1V = -V(P_1)$ and $P^1P_n = (n+1)P_{n+1} - P_1P_n$, where $V = V(1)$. By (2.9-11), $\tau(a((1),0)) = V(P_1)V - V(P_1P_n)$, $\tau(b(0)) = (1/2)(V^2 - V(P_n))$ and $\tau(c(4,0)) = ((n+1)/2)V(P_{n+1})$. By these relations, we have $\tau(P^1b(0)) = -\tau(a((1),0) + (n+1)c(4,0))$. Since τ is monomorphic, we have the desired formula for $P^1b(0)$.

Next, by using Lemmas 1.11, 2.14 and (2.16), we have $\tilde{j}(Sq^4b(0)) = \langle V(P_1),V\rangle + ne^4\cdot V\otimes V = \tilde{j}(a((1),0) + nc(4,0))$ and we can compute $\tilde{j}(Sq^i c(k,0)) = Sq^i\tilde{j}(c(k,0))$ for $i = 1, 2, 4$ and $k = 1, 2$. Since $\tilde{j} : H^{i-1}(F_n; Z_2) \longrightarrow H^i(\Gamma(n,N); Z_2)$ is monomorphic for $i \leqq 8n+6$, we have the desired formulas for $Sq^4b(0)$, $Sq^i c(k,0)$ $(i=1,2,4$ and $k=1,2)$.

To obtain the formulas for $Sq^i c(4k+j,0)$, consider the commutative diagram

$$\begin{array}{ccccc} F_n & \longrightarrow & MSp(n) & \xrightarrow{\ b_n\ } & \Omega^{4N}MSp(n+N) \\ \downarrow{\scriptstyle \tilde{b}_k} & & \downarrow{\scriptstyle b_{n,k}} & & \Vert \\ \Omega^{4k}F_{n+k} & \longrightarrow & \Omega^{4k}MSp(n+k) & \xrightarrow{\ \Omega^{4k}b_{n+k}\ } & \Omega^{4N}MSp(n+N), \end{array}$$

where $\tilde{b}_k$ is the restriction of $b_{n,k}$ and $N > 7n-4$. Then we have the following commutative diagram for $i \leqq 12n-2$:

$$\begin{array}{ccccc} H^{i+4k-1}(F_{n+k}) & \xrightarrow[\cong]{\sigma} & H^{i-1}(\Omega^{4k}F_{n+k}) & \xrightarrow{\mathfrak{b}_k^*} & H^{i-1}(F_n) \\ \downarrow \tau & & & & \downarrow \tau \\ H^{i+4k}(\Omega^{4(N-k)}MSp(n+N)) & & \xrightarrow{\sigma} & & H^{i}(\Omega^{4N}MSp(n+N)), \end{array}$$

where σ's are iterated cohomology suspensions and τ's are the transgressions. Since τ's are monomorphic, we have $\mathfrak{b}_k^*\sigma(b(0)) = c(4k,0)$, $\mathfrak{b}_k^*\sigma(c(i,0)) = c(4k+i,0)$ for $i = 1,2$ and $\mathfrak{b}_k^*\sigma(a((1),0)) = \tilde{\sigma}(UP_1P_{n+k})$ by (2.9-12), where $\tilde{\sigma}$ is the homomorphism in Lemma 2.3 (ii). Hence, by the naturality of the cohomology operation, we have the desired formulas for $Sq^i c(4k+j,0)$. q.e.d.

§3. Symplectic Pontrjagin numbers

For a symplectic cobordism class $u \in \pi_i(MSp)$ and a class $y \in H^j(BSp)$, let $y[u]$ be the Pontrjagin number of u for the class y.

To study the divisibility of some Pontrjagin numbers, consider a fixed element

(0) $x_0 = x + x' \in H^t(F_n)$ ($t=8n+4j-1$ with $j<n$),

where x is one of the classes $a(R,S)$, $b(R)$, $C(4k,R)$ and $d(R)$ given in Proposition 2.15 (i) and x' is a linear combination of another classes. Then we can take the following steps (1)-(4):

(1) Take a basis $\{x_i\}$ of $H^t(F_n)$ which includes x_0, and let $\{\bar{x}_i\}$ be the dual basis of $H_t(F_n)$.

(2) Take a suitable cell decomposition of F_n, and denote its

i-skeleton by $F_n^{(i)}$.

(3) For an integer $\ell \geqq 2$ and the Hurewicz homomorphism $H^{(\ell)} : \pi_t(F_n/F_n^{(t-\ell)}) \longrightarrow H_t(F_n/F_n^{(t-\ell)}) \cong H_t(F_n)$, set

$$H^{(\ell)}(v) = \sum_i k_i^{(\ell)}(v)\bar{x}_i \quad \text{for} \quad v \in \pi_t(F_n/F_n^{(t-\ell)}),$$

where $k_i^{(\ell)}(v)$ are integers.

(4) Let $\alpha(\ell)$ be the greatest common measure of $\{k_0^{(\ell)}(v) \mid v \in \pi_t(F_n/F_n^{(t-\ell)})\}$.

Now we have the following basic lemma.

Lemma 3.1. Assume that the class $x_0 = x + x' \in H^t(F_n)$ in (0) satisfies

$$\tau(x_0) = \sum_T \lambda_T V(P^T) + X,$$

where $\tau : H^t(F_n) \longrightarrow H^{t+1}(\Omega^{4N}MSp(n+N))$ is the transgression in Lemma 2.3 (ii), the coefficient λ_T is a half of an integer and X is a sum of decomposable terms. Then

$$\sum_T \lambda_T P^T[u] \equiv 0 \mod \alpha(\ell) \quad \text{for any} \quad u \in \pi_i(MSp),$$

where $\alpha(\ell)$ is the integer given in the step (4).

Proof. Consider the following commutative diagram:

$$(3.2)\qquad \begin{array}{ccccccc} \pi_{t-4n+1}(MSp) & \xleftarrow{\cong} & \pi_{t+1}(\Omega^{4N}MSp(n+N)) & \xrightarrow{\partial} & \pi_t(F_n) & \xrightarrow{q_*} & \pi_t(F_n/F_n^{(t-\ell)}) \\ \downarrow H & & \downarrow H & & \downarrow H & & \downarrow H^{(\ell)} \\ H_{t-4n+1}(MSp) & \xleftarrow{\sigma} & H_{t+1}(\Omega^{4N}MSp(n+N)) & \xrightarrow{\tau} & H_t(F_n) & \underset{\cong}{\xrightarrow{q_*}} & H_t(F_n/F_n^{(t-\ell)}). \end{array}$$

Here ∂ and τ are the connecting map and the transgression of the fibering (2.1) respectively, σ is the iterated homology suspension, q is the natural projection and H's are the Hurewicz homomorphisms.

For any class $u \in \pi_{t-4n+1}(MSp)$, let $u' \in \pi_{t+1}(\Omega^{4N}MSp(n+N))$ be the class corresponding to u under the isomorphism in (3.2). By the above steps (1)-(3), $H^{(\ell)}q_*\partial(u') = \sum_i k_i^{(\ell)}(q_*\partial(u'))\bar{x}_i$. Taking the Kronecker pairing, we have

$$\langle H^{(\ell)}q_*\partial(u'), x_0\rangle = k_0^{(\ell)}(q_*\partial(u')) \equiv 0 \mod \alpha(\ell).$$

On the other hand, by (3.2) and the assumption, we have

$$\langle H^{(\ell)}q_*\partial(u'), x_0\rangle = \langle H(u'), \tau(x_0)\rangle = \sum_T \lambda_T \langle H(u'), \sigma(UP^T)\rangle = \sum_T \lambda_T P^T[u],$$

and these complete the proof. q.e.d.

By this lemma, if we can take a basis of $H^t(F_n)$ in (1) and a cell decomposition of F_n in (2) which enable us to compute $k_i^{(\ell)}(v)$ in (3) and $\alpha(\ell)$ in (4) for a fixed element x_0 in (0), then we have the divisibility of some Pontrjagin number. Here we shall consider the case $x_0 = a((1),0) + (n+4)c(4,0)$ or $c(4,0)$.

