to be published in
International Journal of Modern Physics A Vol. 9 No. 2 (Jan. 1994)

Probability Distribution Functional for Equal-time Correlation Functions in Curved Space

HIROSHI SUZUKI[★]

Department of Physics, Tohoku University, Sendai 980, Japan

MISAO SASAKI[†]

Department of Physics, Kyoto University, Kyoto 606, Japan

KAZUHIRO YAMAMOTO[‡] AND JUN'ICHI YOKOYAMA[§]

Uji Research Center, Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

ABSTRACT

We present a systematic method to calculate the probability distribution functional (PDF) for spatial configuration of an interacting field in curved spacetime. As an example, we consider PDF for the minimally coupled massive $\lambda\phi^4$-theory up to the first order of the coupling constant and evaluate it both in Minkowski and de Sitter spacetimes. We observe that PDF has an ultraviolet divergence even after the ultraviolet renormalization. This divergence is unavoidable to reproduce *finite* expectation values; thus some kind of regularization is necessary to write down PDF. As an application of it, a scaling law among multi-point correlation functions in the de Sitter space is found.

★ JSPS Junior Scientist Fellow. e-mail address: hsuzuki@tuhep.phys.tohoku.ac.jp
† e-mail address: misao@jpnyitp.bitnet
‡ Also at Department of Physics, Hiroshima University, Higashi-Hiroshima 724, Japan. e-mail address: yamamoto@yisun1.yukawa.kyoto-u.ac.jp
§ e-mail address: yokoyama@yisun1.yukawa.kyoto-u.ac.jp

1. Introduction

It is of great importance to formulate an expression which describes various properties of a quantum field in curved spacetime. Though we have the effective action method to characterize the behavior of the expectation value of a scalar field, yet to be found is the general expression of a probability distribution functional (PDF) that describes the statistical property of its fluctuations at a given instant.

One exception is that of a free scalar field which we know is random-phase Gaussian for a suitably chosen vacuum state, as reviewed below. Consider the Fourier mode expansion of a real scalar field¶

$$\phi(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} q_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{1.1}$$

where the hermiticity requires $q_{-\boldsymbol{k}}(t) = q^*_{\boldsymbol{k}}(t)$. From Wick's theorem and the momentum conservation, we see that

$$\langle 0|\, q_{\boldsymbol{k}_1}(t) \cdots q_{\boldsymbol{k}_n}(t)\, |0\rangle = 0, \tag{1.2}$$

provided all $\boldsymbol{k}_i$ are different. This implies $q_{\boldsymbol{k}}(t)$ with a different $\boldsymbol{k}$ can be regarded as an independent probability variable if we identify the vacuum expectation value with an expectation value in a random process. To find the probability distribution functional (PDF), it is sufficient to consider a specific mode $\boldsymbol{k}$:

$$\langle 0|\, q_{\boldsymbol{k}}(t)^n q_{-\boldsymbol{k}}(t)^m\, |0\rangle = n!\, \langle 0|\, q_{\boldsymbol{k}}(t) q_{-\boldsymbol{k}}(t)\, |0\rangle^n\, \delta_{m,n}. \tag{1.3}$$

Comparing it with the following Gaussian integration formula,

$$\frac{\int_0^\infty dr \int_0^{2\pi} r d\theta\, e^{-a^2r^2} \left(re^{i\theta}\right)^n \left(re^{-i\theta}\right)^m}{\int_0^\infty dr \int_0^{2\pi} r d\theta\, e^{-a^2r^2}} = \frac{n!}{a^{2n}} \delta_{n,m}, \tag{1.4}$$

¶ Throughout this paper, we only consider a spatially flat homogeneous isotropic space. The extension to a general spacetime is straightforward in principle, but actual manipulations will be much more complicated. We also assume $\langle 0|\, \phi(x)\, |0\rangle = 0$.

we find the PDF,

$$P\left[\{q_{\boldsymbol{k}}\};t\right] = \frac{\exp\left[-\frac{1}{2}\int d^3k \, \langle 0|\, q_{\boldsymbol{k}}(t) q_{-\boldsymbol{k}}(t)\, |0\rangle^{-1} \delta^{(3)}(0)\, |q_{\boldsymbol{k}}|^2\right]}{\int \prod_{\boldsymbol{k}} dq_{\boldsymbol{k}} \exp\left[-\frac{1}{2}\int d^3k \, \langle 0|\, q_{\boldsymbol{k}}(t) q_{-\boldsymbol{k}}(t)\, |0\rangle^{-1} \delta^{(3)}(0)\, |q_{\boldsymbol{k}}|^2\right]}, \tag{1.5}$$

reproduces all the spatial correlation functions at time t. As an application of this property of free theory to cosmology, see *e.g.*, [1].

The above demonstration for free theory is simple but it is unclear how such a statistical interpretation can be generalized for interacting field theories. On the other hand, for sensible quantum field theory, we require the micro-causality,

$$[\phi(x), \phi(y)] = 0, \tag{1.6}$$

for spatially separated points x and y. Therefore $\phi(x)$ behaves as a c-number on a spatial hypersurface and the statistical interpretation of the spatial correlation functions should also be possible in interacting field theories. But the general framework to obtain PDF has not appeared to our knowledge.

In this paper, we derive a general expression of PDF for interacting quantum field theory, that can in principle be evaluated by perturbation. The rest of the paper is organized as follows. In the next section, we describe how to obtain PDF whose expectation value reproduces spatial correlation functions in quantum field theory. We utilize the closed-time-path functional formalism, to incorporate the correct definition of the vacuum expectation value in curved space. In section 3, as an example, we calculate PDF for the minimally coupled massive $\lambda\phi^4$-theory, up to the first order of the coupling constant. There we find PDF has an ultraviolet divergence, even after the conventional ultraviolet renormalization. We see that this divergence is unavoidable to reproduce finite expectation values. The closed analytic form of PDF in Minkowski space is also obtained. In section 4, PDF in de Sitter space is considered. In evaluating PDF, we encounter the intriguing fact that perturbative expansion breaks down for a certain range of the mass. As an application of PDF, a scaling law among multi-point correlation functions in the de Sitter space is found. The last section is devoted to conclusion.

2. Formulation

Let us start with the formal definition of PDF,*

$$P[\varphi(\cdot);t] := \langle 0_-|\prod_{\boldsymbol{x}} \delta\left(\phi(\boldsymbol{x},t)-\varphi(\boldsymbol{x})\right)|0_-\rangle\,, \tag{2.1}$$

where $\phi(\boldsymbol{x},t)$ is a Heisenberg field operator, $\varphi(\boldsymbol{x})$ is a c-number field configuration that plays the role of the probability variable, and $|0_-\rangle$ is the vacuum state at $t=-\infty$. Throughout this paper a convention $x=(t,\boldsymbol{x})$, $y=(t',\boldsymbol{y})$ *etc.* is understood. The PDF (2.1) has a desired property,

$$\int\prod_{\boldsymbol{x}} d\varphi(\boldsymbol{x})\, P[\varphi(\cdot);t]\,\varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_n) = \langle 0_-|\,\phi(\boldsymbol{x}_1,t)\cdots\phi(\boldsymbol{x}_n,t)\,|0_-\rangle\,, \tag{2.2}$$

that is, it reproduces the spatial correlation function, or the vacuum expectation value of an operator product at time t. The above naive picture, however, requires a more concrete definition, because it contains a product of field operators at an identical time t. We make the definition (2.1) more precise by introducing a source field $J(\boldsymbol{x})$ as

$$\begin{aligned} &P[\varphi(\cdot);t]\\ &=\int\prod_{\boldsymbol{x}'}\frac{dJ(\boldsymbol{x}')}{2\pi}\langle 0_-|\exp\left[i\int d^3x' J(\boldsymbol{x}')\phi(\boldsymbol{x}',t)\right]|0_-\rangle\exp\left[-i\int d^3x' J(\boldsymbol{x}')\varphi(\boldsymbol{x}')\right]\\ &=\int\prod_{\boldsymbol{x}'}\frac{dJ(\boldsymbol{x}')}{2\pi}\lim_{J(x')\to J(\boldsymbol{x}')\delta(t'-t)}\langle 0_-|\,Te^{iJ\phi}\,|0_-\rangle\, e^{-iJ\varphi}, \end{aligned} \tag{2.3}$$

where $J\phi$ and $J\varphi$ imply, respectively,

$$J\phi := \int d^4x'\, J(x')\phi(x'),\quad J\varphi := \int d^3x'\, J(\boldsymbol{x}')\varphi(\boldsymbol{x}'). \tag{2.4}$$

