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Field Theoretical Description of Quantum Fluctuations in Multi-Dimensional Tunneling Approach

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ABSTRACT

We reformulate the WKB wave function for a multi-dimensional tunneling system, which represents the quasi-ground-state of the metastable vacuum, in a covariant manner. Then we extend the formalism to the case of field theory and develop a systematic method to construct the mode functions which determine the quantum state after tunneling. A clear interpretation of the resulting quantum state is given in the language of the conventional second quantization picture. As a simple example, we apply our method to a scalar field on the background of spatially homogeneous false vacuum decay. The resulting quantum state is found to be highly non-trivial, having some similarity with a thermal state. Some implications of the results are discussed.

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1. Introduction

Our universe is thought to have experienced phase transitions several times and these phase transitions would have given strong influences on the spacetime structure of the universe as well as on its matter content. A case of particular interest is the false vacuum decay during an inflationary stage of the universe, which was recently revived in the context of the extended inflation [1] and subsequently in several other scenarios [2]. Hence it is very important to understand various phenomena associated with such a phase transition. In the case of a first-order phase transition, it proceeds with nucleation of vacuum bubbles. The rate of nucleation and the typical size of a bubble can be evaluated by the Euclidean path integral method developed by Coleman [3] and Callan and Coleman [4]. However, this gives only the lowest WKB picture of the nucleation process. We expect that the drastic change of the state during the false vacuum decay would excite the fields that interact with the tunneling field. This issue was examined by Rubakov with extensive use of the method of Bogoliubov transformation [5]. Although his pioneering work is very much inspiring, it concentrates on evaluating the particle creation rate, but the concept of "particle" is quite ambiguous in the presence of interaction. Further in his approach, it is not quite clear what is assumed as a first principle. Recently, Vachaspati and Vilenkin developed a method to solve the functional Schrödinger equation in a different way [6]. They set a physically reasonable boundary condition on the wave functional to construct it and analyzed the quantum state after tunneling in the context of the $O(4)$-symmetric bubble nucleation. However, in our opinion, it is not quite clear how their quantum state is related to the initial false vacuum state.

In this paper we formulate a method to investigate the quantum state after tunneling in another approach. We reformulate the method of constructing the WKB wave function for a multi-dimensional tunneling system. Keeping in mind that ultimately gravity should be consistently taken into account in the cosmological context, we develop the formalism in a covariant manner so that it will be applicable to that case. We then give the field theoretical interpretation of the wave function in a rather concise form.

This paper is organized as follows. In Sec. 2, we give an alternative derivation of the multi-dimensional tunneling wave function, which is simpler and clearer than the existing ones in the literature [7,8,9]. In Sec. 3, we extend our formalism to the case of field theory and develop a systematic method to construct the mode functions which determine the quantum state after tunneling. Then the result is interpreted in the language of the conventinal second quantization picture. We find the quanfum state after tunneling generally contains a spectrum of field excitations.

In Sec. 4, we consider spatially homogeneous decay of false vacuum as an example of the field theoretical case. The same problem was examined by Rubakov [5]. Here we analyze the quantum state after tunneling in more details by adopting a model which is much simpler but contains the essence. The resulting spectrum of excitations is found to have some similarity with a thermal spectrum. Our result is consistent with Rubakov's analysis, which is shown briefly in Appendix. In Sec. 5, we summarize our formalism and main results obtained from our simple model. A more realistic case of false vacuum decay through an $O(4)$-symmetric vacuum bubble will be discussed in a subsequent paper [10].

2. Multi-dimensional Tunneling Wave Function

In this section, we derive an expression for the multi-dimensional tunneling wave function in a covariant manner. The correspondence to the second quantization picture will be discussed in the next section. Under the WKB approximation, the method to evaluate the tunneling wave function for a multi-dimensional system was developed by Banks, Bender and Wu [7], Gervais and Sakita [8], and Bitar and Chang [9]. Extension of this method to field theory, particularly in connection with instanton physics, was developed by de Vega, Gervais and Sakita [11]. Here we reformulate the multi-dimensional tunneling wave function in an alternative way, which we believe is simpler and clearer than the previous ones.

For generality, we develop the formalism covariantly and consider a system which has the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}(\phi)\dot{\phi}^\alpha \dot{\phi}^\beta - V(\phi) \qquad (\alpha = 0, \cdots, D), \tag{2.1}$$

where ϕ^α are the coordinates of the $(D+1)$-dimensional space of dynamical variables (*i.e.*, superspace) and $g_{\alpha\beta}(\phi)$ is the superspace metric. In this paper, we assume that the signature of the metric is positive definite. Note that it will not be the case when gravity comes into play. The potential $V(\phi)$ is supposed to have a local minimum at $\phi^\alpha = 0$, which is not the absolute minimum, as shown in Fig. 1. We call it the false vacuum origin or the false vacuum minimum throughout this paper. In this section Greek and Latin indices run from 0 to D and from 1 to D, respectively.

The Hamiltonian operator is obtained by replacing the conjugate momentum of the Hamiltonian with the differential operator in the coordinate representation. In general there is ambiguity of the operator ordering. Here we fix it in such a way that the resulting Hamiltonian takes a covariant form;

$$\hat{H} = -\frac{\hbar^2}{2} g^{\alpha\beta}(\phi)\nabla_\alpha \nabla_\beta + V(\phi), \tag{2.2}$$

where $g^{\alpha\beta}(\phi)$ is the inverse of $g_{\alpha\beta}(\phi)$.

Now we construct the *quasi-ground-state* wave function using the WKB approximation, which is defined as the lowest energy state sufficiently localized at the false vacuum origin, *i.e.*, $\phi^\alpha = 0$. Following the WKB ansatz, the wave function is assumed to have the form,

$$\Psi = e^{-\frac{1}{\hbar}(W^{(0)} + \hbar W^{(1)} + \cdots)}, \tag{2.3}$$

which should solve the time-independent Schrödinger equation,

$$\hat{H}\Psi = E\Psi. \tag{2.4}$$

We solve this equation order by order with respect to $\hbar$. We formally divide the energy eigenvalue E into two parts, E_0 and E_1, with $O(\hbar^0)$ and $O(\hbar^1)$, respectively.

The equation in the lowest order of $\hbar$ becomes the Hamilton-Jacobi equation with the energy E_0 ;

$$-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha W^{(0)}\nabla_\beta W^{(0)} + V(\phi) = E_0. \tag{2.5}$$

By setting the relation,

$$\frac{d\phi^\alpha(\tau)}{d\tau} := g^{\alpha\beta}\nabla_\beta W^{(0)}, \tag{2.6}$$

we get the Euclidean equation of motion,

$$0\frac{d^2\phi^\alpha(\tau)}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{d\phi^\beta}{d\tau}\frac{d\phi^\gamma}{d\tau} = g^{\alpha\beta}\nabla_\beta V, \tag{2.7}$$

where $\Gamma^\alpha{}_{\beta\gamma}$ is the connection coefficient of the superspace metric $g_{\alpha\beta}$. Among solutions of the Euclidean equation of motion which start from the false vacuum origin and reach the region outside the potential barrier, there is a solution which gives the minimum action. We call it the tunneling solution and its trajectory the dominant escape path (hereafter DEP). (It is the path of least resistance [7] or the most probable escape path [8,9].) We consider the case when E_0 is chosen to be $V(0)$. Then the tunneling solution is a half way of the so-called *bounce* solution [3]. We can set the Euclidean time so that the tunneling solution leaves the origin, $\phi^\alpha = 0$, at $\tau \to -\infty$, and reaches the turning point at $\tau = 0$, without any loss of generality. For later conveniance, we denote the solution along the DEP as $\phi_0^\alpha(\tau)$.

