A multi-objective discrete reliability optimization problem for dissimilar-unit standby systems

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Abstract. A new methodology for the reliability optimization of a k dissimilar-unit non-repairable cold-standby redundant system is introduced in this paper. Each unit is composed of a number of independent components with generalized Erlang distributions of lifetimes arranged in a series-parallel configuration. We also propose an approximate technique to extend the model to the general types of non-constant hazard functions. To evaluate the system reliability, we apply shortest path technique in stochastic networks. The purchase cost of each component is assumed to be an increasing function of its expected lifetime. There are multiple component choices with different distribution parameters available for being replaced with each component of the system. The objective of the reliability optimization problem is to select the best components, from the set of available components, to be placed in the standby system in order to minimize the initial purchase cost of the system, maximize the system MTTF (mean time to failure), minimize the system VTTF (variance of time to failure) and also maximize the system reliability at the mission time. The goal attainment method is used to solve a discrete-time approximation of the original problem.
Keywords: Reliability optimization-Stochastic networks-Shortest path-Optimal control

1 Introduction

Many fielded systems use cold-standby redundancy as an effective system design strategy. Cold-standby means that the redundant units cannot fail while they are waiting. Space exploration and satellite systems achieve high reliability by using cold-standby redundancy for non-repairable systems, see Sinaki [27]. Space inertial reference units are required to accurately monitor critical information for extended mission times without opportunities for repair. Many other systems use cold-standby redundancy as an effective strategy to achieve high reliability including textile manufacturing systems, see Pandey *et al.* [20], and carbon recovery systems used in fertilizer plants, see Kumar *et al.* [17].

Extensive research has been carried out on the reliability of redundant systems with similar/dissimilar units. Several methods and methodologies have been discussed by Birolini [4] and Srinivasan and Subramanian [28].

Multi-component systems have been analyzed by several authors including Goel *et al.* [11] and Yamashiro [29]. Most of such studies deal with the analysis of a single unit system. Gupta *et al.* [13] investigated a single server two-unit multi-component cold-standby system under the assumption that the cold-standby unit becomes operative instantaneously upon the failure of operative unit. Gupta *et al.* [14] analyzed a two dissimilar-unit multi-component cold-standby system with correlated failures and repairs.

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There has been little research toward the study of k dissimilar-unit multi-component systems because of the complexity in the equations and not getting the results in closed form. Azaron *et al.* [2] developed a new approach to evaluate the reliability function of a class of dissimilar-unit redundant systems with exponentially distributed lifetimes.

In this paper, we present a new methodology for the reliability optimization of a k dissimilar-unit multi-component non-repairable cold-standby redundant system. Each unit is composed of a number of independent components arranged in a series-parallel configuration. The components' lifetimes are assumed to be independent random variables with generalized Erlang distributions. Therefore, this methodology allows non-constant hazard functions. We also propose an approximate technique to extend the model to the case of general lifetime distributions.

The purchase cost of each component is assumed to be an increasing function of its expected lifetime. In other words, it is possible to increase the expected lifetime of each component by placing a more expensive unit in the system. There is a set of component choices with different distribution parameters eligible to be replaced with each component of the system. The problem is to select the best components from these sets. This problem is formulated as a multi-objective discrete optimal control problem that involves four conflicting objective functions. The objective functions are the total costs of the standby system (to be minimized), the mean time to failure of the system (max), the variance of the system lifetime (min), and the system reliability at the given mission time (max). This approach involves the use of graph theory, Markov processes, reliability analysis and multiple objective programming.

There has been little research toward the reliability optimization of non-repairable systems with cold-standby redundancy scheme. The problem has often been solved for non-repairable active redundant systems using dynamic programming (Fyffe *et al.* [8] and Nakagawa and Miyazaki [18]) and integer programming (Bulfin and Liu [5] and Gen *et al.* [9]). Gnedenko and Ushakov [10] presented algorithms to maximize the median time to failure. Nakashima and Yamato [19] solved an analogous problem to maximize the time period where system reliability remains above a preselected value. Their algorithm assumes that components have exponential lifetime, but that the distribution parameters are the decision variables to be determined in addition to the redundancy levels.

The problem of reliability optimization of non-repairable cold-standby redundant systems has received less attention. Albright and Soni [1] have solved a reliability optimization problem for non-repairable systems with standby redundancy. They assumed exponential lifetime and one component choice per subsystem. Robinson and Neuts [22] studied system design for non-repairable systems with cold-standby redundancy. They considered systems with components that have phase-type lifetime distributions. Coit [6] has determined optimal design configurations for non-repairable series-parallel systems with cold-standby redundancy. His problem formulation considers non-constant component hazard functions and imperfect switching. Prasad *et al.* [21] considered the problem of allocating multi-functional redundant components for deterministic and stochastic mission times. In their formulation, there is a limit on the total number of redundant components, which can be used.

There are also a few papers, which consider the multi-objective reliability optimization for either time-independent case, see Sakawa [23], or active redundant systems (Sakawa [24] and Dhingra [7]), and optimize system reliability, cost, weight and volume, for a given mission time. Azaron *et al.* [3] used the surrogate worth trade-off method to find the optimal distribution parameters (continuous decision variables like [18]) in a cold-standby system.