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* Instead of this, one may define PDF by using the wave functional in the Schrödinger picture; $P[\varphi(\cdot);t] := |\Psi[\varphi(\cdot);t]|^2$ where $\Psi[\varphi(\cdot);t]$ is a normalized solution of the time-dependent Schrödinger equation with a suitable initial condition. The reality of PDF is manifest in this representation and it could be evaluated by perturbation. In fact, in the case of Minkowski spacetime for which the vacuum wave functional in the free theory and the perturbations are rather trivial, we have checked this representation reproduces our result given in § 3. In general curved spacetimes, however, it is hardly possible to solve the Schrödinger equation and our approach is more appropriate.

From the definition (2.3), we find

$$\int \prod_{\boldsymbol{x}} d\varphi(\boldsymbol{x})\, P\left[\varphi(\cdot);t\right] \varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_n) = \lim_{t_i\to t} \langle 0_- |\, T\phi(x_1,)\cdots\phi(x_n)\, |0_-\rangle\,. \qquad (2.5)$$

To do the perturbative expansion of (2.3), the path integral representation of the matrix element is useful. However, here we encounter a subtle point: The conventional path integral formalism over time t from $-\infty$ to $+\infty$, which could be made well-defined by the Wick rotation or the $-i\varepsilon$ prescription, results in the transition amplitude between in-vacuum $|0_-\rangle$ and out-vacuum $|0_+\rangle$ at $t = +\infty$, such as$\langle 0_+ |\, Te^{iJ\phi}\, |0_-\rangle$ .In the Minkowski space, fortunately, the in- and out-vacua are identical to each other. In curved space, however, the in-vacuum defined at $t = -\infty$ is generally different from the out-vacuum defined at $t = +\infty$ and they are related to each other by a non-trivial Bogoliubov transformation [2]. Since we are interested in the vacuum expectation value with respect to a specific vacuum, *e.g.*, $|0_-\rangle$, rather than a transition amplitude, we make use of a path integral with a different boundary condition, namely, closed-time-path (CTP) formalism [3] in which the reality of PDF is manifest. For space with higher symmetry, (*e.g.*, Minkowski space with the Poincare group and de Sitter space with the de Sitter group), the vacuum that respects the symmetry may be uniquely singled out. For such a situation, the conventional in-out path integral formalism and the CTP formalism should give the same answer.

According to CTP formalism [3], the matrix element in (2.3) is given by

$$\langle 0_- |\, Te^{iJ\phi}\, |0_-\rangle = Z\left[J,0\right], \qquad (2.6)$$

where the partition function Z is defined by a "closed-time-path" functional integral

$$Z\left[J^+,J^-\right] = \frac{\int \mathcal{D}\phi^+\mathcal{D}\phi^- \exp i\left[S[\phi^+] + J^+\phi^+ - S^*[\phi^-] - J^-\phi^-\right]}{\int \mathcal{D}\phi^+\mathcal{D}\phi^- \exp i\left[S[\phi^+] - S^*[\phi^-]\right]}. \qquad (2.7)$$

Here the path integral of ϕ^+ and ϕ^- is taken over all field configurations that coincide at $t = t^*$. (In practice, one may take $t^* = +\infty$.) The path integral of ϕ^+ proceeds from $t = -\infty$ to $t = t^*$ and that of ϕ^- proceeds backward in time, from $t = t^*$ to $t = -\infty$. It is important to realize that ϕ^+ and ϕ^- are *not* independent fields, because of the boundary condition at $t = t^*$. In (2.7), functional derivatives with respect to J^+ produce the vacuum expectation value of a time-ordered product, while those with respect to J^- an anti-time-ordered product. By construction, ϕ^- always stands on the left of ϕ^+ in expectation values.

It is straightforward to derive a perturbative expansion formula as in the conventional path integral [3]. After taking the limit, $J(x') \to J(\boldsymbol{x}')\delta(t'-t)$ in (2.3), we find

$$
\begin{aligned}
&\lim_{J(x')\to J(\boldsymbol{x}')\delta(t'-t)} Z[J,0] \\
&= \exp\left[\frac{1}{2}\int d^3x d^3y\, iJ(\boldsymbol{x})\Delta_F(\boldsymbol{x}-\boldsymbol{y};t)\, iJ(\boldsymbol{y})\right] \\
&\quad\times \exp\left\{\int d^3x\, iJ(\boldsymbol{x})\left[\Delta_F\cdot\frac{\delta}{\delta\phi^+}+\frac{\delta}{\delta\phi^-}\cdot\Delta\right](\boldsymbol{x},t)\right\} \\
&\quad\times \exp\left\{\frac{1}{2}\frac{\delta}{\delta\phi^+}\cdot\Delta_F\cdot\frac{\delta}{\delta\phi^+}+\frac{\delta}{\delta\phi^-}\cdot\Delta\cdot\frac{\delta}{\delta\phi^+}+\frac{1}{2}\frac{\delta}{\delta\phi^-}\cdot\Delta_D\cdot\frac{\delta}{\delta\phi^-}\right\} \\
&\quad\times \exp i\left\{\int d^4x\,\mathcal{L}_{\rm int}(\phi^+)-\int d^4x\,\mathcal{L}^*_{\rm int}(\phi^-)\right\}\Bigg|_{\phi^+=\phi^-=0},
\end{aligned}
\tag{2.8}
$$

where abbreviations like

$$
\Delta\cdot\frac{\delta}{\delta\phi^+}(\boldsymbol{x},t) := \int d^4y\,\Delta(x,y)\frac{\delta}{\delta\phi^+(y)}
\tag{2.9}
$$

are understood and Δ_F, Δ, and Δ_D are respectively defined by

$$
\begin{aligned}
\Delta_F(x,y) &:= \langle 0_-|\,T\phi(x)\phi(y)\,|0_-\rangle\,,\\
\Delta(x,y) &:= \langle 0_-|\,\phi(x)\phi(y)\,|0_-\rangle\,,\\
\Delta_D(x,y) &:= \langle 0_-|\,\widetilde{T}\phi(x)\phi(y)\,|0_-\rangle\,,
\end{aligned}
\tag{2.10}
$$

with $\widetilde{T}$ being the anti-time ordering. Note that $\Delta^*(x,y) = \Delta(y,x)$. The integration over $J(\boldsymbol{x}')$ in (2.3) now becomes a Gaussian one. We obtain the final expression:

$$
\begin{aligned}
& P\left[\varphi(\cdot);t\right] \\
& \propto \exp\left[-\frac{1}{2}\int d^3x d^3y\, \varphi(\boldsymbol{x})\Delta_F^{-1}(\boldsymbol{x}-\boldsymbol{y};t)\varphi(\boldsymbol{y})\right] \\
& \quad \times \exp\left\{\int d^3x d^3y\, \varphi(\boldsymbol{x})\Delta_F^{-1}(\boldsymbol{x}-\boldsymbol{y};t)\right. \\
& \qquad\qquad \left. \times \left[\Delta_F \cdot \frac{\delta}{\delta\phi^+} + \frac{\delta}{\delta\phi^-}\cdot\Delta\right](\boldsymbol{y},t)\right\} \\
& \quad \times \exp\left\{\int d^3x d^3y \left[\Delta_F \cdot \frac{\delta}{\delta\phi^+} + \frac{\delta}{\delta\phi^-}\cdot\Delta\right](\boldsymbol{x},t)\Delta_F^{-1}(\boldsymbol{x}-\boldsymbol{y};t)\right. \\
& \qquad\qquad \left. \times \left[\Delta_F \cdot \frac{\delta}{\delta\phi^+} + \frac{\delta}{\delta\phi^-}\cdot\Delta\right](\boldsymbol{y},t)\right\} \\
& \quad \times \exp\left\{\frac{1}{2}\frac{\delta}{\delta\phi^+}\cdot\Delta_F\cdot\frac{\delta}{\delta\phi^+} + \frac{\delta}{\delta\phi^-}\cdot\Delta\cdot\frac{\delta}{\delta\phi^+} + \frac{1}{2}\frac{\delta}{\delta\phi^-}\cdot\Delta_D\cdot\frac{\delta}{\delta\phi^-}\right\} \\
& \qquad\qquad \times \exp i\left\{\int d^4x\, \mathcal{L}_{\rm int}(\phi^+) - \int d^4x\, \mathcal{L}^*_{\rm int}(\phi^-)\right\}\Bigg|_{\phi^+=\phi^-=0},
\end{aligned}
\tag{2.11}
$$

where $\Delta_F(\boldsymbol{x}-\boldsymbol{y};t) := \lim_{t'\to t}\Delta_F(x,y)$ and the inverse of it, Δ_F^{-1}, is defined by

$$
\int d^3z\, \Delta_F(\boldsymbol{x}-\boldsymbol{z};t)\Delta_F^{-1}(\boldsymbol{z}-\boldsymbol{y};t) = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}). \tag{2.12}
$$

In the free field theory, only the first line in the right-hand-side of (2.11) survives as in (1.5). From (2.11), we can read off a general rule to evaluate PDF: It is obtained by connecting $\phi(\boldsymbol{x})$ and the spatial correlation functions using Δ_F^{-1} in all possible ways.

In the next section, we apply the above formulation to the minimally coupled massive $\lambda\phi^4$-theory, up to the first order of the coupling constant. We will also obtain the closed analytic form of PDF in Minkowski space.

3. PDF for $\lambda\phi^4$-theory

The minimally coupled massive $\lambda\phi^4$-theory is defined by the action

$$S[\phi] = -\int d^4x \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right). \tag{3.1}$$

The free field operator is decomposed as

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[\hat{a}_{\boldsymbol{k}}\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{k}}\phi^*_{\boldsymbol{k}}(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{3.2}$$

where the mode function $\phi_{\boldsymbol{k}}(t)$ is determined by solving the field equation,

$$\left(-\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu + m^2\right)\phi(x) = 0, \tag{3.3}$$

with a suitable positive frequency condition. In terms of the mode function $\phi_{\boldsymbol{k}}(t)$, two-point functions in (2.10) are expressed as

$$\begin{aligned}\Delta_F(x,y) &= \Delta^*_D(x,y)\\ &= \int\frac{d^3k}{(2\pi)^3}\left[\theta(t-t')\phi_{\boldsymbol{k}}(t)\phi^*_{\boldsymbol{k}}(t') + \theta(t'-t)\phi_{\boldsymbol{k}}(t')\phi^*_{\boldsymbol{k}}(t)\right]e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})},\\ \Delta(x,y) &= \int\frac{d^3k}{(2\pi)^3}\phi_{\boldsymbol{k}}(t)\phi^*_{\boldsymbol{k}}(t')e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}.\end{aligned} \tag{3.4}$$

In particular, the spatial two-point function and its inverse are given, respectively, by

$$\begin{aligned}\Delta_F(\boldsymbol{x}-\boldsymbol{y};t) &= \int\frac{d^3k}{(2\pi)^3}|\phi_{\boldsymbol{k}}(t)|^2e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})},\\ \Delta_F^{-1}(\boldsymbol{x}-\boldsymbol{y};t) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{|\phi_{\boldsymbol{k}}(t)|^2}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}.\end{aligned} \tag{3.5}$$

Applying the perturbation formula (2.11) to the present model, we find

$$
\begin{aligned}
P[\varphi(\cdot);t] &\propto \exp\left[-\frac{1}{2}\int d^3x d^3y\,\varphi(\boldsymbol{x})\Delta_F^{-1}(\boldsymbol{x}-\boldsymbol{y};t)\varphi(\boldsymbol{y})\right]\\
&\times\Bigg\{1+i\Bigg[-\frac{\lambda}{4}\int d^3x_1\cdots d^3x_4 d^4y\,\sqrt{-g(y)}\Delta_F(y,y)\Delta_F(y,x_1)\Delta_F(y,x_2)\\
&\qquad\qquad\times\Delta_F^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_3;t)\Delta_F^{-1}(\boldsymbol{x}_2-\boldsymbol{x}_4;t)\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\\
&\quad-\frac{\delta m^2}{2}\int d^3x_1\cdots d^3x_4 d^4y\,\sqrt{-g(y)}\Delta_F(y,x_1)\Delta_F(y,x_2)\\
&\qquad\qquad\times\Delta_F^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_3;t)\Delta_F^{-1}(\boldsymbol{x}_2-\boldsymbol{x}_4;t)\varphi(\boldsymbol{x}_3)\varphi(\boldsymbol{x}_4)\\
&\quad+\frac{\lambda}{4}\int d^3x_1\cdots d^3x_6 d^4y\,\sqrt{-g(y)}\Delta_F(y,x_1)\Delta_F(y,x_2)\Delta_F(y,x_3)\Delta_F(y,x_4)\\
&\qquad\qquad\times\Delta_F^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_2;t)\Delta_F^{-1}(\boldsymbol{x}_3-\boldsymbol{x}_5;t)\Delta_F^{-1}(\boldsymbol{x}_4-\boldsymbol{x}_6;t)\varphi(\boldsymbol{x}_5)\varphi(\boldsymbol{x}_6)\\
&\quad-\frac{\lambda}{24}\int d^3x_1\cdots d^3x_8 d^4y\,\sqrt{-g(y)}\Delta_F(y,x_1)\Delta_F(y,x_2)\Delta_F(y,x_3)\Delta_F(y,x_4)\\
&\qquad\times\Delta_F^{-1}(\boldsymbol{x}_1-\boldsymbol{x}_5;t)\Delta_F^{-1}(\boldsymbol{x}_2-\boldsymbol{x}_6;t)\Delta_F^{-1}(\boldsymbol{x}_3-\boldsymbol{x}_7;t)\Delta_F^{-1}(\boldsymbol{x}_4-\boldsymbol{x}_8;t)\\
&\qquad\times\varphi(\boldsymbol{x}_5)\varphi(\boldsymbol{x}_6)\varphi(\boldsymbol{x}_7)\varphi(\boldsymbol{x}_8)\Bigg]\\
&\quad-i\left[\Delta_F\to\Delta\right]+O(\lambda^2)\Bigg\}.
\end{aligned}
\tag{3.6}
$$

A diagrammatic representation might be helpful to show the general structure of the perturbation series in (2.11) and (3.6). In Fig. 1, we have depicted diagrams corresponding to each term of (3.6). We have omitted the contribution of the first diagram in Fig. 1, since it is a constant independent of $\varphi(\boldsymbol{x})$, which can be removed by the over-all normalization. The inverse of the spatial two-point function Δ_F^{-1} in (3.5) is represented by the broken line and, the four dimensional propagator, that is either Δ_F or Δ depending on the vertex, by the solid line.