In the lowest order WKB sense, the tunneling process is described by this tunneling solution. Integrating the equation derived from Eqs.(2.5) and (2.6);

$$\frac{dW^{(0)}}{d\tau} = 2\left(V(\phi) - E_0\right), \tag{2.8}$$

the tunneling rate can be naively evaluated by the ratio of the squared amplitude at the turning point to that at the false vacuum origin as

$$\Gamma \sim \exp\left(2(W^{(0)}(-\infty) - W^{(0)}(0))\right). \tag{2.9}$$

Next let us turn to the second order Equation;

$$-g^{\alpha\beta}\nabla_{\alpha}W^{(0)}\nabla_{\beta}W^{(1)}+\frac{1}{2}g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}W^{(0)}=\frac{E_1}{\hbar}. \tag{2.10}$$

If solutions of the Euclidean equation of motion (and also $W^{(0)}(\phi^{\alpha})$) are known with a sufficient number of integral constants in the vicinity of the tunneling solution, we obtain a congruence of solutions in the superspace. Then we can introduce a set of new coordinates $\{\lambda^{\bar{\alpha}}\}:=\{\tau,\lambda^{\bar{n}}\}$ which have one-to-one correspondence to the original coordinates $\{\phi^{\alpha}\}$, where $\{\lambda^{\bar{n}}\}$ are the coordinates labeling different orbits of the congruence. Using these new coordinates, we find

$$g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}W^{(0)}=\frac{\partial}{\partial\tau}\log\left[\det\left(\frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\beta}}}\right)\sqrt{g}\right], \tag{2.11}$$

where $\sqrt{g}$ is the determinant of $g_{\alpha\beta}$. Then Eq.(2.10) is integrated as

$$W^{(1)}=\frac{1}{2}\log\left[\det\left(\frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\beta}}}\right)\sqrt{g}\right]-\frac{E_1}{\hbar}\tau+\text{constant}. \tag{2.12}$$

Therefore the wave function to the second lowest order is formally given by

$$\Psi=\frac{C}{\sqrt{\det\left(\frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\beta}}}\right)\sqrt{g}}}\exp\left[-W^{(0)}(\lambda^{\bar{\alpha}})/\hbar+E_1\tau/\hbar\right]. \tag{2.13}$$

At this point, this wave function is a general one and is not necessarily specific to the *quasi-ground-state* wave function. To get the *quasi-ground-state* wave function, we need to choose a congruence of orbits parametrized by $\lambda^{\bar{n}}$ in the vicinity of the DEP which satisfies an appropriate boundary condition at $\tau\to-\infty$. For this purpose, first (i) we expand the wave function (2.13) around the DEP, and second (ii) we require the thus-expanded wave function to have the correct asymptotic behavior at $\tau\to-\infty$, so that it is correctly matched to the *quasi-ground-state* wave function at $\phi^{\alpha}=0$.

The step (i) can be achieved by using a technique similar to the Fermi-Walker transport of a vector and by deriving an equation similar to the geodesic deviation equation [12]. Consider a set of orthonormal bases $e^\alpha_{[\mu]}(\tau)$ along the DEP; $g_{\alpha\beta}e^\alpha_{[\mu]}e^\beta_{[\nu]} = \delta_{[\mu][\nu]}$, where $[\mu]$ runs through the range $0, 1, \cdots, D$. For notational convenience we introduce another set of indices $(0, \mathbf{a})$ to denote $[\mu]$. We choose $e^\alpha_{\mathbf{0}}$ to be the unit vector tangent to the DEP ;

$$e^\alpha_{\mathbf{0}} := \frac{N^\alpha}{N}, \tag{2.14}$$

where

$$\begin{aligned} N^\alpha &:= \frac{\partial\phi^\alpha}{\partial\tau} = g^{\alpha\beta}\nabla_\beta W^{(0)}, \\ N^2 &:= N_\alpha N^\alpha = \nabla_\alpha W^{(0)}\nabla^\alpha W^{(0)} = 2(V - E_0). \end{aligned} \tag{2.15}$$

If we define a differential operator $D_F/\partial\tau$ for a vector X^α as

$$\frac{D_F}{\partial\tau}X^\alpha := \frac{D}{\partial\tau}X^\alpha + \frac{N^\alpha}{N^2}X_\beta\frac{D}{\partial\tau}N^\beta - \frac{X_\beta N^\beta}{N^2}\frac{D}{\partial\tau}N^\alpha. \tag{2.16}$$

where $D/\partial\tau = N^\alpha\nabla_\alpha$ is the covariant derivative tangent to the DEP, it is easily seen that $D_F e^\alpha_{\mathbf{0}}/\partial\tau = 0$. Hence we can choose all the basis vectors $e^\alpha_{[\mu]}$ along the DEP to satisfy

$$\frac{D_F}{\partial\tau}e^\alpha_{[\mu]} = 0. \tag{2.17}$$

At each point q on the DEP, we can find a hypersurface perpendicular to the DEP, $\Sigma(q)$, which is spanned by all possible geodesics tangent to linear combinations of $e^\alpha_{\mathbf{a}}$ at q at least in a sufficiently small neighborhood of q (see Fig. 2). Then it is known that there exists an exponential map from the tangent space at q of $\Sigma(q)$ to the hypersurface $\Sigma(q)$, on which we can introduce the Riemann normal coordinates $\eta^{\mathbf{a}}$ with the identification $e^\alpha_{\mathbf{a}}\partial/\partial\phi^\alpha = \partial/\partial\eta^{\mathbf{a}}$; *i.e.*, the bases $e^\alpha_{\mathbf{a}}$ becomes coordinate bases. Hence we have

$$\begin{aligned} \left.\frac{\partial W^{(0)}}{\partial\eta^{\mathbf{a}}}\right|_{\eta^{\mathbf{a}}=0} &= W_{;\alpha}e^\alpha_{\mathbf{a}} = N_\alpha e^\alpha_{\mathbf{a}} = 0, \\ \left.\frac{\partial^2 W^{(0)}}{\partial\eta^{\mathbf{a}}\partial\eta^{\mathbf{b}}}\right|_{\eta^{\mathbf{a}}=0} &= W_{;\alpha\beta}e^\alpha_{\mathbf{a}}e^\beta_{\mathbf{b}} =: \Omega_{\mathbf{ab}}, \end{aligned} \tag{2.18}$$

where the semicolon denotes covariant differentiation with respect to the metric

$g_{\alpha\beta}(\phi)$. Consequently $W^{(0)}(\lambda^{\bar{\alpha}})$ is expanded as

$$W^{(0)}(\lambda^{\bar{\alpha}}) = W^{(0)}(\tau) + \frac{1}{2}\Omega_{\mathbf{ab}}\eta^{\mathbf{a}}\eta^{\mathbf{b}} + \cdots. \tag{2.19}$$

Now let us determine the matrix $\Omega_{\mathbf{ab}}$ and $\det\left|\partial\phi^{\alpha}/\partial\lambda^{\bar{\beta}}\right|$ in the expression of the wave function (2.13). First we set

$$z^{\alpha}_{\ \bar{\mu}} := \frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\mu}}}. \tag{2.20}$$