The major limitations in the reliability evaluation and optimization approaches for dissimilar-unit cold-standby systems thus far are:

1. Most available algorithms assume that each unit is composed of a single component, but they also cannot get the results in closed form, see Goel *et al.* [12].
2. Available algorithms that do address dissimilar units multi-component cold-standby systems assume that each unit is composed of a number of components arranged in a series configuration. Although this is a start, there are many more complicated system configurations that should be examined. The problem lies in the difficulty of presenting more complicated structures.
3. Existing system reliability optimization algorithms are most often available for active redundancy. The logarithm of system reliability for an active standby redundant system is a separable function and dynamic programming or integer programming can be used to determine optimal solutions to the problem.
4. Available algorithms that do address cold-standby optimization generally assume similar redundant units and exponential lifetimes.
5. Most available optimization algorithms consider continuous decision variables. In this case, it is difficult in practice to select a component to match a specific distribution parameter.
6. Only one criterion for time-dependent reliability, like maximizing MTTF or maximizing the system reliability at a given mission time is considered in the model. In the reliability optimization problem, one often wishes to lower the risk that systems with short system lifetime are produced, but only maximizing MTTF is not always fit for the requirement, especially when the optimally designed system has a large VTTF. The system reliability at the mission time is another important criterion, which should be considered in the model.

This paper, not only considers the reliability optimization for a complex structure (dissimilar-unit cold-standby system, in which each unit is composed of a number of independent components with non-constant hazard functions arranged in a series-parallel configuration), but also the system is optimized with respect to the four important conflicting objectives.

We formulate the appropriate multi-objective discrete optimal control problem, in which the decision variables are the distribution parameters so that they are to be determined from some discrete sets. The problem formulation is continuous-time, combinatorial and stochastic. We prove that solving the resulting problem by standard optimal control techniques is impossible. Therefore, we do the discretization of time and convert the discrete optimal control problem into an equivalent mixed integer nonlinear optimization problem. Finally, we use the goal attainment technique to solve this new multi-objective problem.

The remainder of this paper is organized in the following way. In section 2, we extend the work of Azaron *et al.* [2] to evaluate the reliability function of a k dissimilar-unit multi-component cold-standby redundant system. In section 3, we present the multi-objective discrete reliability optimization problem. Section 4 presents the computational experiments, and finally we draw the conclusion of the paper, in section 5.

2 Reliability evaluation of dissimilar-unit non-repairable cold-standby systems

A very efficient method to compute the reliability of a system is to express it as a reliability graph, see Shooman [26] for the details. Reliability graphs consist of a set of arcs. Each arc represents a component of the system, while the nodes of the graph tie the arcs together and form the structure. Corresponding with the ith arc of the reliability graph, $i=1,2,\ldots,n$, there is a random variable T_i as the lifetime of the ith component with generalized Erlang distribution of order n_i with parameters $\lambda_{i1}, \lambda_{i2}, ..., \lambda_{i n_i}$. An Erlang distribution of order n_i is a generalized Erlang distribution with $\lambda_{i1} = \lambda_{i2} = ... = \lambda_{i n_i}$. When $n_i = 1$, the underlying distribution becomes exponential with parameter λ_{i1}. T_i, $i=1,2,\ldots,n$, are independent random variables, due to the fact that the components work independently.

By definition, a cut of the graph is a set of arcs, which interrupts all connections between input and output when removed from the graph. A minimal cut is the one that contains no other cuts within it. Each system failure can be represented by the removal of at least one minimal cut from the graph.

As mentioned before, we consider a dissimilar-unit cold-standby system, where each unit is composed of a number of components with series-parallel configuration and not all of its components are set to function at time zero. Initially, only the components of the first path of the reliability graph work. Upon failing one component of this path, the system is switched to the next path and the connection between the input and the output is established through this second path. This process continues until no more connection between the input and the output of the graph exists. In that case, the system fails. In the systems, which we discuss in this paper, the minimal cuts are not coincided with the paths of the reliability graph.

Notations:

T_i: lifetime of the *i*th component of the system, $i=1,2,\ldots,n$,
T: system lifetime,
C_j: *j*th minimal cut of the reliability graph, $j=1,2,\ldots,m$,
P_j: *j*th paths in the directed network,
X_j: failure time of the *j*th minimal cut of the reliability graph, $j=1,2,\ldots,m$,
$R(t)$: reliability function of the system,
$F(t)$: distribution function of shortest path, from the source to the sink node, in directed network.

Lemma 1. For $j=1,2,\ldots,m$, the following relation holds:

$$X_j = \sum_{i \in C_j} T_i \ . \qquad (1)$$

Proof. Taking into account the cold-standby nature of the structure, upon failure of each component of the *j*th minimal cut, the system is switched to the next path. Since this minimal cut is not coincided with any path of the reliability graph, then at any moment only one component of the *j*th minimal cut is activated. Therefore, the failure time of this cut is the sum of all its components. $\square$

To evaluate the reliability function, we construct a directed stochastic network with exponentially distributed arc lengths. There are *m* paths in this network, in which the *j*th path of this directed network corresponds with the *j*th minimal cut of the reliability graph, $j=1,2,\ldots,m$. Clearly, by Lemma 1, the length of each path in this directed network is equal to the failure time of the corresponding cut. For constructing this network, we use the idea that if the lifetime of the *i*th component of the system is distributed according to a generalized Erlang distribution of order n_i with parameters $\lambda_{i1}, \lambda_{i2}, ..., \lambda_{i n_i}$, it can be decomposed to n_i exponential serial arcs with parameters $\lambda_{i1}, \lambda_{i2}, ..., \lambda_{i n_i}$. The following rule describes how to construct the proper directed network.

Rule 1. Arc i belongs to the jth path of the directed network, if and only if $i \in C_j$. If $n_i = 1$, then the length of this arc would be exponentially distributed with parameter λ_{i1}. Otherwise, if $n_i > 1$, then this arc is substituted with n_i exponential serial arcs with parameters $\lambda_{i1}, \lambda_{i2}, ..., \lambda_{in_i}$.