In evaluating (3.6), we have introduced a mass counter term ($m^2 \equiv m_R^2 + \delta m^2$) to compensate the ultraviolet divergence arising from the second diagram in Fig. 1. The divergent quantity $\Delta_F(y,y) = \Delta(y,y)$ is independent of the spatial coordinate $\boldsymbol{y}$ due to the translational invariance. If $\Delta_F(y,y) = \Delta(y,y)$ does not depend on time either, as in Minkowski and de Sitter spacetimes, which we assume, the divergence can be renormalized by a constant mass counter term δm^2. The cross in Fig. 1 implies the counter term δm^2. We will so choose the counter term that it completely cancels the divergence in the second diagram, *i.e.*, on-mass shell renormalization.

In the perturbation series in Fig. 1, we find a new type of diagram that has no analogue in the conventional calculation of Green's functions, namely, a loop closed by a broken line (the fourth diagram). We will see below this diagram has ultraviolet divergence. Note that we have already done renormalization of the conventional ultraviolet divergence that arises from the loops closed by a solid line (the second diagram). This new kind of ultraviolet divergence in PDF, which is present even after the conventional renormalization, however, turns out to be *essential* to obtain *finite* expectation values.

It is often more convenient to express PDF in terms of the Fourier modes $q_{\boldsymbol{k}}$ defined by

$$q_{\boldsymbol{k}} := \int \frac{d^3x}{(2\pi)^{3/2}}\, \varphi(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.7}$$

Using (3.4) and (3.5), PDF in terms of $q_{\boldsymbol{k}}$ is expressed as

$$\begin{aligned}
&P\left[\{q_{\boldsymbol{k}}\};t\right] \\
&\propto \exp\left[-\frac{1}{2}\int d^3k\, \frac{q_{\boldsymbol{k}}q_{-\boldsymbol{k}}}{|\phi_{\boldsymbol{k}}(t)|^2}\right] \\
&\qquad \times \left\{1 + \lambda\Big[\frac{1}{4}\int d^3k\, p_2(\boldsymbol{k};t)\, q_{\boldsymbol{k}}q_{-\boldsymbol{k}} \right. \\
&\qquad\qquad \left. + \frac{1}{24}\int d^3k_1\, d^3k_2\, d^3k_3\, p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;t)\, q_{\boldsymbol{k}_1}q_{\boldsymbol{k}_2}q_{\boldsymbol{k}_3}q_{-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3}\Big] + O(\lambda^2)\right\} \\
&=: \widetilde{P}\left[\{q_{\boldsymbol{k}}\};t\right],
\end{aligned} \tag{3.8}$$

where the functions p_2 and p_4 are given by

$$\begin{aligned}
p_2(\boldsymbol{k};t) &= i\int_t^{\infty} dt'\, I_2(t,t') + i\int_{-\infty}^{t} dt'\, I_2^*(t,t') - i\int_{-\infty}^{\infty} dt'\, I_2(t,t') \\
&= -2\mathrm{Re}\left[i\int_{-\infty}^{t} dt'\, I_2(t,t')\right],
\end{aligned} \tag{3.9}$$

with

$$I_2(t,t') = \sqrt{-g(t')} \int \frac{d^3k'}{(2\pi)^3} \frac{\phi_{k'}(t')^2 \phi^*_{k'}(t)}{\phi_{k'}(t)} \left[\frac{\phi_k(t')}{\phi_k(t)}\right]^2, \tag{3.10}$$

and,

$$\begin{aligned} p_4(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3; t) &= -i \int_t^\infty dt'\, I_4(t,t') - i \int_{-\infty}^t dt'\, I_4^*(t,t') + i \int_{-\infty}^\infty dt'\, I_4(t,t') \\ &= 2\mathrm{Re}\left[i \int_{-\infty}^t dt'\, I_4(t,t') \right], \end{aligned} \tag{3.11}$$

with

$$I_4(t,t') = \sqrt{-g(t')} \frac{\phi_{k_1}(t')\phi_{k_2}(t')\phi_{k_3}(t')\phi_{-k_1-k_2-k_3}(t')}{(2\pi)^3 \phi_{k_1}(t)\phi_{k_2}(t)\phi_{k_3}(t)\phi_{-k_1-k_2-k_3}(t)}. \tag{3.12}$$

In the above expressions, p_2 and p_4 are manifestly real. In the first lines of (3.9) and (3.11), the first two terms in the right hand side are the contributions of Δ_F in (3.6) and, the last term comes from Δ in (3.6). If we started from the conventional in-out formalism instead of the CTP formalism, only the first two terms would appear and hence the reality would not be guaranteed in general. Even for Minkowski space for which both formalisms should give the same answer, the reality therefore is not transparent in the in-out formalism.

For free field theory, $\lambda = 0$, (3.8) reproduces the random-phase Gaussian distribution (1.5), since the two-point function in the Fourier space is given by $\langle 0_- |\, q_{\boldsymbol{k}}(t) q_{-\boldsymbol{k}}(t)\, | 0_- \rangle = |\phi_{\boldsymbol{k}}(t)|^2 \delta^{(3)}(0)$.

At first glance, one may suspect if the term involving p_2 can be absorbed to the free part by redefining the mode function $\phi_{\boldsymbol{k}}(t)$. However, this term turns out to be necessary to obtain correct expectation values. To see the role of p_2, let us evaluate the two-point function $\langle 0_- |\, q_{\boldsymbol{k}_1}(t) q_{\boldsymbol{k}_2}(t)\, | 0_- \rangle$ by using the first order formula (3.8) with (3.9) and (3.11). Since we have already done the renormalization in $O(\lambda)$, the consistency demands that the two-point function should be given by its bare form $|\phi_{\boldsymbol{k}_1}(t)|^2 \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2)$ when expressed in terms of the renormalized mass. By using the fact that a product of $q_{\boldsymbol{k}}$ is factorized to that of pairs $q_{\boldsymbol{k}}q_{-\boldsymbol{k}}$ in the Gaussian integration, we first integrate the right-hand-side of (3.8) over $\{q_{\boldsymbol{k}}\}$,

$$\begin{aligned}
&\int \prod_{\boldsymbol{k}} dq_{\boldsymbol{k}}\, \widetilde{P}\left[\{q_{\boldsymbol{k}}\};t\right] \\
&= \left[\int \prod_{\boldsymbol{k}} dq_{\boldsymbol{k}} \exp\left\{-\frac{1}{2}\int d^3k \frac{q_{\boldsymbol{k}}q_{-\boldsymbol{k}}}{|\phi_{\boldsymbol{k}}(t)|^2}\right\}\right]\left[1+\frac{\lambda}{8}\int d^3k\, p_2(\boldsymbol{k};t)|\phi_{\boldsymbol{k}}(t)|^2\delta^{(3)}(0)\right], \\
&=: N,
\end{aligned} \tag{3.13}$$

to obtain the correctly normalized PDF:

$$P\left[\{q_{\boldsymbol{k}}\};t\right] = \frac{1}{N}\widetilde{P}\left[\{q_{\boldsymbol{k}}\};t\right]. \tag{3.14}$$

Then we find

$$\int \prod_{\boldsymbol{k}} dq_{\boldsymbol{k}}\, P\left[\{q_{\boldsymbol{k}}\};t\right] q_{\boldsymbol{k}_1}q_{\boldsymbol{k}_2} = |\phi_{\boldsymbol{k}_1}(t)|^2\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2), \tag{3.15}$$

as desired. Thus if p_2 were eliminated by hand, it would not give the correct answer.

As we shall see shortly, or may be suspected from the general formula (3.10), p_2 has ultraviolet divergence. However, when $\{q_{\boldsymbol{k}}\}$ is integrated over in (3.15), the ultraviolet divergence in p_2 is canceled by a divergent contribution arising from integration of the p_4-part. The divergent quantity p_2 is thus necessary to reproduce the finite expectation values. Although our analysis here is limited to the first order perturbation, we expect this divergence structure is a generic feature of PDF. Thus when one writes it down, even after the conventional ultraviolet renormalization, some kind of regularization is necessary to give a sensible meaning to divergent quantities, such as p_2. This situation is quite different, say, from the case of the effective action.