Then a straight-forward calculation yields the following equation for $z^{\alpha}_{\ \bar{\mu}}$ along the DEP,

$$\begin{aligned} \frac{D}{\partial\tau}z^{\alpha}_{\ \bar{\mu}} &= N^{\alpha}_{\ ;\beta}z^{\beta}_{\ \bar{\mu}}, \\ \frac{D^2}{\partial\tau^2}z^{\alpha}_{\ \bar{\mu}} &= V^{;\alpha}_{\ \ ;\beta}z^{\beta}_{\ \bar{\mu}} - R^{\alpha}_{\ \sigma\beta\gamma}N^{\sigma}N^{\gamma}z^{\beta}_{\ \bar{\mu}}, \end{aligned} \tag{2.21}$$

where we follow the convention of [12] for the Riemann tensor. The second of the above equations is similar to the geodesic deviation equation, except for the first term on the r.h.s. of it because the DEP is not a geodesic. In deriving this equation, we used the Euclidean equation of motion (2.7), which now should read

$$\frac{D}{\partial\tau}\frac{\partial\phi^{\alpha}}{\partial\tau} - V^{;\alpha} = 0. \tag{2.22}$$

Next we rewrite Eqs.(2.21) in terms of the ordinary partial derivatives along the DEP, we consider the components of $z^{\alpha}_{\ \bar{n}}$ $(\bar{n} = 1, 2, \cdots, D)$ projected in the direction of $e^{\alpha}_{\mathbf{a}}$;

$$z^{\mathbf{a}}_{\ \bar{n}} := e^{\mathbf{a}}_{\alpha}z^{\alpha}_{\ \bar{n}} := K^{\mathbf{a}}_{\ \mathbf{b}}(\tau)\chi^{\mathbf{b}}_{\ \bar{n}}, \tag{2.23}$$

where $\chi^{\mathbf{b}}_{\ \bar{n}}$ is a τ-independent matrix introduced as a normalization factor of $K^{\mathbf{a}}_{\ \mathbf{b}}(\tau)$.

Then it is straightforward to find the equations for $z^{\mathbf{a}}{}_{\bar{n}}$ along the DEP;

$$\frac{d}{d\tau} z^{\mathbf{a}}{}_{\bar{n}} = e^{\alpha}_{\mathbf{a}} W_{;\alpha\beta} e^{\beta}_{\mathbf{b}} z^{\mathbf{b}}{}_{\bar{n}} = \Omega_{\mathbf{ab}} z^{\mathbf{b}}{}_{\bar{n}}, \tag{2.24}$$

$$\frac{d^2}{d\tau^2} z^{\mathbf{a}}{}_{\bar{n}} = V_{;\mathbf{ab}} z^{\mathbf{b}}{}_{\bar{n}} - N^2 R_{\mathbf{a0b0}} z^{\mathbf{b}}{}_{\bar{n}} - 3N^{-2} V_{;\mathbf{a}} V_{;\mathbf{b}} z^{\mathbf{b}}{}_{\bar{n}}, \tag{2.25}$$

where

$$\begin{aligned} V_{;\mathbf{ab}} &:= e^{\alpha}_{\mathbf{a}} V_{;\alpha\beta} e^{\beta}_{\mathbf{b}}, \\ R_{\mathbf{a0b0}} &:= e^{\alpha}_{\mathbf{a}} e^{\beta}_{\mathbf{b}} R_{\alpha\sigma\beta\gamma} N^{\sigma} N^{\gamma}, \\ V_{;\mathbf{a}} &:= e^{\alpha}_{\mathbf{a}} V_{;\alpha}. \end{aligned} \tag{2.26}$$

Note that the matrix $K^{\mathbf{a}}{}_{\mathbf{b}}$ defined in Eq.(2.23) satisfies exactly the same equations as $z^{\mathbf{a}}{}_{\bar{n}}$ does.

Using Eq.(2.24), we express $\Omega_{\mathbf{ab}}$ in terms of $K^{\mathbf{a}}{}_{\mathbf{b}}$; multiplying the both sides of the equation by the inverse of $z^{\mathbf{a}}{}_{\bar{n}}$, we find

$$\Omega_{\mathbf{ab}} = \dot{z}_{\mathbf{a}\bar{n}} (z^{-1})^{\bar{n}}_{\mathbf{b}} = \dot{K}_{\mathbf{a}}{}^{\mathbf{c}} (K^{-1})_{\mathbf{cb}}, \tag{2.27}$$

where the dot denotes τ-differentiation. We also express $\det\left|\partial\phi^{\alpha}/\partial\lambda^{\bar{\beta}}\right|$ in terms of $K^{\mathbf{a}}{}_{\mathbf{b}}$. In order to do so, we write down the superspace line element in the coordinates $\{\lambda^{\bar{\mu}}\} = \{\tau, \lambda^{\bar{n}}\}$ in two different ways;

$$\begin{aligned} ds^2 = g_{\alpha\beta} \frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\mu}}} \frac{\partial\phi^{\beta}}{\partial\lambda^{\bar{\nu}}} d\lambda^{\bar{\mu}} d\lambda^{\bar{\nu}} &= \left(e^{0}_{\alpha} e_{0\beta} + e^{\mathbf{a}}_{\alpha} e_{\mathbf{a}\beta}\right) \frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\mu}}} \frac{\partial\phi^{\beta}}{\partial\lambda^{\bar{\nu}}} d\lambda^{\bar{\mu}} d\lambda^{\bar{\nu}} \\ &= N^2 d\tau^2 + \delta_{\mathbf{ab}} z^{\mathbf{a}}{}_{\bar{n}} z^{\mathbf{b}}{}_{\bar{m}} d\lambda^{\bar{n}} d\lambda^{\bar{m}}. \end{aligned} \tag{2.28}$$

Then equating the volume elements in the two expressions, we find

$$\begin{aligned} \sqrt{g} \left| \det\left(\frac{\partial\phi^{\alpha}}{\partial\lambda^{\bar{\mu}}}\right) \right| &= N \left| \det z^{\mathbf{a}}{}_{\bar{n}} \right| \\ &= \sqrt{2(V(\phi(\tau)) - E_0)}\, \left| \det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau) \right| \left| \det \chi^{\mathbf{c}}{}_{\bar{n}} \right|. \end{aligned} \tag{2.29}$$

Substituting Eqs.(2.29) and (2.19) into (2.13), we arrive at a desired expression for the *quasi-ground-state* wave function,

$$\Psi = \frac{C\, e^{-W^{(0)}(\tau)/\hbar} e^{E_1 \tau/\hbar}}{\left(\sqrt{2(V(\phi_0(\tau)) - E_0)}\,|\det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)|\,|\det \chi^{\mathbf{c}}{}_{\bar{n}}|\right)^{1/2}} \times \exp\left[-\frac{1}{2\hbar}\eta^{\mathbf{a}}\eta^{\mathbf{b}}\Omega_{\mathbf{ab}}(\tau)\right]. \tag{2.30}$$

This wave function is the same as that given by Gervais and Sakita [8], assuming that $\eta^{\mathbf{a}}$ is $O(\hbar^{1/2})$.

Now we turn to the step (ii), *i.e.*, the matching condition for the wave function. As we are interested in the *quasi-ground-state* wave function which would be the ground-state wave function if the false vacuum were the absolute minimum, we assume that the system can be well approximated by a collection of harmonic oscillators near the false vacuum minimum and *quasi-ground-state* wave function there can be approximated by the ground-state wave function for this collection of harmonic oscillators.