Example 1. To operate the accounting activities of a firm, either one computer or one calculator is needed. The calculator needs one battery to do the required operations. However, there are two batteries available in the system to function as standby. At the beginning, the system may start with the computer. If it fails, then, the calculator with one battery is doing the necessary operations. In that case, if the calculator fails so does the system. However, if the battery fails, the calculator works with the standby one. In fact, if either calculator or the second battery fails, then the operation comes to the end.

The system can be represented by a reliability graph, as depicted in Figure 1, in which arc 1 represents the computer, arc 2 represents the calculator, arc 3 and arc 4 represent the first and the second battery, respectively.

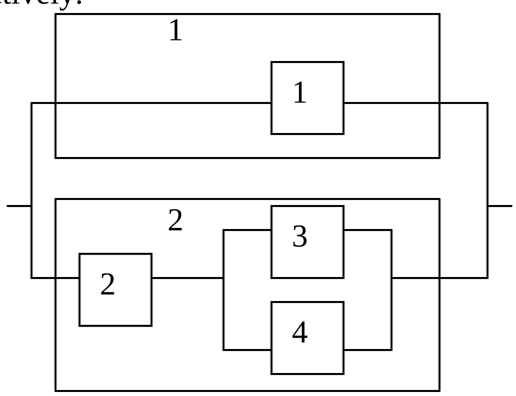

Fig. 1. Reliability graph of example 1.

This is an example of a two dissimilar-unit multi-component cold-standby system. The first unit is the computer, but the second unit is composed of a calculator and two batteries. It is assumed that the elements lifetimes in this example are all exponentially distributed.

This reliability graph has three paths. P_1=(1) is corresponding with the first active unit (computer), and P_2=(2,3) and P_3=(2,4) are two paths corresponding with the second unit. Even if we change the order of paths corresponding with the second unit, the final result will not change, because of the memoryless property of the elements lifetimes. Two minimal cut sets of the reliability graph are C_1=(1,2) and C_2=(1,3,4). From Lemma 1, the failure times of the minimal cuts are

$$X_1 = T_1 + T_2,$$
$$X_2 = T_1 + T_3 + T_4.$$

Therefore, we construct the directed network following Rule 1, as depicted in Figure 2. This network has two paths, P_1=(1,2) and P_2=(1,3,4).

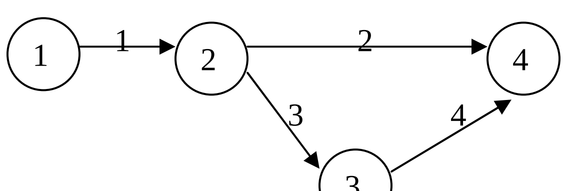

Fig. 2. Directed network of example 1.

Theorem 1. The system lifetime is given by

$$T = \min_{j=1,2,\ldots,m} \{ X_j \}. \quad (2)$$

Proof. Upon the failure of the first minimal cut of the reliability graph of system, all connections between the input and the output are interrupted, and consequently the system fails. Therefore, the lifetime of the system would be equal to the failure time of the first minimal cut, which results in (2). □

Corollary 1. The reliability function of the system is given by

$$R(t)=1-F(t). \quad (3)$$

Proof. Relation (3) follows from the definitions of *R(t)* and *F(t)*. □

2.1 Shortest path analysis in directed networks

Kulkarni's method [16] is applied to obtain the distribution function of shortest path, from the source to the sink node, in the directed network, and accordingly the reliability function of the cold-standby system.

Let $G=(V,A)$ be a directed network, in which V and A represent the sets of nodes and arcs of the network, respectively. Let s and t represent the source and the sink nodes of this network, respectively. The length of arc $(u,v)\in A$ is indicated by $T_{(u,v)}$, which is an exponential random variable with parameter $\lambda_{(u,v)}$.

For constructing the proper stochastic process, it is convenient to visualize the stochastic network as a communication network with the nodes as stations capable of receiving and transmitting messages and arcs as one-way communication links connecting pairs of nodes. The messages are assumed to travel at a unit speed so that $T_{(u,v)}$ denotes the travel time from node u to v. As soon as a node receives a message over one of the incoming arcs, it transmits it along all the outgoing arcs and then disables itself, *i.e.*, loses the ability to receive and transmit the future messages. This process continues until the message reaches the sink node t. Now, at any time there may be some nodes and arcs in the stochastic network that are "useless" for the progress of the message towards the sink node, *i.e.*, even if the messages are received and transmitted by these nodes and carried by these arcs, the message can only reach disabled nodes. It is assumed that all such "useless" nodes are also disabled and the messages traveling on such arcs are aborted. Now, let $X(t)$ be the set of all disabled nodes at time t. $X(t)$ is called the state of the network at time t.

Definition 1: To describe the evolution of the stochastic process $\{X(t), t\geq 0\}$, for each $X\subset V$, where $s\in X$ and $t\in \overline{X}=V-X$, we define the following sets:

1. $\overline{X}_1 \subset \overline{X}$, set of nodes not included in X with the property that each path which connects any node of this set to the sink node t, contains at least one member of X.
2. $S(X)=X\cup \overline{X}_1$.

Definition 2:

$$\Omega=\{X\subset V/s\in X,\ t\in \overline{X},\ X=S(X)\},$$
$$\Omega^{*}=\Omega\cup V. \quad (4)$$

Example 2. In the directed network depicted in Figure 3, if we consider X=(1,2), then $\overline{X}_1=\phi$, and $S(X)$=(1,2). However, if we consider X=(1,3,4), then the only path that connects node

$(2) \in \overline{X}$ to node (5) passes through node (4), which belongs to X. Therefore, $\overline{X}_1$ =(2), and $S(X)$=(1,2,4). In this example, Ω^*={(1),(1,2),(1,3),(1,2,3),(1,2,4),(1,2,3,4),(1,2,3,4,5)}.