As an explicit demonstration of the above results, let us examine the case in

Minkowski space. The mode function is given by

$$\phi_{\boldsymbol{k}}(t) = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} e^{-i\omega_{\boldsymbol{k}} t}, \quad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2}, \tag{3.16}$$

where and hereafter we denote the renormalized mass m_R^2 by m^2 for simplicity. The mass counter term δm^2 in (3.6) is given by

$$\delta m^2 = -\frac{\lambda}{2}\Delta_F(y,y) = -\frac{\lambda}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_{\boldsymbol{k}}}. \tag{3.17}$$

Then it is straightforward to evaluate p_2 and p_4 in (3.9) and (3.11),

$$\begin{aligned} p_2(\boldsymbol{k};t) &= \frac{1}{2}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\omega_{\boldsymbol{k}'}(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}})}, \\ p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;t) &= \frac{-2}{(2\pi)^3}\frac{1}{\omega_{\boldsymbol{k}_1}+\omega_{\boldsymbol{k}_2}+\omega_{\boldsymbol{k}_3}+\omega_{-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3}}. \end{aligned} \tag{3.18}$$

In the above, p_2 has a linear divergence as noted previously. Since p_4 is negative definite, the distribution of the mode $q_{\boldsymbol{k}}$ tends to concentrate near the origin if $\lambda > 0$. This is in accordance with an intuitive expectation that a repulsive force should suppress the amplitude of fluctuations.

Since the in-vacuum is identical to the out-vacuum in Minkowski space, the third integral in the first lines in (3.9) and (3.11), that are absent in the conventional in-out formalism, should be vanishing. In fact we find from (3.16),

$$\begin{aligned} &\int_{-\infty}^{\infty} dt' \sqrt{-g(t')}\, \phi_{\boldsymbol{k}_1}(t')\phi_{\boldsymbol{k}_2}(t')\phi_{\boldsymbol{k}_3}(t')\phi_{\boldsymbol{k}_4}(t') \\ &= \frac{2\pi}{\sqrt{2\omega_{\boldsymbol{k}_1}}\sqrt{2\omega_{\boldsymbol{k}_2}}\sqrt{2\omega_{\boldsymbol{k}_3}}\sqrt{2\omega_{\boldsymbol{k}_4}}}\delta(\omega_{\boldsymbol{k}_1}+\omega_{\boldsymbol{k}_2}+\omega_{\boldsymbol{k}_3}+\omega_{\boldsymbol{k}_4}), \end{aligned} \tag{3.19}$$

which vanishes for $m^2 > 0$. Even for $m^2 = 0$, the only contribution comes from the zero mode, $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \boldsymbol{k}_3 = \boldsymbol{k}_4 = 0$. But in (3.8), the zero mode is always accompanied with vanishing measure; *e.g.*, $d^3k_1 = k_1^2 dk_1 d\phi d\theta = 0$ for $\boldsymbol{k}_1 = 0$. Therefore the third integrals in (3.9) and (3.11) vanish as expected.

4. $\lambda\phi^4$-theory in de Sitter space

Now we consider the $\lambda\phi^4$-theory in de Sitter spacetime whose metric is given by

$$ds^2 = \frac{1}{(-H\eta)^2}(-d\eta^2 + d\boldsymbol{x}^2), \tag{4.1}$$

where H is the Hubble parameter and η is the conformal time. In Fig. 2, the Penrose diagram of de Sitter space is depicted. To define the time ordering consistent with the causal structure in Fig. 2, we introduce a time variable $t = -1/\eta$.* Then an event at time t is in causal future of a certain event at t' if $t > t'$. Therefore the time ordering with respect to the new time t is appropriate, except perhaps at $t = 0$ ($\eta = \pm\infty$) where the hypersurface becomes null. However, it turns out that this causes no trouble. In particular our formulas (3.9)-(3.12) in terms of t are valid as they are.

In de Sitter space the most natural vacuum is the so-called Euclidean vacuum, which is de Sitter invariant and for which the short distance behavior of the field is identical to that in Minkowski space [4]. Then the positive frequency mode function is given by

$$\phi_{\boldsymbol{k}}(\eta) = e^{i\theta}\frac{\sqrt{\pi}}{2}H(-\eta)^{3/2}e^{i\nu\pi/2}H_\nu^{(1)}(-k\eta), \tag{4.2}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind, $\nu = \sqrt{9/4 - m^2/H^2}$, and $k = |\boldsymbol{k}|$. For later convenience, we also introduce a new parameter $c := 3/2 - \nu$ to represent the mass. The phase factor $e^{i\theta}$ is not determined from the commutation relation but can be fixed by imposing a condition $\phi_k(-\eta) \equiv \phi_k(e^{\pi i}\eta) = \phi_k^*(\eta)$ for $\eta < 0$ [4]; then we have $e^{i\theta} = \pm e^{-i\pi/4}$. The mass counter term in (3.6) is given by

$$\delta m^2 = -\frac{\lambda}{2}\Delta_F(y,y) = -\frac{\lambda H^2}{16\pi}\int_0^\infty d\zeta\, \zeta^2 \left|H_\nu^{(1)}(\zeta)\right|^2, \tag{4.3}$$

where we have set $\zeta = k|\eta|$. Note that $\Delta_F(y,y)$ is a (divergent) constant, as it

---

* Do not confuse this t with the proper time $\tilde{t}$, by which the invariant length reads, $ds^2 = -d\tilde{t}^2 + e^{2H\tilde{t}}d\boldsymbol{x}^2$ and, $-\infty \le \tilde{t} \le \infty$ covers only the left upper-half of the maximally extended de Sitter space in Fig. 2.

should be from the de Sitter invariance [4].

All we have to do is the integrations in (3.9) and (3.11). We first note they have a common structure:

$$\int_{-\infty}^{t} dt'\, I_j(t,t') = \left(\int_{+0}^{+\infty} d\eta' + \int_{-\infty}^{\eta} d\eta' \right) I_j(\eta,\eta'), \tag{4.4}$$

where $j = 2$ or 4, and we have assumed the hypersurface with the time t is in the expanding phase of the de Sitter space, *i.e.*, $\eta < 0$. The integrand $I_j(\eta,\eta')$ contains the following common form,

$$\begin{aligned} &\left(\int_{+0}^{+\infty} d\eta' + \int_{-\infty}^{\eta} d\eta' \right) \sqrt{-g(\eta')}\phi_{k_1}(\eta')\phi_{k_2}(\eta')\phi_{k_3}(\eta')\phi_{k_4}(\eta') \\ &= e^{4i\theta} e^{2i\pi\nu} \frac{\pi^2}{16} \left(\int_{+0}^{+\infty} d\eta' + \int_{-\infty}^{\eta} d\eta' \right) (-\eta')^2 H_\nu^{(1)}(-k_1\eta') H_\nu^{(1)}(-k_2\eta') \\ &\qquad \times H_\nu^{(1)}(-k_3\eta') H_\nu^{(1)}(-k_4\eta'). \end{aligned} \tag{4.5}$$

The asymptotic behavior of the Hankel function near the origin is

$$H_\nu^{(1)}(z) \sim \frac{-i}{\sin\nu\pi} \frac{1}{\Gamma(-\nu+1)} \left(\frac{z}{2}\right)^{-\nu} \left[1 - \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)} e^{-i\nu\pi} \left(\frac{z}{2}\right)^{2\nu} \right]. \tag{4.6}$$

Therefore the integrand in (4.5) behaves as $\sim |\eta'|^{2-4\nu} = |\eta'|^{4c-4}$ near $\eta' = 0$. We see that the necessary condition for the integral (4.5) to be finite is

$$\mathrm{Re}[c] > \frac{3}{4}, \quad \text{or} \quad m^2 > \frac{27}{16} H^2. \tag{4.7}$$

Surprisingly, the convergence of the integral (4.5) imposes a lower bound to the mass parameter m^2 in the de Sitter space.