Specifically we assume that the potential and the superspace metric have the following asymptotic forms, respectively, near $\phi^\alpha = 0$;

$$V(\phi^\alpha) - E_0 \to \frac{1}{2}(\omega^2)_{\alpha\beta}\phi^\alpha\phi^\beta, \qquad g_{\alpha\beta} \to g^{(0)}_{\alpha\beta}. \tag{2.31}$$

Here $g^{(0)}_{\alpha\beta}$ is a constant positive definite metric and $\omega_{\alpha\beta}$ is assumed to be a positive definite matrix. As we can set $g^{(0)}_{\alpha\beta} = \delta_{\alpha\beta}$ without loss of generality, we do so. The ground-state wave function for this system is

$$\Psi = \left(\det\frac{\omega}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\omega_{\alpha\beta}\phi^\alpha\phi^\beta\right), \tag{2.32}$$

which should be matched to the WKB wave function (2.30).

From the assumption (2.31), the Euclidean equation of motion (2.22) at $\tau \to -\infty$ takes the following form,

$$\frac{\partial^2}{\partial\tau^2}\phi^\alpha = (\omega^2)_{\alpha\beta}\phi^\beta. \tag{2.33}$$

Hence with the boundary condition that $\phi^\alpha(\tau) \to 0$ as $\tau \to -\infty$, the relevant solution which describes a congruence along the DEP is given by

$$\phi^\alpha = \left(e^{\omega\tau}\right)^\alpha{}_\beta C^\beta,$$

where C^β are some constants. Integrating the equation $\partial W^{(0)}/\partial\phi^\alpha = \partial\phi^\alpha/\partial\tau = \omega_{\alpha\beta}\phi^\beta$, we get

$$W^{(0)}(\phi^\alpha) = \frac{1}{2}\omega_{\alpha\beta}\phi^\alpha\phi^\beta, \tag{2.34}$$

where we have set $W^{(0)} = 0$ at $\phi^\alpha = 0$. This implies $\Omega_{\mathbf{ab}} \to \omega_{\alpha\beta}e^\alpha_{\mathbf{a}}e^\beta_{\mathbf{b}}$. Then from Eq.(2.27), the asymptotic boundary condition that $K^{\mathbf{a}}{}_{\mathbf{b}}$ should satisfy becomes

$$K^{\mathbf{a}}{}_{\mathbf{b}} = \left(e^{\bar\omega\tau}\right)^{\mathbf{a}}{}_{\mathbf{c}}\,\kappa^{\mathbf{c}}{}_{\mathbf{b}}, \tag{2.35}$$

where $\kappa^{\mathbf{c}}{}_{\mathbf{b}}$ are constants and $\bar\omega$ is a matrix defined by

$$\bar\omega_{\mathbf{ab}} := e^\alpha_{\mathbf{a}}\omega_{\alpha\beta}e^\beta_{\mathbf{b}}.$$

In particular, using the fact that

$$e^\alpha_{\mathbf{0}}\omega_{\alpha\beta}e^\beta_{\mathbf{0}} = \frac{1}{N^2}\omega_{\alpha\beta}\dot\phi^\alpha\dot\phi^\beta = \frac{1}{2}\frac{\partial}{\partial\tau}\log N^2,$$

we can readily show that the following equality holds in the asymptotic region,

$$\left|\det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)\right| = \frac{e^{\mathrm{Tr}\,\omega\,\tau}}{\sqrt{2(V(\phi_0(\tau)) - E_0)}}, \tag{2.36}$$

where we have used the arbitrariness of $\chi^{\mathbf{a}}_{\bar n}$ (or of $\kappa^{\mathbf{a}}{}_{\mathbf{b}}$) to normalize $K^{\mathbf{a}}{}_{\mathbf{b}}$.

Substituting (2.34) and (2.36) into (2.30), and comparing the result with the harmonic oscillator wave function (2.32), we find

$$E_1 = \frac{\hbar}{2}\mathrm{Tr}\,\omega, \qquad \frac{C}{\sqrt{\left|\det\chi^{\mathbf{c}}{}_{\bar n}\right|}} = \left(\det\frac{\omega}{\pi}\right)^{1/4}.$$

Thus E_1 is the vacuum fluctuation energy of the false vacuum. Finally we obtain the WKB *quasi-ground-state* wave function to the second order as

$$
\begin{aligned}
\Psi = & \frac{\left(\det \omega/\pi\right)^{1/4}}{\left[2\left(V(\phi_0(\tau)) - E_0\right)\right]^{1/4} \sqrt{\left|\det K^{\mathbf{a}}{}_{\mathbf{b}}(\tau)\right|}} \\
& \times \exp\left(-\frac{1}{\hbar}\int_{-\infty}^{\tau} d\tau' 2\left(V(\phi_0(\tau)) - E_0\right) + \frac{1}{2}\mathrm{Tr}\,\omega\tau\right) \\
& \times \exp\left(-\frac{1}{2\hbar}\Omega_{\mathbf{ab}}\eta^{\mathbf{a}}\eta^{\mathbf{b}}\right),
\end{aligned}
\tag{2.37}
$$

where $\Omega_{\mathbf{ab}}$ is expressed in terms of $K^{\mathbf{a}}{}_{\mathbf{b}}$, and it is determined by solving Eq.(2.25) with the boundary condition (2.35), *i.e.*, the exponentially decreasing solution as $\tau \to -\infty$.

Thus we have found the WKB wave function in the forbidden region. However, what we really want to know is the wave function beyond the turning point. Following the conventional terminology, we call this classically allowed region the Lorentzian region, while the classically forbidden region the Euclidean region. The construction of the general form of the Lorentzian wave function is not much different from that of the Euclidean wave function. The essential issue is the matching condition at the turning point at which the WKB approximation breaks down. Nevertheless, in the case that the potential depends only on τ but not on $\eta^{\mathbf{a}}$ around the turning point with a good accuracy, the matching problem reduces to that in the case of one-dimensional quantum system. Hence, the Lorentzian wave function will be simply given by the analytic continuation of the Euclidean wave function, *i.e.*, replacing the Euclidean time parameter τ by the Lorentzian time t with $\tau \to it$. The matching problem for a general case has not been formulated so far and we hope to come back to this issue in a future publication.

3. Interpretation in the Second Quantization Picture

Now we discuss the case of a quantum field coupled to the tunneling field by applying the formalism developed in the previous section. Originally this problem was examined by Rubakov [5], focusing on the particle creation during the tunneling. However, as noted before, the concept of particle is quite ambiguous in the presence of the interaction as in the case of field theory on a curved spacetime which has no asymptotically stationary region. Hence we focus on the quantum state itself and as a specific example of the observables after tunneling, we derive the equal-time two-point correlation function from the wave functional. Then we interpret the results in the conventional second quantization picture.