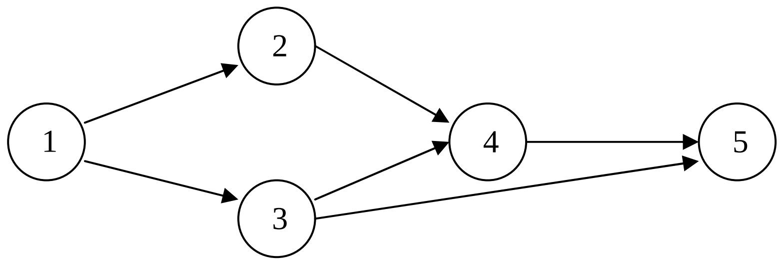

Fig. 3. Directed network of example 2.

Definition 3: If $X \subset V$ such that $s \in X$ and $t \in \overline{X}$, then a cut is defined as:

$$C\left(X,\overline{X}\right) = \{(u,v) \in A / u \in X, v \in \overline{X}\}. \tag{5}$$

There is a unique minimal cut contained in $C\left(X,\overline{X}\right)$, denoted by $C(X)$. If $X \in \Omega$, then,

$$C\left(X,\overline{X}\right) = C(X).$$

It is shown that $\{X(t), t \geq 0\}$ is a continuous-time Markov process with state space Ω^* and the infinitesimal generator matrix $Q=[q(X,Y)]$ $(X,Y \in \Omega^*)$, see [16] for the details, where

$$q(X,Y) = \begin{cases} \sum_{(u,v)\in C(X)} \lambda_{(u,v)} & if Y = S(X \cup \{v\}), \\ -\sum_{(u,v)\in C(X)} \lambda_{(u,v)} & if Y = X, \\ 0 & otherwise. \end{cases} \tag{6}$$

We assume that the states in Ω^* are numbered $1,2,\ldots,N=\left|\Omega^*\right|$ so that Q matrix is upper triangular. State *1* is the initial state, and state N is the final (absorbing) state. In example 2, state *1* is (1), and state *7* is (1,2,3,4,5).

Let T represent the length of the shortest path in the directed network. Clearly,

$$T = min\ \{t>0: X(t)=N\ /\ X(0)=1\}. \tag{7}$$

Therefore, the length of the shortest path in the directed network would be equal to the time until $\{X(t), t \geq 0\}$ gets absorbed in the final state N, starting from state *1*.

Chapman-Kolmogorov backward equations can be applied to compute $F(t)=P\{T \leq t\}$. If we define:

$$P_i(t)=P\{X(t)=N\ /\ X(0)=i\} \quad i=1,2,\ldots,N, \tag{8}$$

then, $F(t)=P_1(t)$.

The system of differential equations for the vector $P(t)=[P_1(t), P_2(t),\ldots, P_N(t)]^T$ is given by

$$\begin{aligned} \dot{P}(t) &= QP(t), \\ P(0) &= [0,0,\ldots,1]^T, \end{aligned} \tag{9}$$

where $P(t)$ represents the state vector of the system and Q is the infinitesimal generator matrix. By taking advantage of the upper triangular nature of Q, the differential equations (9) can be easily solved. After computing $F(t)$, the system reliability can be computed from equation (3).

3 Multi-objective discrete reliability optimization problem

In this section, we develop a multi-objective discrete model to select the best components from the set of available components to be placed in the cold-standby system. In fact, we may increase the expected lifetime of each component by placing a more expensive component in the system. In that case, the mean time to failure of the system will be increased. However, clearly it causes the initial purchase cost of the system to be increased, accordingly. Consequently, an appropriate trade-off between cost and reliability is required.

To achieve the above-mentioned goals, we develop a multi-objective problem, in which four objectives are sought simultaneously, minimizing initial purchase cost, maximizing MTTF, minimizing VTTF and also maximizing system reliability *R(u)* at the given mission time, *u*.

The purchase cost of each component is assumed to be an increasing function of its expected lifetime. The expected lifetime of a component with generalized Erlang distribution of order n_i and the parameters $(\lambda_{i1}, \lambda_{i2}, ..., \lambda_{in_i})$ is equal to $\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}}$. Therefore, C or the initial purchase cost of the standby system is given by

$$C = \sum_{i=1}^{n} g_i \left(\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}} \right). \tag{10}$$

where $g_i \left(\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}} \right)$ is an increasing function respect to $\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}}$.

MTTF and VTTF are given by

$$\text{MTTF} = \int_0^{\infty} (1 - P_1(t))dt\,, \tag{11}$$

$$\text{VTTF} = \int_0^{\infty} t^2 \dot{P}_1(t)dt - \left[\int_0^{\infty} t\, \dot{P}_1(t)dt \right]^2. \tag{12}$$

Considering S_{ij} as the set of different values of λ_{ij} $(\lambda_{ij} \in S_{ij})$, corresponding with the *j*th distribution parameter of available functionally equivalent components eligible to be replaced with the *i*th component, the infinitesimal generator matrix Q would be a function of the control vector $\lambda = \left[\lambda_{ij}, i = 1,2,...,n,\ j = 1,2,...,n_i \right]^T$. Therefore, the dynamic model would be

$$\begin{aligned} &\dot{P}(t) = Q(\lambda)P(t), \\ &P_i(0) = 0 \quad i = 1,2,\dots,N-1, \\ &P_N(t) = 1. \end{aligned} \tag{13}$$