It is *not* a specious condition that appears accidentally in (4.5) but intrinsic to the $\lambda\phi^4$-theory in de Sitter space. The integration over η' in (4.5) corresponds to that over the position of the $\lambda\phi^4$-vertex. Therefore the divergence of (4.5) implies that of the tree-level spatial four-point correlation function in the configuration space, as may be observed from the structure of p_4 (the fifth diagram in Fig. 1).

This type of divergence in the spatial correlation function is not a peculiarity of the $\lambda\phi^4$-theory. In ϕ^n-theory, the condition (4.7) is generalized to $\mathrm{Re}[c] > 3/n$ or $m^2 > 9(1-1/n)H^2/n$ [6]. Though ϕ^n-theory with $n \geq 5$ is not renormalizable, the crucial point is that the divergence exists even at the tree diagram. Note that a spatial correlation function is observable since it is a vacuum expectation value of a product of hermite operators. The divergence comes from the integration near $\eta' = 0$, which originates in the extreme expansion of the volume factor in de Sitter space so that it is a kind of infrared divergence in nature.

This lower bound was essentially aware of already by Tagirov [5] in a slightly different context. However, since he included the conformal coupling $-R\phi^2/6$ in the Lagrangian from the beginning, the condition (4.7) itself had been hidden behind it. In de Sitter space, the curvature coupling $-\xi R\phi^2$ is effectively absorbed in a change of the mass parameter, $m^2 \to m^2 + 12\xi H^2$. Thus our mass parameter m^2 is related to that in [5], M^2, as $m^2 = 2H^2 + M^2$. For $\lambda\phi^4$-theory, the condition (4.7) is thus always satisfied for $M^2 \geq 0$. For $\lambda\phi^3$-theory, on the other hand, the divergence appears at $M^2 = 0$ as was pointed out by Tagirov. Incidentally, another of his statement that a similar divergence appears in $\lambda\phi^3$-theory irrespective of the mass is incorrect [6]. It seems that this divergence has not been received much attention in the literature. The detailed discussion on this divergence will be given elsewhere [6]. In the present paper, we consider only the case that satisfies (4.7).

Let us return to the calculation of p_2 and p_4. We first show that it is possible to convert the integration region in (4.4) as

$$\int_{-\infty}^{t} dt'\, I_j(t,t') = -\int_{t}^{+\infty} dt'\, I_j(t,t'), \tag{4.8}$$

or

$$\left(\int_{+0}^{+\infty} d\eta' + \int_{-\infty}^{\eta} d\eta'\right) I_j(\eta,\eta') = -\int_{\eta}^{-0} d\eta'\, I_j(\eta,\eta'), \tag{4.9}$$

which is equivalent to the following equality.

$$\int_{-\infty}^{+\infty} dt'\, I_j(t,t') = \int_{-\infty}^{+\infty} d\eta'\, I_j(\eta,\eta') = 0. \tag{4.10}$$

For this being the case, it is sufficient to show that

$$\int_{-\infty}^{+\infty} d\eta'\, (-\eta')^2 H_\nu^{(1)}(-k_1\eta') H_\nu^{(1)}(-k_2\eta') H_\nu^{(1)}(-k_3\eta') H_\nu^{(1)}(-k_4\eta') = 0. \tag{4.11}$$

Let us consider an integration contour (C_1+C_2) in Fig. 3, instead of the integration on the real axis (C_1) in (4.11). Since $H_\nu^{(1)}(z)$ is analytic in the upper half plane of z, there is no pole singularity within the integration contour for $\mathrm{Re}[c] > 3/4$. Hence the integral (4.11) is equal to

$$\begin{aligned} &- \frac{4e^{2ci\pi}}{\pi^2\sqrt{k_1}\sqrt{k_2}\sqrt{k_3}\sqrt{k_4}} \int_{C_2} d\eta'\, e^{-i(k_1+k_2+k_3+k_4)\eta'} \\ &= \frac{8e^{2ci\pi}}{\pi\sqrt{k_1}\sqrt{k_2}\sqrt{k_3}\sqrt{k_4}} \delta(k_1+k_2+k_3+k_4), \end{aligned} \tag{4.12}$$

where we have used the asymptotic form of the Hankel function at $|z| \to \infty$,

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i[z-(2\nu+1)\pi/4]}. \tag{4.13}$$

From (4.12), we can repeat the same argument as in Minkowski case. Thus (4.10) is established.

Actually the relation (4.10) is expected. In de Sitter space, the in- and out-vacua are the same due to the de Sitter invariance. The conventional formalism and the CTP formalism thus should give the same answer and this requires the third integrals in (3.9) and (3.11) to vanish. Then from (4.9), p_2 and p_4 read

$$\begin{aligned} p_2(\boldsymbol{k};\eta) &= 2\mathrm{Re}\left[i\int_{\eta}^{-0} d\eta'\, I_2(\eta,\eta') \right], \\ p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta) &= -2\mathrm{Re}\left[i\int_{\eta}^{-0} d\eta'\, I_4(\eta,\eta') \right]. \end{aligned} \tag{4.14}$$

Now for $c = 1$ or $m^2 = 2H^2$ (*i.e.*, $\nu = 1/2$), it is essentially equivalent to a conformally coupled massless field; $m^2 = 0$, $\xi = 1/6$. Hence the mode function (4.2) is conformal to that in Minkowski space;

$$\phi_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}(-H\eta), \tag{4.15}$$

and p_2 and p_4 (4.14) can be evaluated analytically. Using (4.15), we find

$$\begin{aligned} p_2(\boldsymbol{k};\eta) &= \frac{1}{2}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{k'(k'+k)}\left\{1-\cos\left[2(k'+k)(-\eta)\right]\right\}(-H\eta)^{-2}, \\ p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta) &= \frac{-2}{(2\pi)^3}\frac{1}{k_1+k_2+k_3+k_4} \\ &\quad\times\left\{1-\cos\left[(k_1+k_2+k_3+k_4)(-\eta)\right]\right\}(-H\eta)^{-4}, \end{aligned} \tag{4.16}$$

where $k_4 := |\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3|$. A notable fact is that p_2 and p_4 in this case are *not* conformal to those in Minkowski space even though the field itself is locally conformal invariant. The difference is due to the cosine factors in the above expressions, without which they would be conformal to (3.18) with $m^2 = 0$. The origin of the cosine factors is the presence of the integration boundary $\eta' = -0$ in (4.14), thus they originate in the global structure of de Sitter space which differs from that of Minkowski space.

For $c \neq 1$, we have been unable to obtain the exact analytic form of p_2 and p_4. However, for wavenumbers which satisfy $-k\eta \ll 1$, they can be evaluated by using the asymptotic form of the Hankel function (4.6) at $z = 0$. We obtain for

$3/4 < c < 3/2$, or $27H^2/16 < m^2 < 9H^2/4$,

$$
\begin{aligned}
&p_2(\boldsymbol{k};\eta) \\
&\quad\simeq \frac{-\pi}{H^2}\frac{1}{\cos c\pi}\frac{1}{\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(\frac{5}{2}-c\right)}\left(\frac{1}{4c-3}-\frac{1}{2c}\right)\int\frac{d^3k'}{(2\pi)^3}\left[1+\left(\frac{k}{k'}\right)^{3-2c}\right], \\
&p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta) \simeq \frac{2}{(2\pi)^3H^4}\cos c\pi\frac{\Gamma\left(c-\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}-c\right)}\left(\frac{1}{4c-3}-\frac{1}{2c}\right)(-\eta)^{-2c} \\
&\qquad\times\left[\left(\frac{k_1}{2}\right)^{3-2c}+\left(\frac{k_2}{2}\right)^{3-2c}+\left(\frac{k_3}{2}\right)^{3-2c}+\left(\frac{k_4}{2}\right)^{3-2c}\right].
\end{aligned}
\tag{4.17}
$$

Note that $p_2 > 0$ while $p_4 < 0$ also in this parameter region. As discussed in detail in the middle of this section, PDF is not defined perturbatively for $c \leq 3/4$ due to the divergence at $\eta' = 0$. Reflecting it, our results (4.17) blow up at $c = 3/4$ when approaching from above.