Let us consider the system which consists of two fields, *i.e.*, the tunneling field σ and another field ϕ coupled to it. Explicitly, we assume the Hamiltonian of the form,

$$H = H_\sigma + H_\phi, \tag{3.1}$$

where

$$\begin{aligned} H_\sigma &:= \int d^3\mathbf{x}\Big[\,\frac{1}{2}p_\sigma^2 + \frac{1}{2}(\nabla\sigma)^2 + U(\sigma)\Big], \\ H_\phi &:= \int d^3\mathbf{x}\Big[\,\frac{1}{2}p_\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2(\sigma)\phi^2\Big]. \end{aligned} \tag{3.2}$$

Here p_σ and p_ϕ are the conjugate momenta of σ and ϕ, respectively, and we assume the interaction between the two fields is described by the σ-dependent mass term of ϕ; $m^2(\sigma)\phi^2/2$. We denote the tunneling solution by $\sigma_0(\boldsymbol{x};\tau)$ whose spatial configuration is completely determined by the single parameter τ. In particular, $\sigma_0(\boldsymbol{x};-\infty) = \sigma_F$ where σ_F is the value of σ at the false vacuum minimum. We neglect the fluctuations of the tunneling field itself, though ϕ may be regarded as the fluctuating σ-field if one replaces $m^2(\sigma)$ by $U''(\sigma_0)$.

To apply the previous formalism to the field theory, we make the following correspondence,

$$\phi_0^\alpha(\tau) \to \sigma_0(\boldsymbol{x};\tau), \qquad \eta^{\mathbf{a}} \to \phi(\boldsymbol{x}). \tag{3.3}$$

Thus, as much as the fluctuating degrees of freedom are concerned, the extention to the field theory is done by replacing the suffix a with the spatial coordinates $\boldsymbol{x}$.

To find the ***quasi-ground-state*** wave functional, we have to solve for the matrix $K^{\mathrm{a}}{}_{\mathrm{b}}(\tau)$, which we denote in the present case as $K(\boldsymbol{x},\boldsymbol{y};\tau)$. Then, Eq.(2.25) now reads

$$\Big[\frac{\partial^2}{\partial\tau^2}+\Delta_{\boldsymbol{x}}-m^2(\sigma_0(\boldsymbol{x},\tau))\Big]K(\boldsymbol{x},\boldsymbol{y};\tau)=0. \tag{3.4}$$

The boundary condition is that it decreases exponetially at $\tau\to-\infty$. Instead of directly dealing with $K(\boldsymbol{x},\boldsymbol{y};\tau)$, we consider a complete set of mode functions $g_{\boldsymbol{k}}(x)$ (hereafter $x,y,\cdots$ represent $(\boldsymbol{x},\tau),(\boldsymbol{y},\tau),\cdots$) which satisfy the field equation,

$$\Big[\frac{\partial^2}{\partial\tau^2}+\Delta-m^2(\sigma_0(\boldsymbol{x},\tau))\Big]g_{\boldsymbol{k}}(x)=0, \tag{3.5}$$

with the condition that they decrease exponentially at $\tau\to-\infty$. For convenience, we assume the eigenvalue index $\boldsymbol{k}$ to represent the eigenvalue of a spatial harmonic function $Y_{\boldsymbol{k}}(\boldsymbol{x})$; *i.e.*, at $\tau\to-\infty$, we assume $g_{\boldsymbol{k}}(x)$ to have the form,

$$g_{\boldsymbol{k}}(x)\to e^{\omega_-\tau}Y_{\boldsymbol{k}}(\boldsymbol{x});\qquad \omega_-:=\sqrt{k^2+m^2(\sigma_F)}, \tag{3.6}$$

where $k^2=|\boldsymbol{k}|^2$ and $Y_{\boldsymbol{k}}(x)$ satisfies

$$\left[\Delta+k^2\right]Y_{\boldsymbol{k}}(\boldsymbol{x})=0.$$

Then $K(\boldsymbol{x},\boldsymbol{y};\tau)$ can be expanded as

$$K(\boldsymbol{x},\boldsymbol{y};\tau)=\sum_{\boldsymbol{k}}g_{\boldsymbol{k}}(x)Y_{\boldsymbol{k}}(y). \tag{3.7}$$

In the expression for the wave function given in the previous section, (2.37), information of the quantum fluctuations around the DEP is essentially contained in the last Gaussian factor. Hence in the present case, we write the wave functional as

$$\Psi = F(\tau)\exp\left[-\frac{1}{2}\int d^3x d^3y \phi(x)\Omega(x,y;\tau)\phi(y)\right], \tag{3.8}$$

where $F(\tau)$ depends only on the parameter τ and

$$\Omega(x,y;\tau) = \int d^3z \dot{K}(x,z;\tau)K^{-1}(z,y;\tau), \tag{3.9}$$

and focus on the Gaussian factor. Using (3.7), $\Omega(x,y;\tau)$ can be expressed in terms of $g_k(x)$ as

$$\Omega(x,y;\tau) = \sum_k \dot{g}_k(x)g_k^{-1}(y), \tag{3.10}$$

where $g_k^{-1}(y)$ is the inverse of $g_k(x)$ such that

$$\sum_k g_k^{-1}(x)g_k(y) = \delta^3(x-y), \qquad \int d^3x\, g_k^{-1}(x)g_p(x) = \delta_{kp}.$$

Thus, to obtain the Euclidean wave functional, all we need to know are the mode functions $g_k(x)$.

Once we obtain the Euclidean wave functional, the remaining task is to derive the Lorentzian wave functional by matching these two at the turning point $\tau = 0$. We denote the latter by Ψ_L while the former by Ψ. As noted in the end of the previous section, this matching procedure can be quite complicated in general, but in the present case Ψ_L is obtained by the analytic continuation of Ψ, since the matching involves only one degree of freedom that corresponds to the DEP. Introducing a function $v_k(x)$ in the Lorentzian region, the complex conjugate of which, $v_k^*(x)$, is the analytic continuation of $g_k(x)$ with $\tau \to it$, we find

$$\Psi_L = F(it)\exp\left[-\frac{1}{2}\int d^3x d^3y \phi(x)\Omega(x,y;t)\phi(y)\right], \tag{3.11}$$

with

$$\Omega(x,y;t) = -i\sum_k \dot{v}_k^*(x)v_k^{*-1}(y), \tag{3.12}$$

where and in what follows the dot denotes the differentiation with respect to t and $x, y, \cdots$ represent $(x,t),(y,t),\cdots$.

Now we investigate the nature of the quantum state described by Ψ_L. For this purpose, let us consider the equal-time two-point correlation function. It is expressed as

$$
\begin{aligned}
G^{+}(x,y) &= \frac{\int d\phi(\cdot)\Psi_L^*\{\phi(x),\phi(y)\}\Psi_L}{\int d\phi(\cdot)\Psi_L^*\Psi_L} \\
&= \Big(\Omega(x,y;t)+\Omega^*(x,y;t)\Big)^{-1} + (x \leftrightarrow y), \qquad (3.13)\\
&= \Big(i\sum_k \dot{v}_k(x)v_k^{-1}(y) - i\sum_k \dot{v}_k^*(x)v_k^{*\,-1}(y)\Big)^{-1} + (x \leftrightarrow y).
\end{aligned}
$$

This expression, as it is, does not give us much information. The reason is that although the functions $v_k(x)$ form a complete set, they are not properly orthonormalized. Hence, in order to rewrite Eq.(3.13) in a more comprehensible form, we introduce a set of normal mode functions $u_q(x)$, each of which is a linear combination of $v_k(x)$,

$$
u_q(x) = \sum_k \Lambda_q{}^k v_k(x), \qquad (3.14)
$$

and are normalized as

$$
-i\int d^3\mathbf{z}\Big(u_q(z)\dot{u}_{q'}^*(z) - \dot{u}_q(z)u_{q'}^*(z)\Big) = \delta_{qq'}. \qquad (3.15)
$$