Considering *R* as the system reliability at the mission time *u*, the appropriate multi-objective discrete optimal control problem is

$$Min\ \ C = \sum_{i=1}^{n} g_i \left(\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}} \right)$$

$$Max\ \ \text{MTTF} = \int_0^{\infty} (1 - P_1(t))dt$$

$$Min\ \ \text{VTTF} = \int_0^{\infty} t^2 \dot{P}_1(t)dt - \left[\int_0^{\infty} t\, \dot{P}_1(t)dt \right]^2$$

$$
\begin{aligned}
&Max\ \ R = 1 - P_1(u)\\
&s.t.:\\
&\qquad \dot{P}(t) = Q(\lambda)P(t)\\
&\qquad P_i(0) = 0 \quad i = 1,2,\ldots,N-1\\
&\qquad P_N(t) = 1\\
&\qquad \lambda_{ij} \in S_{ij} \quad i = 1,2,\ldots,n,\ j = 1,2,\ldots,n_i
\end{aligned} \tag{14}
$$

We try to solve problem (14), optimally, using the Maximum Principle, see Sethi and Thompson [25] for the details. For simplicity, we consider only one objective function, for example MTTF= $\int_0^\infty (1 - P_1(t))dt$, in the model.

Considering S as the set of allowable controls, which consists of the last set of constraints of problem (14) ($\lambda \in S$), and N-vector $\mu(t)$ as the adjoint vector function, the Hamiltonian function would be

$$H(\mu(t), P(t), \lambda) = \mu(t)^T Q(\lambda) P(t) + 1 - P_1(t)\ . \tag{15}$$

In (15), $\mu(t)$ plays the role of Lagrange multipliers in nonlinear optimization, but in optimal control theory. Then, we write the adjoint equations and terminal conditions, which are

$$
\begin{aligned}
&-\dot{\mu}(t)^T = \mu(t)^T Q(\lambda) + [-1,0,...,0],\\
&\mu(T)^T = 0, \qquad T \to \infty .
\end{aligned} \tag{16}
$$

If we could compute $\mu(t)$ from (16), we could maximize the Hamiltonian function subject to $\lambda \in S$ in order to get the optimal control λ^*, and solve the problem optimally. Unfortunately, the adjoint equations (16) are dependent on the unknown control vector, λ, and therefore they cannot be solved directly.

If we could also maximize the Hamiltonian function (15), subject to $\lambda \in S$, for an optimal control function in closed form as $\lambda^* = f(P^*(t), \mu^*(t))$, then we could substitute this into the state equations, $\dot{P}(t) = Q(\lambda)P(t)$, $P(0) = [0,0,...,1]^T$, and adjoint equations (16) to get a set of differential equations, which is a two-point boundary value problem. Unfortunately, we cannot obtain λ^* by differentiating H respect to λ , because λ is a discrete vector, and consequently λ^* cannot be obtained in closed form.

According to the two mentioned points, it is impossible to solve the optimal control problem (14), optimally, even in the case of single objective problem. Relatively few optimal control problems can be solved optimally. Therefore, we try to solve this problem, approximately. To do that, we do the discretization of time and convert the multi-objective discrete optimal control problem into an equivalent multi-objective mixed integer nonlinear programming one. In other words, we transform the differential equations to the equivalent difference equations as well as transform the integral terms into equivalent summation terms. To follow this approach, the time interval is divided into K equal portions with the length of Δt. If Δt is sufficiently small, it can be assumed that $P(t)$ varies only in times $0, \Delta t, \ldots, (K-1)\Delta t$. Consider $P(k\Delta t)$ as $P(k)$, the continuous-time system $\dot{P}(t) = Q(\lambda)P(t)$ is approximated as the following discrete-time system:

$$P(k+1) = P(k) + Q(\lambda)P(k)\Delta t \qquad k = 0,1,\ldots,K-1. \tag{17}$$

Similarly, MTTF and VTTF are approximated as:

$$MTTF_a = \sum_{k=0}^{K} \left(1 - P_1(k)\right)\Delta t \,, \tag{18}$$

$$VTTF_a = \sum_{k=0}^{K-1} (k\Delta t)^2 \left(P_1(k+1) - P_1(k)\right) - \left[\sum_{k=0}^{K-1} k\Delta t\left(P_1(k+1) - P_1(k)\right)\right]^2 . \tag{19}$$

Since each $P_i(k)$, for $i=1,2,\dots,N-1$, $k=1,2,\dots,K$ is a distribution function, then we should consider the following constraints in the discrete-time approximation problem.

$$P_i(k) \le 1 \qquad i=1,2,\dots,N-1,\ k=1,2,\dots,K. \tag{20}$$

3.1 Goal attainment method

This method requires setting up a goal and weight, b_j and c_j ($c_j \ge 0$) for $j=1,2,3,4$, for the four indicated objective functions. c_j relates the relative under-attainment of the b_j. For under-attainment of the goals, a smaller c_j is associated with the more important objectives. c_j, $j=1,2,3,4$, are generally normalized so that $\sum_{i=1}^{4} c_j = 1$. Considering $\left[\frac{u}{\Delta t}\right]$ as the integer part of $\frac{u}{\Delta t}$, the appropriate goal attainment formulation would be

$$\begin{aligned}
&Min\ \ z\\
&s.t.:\\
&\sum_{i=1}^{n} g_i\left(\sum_{j=1}^{n_i} \frac{1}{\lambda_{ij}}\right) - c_1 z \le b_1\\
&\sum_{k=0}^{K} \left(1 - P_1(k)\right)\Delta t + c_2 z \ge b_2\\
&\sum_{k=0}^{K-1} (k\Delta t)^2 \left(P_1(k+1) - P_1(k)\right) - \left[\sum_{k=0}^{K-1} k\Delta t\left(P_1(k+1) - P_1(k)\right)\right]^2 - c_3 z \le b_3\\
&1 - P_1\left(\left[\frac{u}{\Delta t}\right]\right) + c_4 z \ge b_4\\
&P(k+1)=P(k)+Q(\lambda)P(k)\Delta t \qquad k=0,1,\dots,K-1\\
&P_i(0)=0 \qquad i=1,2,\dots,N-1\\
&P_N(k)=1 \qquad k=0,1,\dots,K\\
&P_i(k) \le 1 \qquad i=1,2,\dots,N-1,\ k=1,2,\dots,K\\
&\lambda_{ij} \in S_{ij} \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i\\
&z \ge 0
\end{aligned} \tag{21}$$

The optimal solution using this formulation is fairly sensitive to b and c. Depending upon the values for b, it is possible that c does not appreciably influence the optimal solution. Instead, the optimal solution can be determined by the nearest Pareto-optimal solution from b. This might require that c be varied parametrically to generate a set of Pareto-optimal solutions.