We remark that F_n is (8n-2)-connected. By Proposition 2.15 and Lemmas 2.18 and 2.19, we have

Lemma 3.3. (i) For $n \geqq 1$,
$H^{8n-1}(F_n) = Z\langle b(0)\rangle$, $H^{8n}(F_n) = 0$, $H^{8n+1}(F_n) = Z_2\langle c(2,0)\rangle$.

(ii) For $n \geqq 2$,
$H^{8n+2}(F_n) = H^{8n+4}(F_n) = 0$, $H^{8n+3}(F_n) = Z\langle a'\rangle \oplus Z\langle c(4,0)\rangle$, $H^{8n+5}(F_n) = Z_2\langle c(5,0)\rangle$,
where $a' = a((1),0) + (n+4)c(4,0)$.

(iii) $Sq^4 b(0) = a'$, $P^1 b(0) = -a'$ for $p = 3$, $Sq^i b(0) = 0$ for $1 \leqq i \leqq 7$ and $i \neq 4$, $Sq^i c(1,0) = c(2,0)$ if $i = 1$, 0 if $i = 2$, $nc(5,0)$ if $i = 4$, $Sq^i c(2,0) = 0$ if $i = 1$, $c(4,0)$ if $i = 2$,

$nc(6,0)$ if $i = 4$, $Sq^i(a') = Sq^i c(4,0) = 0$ for $1 \leqq i \leqq 3$.

By this lemma, we have immediately

Lemma 3.4. Let $n \geqq 2$. Then we can take a complex K given by $K = S^{8n-1} \cup_{\phi_0} e^{8n} \cup_{\phi_1} e^{8n+1} \cup_{\psi_1 \vee \psi_2} (e_1^{8n+3} \vee e_2^{8n+3}) \cup_{\phi_4} e^{8n+4} \cup_{\phi_5} e^{8n+5}$ and a map $f : K \longrightarrow F_n$, which satisfy the following (i)-(ii):

(i) $f_* : H_i(K) \longrightarrow H_i(F_n)$ is isomorphic for $i \leqq 8n+4$.

(ii) The cells e_1^{8n+3} and e_2^{8n+3} correspond to the cohomology classes $f^*(a')$ and $f^*(c(4,0))$ respectively.

Proposition 3.5. Let $n \geqq 2$. Then

(i) $\pi_{8n-1}(F_n) = Z$, $\pi_{8n}(F_n) = \pi_{8n+1}(F_n) = Z_2 \oplus Z_2$, $\pi_{8n+2}(F_n) = 0$, $\pi_{8n+3}(F_n) = Z \oplus Z$ (resp. $Z \oplus Z \oplus Z_2$) if n is odd (resp. even).

(ii) We can take a basis $\{u(3), v(3)\}$ of a free part of $\pi_{8n+3}(F_n)$ to satisfy $H(u(3)) = 24\bar{a}'$ and $H(v(3)) = 4\bar{c}(4,0)$, where $H : \pi_{8n+3}(F_n) \longrightarrow H_{8n+3}(F_n)$ is the Hurewicz homomorphism and $\{\bar{a}', \bar{c}(4,0)\}$ is the dual basis of $\{a', c(4,0)\}$ in Lemma 3.3.

Proof. By Lemma 3.4, we prove the proposition for K in Lemma 3.4 instead of F_n.

It is obvious that $\pi_{8n-1}(K) = Z$ with $K^{(8n)} = S^{8n-1} \vee S^{8n}$. If $q_1 : K^{(8n)} \longrightarrow S^{8n-1}$ and $q_2 : K^{(8n)} \longrightarrow S^{8n}$ are the respective projections, then $q_1\phi_1$ is homotopic to the constant map and $\deg q_2\phi_1 = 2$ since $Sq^2 b(0) = 0$ and $Sq^1 c(1,0) = c(2,0)$ by Lemma 3.3 (iii). Hence we have $K' = K^{(8n+1)} = K^{(8n+2)} = S^{8n-1} \vee (S^{8n} \cup_2 e^{8n+1})$, and the split exact sequence

$$(*) \quad 0 \longrightarrow \pi_{8n+i}(S^{8n-1}) \underset{q_*}{\overset{}{\rightleftarrows}} \pi_{8n+i}(K') \xrightarrow{p_*} \pi_{8n+i}(S^{8n} \cup_2 e^{8n+1}) \longrightarrow 0$$

where p and q are the projections. Therefore $\pi_{8n}(K) = \pi_{8n+1}(K) = Z_2 \oplus Z_2$. Furthermore

(3.6) $\pi_{8n+2}(K') \cong \pi_{8n+2}(S^{8n-1}) \oplus \pi_{8n+2}(S^{8n} \cup_2 e^{8n+1}) = Z_{24} \oplus Z_4$ (cf. [2; 4.1]), and, for the attaching maps ψ_1 and ψ_2, $q_*\psi_1$ and $p_*\psi_2$ generate the first and second summands respectively, and the order $p_*\psi_1$ and $q_*\psi_2$ are divisors of 2 and 4 respectively.

To prove the latter half of (3.6), we consider the commutative diagram

$$\begin{array}{ccccc}
S^{8n+2} & \longrightarrow & S^{8n-1} & \longrightarrow & S^{8n-1} \cup_{q\psi_k} e^{8n+3} \\
\| & & \uparrow q & & \uparrow \tilde{q} \\
S^{8n+2} & \xrightarrow{\psi_k} & K^{(8n+1)} & \longrightarrow & K^{(8n+1)} \cup_{\psi_k} e^{8n+3} \\
\| & & \downarrow \pi & & \downarrow \tilde{\pi} \\
S^{8n+2} & \longrightarrow & S^{8n+1} & \longrightarrow & S^{8n+1} \cup_{\pi\psi_k} e^{8n+3}
\end{array}$$

for $k = 1$ and 2, where π is the natural projection, and $\tilde{q}$ and $\tilde{\pi}$ are the maps defined by q and π respectively. Consider the mod 2 and mod 3 cohomology groups of this diagram. Then, since $Sq^4 b(0) = a'$ and $P^1 b(0) = -a'$ for $p = 3$ by Lemma 3.3 (iii), we see that $q_*\psi_1$ is a generator of $\pi_{8n+2}(S^{8n-1}) = Z_{24}$ and the order of $q_*\psi_2$ is a divisor of 4. Since $Sq^2 c(2,0) = c(4,0)$ by Lemma 3.3 (iii), $\pi_*\psi_1 = 0$ and $\pi_*\psi_2 \neq 0$ in $\pi_{8n+2}(S^{8n+1}) = Z_2$. Hence the order of $p_*\psi_1$ is at most 2 and $p_*\psi_2$ is a generator of $\pi_{8n+2}(K'/S^{8n-1}) = \pi_{8n+2}(S^{8n} \cup_2 e^{8n+1}) = Z_2$, by the fact that $\pi'_* : \pi_{8n+2}(S^{8n+2} \cup_2 e^{8n+1}) \longrightarrow \pi_{8n+2}(S^{8n+1})$ is epimorphic where π' is the restriction of π (cf. [2; 4.1]). These imply the latter half of (3.6).