We note that, in principle, these functions can be constructed by Schmidt's orthogonalization procedure, though in practice it may be formidable to do so. Contracting the both sides of the above normalization condition by the inverse of $u_q(x)$ and $u_{q'}^*(y)$, we find

$$
-i\sum_q\Big(\dot{u}_q^*(x)u_q^{*\,-1}(y) - u_q^{-1}(x)\dot{u}_q(y)\Big) = \sum_q u_q^{-1}(x)u_q^{*\,-1}(y). \qquad (3.16)
$$

Then since $\sum_k \dot{v}_k^*(x)v_k^{*\,-1}(y) = \sum_q \dot{u}_q(x)u_q^{-1}(y)$, the equal-time two-point function is expressed in terms of $u_q(x)$ as

$$
G^{+}(x,y) = \sum_q\Big(u_q(x)u_q^*(y) + u_q^*(x)u_q(y)\Big). \qquad (3.17)
$$

This coincides with the one for the Heisenberg state $|\Phi\rangle$ defined by

$$\hat{a}_q|\Phi\rangle = 0, \qquad \text{for } \forall q, \tag{3.18}$$

when the field operator $\hat{\phi}(x)$ is expanded as

$$\hat{\phi}(x) = \sum_q \left(\hat{a}_q u_q(x) + \hat{a}_q^\dagger u_q^*(x) \right). \tag{3.19}$$

where $\hat{a}_q$ $(\hat{a}_q^\dagger)$ is the annihilation (creation) operator associated with the mode function $u_q(x)$ $(u_q^*(x))$.

Thus the quantum state after tunneling is the "vacuum" with respect to $\hat{a}_q$ for which $u_q(x)$ plays the role of the "positive frequency" function. Note that this mode function $u_q(x)$ is generally different from the true positive frequency function after tunneling, say $w_q(x)$, if it can be defined. This will be the case when $m^2(\sigma)$ approaches a constant sufficiently fast after tunneling. Then $u_q(x)$ and $w_q(x)$ will be related to each other by a non-trivial Bogoliubov transformation. This implies the quantum state after tunneling contains a spectrum of excitations of the field ϕ. In the next section, we give an explicit example of such a case.

4. Spatially Homogeneous Decay

In this section, we consider a specific example of the tunneling in field theory discussed in the previous section. The aim is to demonstrate the significance of our formalism and to show how non-trivial the resulting quantum state after tunneling will be, as well as to clarify its relation to the previous work by Rubakov [5]. For simplicity, we consider the case when the tunneling solution is spatially homogeneous; $\sigma = \sigma_0(\tau)$. This situation can be realized if we consider a spatially closed universe, or it may be regarded as the limiting case of a sufficiently large vacuum bubble compared to the scale of interest. Further, for simplicity, we restrict the

σ-dependence of the ϕ-field mass, $m^2(\sigma)$, to be that given by a step function,

$$m^2(\sigma) = \begin{cases} m_-^2 & (\sigma < \tilde{\sigma}) \\ m_+^2 & (\sigma > \tilde{\sigma}). \end{cases} \tag{4.1}$$

We assume the false vacuum origin is in the region $\sigma_0(-\infty) = \sigma_F < \tilde{\sigma}$ and introduce the parameter time $\tilde{\tau}(< 0)$ which denotes the time at which the mass changes; $\sigma_0(\tilde{\tau}) = \tilde{\sigma}$.

Since we have

$$\begin{aligned} \left[\partial_\tau^2 + \Delta - m_+^2\right] g_{\boldsymbol{k}}(x) = 0 \qquad (\tau > \tilde{\tau}), \\ \left[\partial_\tau^2 + \Delta - m_-^2\right] g_{\boldsymbol{k}}(x) = 0 \qquad (\tau < \tilde{\tau}), \end{aligned} \tag{4.2}$$

the (unnormalized) mode function $g_{\boldsymbol{k}}(x)$ satisfying the boundary condition is easily obtained as

$$g_{\boldsymbol{k}}(x) = \begin{cases} e^{\omega_- \tau} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} & (\tau < \tilde{\tau}), \\ \left(A_{\boldsymbol{k}} e^{\omega_+ \tau} + B_{\boldsymbol{k}} e^{-\omega_+ \tau}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} & (\tau > \tilde{\tau}), \end{cases} \tag{4.3}$$

where $\omega_\pm = \sqrt{k^2 + m_\pm^2}$ and

$$\begin{aligned} A_{\boldsymbol{k}} &= \frac{1}{2\omega_+}(\omega_+ + \omega_-) e^{-(\omega_+ - \omega_-)\tilde{\tau}}, \\ B_{\boldsymbol{k}} &= \frac{1}{2\omega_+}(\omega_+ - \omega_-) e^{(\omega_+ + \omega_-)\tilde{\tau}}. \end{aligned} \tag{4.4}$$

Fortunately, in the present model, the orthonormal mode functions $u_{\boldsymbol{k}}(x)$ in the Lorentzian region are readily obtained from the analytic continuation of $g_{\boldsymbol{k}}(x)$;

$$\begin{aligned} u_{\boldsymbol{k}}(x) &= \frac{A_{\boldsymbol{k}} e^{-i\omega_+ t} + B_{\boldsymbol{k}} e^{i\omega_+ t}}{\sqrt{2\omega_+ (A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2)}} \frac{e^{ikx}}{(2\pi)^{3/2}} \\ &= \frac{A_{\boldsymbol{k}}}{\sqrt{A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2}} w_{\boldsymbol{k}}(x) + \frac{B_{\boldsymbol{k}}}{\sqrt{A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2}} w_{\boldsymbol{k}}^*(x), \end{aligned} \tag{4.5}$$

where $w_{\boldsymbol{k}}(x)$ is the usual Minkowski positive frequecy mode function,

$$w_{\boldsymbol{k}}(x) = \frac{e^{-i\omega_+ t + i\boldsymbol{k}\cdot\boldsymbol{x}}}{\sqrt{2\omega_+(2\pi)^3}}. \tag{4.6}$$

In this model, since the field after tunneling is a simple massive scalar field, there is no ambiguity in the concept of a particle. This allows us to compare our result with that of Rubakov [5]. The number of created particles is definitely estimated as

$$\begin{aligned} N_{\boldsymbol{k}} := |\beta_{\boldsymbol{k}}|^2 &:= \frac{B_{\boldsymbol{k}}^2}{A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2} \\ &= \frac{(\omega_+ - \omega_-)^2}{(\omega_+ + \omega_-)^2 e^{-4\omega_+\tilde{\tau}} - (\omega_+ - \omega_-)^2}. \end{aligned} \tag{4.7}$$

In Appendix, it is shown that this agrees with the result obtained in Rubakov's formalism.

Here, we note that the above particle spectrum differs from that in the case of particle creation due to a sudden change of the mass in the real Lorentzian spacetime. The latter would be the case if the false vacuum decay were considered in the classical picture and were assumed to occur suddenly at, say $t = 0$; in this case, the number of created particles would be given by $N_{\boldsymbol{k}}$ of Eq.(4.7) with $\tilde{\tau} = 0$.