For solving the goal attainment formulation (21), we define the new 0-1 decision variables y_{ijk}. Let α_{ijk} represent the kth member of S_{ij}, $k=1,2,\dots,|S_{ij}|$. Then, $\lambda_{ij} \in S_{ij}$, $i=1,2,\dots,n$, $j=1,2,\dots,n_i$ in (21) should be replaced with the constraints (22) and (23).

$$\lambda_{ij} = \sum_{k=1}^{|S_{ij}|} \alpha_{ijk} y_{ijk} \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i\,. \tag{22}$$

$$\sum_{k=1}^{|S_{ij}|} y_{ijk} = 1 \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i\,. \tag{23}$$

Finally, the following mixed integer nonlinear programming problem would be approximately equivalent to the original model and from which $\lambda^* = \left[\lambda^*_{ij}, i = 1,2,...,n,\ j = 1,2,...,n_i\right]^T$ or the optimal control vector is obtained.

$$\begin{aligned}
&\mathit{Min}\ \ z \\
&\mathit{s.t.:} \\
&\sum_{i=1}^{n} g_i\left(\sum_{j=1}^{n_i}\frac{1}{\lambda_{ij}}\right) - c_1 z \le b_1 \\
&\sum_{k=0}^{K}\left(1 - P_1(k)\right)\Delta t + c_2 z \ge b_2 \\
&\sum_{k=0}^{K-1}(k\Delta t)^2\left(P_1(k+1) - P_1(k)\right) - \left[\sum_{k=0}^{K-1} k\Delta t\left(P_1(k+1) - P_1(k)\right)\right]^2 - c_3 z \le b_3 \\
&1 - P_1\left(\left[\frac{u}{\Delta t}\right]\right) + c_4 z \ge b_4 \\
&P(k+1)=P(k)+Q(\lambda)P(k)\Delta t \qquad k=0,1,\dots,K-1 \\
&P_i(0)=0 \qquad i=1,2,\dots,N-1 \\
&P_N(k)=1 \qquad k=0,1,\dots,K \\
&P_i(k) \le 1 \qquad i=1,2,\dots,N-1,\ k=1,2,\dots,K \\
&\lambda_{ij} = \sum_{k=1}^{|S_{ij}|} \alpha_{ijk} y_{ijk} \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i \\
&\sum_{k=1}^{|S_{ij}|} y_{ijk} = 1 \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i \\
&y_{ijk} \in \{0,1\} \qquad i=1,2,\dots,n,\ j=1,2,\dots,n_i,\ k = 1,2,...,|S_{ij}| \\
&z \ge 0
\end{aligned} \tag{24}$$

If we consider the initial and terminal state conditions for $P(k)$ implicitly, and substitute each $P_i(0)$ and $P_N(k)$ with zero and one, respectively, the mixed integer nonlinear programming problem (24) would have $K(N-1)+1$ continuous decision variables and $\sum_{i=1}^{n}\sum_{j=1}^{n_i}|S_{ij}|$ 0-1 decision variables.

For estimating the length of the time interval, we consider each λ_{ij} as the median of S_{ij} for $i=1,2,\dots,n,\ j=1,2,\dots,n_i$. Then, we solve the system of differential equations (9), analytically, to obtain $P_1(t)$, according to the values of λ_{ij} taken from the previous step. A good estimation for the length of the time interval is given by $\hat{T}$, in which $P_1(\hat{T})$ should be greater than or equal $1-\varepsilon$. We consider ε equal to *0.01*, in this paper, and consequently, $\hat{T}$ can be computed by solving the nonlinear equation $P_1(\hat{T})=0.99$, numerically.

A computer program was written in order to evaluate our algorithm on some problems with different sizes and investigate the trade-off between the accuracy (correctness) and the computational time in each problem. At the beginning of the algorithm, we consider $K=10$ and $\Delta t = \hat{T}/10$. In an accurate solution $P_1(k)$ should approach *1*. Otherwise, the value of Δt is increased in order to obtain a more accurate solution.

After solving the problem (24) and obtaining λ^*, we compute $P_1(t)$ by solving the system of differential equations with constant coefficients (9), analytically. Then, we compute the exact MTTF and VTTF from (11) and (12), respectively. The Percentage Difference between the approximated MTTF taken from (18) and the exact MTTF taken from (11) (PD.M) and also the Percentage Difference between the approximated VTTF taken from (19) and the exact VTTF taken from (12) (PD.V), or the absolute differences between the approximated values and the exact values divided by the exact ones, can be considered as two important criteria for the accuracy of the discrete-time approximated solution. As K is increased and Δt is decreased, PD.M and PD.V approach zero. Therefore, the approximated discrete lifetime distribution approaches to the exact distribution, because of matching the first two moments, and consequently, the optimal solution of the discrete-time problem (24) approaches to the goal attainment formulation of the original optimal control problem (14).

In the next steps of the algorithm, we replace K with $K+10$ and each new value for Δt should be considered, such that the length of the time interval ($\hat{T} = K\Delta t$) remains unchanged, in order to investigate the trade-off between optimality and computational time.