Consider the exact sequence

$$\pi_{8n+3}(S_1^{8n+2} \vee S_2^{8n+2}) \xrightarrow{(\psi_1 \vee \psi_2)_*} \pi_{8n+3}(K') \xrightarrow{i_*} \pi_{8n+3}(K^{(8n+3)})$$
$$\xrightarrow{\partial} \pi_{8n+2}(S_1^{8n+2} \vee S_2^{8n+2}) \xrightarrow{(\psi_1 \vee \psi_2)_*} \pi_{8n+2}(K').$$

Then $\mathrm{Im}\,\partial = \mathrm{Ker}\,(\psi_1 \vee \psi_2)_* = Z\langle 24\iota_1\rangle \oplus Z\langle 4\iota_2\rangle$ by (3.6), where ι_j is a generator of $\pi_{8n+2}(S_j^{8n+2})$, $j = 1, 2$. Further studying the first $(\psi_1 \vee \psi_2)_*$ by (*) and the latter half of (3.6), we see that $\pi_{8n+3}(K^{(8n+3)}) = Z \oplus Z \oplus Z_2$. On the other hand, the attaching map ϕ_4 is contained in $\mathrm{Ker}\,\partial = \mathrm{Im}\, i_* = Z_2$ by the last equalities in Lemma 3.3 (iii). Furthermore, since $Sq^4 c(1,0) = nc(5,0)$ by Lemma 3.3 (iii), we see that $\phi_4 \neq 0$ if n is odd and $\phi_4 = 0$ if n is even.

By the above exact sequence, we can take a basis $\{u(3), v(3)\}$ of $\pi_{8n+3}(K)/\mathrm{Tor} = \pi_{8n+3}(K^{(8n+3)})/\mathrm{Tor}$ to satisfy $\partial u(3) = 24\iota_1$ and $\partial v(3) = 4\iota_2$. These imply that $H(u(3)) = 24\bar{a}'$ and $H(v(3)) = 4\bar{c}(4,0)$, and we complete the proof. q.e.d.

Remark 3.7. For $n = 1$, $\pi_{8n+i}(F_n)$ $(i=-1,0,1)$ are the same as the ones given in Proposition 3.5.

By Lemmas 3.1, 3.3 and Proposition 3.5, we have the following

Theorem 3.8. Let $n \geqq 1$. Then

(i) $P_n[u] \equiv 0 \mod 8$ for any $u \in \pi_{4n}(MSp)$.

(ii) $P_1P_n[u] - ((n+4)/2)P_{n+1}[u] \equiv 0 \mod 24$ for any $u \in \pi_{4n+4}(MSp)$.

Proof. For $n = 1$, (i) and (ii) follow from the results of [7], [6] on $\pi_4(MSp)$ and $\pi_8(MSp)$. Let $n \geqq 2$. We consider the case that $x_0 = a'$ or $c(4,0)$ and $\ell = 5$ in Lemma 3.1. By Lemma 3.3 (ii), we can take a basis $\{a', c(4,0)\}$ of $H^{8n+3}(F_n)$. When $x_0 = c(4,0)$, we see that $\alpha(5)$ is a multiple of 4 by Proposition 3.5 and $\tau(c(4,0)) = (1/2)V(P_{n+1})$ by (2.11), hence (i) follows from Lemma 3.1. When $x_0 = a'$, $\alpha(5)$ is a multiple of 24 by Proposition 3.5 and $\tau(a') = -V(P_1P_n) + ((n+4)/2)V(P_{n+1}) + V(P_1)V$ by (2.9) and (2.11), hence (ii) follows from Lemma 3.1. q.e.d.

Remark 3.9. In addition to Proposition 3.5 (i), the homotopy groups $\pi_i(F_n)$ can be determined for $i \leqq 8n+6$ by the information of S.Oka.

§4. Homotopy groups of MSp(n)

In the rest of this note, we study the homotopy groups $\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ for $n \geqq 1$.

Consider the homotopy exact sequence of the fibering (2.1):

$$(4.1)\quad \cdots \longrightarrow \pi_i(MSp) \xrightarrow{\partial} \pi_{i+4n-1}(F_n) \longrightarrow \pi_{i+4n-1}(MSp(n)) \xrightarrow{b_{n*}} \pi_{i-1}(MSp) \xrightarrow{\partial} \pi_{i+4n-2}(F_n) \longrightarrow \cdots,$$

where we identify $\pi_i(MSp)$ with $\pi_{i+4n}(\Omega^{4N}MSp(n+N))$ since $N > 3n$. Because F_n is (8n-2)-connected, b_{n*} is isomorphic for $i \leqq 4n-2$ and epimorphic for $i = 4n$.

Proposition 4.2. (i) For $\partial : \pi_{4n}(MSp) \longrightarrow \pi_{8n-1}(F_n) = Z$ $(n \geqq 1)$ (see Prop.3.5 (i) and Remark 3.7), it holds

$$\partial u = \pm(1/2)P_n[u] \quad \text{for any } u \in \pi_{4n}(MSp).$$

(ii) For $\partial : \pi_{4n+4}(MSp) \longrightarrow \pi_{8n+3}(F_n) = Z \oplus Z \oplus \mathrm{Tor}$ $(n \geqq 2)$ (see Prop.3.5 (i)), it holds

$$\partial u = (1/24)(-P_1P_n[u] + ((n+4)/2)P_{n+1}[u])u(3) + ((1/8)P_{n+1}[u])v(3) + x$$

for any $u \in \pi_{4n+4}(MSp)$, by taking the basis $\{u(3), v(3)\}$ of a free part of $\pi_{8n+3}(F_n)$ (see Prop.3.5 (ii)), where x is 0 if n is odd and an element in the summand Z_2 if n is even.

Proof. We shall prove (ii). (i) is proved similarly.

By Proposition 3.5 (ii), the basis $\{u(3), v(3)\}$ of the free part of $\pi_{8n+3}(F_n)$ is taken so that $H(u(3)) = 24\bar{a}'$ and $H(v(3)) = 4\bar{c}(4,0)$, where $\{\bar{a}', \bar{c}(4,0)\}$ is the dual basis for the basis $\{a', c(4,0)\}$ of $H^{8n+3}(F_n)$ in Lemma 3.3 (ii). For any class $u \in \pi_{4n+4}(MSp) = \pi_{8n+4}(\Omega^{4N}MSp(n+N))$, set $\partial u = ku(3) + \ell v(3) + x$ for some integers k, ℓ and some torsion element x. Then, by taking the Kronecker pairing,

$$24k = \langle H(\partial u), a' \rangle = \langle H(u), \tau(a') \rangle = \langle H(u), -V(P_1P_n) + ((n+4)/2)V(P_{n+1}) \rangle$$
$$= -P_1P_n[u] + ((n+4)/2)P_{n+1}[u], \quad \text{and}$$
$$4\ell = \langle H(\partial u), c(4,0) \rangle = \langle H(u), \tau(c(4,0)) \rangle = \langle H(u), (1/2)V(P_{n+1}) \rangle$$
$$= (1/2)P_{n+1}[u],$$

where we use the equality $\tau(a') = V(P_1)V - V(P_1P_n) + ((n+4)/2)V(P_{n+1})$, and $\tau(c(4,0)) = (1/2)V(P_{n+1})$ by (2.9) and (2.11). Hence we have the desired result. q.e.d.

The Pontrjagin number $P_n[u]$ is a multiple of 8 for any $u \in \pi_{4n}(MSp)$ $(n \geqq 1)$ by Theorem 3.8 (i). Thus we set

(4.3) $m(n) = \text{g.c.m.}\ \{(1/8)P_n[u] \mid u \in \pi_{4n}(MSp)\}$ for $n \geqq 1$.

Corollary 4.4. The kernel of the epimorphism $b_{n*} : \pi_{8n-1}(MSp(n)) \longrightarrow \pi_{4n-1}(MSp)$ is a cyclic group of order $4m(n)$ generated by the Whitehead product $[i,i]$ for the homotopy class i of the natural inclusion $i : S^{4n} \longrightarrow MSp(n)$.