For $m^2 > 9H^2/4$ or $\nu = 3/2 - c =: ib$ with a positive b, the asymptotic form of PDF could be evaluated similarly, but it is too complicated to be illustrative. Only for the infinitely large mass limit $m^2 \gg H^2$, or $b \gg 1$, we have a simplified form,

$$
\begin{aligned}
p_2(\boldsymbol{k};\eta) &\simeq \frac{\pi}{H^2}\frac{16\pi b^2 e^{-\pi b}}{16b^2+9}\int\frac{d^3k'}{(2\pi)^3}\cdot 1, \\
p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta) &\simeq \frac{-1}{(2\pi)^3H^4}\frac{8b}{16b^2+9}(-\eta)^{-3}.
\end{aligned}
\tag{4.18}
$$

Thus p_2 and p_4 become independent of the wavenumber. Further, as $b \to \infty$, p_2 and p_4 vanish, *i.e.*, the correction due to the interaction disappears. This may be expected because in this limit the mass term dominates the action.

Our PDF gives the spatial correlation functions as the expectation values of a random process. Therefore if PDF has some (approximate) symmetries, we could argue a general (approximate) property of the spatial correlations not restricted to a particular n-point function. This is an advantage to consider PDF instead of the correlation functions themselves.

As an example, we consider a scaling law of the correlation functions and its violation due to the interaction. We first consider the case of $m^2 < 9H^2/4$

($c < 3/2$), for which the infrared behavior of the mode function, $\phi_{\boldsymbol{k}}(\eta)$ at $-k\eta \ll 1$, reads

$$|\phi_{\boldsymbol{k}}(\eta)|^2 \sim \frac{\pi}{4} H^2(-\eta)^3 \frac{1}{\cos^2 c\pi \Gamma(c-\frac{1}{2})^2} \left(\frac{-k\eta}{2}\right)^{2c-3}. \tag{4.19}$$

Therefore, in the case of the free field theory, $\lambda = 0$, PDF (3.8) for $-k\eta \ll 1$ is invariant under a substitution, $q_{\boldsymbol{k}} \to s^{c-3} q_{\boldsymbol{k}/s}$, *i.e.*, $P[\{s^{c-3}q_{\boldsymbol{k}/s}\};\eta] \simeq P[\{q_{\boldsymbol{k}}\};\eta]$. From this scale invariance, we obtain a corresponding "Ward identity";

$$\langle 0_- | \, q_{s\boldsymbol{k}_1} \cdots q_{s\boldsymbol{k}_n} \, |0_-\rangle \simeq s^{(c-3)n} \langle 0_- | \, q_{\boldsymbol{k}_1} \cdots q_{\boldsymbol{k}_n} \, |0_-\rangle \,, \tag{4.20}$$

for $-k_i\eta \ll 1$, or in terms of the configuration space correlation functions,

$$\langle 0_- | \, \varphi(\boldsymbol{x}_1) \cdots \varphi(\boldsymbol{x}_n) \, |0_-\rangle \simeq s^{cn} \langle 0_- | \, \varphi(s\boldsymbol{x}_1) \cdots \varphi(s\boldsymbol{x}_n) \, |0_-\rangle \,, \tag{4.21}$$

for $-|\boldsymbol{x}_i|\eta \gg 1$. For a sufficiently small c, this relation shows almost scale-invariant behavior of the spectrum.

In the presence of the interaction, p_4-part of the probability distribution in (3.8) is not invariant under the substitution $q_{\boldsymbol{k}} \to s^{c-3} q_{\boldsymbol{k}/s}$. This breaking follows from the explicit form in (4.17). Unlike the free theory, we find

$$\begin{aligned} &P[\{s^{c-3}q_{\boldsymbol{k}/s}\};\eta] \\ &\simeq P[\{q_{\boldsymbol{k}}\};\eta] \\ &\quad \times \left[1 + (s^{2c}-1)\frac{\lambda}{24}\int d^3k_1 d^3k_2 d^3k_3 \, p_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta)\, q_{\boldsymbol{k}_1} q_{\boldsymbol{k}_2} q_{\boldsymbol{k}_3} q_{-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3}\right], \end{aligned} \tag{4.22}$$

up to $O(\lambda^2)$. As a result, we have the following relation instead of (4.20),

$$\begin{aligned}
&\langle 0_-|\, q_{sk_1} \cdots q_{sk_n} \,|0_-\rangle \\
&= s^{(c-3)n} \Big[\langle 0_-|\, q_{k_1} \cdots q_{k_n} \,|0_-\rangle \\
&\quad + (s^{2c}-1)\frac{\lambda}{24} \int d^3k d^3k' d^3k''\, p_4(k, k', k''; \eta) \\
&\qquad \times \Big(\langle 0_-|\, q_{k_1} \cdots q_{k_n} q_k q_{k'} q_{k''} q_{-k-k'-k''} \,|0_-\rangle \\
&\qquad\quad - \langle 0_-|\, q_{k_1} \cdots q_{k_n} \,|0_-\rangle \langle 0_-|\, q_k q_{k'} q_{k''} q_{-k-k'-k''} \,|0_-\rangle \Big) \Big] \\
&\quad + O(\lambda^2).
\end{aligned} \tag{4.23}$$

The last two terms are the effect of the interaction. For $m^2 \geq 9H^2/4$, we do not see any apparent scaling behavior at large scale, $-k\eta \ll 1$. Only for $m^2 \gg H^2$, from (4.18), we have a relation that is identical with the above expressions (4.22) and (4.23) with $c = 3/2$. Although our consideration here relies on the explicit form of p_2 and p_4, it may be further put forward by using the general form of PDF in (2.11) and the scaling behavior of the two-point functions.

5. Conclusion

In the present paper we have formulated the probability distribution functional for equal-time spatial configurations of a scalar field $\phi(x, t)$ in curved spacetime in terms of the closed-time path functional formalism. The PDF thus obtained reproduces spatial correlation functions as the expectation values. As a specific example, we have considered $\lambda\phi^4$ self-interacting field and perturbatively calculated PDF up to the first order of λ both in Minkowski and de Sitter spacetimes. We have seen it has an ultraviolet divergence even after the conventional ultraviolet renormalization, which is unavoidable to reproduce finite expectation values.

Although our formula for PDF is admittedly complicated, it has an advantage that one can extract information of arbitrary higher-order correlation functions at one time, such as their symmetry behavior discussed in section 4.

Our approach is in principle applicable to describe the statistics of various types of primordial fluctuations predicted in inflationary cosmology [7] without any *ad hoc* assumptions. Since an inflationary phase is well-approximated by de Sitter spacetime, results presented in section 4 are relevant to it. Care must be taken, however, to the boundary of integration. The inflationary expanding universe may emerge after, say, the radiation-dominated universe. Then only the left upper-half of the maximally extended de Sitter space in Fig. 2 should be taken into account, so that the conformal time runs only over $-\infty < \eta < 0$ and the integration region in (4.5) is restricted to $\int_{-\infty}^{\eta} d\eta'$. The resulting PDF in this case will differ significantly from that in the maximally extended de Sitter space. For example, in the case of $c = 1$ which corresponds to the conformal invariant field, the cosine factors in p_2 and p_4 given in Eq. (4.16) will be absent; *i.e.*, the resulting PDF will be precisely conformal to that in Minkowski space. This is because the region $-\infty < \eta' < \eta$ is totally conformal to a region $-\infty < t' < t$ in Minkowski space, where t denotes the usual Minkowski time coordinate with an arbitrary value. Furthermore, we will have no constraint on the mass parameter such as (4.7) in this case, since we no longer have any divergence that have arisen in the integral at $\eta' = 0$. This crucial dependence of PDF on the global structure of spacetime may lead to some interesting theoretical as well as observational consequences.