The proposed methodology is easily generalized, in which not only the scale parameters (λ_{ij}) but also the shape parameters (n_i) are also considered as the design variables. In real-world problems, the designers sometimes use fundamentally different designs or technologies with different shape parameters, because the failure mechanisms would be different. In this case, we first solve the optimization problem (24), for all combinations of n_i for $i=1,2,\ldots,n$. Then, the optimal n_i^*, $i=1,2,\ldots,n$, would be related to that combination, which results the minimum z of the problem (24). It should also be noted that the infinitesimal generator matrix for each combination of n_i would be different from the other combinations, and this matter clearly increases the complexity of the problem.

4. Computational experiments

For showing the numerical stability of the theoretical developments of the paper, we solve two numerical examples, and investigate the trade-off between the accuracy and the computational time in each of them. In both examples, Saaty' method of pair wise comparisons, see Hwang and Yoon [16], is used to compute the weights.

4.1. Case I

For space exploring, there are two space shuttles, which are depicted as in Figure 1. In this system, there are two non-repairable dissimilar units in a cold-standby redundancy scheme. At the beginning, the operating unit is unit 1, which is composed of shuttle A (component 1). When this shuttle fails, the redundant unit 2, which is composed of shuttle B (component 2), central controller I (component 3) and central controller II (component 4), as the cold-standby redundant components, arranged in a series-parallel configuration, is put into operation. Table 1 shows the characteristics of the components.

Table 1. Characteristics of the components of Case I.

i	Distribution	Parameters	Purchase cost
1	Generalized Erlang	$(\lambda_{11},\lambda_{12})$	$2\left(\frac{1}{\lambda_{11}}\right)+3\left(\frac{1}{\lambda_{12}}\right)+4$
2	Generalized Erlang	$(\lambda_{21},\lambda_{22},\lambda_{23})$	$\left(\frac{1}{\lambda_{21}}\right)+5\left(\frac{1}{\lambda_{22}}\right)+2\left(\frac{1}{\lambda_{23}}\right)+3$
3	Exponential	λ_{31}	$6\left(\frac{1}{\lambda_{31}}\right)^2+5$
4	Exponential	λ_{41}	$3\left(\frac{1}{\lambda_{41}}\right)^2+7$

The cost unit is in million dollars and the time unit is in year. The mission time, u, is assumed to be equal 2 years. It is also assumed that $S_{ij}=\{1,1.1,1.2,\ldots,2\}$ for $i=1,2,3,4$, $j=1,2,\ldots,n_i$. We set the goals for the initial purchase cost, MTTF, VTTF and the system reliability at the mission time as $b_1=30$, $b_2=2.7$, $b_3=0.5$ and $b_4=0.7$, respectively. Since one year deviation from the system MTTF is known to be 20, 0.5 and 5 times as important as one million dollars deviation from the initial purchase cost, one year deviation from the system VTTF, and also one unit deviation from the system reliability, respectively, then $c_1=0.7547$, $c_2=0.0377$, $c_3=0.0189$, $c_4=0.1887$. The objective is to select the best components, from the set of available components, to be placed in this 2 dissimilar-unit multi-component non-repairable cold-standby redundant system.

First, we construct the proper directed network following Rule 1, as depicted in Figure 4. The stochastic process $\{X(t),t\geq 0\}$ related to the shortest path analysis of this directed network has 9 states in the order of $\Omega^*=\{(1),(1,2),(1,2,3),(1,2,3,4),(1,2,3,5),(1,2,3,4,5),(1,2,3,4,6),(1,2,3,4,5,6),(1,2,3,4,5,6,7)\}$. Table 2 shows matrix $Q(\lambda)$.

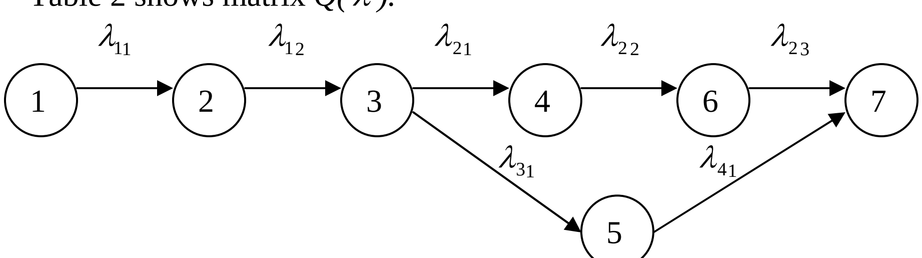

Fig. 4. Directed network of Case I.

Table 2. Matrix *Q(* λ *)* corresponding with Case I.

State	1	2	3	4	5	6	7	8	9
1	$-\lambda_{11}$	λ_{11}	0	0	0	0	0	0	0
2	0	$-\lambda_{12}$	λ_{12}	0	0	0	0	0	0
3	0	0	$-(\lambda_{21}+\lambda_{31})$	λ_{21}	λ_{31}	0	0	0	0
4	0	0	0	$-(\lambda_{22}+\lambda_{31})$	0	λ_{31}	λ_{22}	0	0
5	0	0	0	0	$-(\lambda_{21}+\lambda_{41})$	λ_{21}	0	0	λ_{41}
6	0	0	0	0	0	$-(\lambda_{22}+\lambda_{41})$	0	λ_{22}	λ_{41}
7	0	0	0	0	0	0	$-(\lambda_{23}+\lambda_{31})$	λ_{31}	λ_{23}
8	0	0	0	0	0	0	0	$-(\lambda_{23}+\lambda_{41})$	$\lambda_{23}+\lambda_{41}$
9	0	0	0	0	0	0	0	0	0

The length of the time interval is approximated as $\hat{T}=5$. Therefore, we consider *K=10* and Δt *=0.5*, at the beginning. Then, we formulate the proper multi-objective reliability optimization problem according to (24). For this problem, there are 77 0-1 decision variables. The number of prospective solutions to the problem is larger than 1.51^{23}. For investigating the trade-off between the accuracy and the Computational Time (C.T.) (mm:ss) on a PC Pentium IV 2.1 GHz Processor, we also solve the problem for *K=20,30,40,50,500*, and compute the values of PD.M and PD.V in each case. Table 3 shows the results. In this table, all solutions are Pareto-optimal and satisfy the necessary condition ($P_1(K)\geq 0.99$).