Proof. By Proposition 4.2 (i), the definition (4.3) and the exact sequence (4.1), we see that $\operatorname{Ker} b_{n*}$ is a cyclic group of order $4m(n)$. Consider the commutative diagram

$$\begin{array}{ccccc}
F(i_1) & \xrightarrow{j'} & S^{4n} & \xrightarrow{i_1} & S^{4n+1} \\
\downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\
F_n & \xrightarrow{j} & MSp(n) & \xrightarrow{b_n} & \Omega^{4N}MSp(n+N).
\end{array}$$

Here i_1 denotes the natural inclusion and $F(i_1)$ is the fiber, and i' is the composition $\Omega S^{4n+1} \longrightarrow \Omega^{4N}S^{4(n+N)} \longrightarrow \Omega^{4N}MSp(n+N)$ of the natural inclusions. It holds that $\pi_{8n-1}(F(i_1)) = Z$ and $j'_*(1) = \pm[\iota,\iota]$ for a generator $\iota \in \pi_{4n}(S^{4n})$ by the definition of the Whitehead product. Since $\bar{i}_* : \pi_{8n-1}(F(i_1)) \longrightarrow \pi_{8n-1}(F_n)$ is isomorphic, the kernel of $b_{n*} : \pi_{8n-1}(MSp(n)) \longrightarrow \pi_{4n-1}(MSp)$ is generated by $[i,i]$ by the naturality. q.e.d.

Let $MU(2n)$ be the Thom space of the universal complex vector bundle over $BU(2n)$, and consider the map $c : MSp(n) \longrightarrow MU(2n)$ induced by the inclusion $Sp(n) \subset U(2n)$. Then we have the following corollary, where $\nu_2(y)$ is the exponent of 2 in the prime power decomposition of y:

Corollary 4.5. Assume that

(a) n is not a power of 2 and $2\pi_{4n-1}(MSp) = 0$, or

(b) $\nu_2(m(n)) + 2 = \nu_2(|\pi_{8n-1}(MU(2n))|)$.

Then the epimorphism $b_{n*} : \pi_{8n-1}(MSp(n)) \longrightarrow \pi_{4n-1}(MSp)$ is split, that is, it holds

$$\pi_{8n-1}(MSp(n)) \cong Z_{4m(n)} \oplus \pi_{4n-1}(MSp).$$

Proof. Let $\tilde{F}_{2n} \longrightarrow MU(2n) \xrightarrow{\tilde{b}_{2n}} \Omega^{4N}MU(2n+2N)$ be the fibering defined by the same way as (2.1), and consider the commutative diagram

$$\begin{array}{ccccc}
F_n & \longrightarrow & MSp(n) & \xrightarrow{b_n} & \Omega^{4N}MSp(n+N) \\
\downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Omega^{4N}c} \\
\tilde{F}_{2n} & \longrightarrow & MU(2n) & \xrightarrow{\tilde{b}_{2n}} & \Omega^{4N}MU(2n+2N)
\end{array}$$

induced by c. We remark that $\tilde{F}_{2n}$ is $(8n-2)$-connected. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
\pi_{4n}(MSp) & \xrightarrow{\partial} & \pi_{8n-1}(F_n) & \longrightarrow & \pi_{8n-1}(MSp(n)) & \xrightarrow{b_{n*}} & \pi_{4n-1}(MSp) & \longrightarrow & 0 \\
\downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c'_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} & & \\
\pi_{4n}(MU) & \xrightarrow{\tilde{\partial}} & \pi_{8n-1}(\tilde{F}_{2n}) & \longrightarrow & \pi_{8n-1}(MU(2n)) & \xrightarrow{\tilde{b}_{2n*}} & 0. & &
\end{array}$$

In the first place, we notice that c'_* is isomorphic. By E.Rees and E.Thomas [11;§2], $H^{8n-1}(\tilde{F}_{2n})$ is Z generated by α_1 which satisfies

$$\tilde{\tau}(\alpha_1) = (1/2)(\tilde{\sigma}(\tilde{U}c_{2n}) - (\tilde{\sigma}(\tilde{U}))^2),$$

where $\tilde{\tau} : H^{8n-1}(\tilde{F}_{2n}) \longrightarrow H^{8n}(\Omega^{4N}M)$ and $\tilde{\sigma} : H^{8n+4N}(M) \longrightarrow H^{8n}(\Omega^{4N}M)$ are the transgression and the iterated suspension respectively, and $\tilde{U} \in H^{4n+4N}(M)$ is the Thom class $(M=MU(2n+2N))$. The above equality and $\tau(b(0)) = (1/2)(V^2 - V(P_n))$ of (2.10) imply $\tau c'(\alpha_1) = -\tau(b(0))$ and so $c'^*(\alpha_1) = -b(0)$, because $c^*(c_{2n}) = \pm P_n$, $c^*(\tilde{U}) = \pm U$ and τ is monomorphic. Thus $c'^* : H^{8n-1}(\tilde{F}_{2n}) \longrightarrow H^{8n-1}(F_n)$ is isomorphic and so is c'_* in the diagram.

Furthermore $\pi_{8n-1}(MU(2n)) \cong \text{Coker}\ \tilde{\partial}$ is a cyclic group of order 2^β where $\beta = \rho_0(2n)-1$ by [11; Th.A], and Coker $\partial = Z_{4m(n)}$ by Corollary 4.4. Thus we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Coker}\ \partial\ (=Z_{4m(n)}) & \longrightarrow & \pi_{8n-1}(MSp(n)) & \xrightarrow{b_{n*}} & \pi_{4n-1}(MSp) & \longrightarrow & 0 \\ (*) & & \downarrow c'' & & \downarrow c_* & & & & \\ & & \text{Coker}\ \tilde{\partial}\ (=Z_{2^\beta}) & \cong & \pi_{8n-1}(MU(2n)), & & & & \end{array}$$

where c'' is the epimorphism induced by c'_*.

When (b) holds, c'' induces the isomorphism of the 2-torsion parts, hence the upper sequence in (*) splits because $\pi_{4n-1}(MSp)$ is a 2-torsion group (cf. [15; 20.40]).

Now we assume that (a) holds. Then $\beta \neq 0$ by the definition of $\rho_0(2n)$ ([11; Th.A]) and $\pi_{4n-1}(MSp) \otimes Z_2 = \pi_{4n-1}(MSp)$. Hence, by tensoring Z_2 to (*), we have the split exact sequence $0 \longrightarrow Z_2 \longrightarrow \pi_{8n-1}(MSp(n)) \otimes Z_2 \longrightarrow \pi_{4n-1}(MSp) \longrightarrow 0$. Therefore the upper sequence in (*) splits as desired. q.e.d.

We shall prove the following theorems in the next section by preparing some symplectic cobordism classes.

Theorem 4.6. For the integer $m(n)$ in (4.3), the following (i) and (ii) hold:

(i) $m(n)$ is a power of 2 for $n \neq 1, 3$, and $m(1) = m(3) = 3$.

(ii) $m(n) = 1$ if $n = 2^k+2^\ell-1$ or 2^k+2^ℓ $(k,\ell \geqq 0)$ and $n \neq 1, 3$.

Theorem 4.7. (i) $\pi_{8n+3}(MSp(n))$ $(n \geqq 3)$ has no p-torsion for any odd prime p.

(ii) $b_{n*} : \pi_{8n+3}(MSp(n)) \longrightarrow \pi_{4n+3}(MSp)$ is epimorphic for $n \geqq 1$.

(iii) If $n = 2^k+2^\ell-1$ $(k,\ell \geqq 1)$, then b_{n*} in (ii) is isomorphic, i.e., $\pi_{8n+3}(MSp(n)) \cong \pi_{4n+3}(MSp)$.

§5. Symplectic cobordism classes

In this section, we examine the characteristic numbers of some symplectic cobordism classes to prove Theorems 4.6 and 4.7.