Let us consider some implications of Eq.(4.7). First note that $\tilde{\tau} < 0$, hence $N_{\boldsymbol{k}}$ decreases exponentially as the absolute value of $\tilde{\tau}$ becomes large. In particular, in the limit $\omega_+ \gg \omega_-$, which holds if $m_-^2 \ll m_+^2$ and $k^2 \lesssim m_+^2$, $N_{\boldsymbol{k}}$ takes the same form as the thermal distribution with temperature $T = 1/(4|\tilde{\tau}|)$. However, the behavior in the large momentum limit differs from the thermal spectrum as

$$N_{\boldsymbol{k}} \simeq \frac{1}{(4k^2/\Delta m^2)^2\, e^{4\omega_+|\tilde{\tau}|} - 1} \qquad \text{for } k^2 \gg m_\pm^2\,, \tag{4.8}$$

where $\Delta m^2 := m_+^2 - m_-^2$.

To gain a bit more insight into the quantum state after tunneling, let us consider the case when the mass difference between the true and false vacua is small; $|\Delta m^2| \ll m_+^2$. Then, the number of created particles becomes

$$N_k \simeq \left(\frac{\Delta m^2}{4\omega_+^2}\right)^2 e^{4\omega_+\tilde{\tau}}. \tag{4.9}$$

and the energy density due to quantum fluctuations of the ϕ-field is given by

$$\begin{aligned}\mathcal{E} &= \int_0^\infty \frac{d^3k}{(2\pi)^3}\omega_+ N_k \\ &= \frac{\Delta m^4}{32\pi^2}e^{-4m_+|\tilde{\tau}|}\int_0^\infty dx\frac{\sqrt{x(x+2x_0)}}{(x+x_0)^2}e^{-4x},\end{aligned} \tag{4.10}$$

where $x_0 := m_+|\tilde{\tau}|$. Thus, the energy density generated through the tunneling is of $O(\Delta m^4)$ if $x_0 \lesssim 1$, while it becomes negligibly small if $x_0 \gg 1$. Presumably, $1/|\tilde{\tau}|$ is related to a certain mass scale M associated with the tunneling field σ. Hence, the particle creation is expected to be rather significant for models with $m_+ \lesssim M$. We note that this conclusion qualitatively holds for general values of the masses $m_\pm^2$ as well, though it has been derived by assuming $|\Delta m^2| \ll m_+^2$.

5. Summary and Discussion

We have developed the formulation to determine the quantum state after the tunneling in field theory. Our approach is based on the method of the multi-dimensional wave function. The formulation can be applied to a system in which the state before tunneling is the quasi-ground-state of false vacuum.

The crucial procedure in our formalism is the method to construct appropriate mode functions of a quantum scalar field ϕ coupling to the tunneling field σ. This is done as follows. First, we solve the linearized field equation for ϕ in the background of the Euclidean classical tunneling solution $\sigma = \sigma_0(\boldsymbol{x}, \tau)$ with the condition that the field goes to zero exponetially as the Euclidean time τ goes to $-\infty$, and construct a set of Euclidean mode functions $g_{\boldsymbol{k}}(x)$. Then the Lorentzian mode functions $v_{\boldsymbol{k}}(x)$, which describe the quantum state after tunneling, are obtained by the analytic continuation of $g_{\boldsymbol{k}}(x)$ with $\tau \to it$ and by taking their complex conjugates. As these Lorentzian mode functions are not in general orthonormalized, if necessary we have to construct a new set of orthonormalized mode functions $u_{\boldsymbol{k}}(x)$ by a suitable linear transformation of the original ones.

The resulting quantum state after tunneling is most conveniently described in the Heisenberg picture. Namely, if we represent the field operator as $\hat{\phi}(x) = \sum_{\boldsymbol{k}}\left(\hat{a}_{\boldsymbol{k}} u_{\boldsymbol{k}}(x) + \hat{a}^{\dagger}_{\boldsymbol{k}} u^{*}_{\boldsymbol{k}}(x)\right)$, the state is identical to the "vacuum" state annihilated by the operator $\hat{a}_{\boldsymbol{k}}$. This state is not the true vacuum state of the field ϕ, but is in general a highly excited state.

In order to demonstrate the significance of our formalism and to show how non-trivial the resulting quantum state can be, we have applied it to an example of spatially homogeneous decay of false vacuum. For simplicity, We have evaluated excitations of a scalar field whose mass square m^2 is a step function of the tunneling field and undergoes a discrete change during the tunneling process. The resulting spectrum of excitations has some similarity with a thermal spectrum with its temperature presumably given by a certain mass scale M associated with the tunneling field. However, the high momentum distribution is more suppressed than the latter. As a result, the generated total energy density is determined not by M but by the difference of m^2 before and after the tunneling; $\mathcal{E} \sim \Delta m^4$.

We should admit the objection, however, that the model analyzed in the present paper is oversimplified and is not directly applicable to a realistic case of false vacuum decay. In order to show further advantages of our formalism and to derive results which have more realistic implications, it is necessary to investigate a model in which the Euclidean tunneling solution is an $O(4)$-symmetric bubble. This issue is tackled in another paper [10].

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APPENDIX

In this appendix, we show that our estimate of the particle creation given in Sec. 4 agrees with the result obtained in Rubakov's approach [5]. We omit the details and show only the resulting expression for the particle number in his approach. In Rubakov's formalism, the number of created particles is given by

$$N_{\boldsymbol{k}} = \left(\frac{D^2}{1-D^2}\right)_{\boldsymbol{k}\boldsymbol{k}}, \tag{A1}$$

where D is a matrix given in Eq.(3.19) of [5]. In the case of spatially homogeneous decay of false vacuum, each $\boldsymbol{k}$-mode decouples and the matrix D becomes diagonal. Hence we can treat each mode separately. Since our mode function $g_{\boldsymbol{k}}$ corresponds to Rubakov's mode function g_α defined in Eq.(3.8) of [5], we find the diagonal component of D with the wavenumber $\boldsymbol{k}$ is expressed in terms of $g_{\boldsymbol{k}}$ as

$$D_{\boldsymbol{k}} = -\left.\frac{\dot{g}_{\boldsymbol{k}} - \omega_+ g_{\boldsymbol{k}}}{\dot{g}_{\boldsymbol{k}} + \omega_+ g_{\boldsymbol{k}}}\right|_{\tau=0} = \frac{\tilde{\omega} - \omega_+}{\tilde{\omega} + \omega_+}, \tag{A2}$$

where $\tilde{\omega} := \partial_\tau g_{\boldsymbol{k}}/g_{\boldsymbol{k}}|_{\tau=0}$. For the model of Sec. 4, we have

$$\tilde{\omega} = \frac{A_{\boldsymbol{k}} - B_{\boldsymbol{k}}}{A_{\boldsymbol{k}} + B_{\boldsymbol{k}}}\omega_+ .$$

Hence the number spectrum of created particles in Rubakov's formalism, Eq.(A1), is calculated to be

$$N_{\boldsymbol{k}} = \frac{(\tilde{\omega} - \omega_+)^2}{4\tilde{\omega}\omega_+} = \frac{B_{\boldsymbol{k}}^2}{A_{\boldsymbol{k}}^2 - B_{\boldsymbol{k}}^2}. \tag{A3}$$

This is exactly in agreement with the result given in Eq.(4.7).

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FIGURE CAPTIONS

Fig. 1 Illustration of a potential for a multi-dimensional tunneling system. The oordinates $(\phi^2, \phi^3, \cdots, \phi^D)$ are suppressed.