Acknowledgements: We would like to thank B. Allen, H. Ishihara, M. Morikawa and B. Spokoiny for discussions. This work was supported in part by the Grant-in Aid for Scientific Research from the Ministry of Education, Science, and Culture No. 02640231.

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FIGURE CAPTIONS

Fig. 1 Diagrammatic representation of the terms in (3.6) for the probability distribution functional of $\lambda\phi^4$-theory.

Fig. 2 Penrose diagram of the maximally extended de Sitter space. The solid lines represent hypersurfaces with constant η (or t).

Fig. 3 Integration contour of (4.12) on the complex η'-plane.

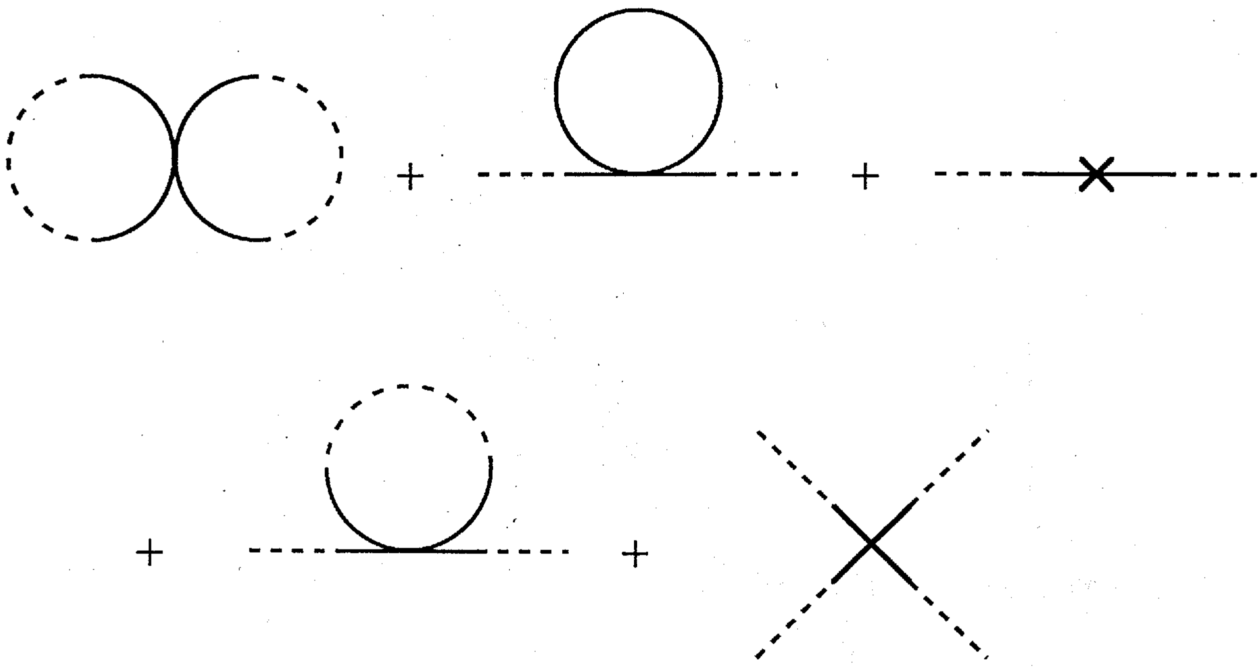

Fig. 1

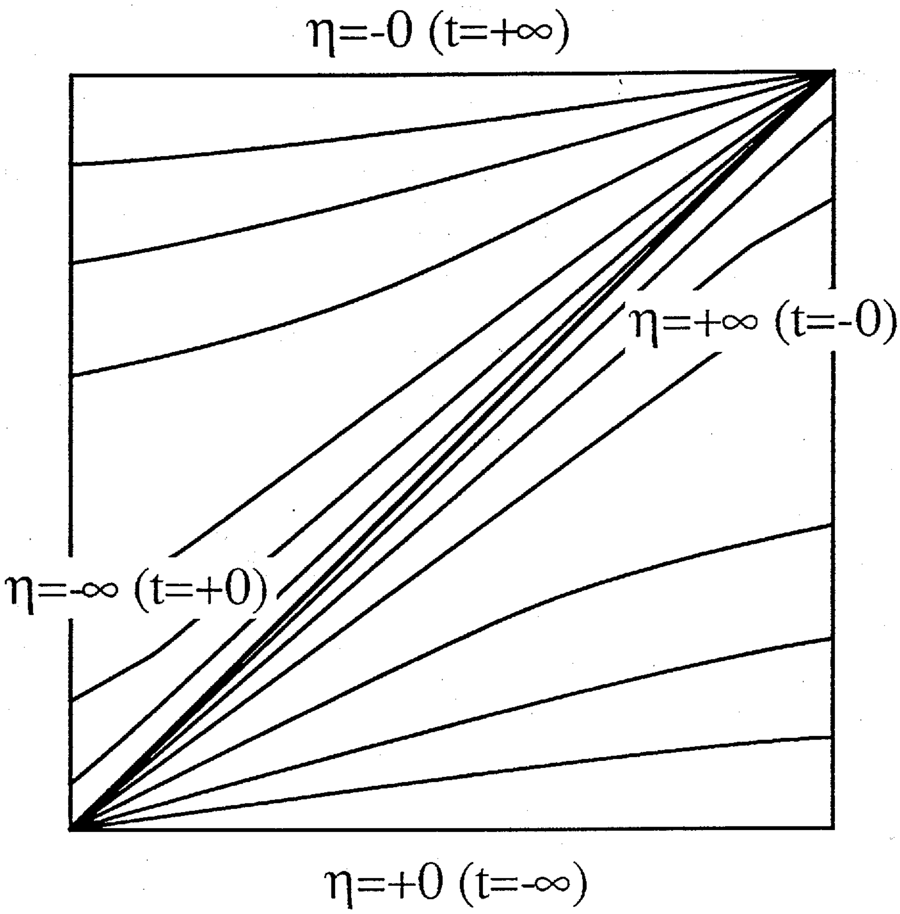

Fig. 2

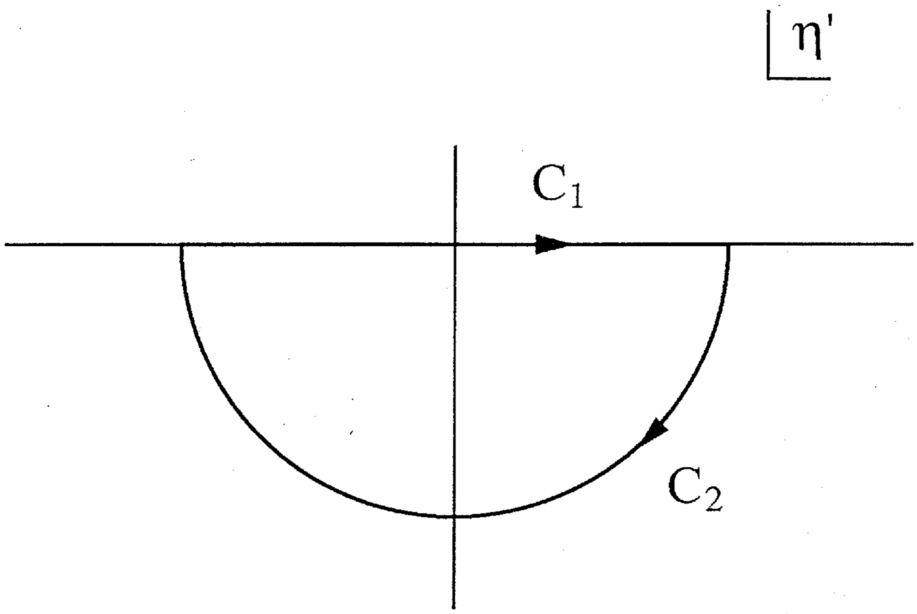

Fig. 3