Let ξ_1 be the universal symplectic line bundle over the quaternionic projective space $HP^\infty = BSp(1)$, and $\xi_1 \otimes_C \xi_1 \otimes_C \xi_1$ be the tensor product of ξ_1 over $HP^\infty \times HP^\infty \times HP^\infty$ by taking ξ_1 as the complex vector bundle. Then it is a symplectic vector bundle ξ_1^3 (cf. [14]), and so we denote its classifying map by

(5.1) $$\phi : Y = HP^\infty \times HP^\infty \times HP^\infty \longrightarrow BSp.$$

Let $P_1^{MSp} \in MSp^4(BSp)$ be the universal first Pontrjagin class and $P_1^{MSp}(\xi_1) \in MSp^4(HP^\infty)$ be the Euler class of ξ_1 in the symplectic cobordism theory. By using the respective projections $q_i : Y \longrightarrow HP^\infty$ $(i=1,2,3)$ onto the three factors, we set $X_i = q_i^* P_1^{MSp}(\xi_1) \in MSp^4(Y)$. Then it holds $MSp^*(Y) = MSp^*[\![X_1, X_2, X_3]\!]$, and we have an expansion

(5.2) $$\phi^*(P_1^{MSp}) = P_1^{MSp}(\xi_1^3) = \sum_{i,j,k \geqq 0} a_{ijk} X_1{}^i X_2{}^j X_3{}^k$$

for some cobordism classes

(5.3) $$a_{ijk} \in \pi_{4(i+j+k-1)}(MSp).$$

We shall consider the Pontrjagin numbers $P_{i+j+k-1}[a_{ijk}]$ and

$P_1 P_{i+j+k-2}[a_{ijk}]$.

For $E = H$ or MSp, let $\beta_j^E \in E_{4j}(HP^\infty)$ be the dual class of $(P_1^E(\xi_1))^j$ where $P_1^E(\xi_1)$ is the Euler class of ξ_1. Then the following hold (cf. [8],[15; §16]):

(5.4) $E_*(HP^\infty)$ is a free $\pi_*(E)$-module with basis consisting of $\{\beta_j^E \mid j \geqq 0\}$, and

$$E_*(MSp) \cong \pi_*(E)[b_1^E, b_2^E, \cdots], \quad b_j^E = i_*\beta_{j+1}^E \in E_{4j}(MSp),$$

where $i : HP^\infty \longrightarrow \Sigma^4 MSp$ is the natural inclusion.

Consider the classes $x_i = q_i^* P_1(\xi_1) \in H^4(Y)$, $i = 1,2,3$, where $q_i : Y \longrightarrow HP^\infty$ are the respective projections onto the three factors. Then $H^*(Y) = Z[[x_1, x_2, x_3]]$, and we have the following lemma, where $P^{\Delta i} \in H^{4i}(BSp)$ denotes the primitive class defined inductively by $P^{\Delta i} = \sum_{j=1}^{i-1}(-1)^{j+1}P_j P^{\Delta i-j} + (-1)^{i+1} i P_i$:

Lemma 5.5. For the induced homomorphism $\phi^* : H^*(BSp) \longrightarrow H^*(Y)$ of ϕ in (5.1),

$$\phi^*(P^{\Delta i}) = 4\sum((2i)!/(2k)!(2\ell)!(2m)!)x_1^k x_2^\ell x_3^m,$$

where the summation is taken over $k,\ell,m \geqq 0$ with $k+\ell+m = i$.

Proof. Let $c_i \in H^{2i}(BU)$ be the i-th Chern class, and $c^{\Delta i} \in H^{2i}(BU)$ be the primitive class defined by $c^{\Delta i} = \sum_{j=1}^{i-1}(-1)^{j+1}c_j c^{\Delta i-j} + (-1)^{i+1} i c_i$. Then, for the canonical map $c : BSp \longrightarrow BU$, it holds $c^*(c^{\Delta 2i}) = 2P^{\Delta i}$ by the definitions of $P^{\Delta i}$ and $c^{\Delta j}$. Hence, by the definition of ϕ,

$$2\phi^*(P^{\Delta i}) = 2P^{\Delta i}(\xi_1^3) = c^{\Delta 2i}(\xi_1 \otimes_C \xi_1 \otimes_C \xi_1) \quad \text{in } H^{4i}(Y).$$

Let η be the canonical complex line bundle over CP^∞, and $\bar{\eta}$ be the conjugate bundle of η. Then $\eta \oplus \bar{\eta}$ is a symplectic line bundle over CP^∞, and we denote its classifying map by $q : CP^\infty \longrightarrow HP^\infty$. Set $Z = CP^\infty \times CP^\infty \times CP^\infty$. Then it holds $H^*(Z) = Z[[y_1, y_2, y_3]]$, where $y_i = \pi_i^* c_1(\eta) \in H^2(Z)$ $(i=1,2,3)$ for the respective projections $\pi_i : Z \longrightarrow CP^\infty$ onto the three factors. For the homomorphism $(q \times q \times q)^* : H^*(Y) \longrightarrow H^*(Z)$, we see that

$$(q\times q\times q)^*(c^{\Delta 2i}(\xi_1 \otimes_C \xi_1 \otimes_C \xi_1)) = c^{\Delta 2i}((\eta\oplus\bar{\eta})\otimes_C(\eta\oplus\bar{\eta})\otimes_C(\eta\oplus\bar{\eta}))$$
$$= 2\{(y_1+y_2+y_3)^{2i}+(y_1+y_2-y_3)^{2i}+(y_1-y_2+y_3)^{2i}+(y_1-y_2-y_3)^{2i}\}$$
$$= 8\textstyle\sum_{k+\ell+m=i}((2i)!/(2k)!(2\ell)!(2m)!)y_1^{2k}y_2^{2\ell}y_3^{2m},$$

by using the equality $c^{\Delta i}(\sum_k \zeta_k) = \sum_k (c_1(\zeta_k))^i$ for line bundles ζ_k. Since $(q\times q\times q)^*$ is monomorphic and $(q\times q\times q)^*(x_k) = y_k^2$ for $k = 1,2,3$, we have the desired result by the above equalities. q.e.d.

Let $(b)_\ell^k \in H_{4\ell}(MSp)$ denote the 4ℓ-dimensional part of $b^k = (1+b_1+b_2+\cdots)^k$, i.e.,

(5.6) $(1+b_1x+b_2x^2+\cdots)^k = \sum_{\ell \geqq 0}(b)_\ell^k x^\ell$, where $b_j = b_j^H$ in (5.4),

and let $H : \pi_*(MSp) \longrightarrow H_*(MSp)$ be the Hurewicz homomorphism.

Proposition 5.7. For any non negative integer r, s, t,

$$\textstyle\sum H(a_{ijk})(b)_{r-i}^i(b)_{s-j}^j(b)_{t-k}^k = 4((2(r+s+t)!/(2r)!(2s)!(2t)!)b_{r+s+t-1},$$

where the summation is taken over all $i,j,k \geqq 0$ with $i \leqq r$, $j \leqq s$, $k \leqq t$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} MSp^*(BSp) & \xrightarrow{\bar{h}} & (H \wedge MSp)^*(BSp) & \cong & H_*(MSp) \otimes H^*(BSp) \\ \downarrow \phi^* & & \downarrow \phi^* & & \downarrow 1 \otimes \phi^* \\ MSp^*(Y) & \xrightarrow{\bar{h}} & (H \wedge MSp)^*(Y) & \cong & H_*(MSp) \otimes H^*(Y), \end{array}$$

where $\bar{h}$ denotes the Boardman homomorphism. Then we have

(5.8) $\quad (1\otimes\phi^*)\bar{h}(P_1^{MSp}) = \bar{h}\phi^*(P_1^{MSp}).$

The following relation holds (cf. [1],[8;(5.1)]):

(5.9) $\quad \bar{h}(P_1^{MSp}) = \sum_{i\geqq 1} b_{i-1} P^{\Delta_i},$

where P^{Δ_i} is the primitive class in Lemma 5.6. By (5.9) and Lemma 5.5,

$$(1\otimes\phi^*)\bar{h}(P_1^{MSp}) = 4\sum_{r+s+t=i\geqq 1}((2i)!/(2r)!(2s)!(2t)!)b_{i-1}x_1{}^r x_2{}^s x_3{}^t.$$

On the other hand, by (5.9) and (5.6),

$$\bar{h}(X_k^j) = (\sum_{i\geqq 1} b_{i-1} x_k{}^i)^j = \sum_{s\geqq j}(b)^j_{s-j} x_k^s.$$

By (5.2) and this equality, we have

$$\bar{h}\phi^*(P_1^{MSp}) = \sum_{r+s+t\geqq 1}(\sum H(a_{ijk})(b)^i_{r-i}(b)^j_{s-j}(b)^k_{t-k})x_1{}^r x_2{}^s x_3{}^t.$$

Therefore, we have the proposition by (5.8). q.e.d.