Fig. 2 Illustration of a hypersurface orthogonal to the DEP at q, spanned by all the geodesics tangent to linear combinations of basis vectors $e_{\boldsymbol{a}}^{\alpha}$ $(\boldsymbol{a} = 1, 2, \cdots, D)$ at q.

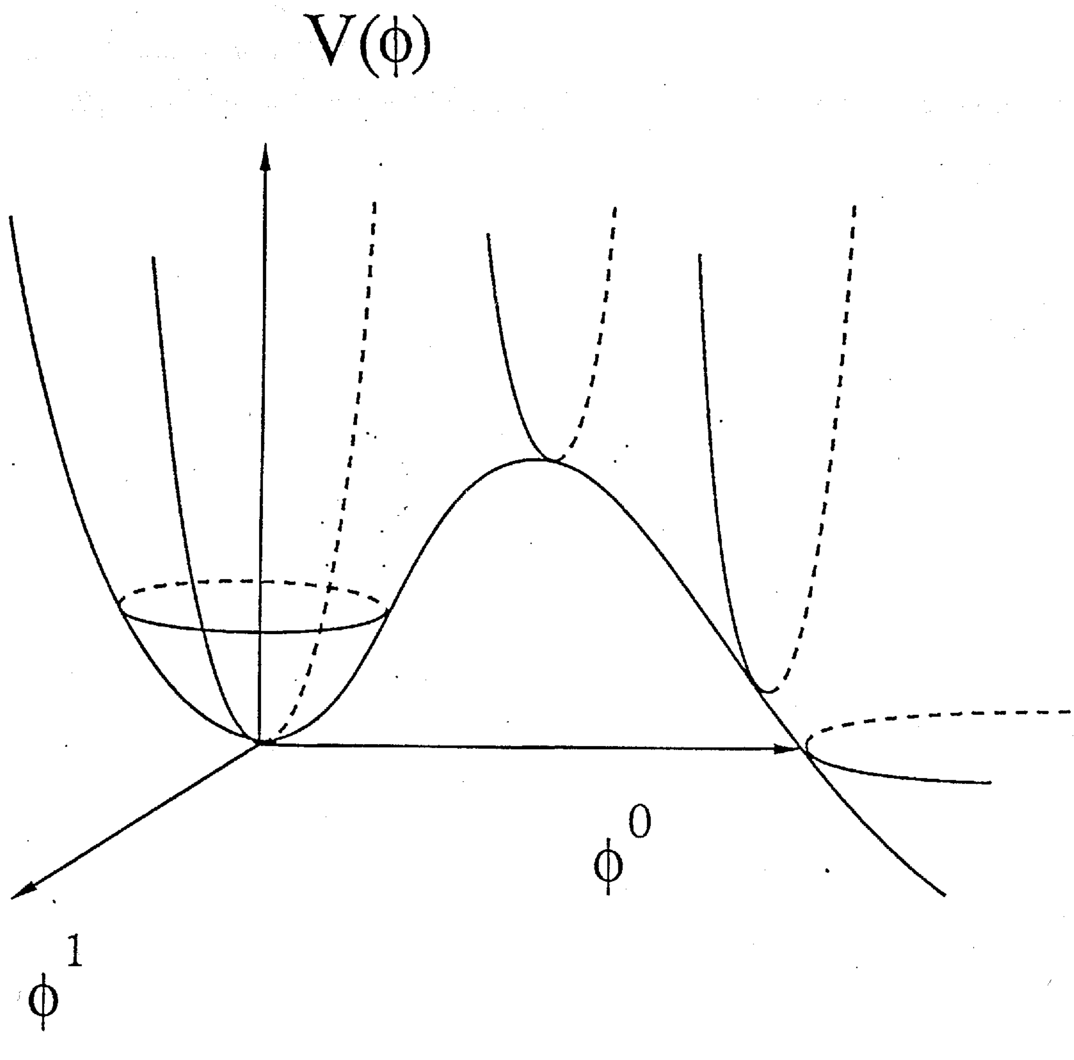

Fig.1

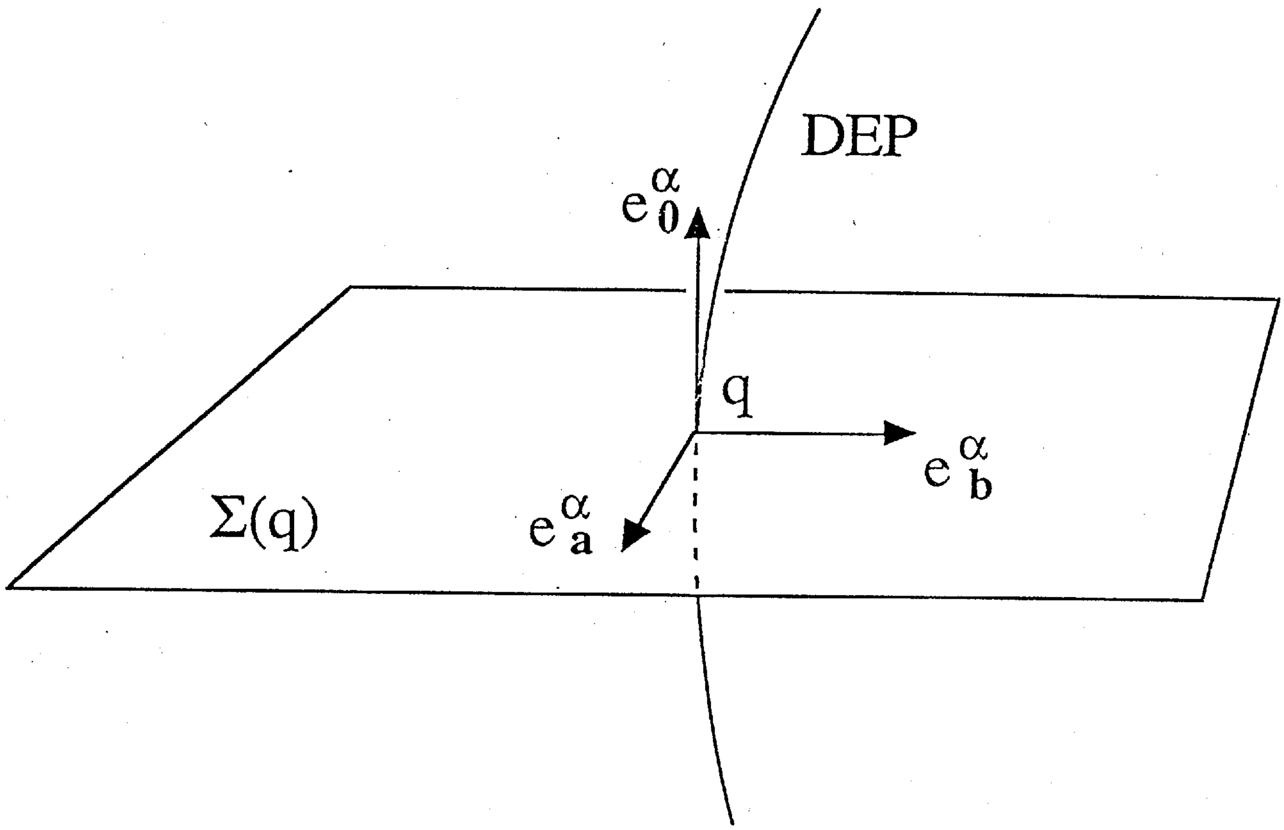

Fig.2