For any class $u \in \pi_{4n}(MSp)$, its Hurewicz image $H(u)$ can be written as

$$H(u) = \sum\lambda_{(r_1,\cdots,r_j)} b_1{}^{r_1}\cdots b_j{}^{r_j} \in H_*(MSp) = Z[b_1,b_2,\cdots].$$

For our purpose, we denote simply the coefficients $\lambda_{(n)}$ and $\lambda_{(n-2,1)}$ of $b_1{}^n$ and $b_1{}^{n-2}b_2$ by (u) and $\langle u\rangle$, respectively. Then we have

(5.10) $\quad P_n[u] = (u), \quad P_1P_{n-1}[u] = \langle u\rangle + n(u) \quad$ for $n \geqq 1.$

These formulas can be proved by the same proof as that for MU given in [1; pp.10-11], [15; pp.401-402].

By comparing the coefficients of $b_1^{r+s+t-1}$ or $b_1^{r+s+t-3}b_2$ in the both sides of the equality in Proposition 5.7, and by the above notations of ( ) and < >, we have the following

Lemma 5.11. For $r,s,t \geqq 0$, the following hold, where summations are taken over $i,j,k \geqq 0$ with $i \leqq r$, $j \leqq s$, $k \leqq t$.

(i)

$$\sum (a_{ijk}) \binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k} = \begin{cases} 4 & \text{if } \{r,s,t\} = \{1,0,0\} \text{ or } \{2,0,0\}, \\ 24 & \text{if } \{r,s,t\} = \{1,1,0\}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$\sum \{ \langle a_{ijk} \rangle \binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k} + (a_{ijk}) \left(i\binom{i-1}{r-i-2}\binom{j}{s-j}\binom{k}{t-k} + j\binom{i}{r-i}\binom{j-1}{s-j-2}\binom{k}{t-k} \right.$$
$$\left. + k\binom{i}{r-i}\binom{j}{s-j}\binom{k-1}{t-k-2} \right) \} = \begin{cases} 4 & \text{if } \{r,s,t\} = \{3,0,0\}, \\ 60 & \text{if } \{r,s,t\} = \{2,1,0\}, \\ 360 & \text{if } r = s = t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.12. (i) For $i,j \geqq 1$,

$$(a_{ij0}) \equiv \begin{cases} 8 \mod 16 & \text{if } i \text{ and } j \text{ are powers of } 2, \\ 0 \mod 16 & \text{otherwise.} \end{cases}$$

(ii) For $i,j,k \geqq 1$,

$(a_{ijk}) = 0$ and

$$\langle a_{ijk} \rangle \equiv \begin{cases} 8 \mod 16 & \text{if } i,j,k \text{ are powers of } 2, \\ 0 \mod 16 & \text{otherwise.} \end{cases}$$

We shall prove Proposition 5.12 by preparing the following two lemmas:

Lemma 5.13. (i) $(a_{ijk}) = 0$ if $i,j,k \geqq 1$.

(ii)
$$\sum \langle a_{ijk}\rangle \binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k} = \begin{cases} 360 & \text{if } r = s = t = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r,s,t \geqq 1$, where the summation is taken over all $i,j,k \geqq 1$ with $i \leqq r$, $j \leqq s$, $k \leqq t$.

Proof. (i) By Lemma 5.11 (i), $\sum (a_{ijk})\binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k} = 0$ for any $r,s,t \geqq 1$, where the summation is taken over all $i,j,k \geqq 1$ with $i \leqq r$, $j \leqq s$, $k \leqq t$. Therefore we see (i) by the induction on $i+j+k$.

(ii) (i) and Lemma 5.11 (ii) imply (ii). q.e.d.

Lemma 5.14. (i) $\langle a_{ijk}\rangle$ is a multiple of $\langle a_{111}\rangle = 360$ for any $i,j,k \geqq 1$.

(ii) $\langle a_{ijk}\rangle = \langle a_{i'j'k'}\rangle$ for any permutation (i',j',k') of (i,j,k).

(iii) $\sum_{i=1}^{r}\langle a_{ist}\rangle\binom{i}{r-i} = 0$ for $r \geqq 2$ and $s,t \geqq 1$.

(iv) Set $m_{ijk} = \langle a_{ijk}\rangle/360$. Then $m_{rst} = m_{r11}m_{s11}m_{t11}$ for $r,s,t \geqq 1$.

(V)
$$m_{r11} = \begin{cases} 1 \mod 2 & \text{if } r \text{ is a power of } 2, \\ 0 \mod 2 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 5.13 (ii), we can prove (i) and (ii) by the induction on $i+j+k$, and (iii) by the induction on $s+t$. We can prove (iv) inductively on $r+s+t$ by using (iii) and (ii), and (V) inductively on r by using (iii) and the fact that $\binom{2^i}{r-2^i}$ for $r > 2^i$ is odd if and only if $r = 2^{i+1}$. q.e.d.

Proof of Proposition 5.12. (ii) The first equality is proved in Lemma 5.13 (i). The second equality is an immediate consequence of Lemma 5.14 (iv),(v).

(i) By Lemma 5.11 (i), we see that for $r,s \geqq 1$,

$$\sum_{1\leqq i\leqq r,1\leqq j\leqq s}(a_{ij0})\binom{i}{r-i}\binom{j}{s-j} = \begin{cases} 24 & \text{if } r = s = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By using this equality instead of Lemma 5.13 (ii), we can prove (i) by the same way as the above proof of the second equality in (ii).

q.e.d.

Now we consider the another example of symplectic cobordism classes defined by R.E.Stong [14] and N.Ray [9]. We follow the methods due to N.Ray.

The complex projective space CP^{2i-1} is a weakly almost symplectic manifold (see[14]), and so is the product $\Pi_{i=1}^{2r}CP^{2n_i-1}$. Consider the composition

$$f : \Pi_{i=1}^{2r}CP^{2n_i-1} \xrightarrow{j} (CP^\infty)^{2r} \xrightarrow{m} CP^\infty \xrightarrow{q} HP^\infty,$$

where j and q are canonical maps and m is the classifying map of the tensor product of $2r$ copies of η, we have a bordism class

$$[\Pi_{i=1}^{2r}CP^{2n_i-1}, f] \in MSp_{4(n-r)}(HP^\infty)$$

for $n = \sum_{i=1}^{2r}n_i$. By (5.4), we have an expansion

$$[\Pi_{i=1}^{2r}CP^{2n_i-1}, f] = \sum_{k=1}^{n-r}a_k(n_1,\cdot\cdot,n_{2r})\beta_k^{MSp}$$

for some classes

(5.15) $a_k(n_1,\cdot\cdot,n_{2r}) \in \pi_{4(n-r-k)}(MSp)$ $(n=\sum_{i=1}^{2r}n_i)$.

By the result of N.Ray [9; (3.1),(3.2)] for the computation of the Hurewicz image of these classes $a_k(n_1,\cdots,n_{2r})$, we have the following proposition, where $s_n(j_1,\cdots,j_{2r})$ denotes the coefficient of $\Pi_{i=1}^{2r} x_i^{2(n_i-j_i)-1}$ in the expansion of $(\sum_{i=1}^{2r} x_i)^{2(n-r-j)}$.

Proposition 5.16. For the Hurewicz homomorphism $H : \pi_{4(n-r-k)}(MSp) \longrightarrow H_{4(n-r-k)}(MSp)$, it holds

$$H(a_k(n_1,\cdots,n_{2r})) = \sum s_n(j_1,\cdots,j_{2r})(b)_{j_1}^{-n_1}\cdots(b)_{j_{2r}}^{-n_{2r}}(b)_{n-r-k-j}^{k},$$

where $n = \sum_{i=1}^{2r} n_i$, $j = \sum_{i=1}^{2r} j_i$ and the summation is taken over all $j_i \geqq 0$.

We notice that the coefficients of $b_1{}^{\ell}$ and $b_1{}^{\ell-2}b_2$ in the 4ℓ-dimensional component $(b)_{\ell}^{-m}$ of $(b)^{-m}$ are $(-1)^{\ell}\binom{m+\ell-1}{m-1}$ and $(-1)^{\ell-1}(\ell-1)\binom{m+\ell-2}{m-1}$ respectively. Therefore, by comparing the coefficients of $b_1{}^{n-r-k}$ and $b_1{}^{n-r-k-2}b_2$ in the both sides of the above equality, and by using the notations ( ) and < > in (5.10), we see the following

Lemma 5.17. For $n_i \geqq 1$ $(1 \leqq i \leqq 2r)$, the following hold, where $n = \sum_{i=1}^{2r} n_i$, $j = \sum_{i=1}^{2r} j_i$ and the summations are taken over $j_i \geqq 0$ with $j_i \leqq n_i - 1$ $(1 \leqq i \leqq 2r)$:

(i) $(a_k(n_1,\cdots,n_{2r})) = \sum(-1)^j\binom{k}{n-r-k-j}s_n(j_1,\cdots,j_{2r})\Pi_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1}$.

(ii) $\langle a_k(n_1,\cdots,n_{2r})\rangle = \sum(-1)^j s_n(j_1,\cdots,j_{2r})\Pi_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1}$
$\{(n-r-k-j-1)\binom{k}{n-r-k-j-1} - \binom{k}{n-r-k-j}\sum_{\ell=1}^{2r}(j_\ell-1)j_\ell/(n_\ell+j_\ell-1)\}$.

When $k = 1$, we have the following

Proposition 5.18. For $n-r \geqq 2$, the following hold, where

$n = \sum_{i=1}^{2r} n_i$:

(i)

$$P_{n-r-1}[a_1(n_1,\cdots,n_{2r})] = \begin{cases} (-1)^{n+1}2\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1} & \text{if } r=1 \text{ and } n_1,n_2 \geqq 2, \\ (-1)^n 24\Pi_{i=1}^4\binom{2n_i-2}{n_i-1} & \text{if } r=2 \text{ and } n_i \geqq 1\ (1\leqq i\leqq 4), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $P_1P_{n-r-2}[a_1(n_1,\cdots,n_{2r})] =$

$$\begin{cases} (-1)^{n+1}(n-5)\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1} & \text{if } r=1 \text{ and } n_1,n_2\geqq 2, \\ (-1)^n 12(n-18+4\ell)\Pi_{i=1}^4\binom{2n_i-2}{n_i-1} & \text{if } r=2,\ 0\leqq \ell\leqq 4 \text{ and } \ell \text{ numbers of } n_i\text{'s} \\ & \text{are equal to 1 and the other } n_i\text{'s} \\ & \text{are more than 2,} \\ (-1)^n 720\Pi_{i=1}^6\binom{2n_i-2}{n_i-1} & \text{if } r=3 \text{ and } n_i\geqq 1\ (1\leqq i\leqq 6), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) The equality in Lemma 5.17 (i) for $k = 1$ is

$$(a_1(n_1,\cdots,n_{2r})) = \sum(-1)^j s_n(j_1,\cdots,j_{2r})\Pi_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1},$$

where $n = \sum_{i=1}^{2r} n_i$, $j = \sum_{i=1}^{2r} j_i$ and the summation is taken over $j_i\geqq 0$ $(1\leqq i\leqq 2r)$ with $j_i\leqq n_i-1$ and $j = n-r-1,\ n-r-2$. Therefore the left hand side is 0 if $r\geqq 3$, because $j\leqq n-2r$.

Let $r = 1$. If $n_1 = 1$ or $n_2 = 1$, then $P_{n-2}[a_1(n_1,n_2)] = 0$ is clear. For the case $n_1,n_2 \geqq 2$, the summation in the above equality is taken over $(j_1,j_2) = (n_1-1,n_2-1)$, (n_1-1,n_2-2) and (n_1-2,n_2-1), and then $s_n(j_1,j_2) = 2$, 4 and 4 respectively. Hence we have

$$(a_1(n_1,n_2)) = (-1)^{n-1}2\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1}$$

for $n_1,n_2 \geqq 2$. Since $P_{n-2}[a_1(n_1,n_2)] = (a_1(n_1,n_2))$ by (5.10), we have the desired equality for $P_{n-2}[a_1(n_1,n_2)]$.

For the case $r = 2$, the summation in the first equality is taken over $j_i = n_i-1$ $(1 \leqq i \leqq 4)$ only, and then $s_n(j_1,\cdots,j_4) = 24$. Thus we have the desired equality for $P_{n-3}[a_1(n_1,n_2,n_3,n_4)]$.

(ii) By the equality in Lemma 5.17 (ii), and by a similar argument in (i), we see that $\langle a_1(n_1,\cdots,n_{2r})\rangle = 0$ if $r \geqq 4$, and that, for $r = 1$,

$$\langle a_1(n_1,n_2)\rangle = \begin{cases} (-1)^n(n+1)\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1} & \text{if } n_1,n_2 \geqq 2, \\ 0 & \text{otherwise.} \end{cases}$$

By (5.10), $P_1P_{n-3}[a_1(n_1,n_2)] = \langle a_1(n_1,n_2)\rangle + (n-2)(a_1(n_1,n_2))$ for $n \geqq 3$. Hence we have the desired equality for $P_1P_{n-3}[a_1(n_1,n_2)]$ by the above equalities for $\langle a_1(n_1,n_2)\rangle$ and $(a_1(n_1,n_2))$. We omit the proof of the equalities for $P_1P_{n-r-2}[a_1(n_1,\cdots,n_{2r})]$ $(r=2,3)$, since we can prove them similarly and we shall not use them.

q.e.d.

Corollary 5.19. $P_{n-r-1}[a_1(n_1,\cdots,n_{2r})]$ $(n-r \geqq 2)$ is congruent to

$$\begin{cases} 8 \mod 16 & \text{if } r=1 \text{ and } n_1-1,n_2-1 \text{ are powers of 2, or} \\ & \phantom{\text{if }} r=2 \text{ and } n_1=n_2=n_3=n_4=1, \\ 0 \mod 16 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 5.18 (i), $P_{n-r-1}[a_1(n_1,\cdots,n_{2r})] = 0$ for $n-r \geqq 2$ unless $r = 1$ and $n_1,n_2 \geqq 2$, or $r = 2$. We prove the corollary for $r = 1$ $(n_1,n_2 \geqq 2)$ and $r = 2$. We notice that $\nu_2(\binom{m}{n}) = \alpha(n) + \alpha(m-n) - \alpha(m)$ (cf.[10;(6)]), where $\alpha(y)$ is the number of 1's in the dyadic expansions of y. Thus, by Proposition 5.18 (i), we have

$$\nu_2(|P_{n-2}[a_1(n_1,n_2)]|) = 1 + \alpha(n_1-1) + \alpha(n_2-1) \quad (n_1,n_2 \geqq 2), \text{ and}$$
$$\nu_2(|P_{n-3}[a_1(n_1,n_2,n_3,n_4)]|) = 3 + \sum_{i=1}^{4}\alpha(n_i-1) \quad (n_1,n_2,n_3,n_4 \geqq 1).$$

Hence $\nu_2(|P_{n-2}[a_1(n_1,n_2)]|)$ is at least 3, and is 3 if and only if n_1-1 and n_2-1 are powers of 2. Also $\nu_2(|P_{n-3}[a_1(n_1,n_2,n_3,n_4)]|)$ is 3 if and only if $n_i = 1$ $(1 \leqq i \leqq 4)$. These complete the proof.

q.e.d.

Now we can prove the following theorem which is Theorem III (i):

Theorem 5.20. $\pi_{8n+3}(MSp(n))$ $(n \geqq 3)$ has no p-torsion for any odd prime p.

Proof. Let $Q_p = \{\ell/m \mid (m,p) = 1\} \subset Q$. Tensoring Q_p to (4.1) for $i = 4n+4$ $(n \geqq 2)$, we have the exact sequence

$$\pi_{4n+4}(MSp) \otimes Q_p \xrightarrow{\partial\otimes 1} Q_p \oplus Q_p \longrightarrow \pi_{8n+3}(MSp(n)) \otimes Q_p \longrightarrow 0,$$

since $\pi_{8n+3}(F_n) \otimes Q_p \cong Q_p \oplus Q_p$ $(n \geqq 2)$ by Proposition 3.5 (i) and $\pi_{4n+3}(MSp)$ is a 2-torsion group. Therefore it is sufficient to show that

(5.21) $\partial\otimes 1 : \pi_{4n+4}(MSp) \otimes Q_p \longrightarrow Q_p \oplus Q_p$ is epimorphic for $n \geqq 3$.

Set $y_1 = a_1(1,1,1,1)$, $y_i = a_1(2,i)$, $2 \leqq i \leqq 6$, and $z = a_1(3,3)$. Then, by using Proposition 4.2, the equalities

$$(5.22) \qquad \begin{aligned} P_{k+\ell}[uv] &= P_k[u]P_\ell[v], \\ P_1P_{k+\ell-1}[uv] &= P_1P_{k-1}[u]P_\ell[v] + P_k[u]P_1P_{\ell-1}[v] \end{aligned}$$

for $u \in \pi_{4k}(MSp)$, $v \in \pi_{4\ell}(MSp)$ $(k,\ell \geqq 1)$ and Proposition 5.18, we see the following equalities for $k \geqq 0$, where $(a,b) = au(3) + bv(3) +$ a torsion element:

$$\partial(y_1y_2^{k+1}) = ((-1)^{k+1}8^k\cdot 4\cdot(3k+5),\ (-1)^{k+1}8^{k+1}\cdot 3),$$
$$\partial(y_2^{k+2}) = ((-1)^k 8^k\cdot 4\cdot(k+3),\ (-1)^k 8^{k+1}),$$
$$\partial(y_2^k y_3) = ((-1)^k 8^{k-1}\cdot 4\cdot 3\cdot(k+2),\ (-1)^k 8^k\cdot 3),$$
$$\partial(y_2^k y_4) = ((-1)^{k+1}8^k\cdot 5\cdot(k+2),\ (-1)^{k+1}8^k\cdot 2\cdot 5),$$
$$\partial(y_2^k z) = ((-1)^{k+1}8^{k-1}\cdot 4\cdot 9\cdot(k+2),\ (-1)^{k+1}8^k\cdot 9),$$
$$\partial(y_2^k y_5) = ((-1)^k 8^{k-1}\cdot 4\cdot 5\cdot 7\cdot(k+2),\ (-1)^k 8^k\cdot 5\cdot 7),$$
$$\partial(y_2^k y_6) = ((-1)^{k+1}8^k\cdot 7\cdot 9\cdot(k+2),\ (-1)^{k+1}8^k\cdot 2\cdot 7\cdot 9).$$

Therefore, for $\partial\otimes 1$ in (5.21), we have the following equalities:

When $n = 2k-1$ with $k \geqq 2$ and $p \neq 5$,

$$(-1)^k\partial\otimes 1((1/8^{k-2}\cdot 4)y_2^k + (1/8^{k-2}\cdot 5)y_2^{k-2}y_4) = (1,0),$$
$$(-1)^{k+1}\partial\otimes 1((k/8^{k-1})y_2^k + ((k+1)/8^{k-2}\cdot 2\cdot 5)y_2^{k-2}y_4) = (0,\ 1);$$

when $n = 2k-1$ for $k \geqq 3$ and $p = 5$,

$$(-1)^k\partial\otimes 1((1/8^{k-1})y_2^k - (1/8^{k-3}\cdot 2\cdot 7\cdot 9)y_2^{k-3}y_6) = (1,\ 0),$$
$$(-1)^{k+1}\partial\otimes 1(((k-1)/8^{k-1}\cdot 2)y_2^k - ((k+1)/8^{k-3}\cdot 4\cdot 7\cdot 9)y_2^{k-3}y_6) = (0,\ 1);$$

when $n = 3$ and $p = 5$,

$$\partial\otimes 1((1/4)y_2^2 + (2/9)z) = (1,\ 0),\quad \partial\otimes 1(-(1/4)y_2^2 - (1/3)z) = (0,\ 1);$$

when $n = 2k$ for $k \geqq 2$,

$$(-1)^{k+1}\partial\otimes 1((1/8^{k-1}\cdot 4)y_1y_2^k + (1/8^{k-2}\cdot 4)y_2^{k-1}y_3) = (1,0),$$
$$(-1)^k\partial\otimes 1(((k+36)/8^k)y_1y_2^k + ((k+24)/8^{k-1})y_2^{k-1}y_3 - (1/8^{k-2})y_2^{k-2}y_5) = (0,1).$$

These equalities imply (5.21), and we have the desired result. q.e.d.

Now we prove Theorems 4.6 and 4.7.

Proof of Theorem 4.6. (i) By Proposition 5.18 and (5.22),

$P_{2i}[a_1(2,2)^i] = (-8)^i \ (i \geqq 1)$, $P_{2i+1}[a_1(2,2)^{i-1}a_1(2,3)] = (-1)^{i-1}3\cdot 8^i$ $(i \geqq 1)$, $P_{2i+1}[a_1(2,2)^{i-2}a_1(2,5)] = (-1)^i 35\cdot 8^{i-1} \ (i \geqq 2)$.

Therefore $m(2i)$ $(i \geqq 1)$ is a power of 2 by the first equality, and $m(2i+1)$ $(i \geqq 2)$ is so by the last two ones. $m(1) = m(3) = 3$ follows from the result of [7], [6] on $\pi_4(MSp)$ and $\pi_{12}(MSp)$.

(ii) The desired result for $n = 2^k+2^\ell-1$ (resp. 2^k+2^ℓ) follows immediately from (i) and the fact that $P_n[a_{2^k2^\ell 0}]$ (resp. $P_n[a_1(2^k+1, 2^\ell+1)]$) is not a multiple of 16 by (5.10) and Proposition 5.12 (i) (resp. Corollary 5.19). q.e.d.

Proof of Theorem 4.7. (i) is proved in Theorem 5.20.

(ii) If $n = 1$, then $\pi_7(MSp) = 0$ by [7], and (ii) is trivial. If $n \geqq 2$, then $\pi_{8n+2}(F_n) = 0$ by Proposition 3.5 (i). Thus (ii) follows from the exact sequence (4.1).

(iii) Consider the exact sequence (4.1) for $i = 4n+4$ and $n = 2^k+2^\ell-1$ with $k, \ell \geqq 1$:

$$\pi_{4n+4}(MSp) \xrightarrow{\partial} Z \oplus Z \longrightarrow \pi_{8n+3}(MSp(n)) \xrightarrow{b_{n*}} \pi_{4n+3}(MSp) \longrightarrow 0,$$

where we identify $\pi_{8n+3}(F_n)$ with $Z \oplus Z$ by Proposition 3.5 (i). By Propositions 4.2 (ii), 5.12 (ii) and Corollary 5.19, we have

$$\partial(a_{2^k2^\ell 1}) = (x, 0) \quad \text{and} \quad \partial(a_1(2^k+1, 2^\ell+1)) = (x', y)$$

for some integer x' and some odd integers x and y. These imply that Coker ∂ is a finite group and has no 2-torsion. By Theorem 5.20, Coker ∂ has no p-torsion for any odd prime p, hence Coker $\partial = 0$, and b_{n*} is isomorphic. q.e.d.

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