Table 3. Trade-off results in Case I.

No.	*C*	$MTTF_a$	$VTTF_a$	R_a	PD.M %	PD.V %	*K*	Δt	C.T.
1	*37.368*	*2.499*	*0.164*	*0.766*	*0*	*87.98*	*10*	*0.5*	*52*
2	*36.101*	*2.394*	*0.653*	*0.677*	*2.44*	*48.3*	*20*	*0.25*	*5:42*
3	*35.839*	*2.296*	*0.702*	*0.561*	*1.54*	*37.88*	*30*	*0.167*	*16:11*
4	*35.5*	*2.227*	*0.733*	*0.537*	*1.68*	*30.91*	*40*	*0.125*	*18:38*
5	*35.206*	*2.184*	*0.749*	*0.522*	*1.75*	*26.49*	*50*	*0.1*	*22:49*
6	*31.134*	*1.977*	*0.78*	*0.443*	*1.73*	*9.19*	*500*	*0.01*	*48:34*

Table 4 shows λ^*_{ij} for *i=1,2,3,4, j=1,2,…,* n_i, considering *K=500* and Δt *=0.01.* In this case, PD.M and PD.V are almost equal to *1.7%* and *9%*, respectively. Therefore, the accuracy of this solution is acceptable, but access to this level of accuracy still needs a relatively long computational time (about *48* minutes).

Table 4. λ^*_{ij} for $i=1,2,3,4$, $j=1,2,\ldots,n_i$, in Case I.

λ^*_{11}	λ^*_{12}	λ^*_{21}	λ^*_{22}	λ^*_{23}	λ^*_{31}	λ^*_{41}
2	2	2	2	2	1.3	1.2

4.2. Case II

Case II, which is depicted in Figure 5, shows the controller system of a spacecraft. In this system, there are three non-repairable dissimilar units in a cold-standby redundancy scheme.

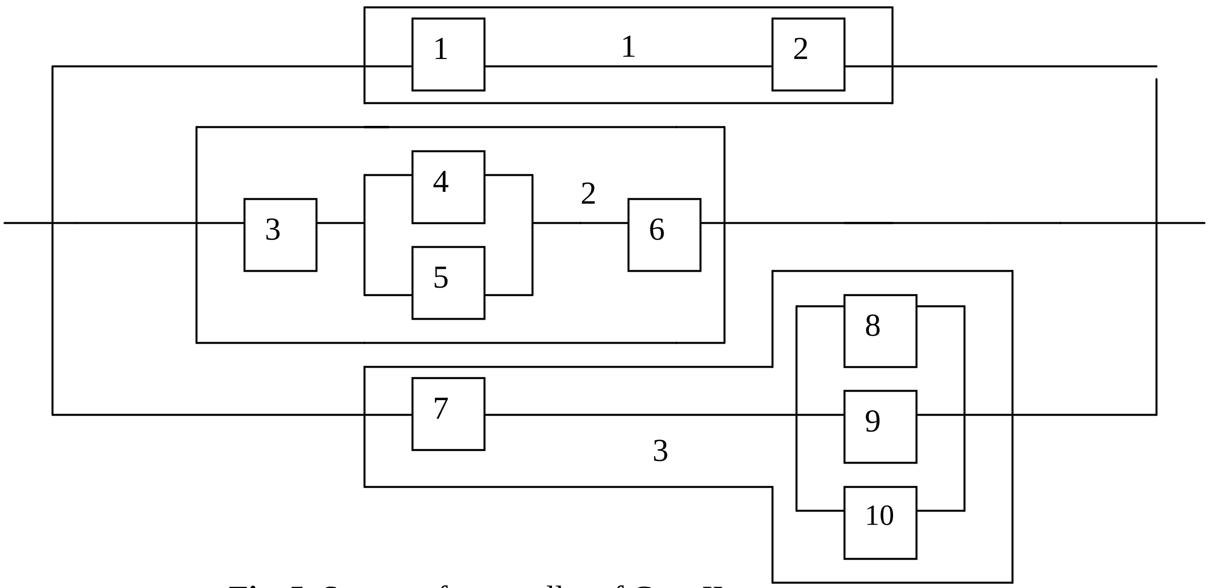

Fig. 5. Spacecraft controller of Case II.

At the beginning, the operating unit is unit 1, which is composed of a laptop computer (component 1) and a power supply (component 2) arranged in a series configuration. When this unit fails, the redundant unit 2, which is composed of PC I (component 3), CD drive I (component 4) and CD drive II (component 5), as the cold-standby redundant components, and also a monitor (component 6), arranged in a series-parallel configuration, is put into operation. If unit 2 fails, then the redundant unit 3, which is composed of PC II (component 7) and hard drive I (component 8), hard drive II (component 9) and hard drive III (component 10), as the cold standby redundant components, arranged in a series-parallel configuration, goes into operation.

In all components, except component 2, the shape parameters (n_i) are considered equal to 1, because we suppose that the replacements have the same failure mechanisms. For component 2, it is supposed that there is also another replacement with Generalized Erlang distribution lifetime of order $n_2=2$ and the parameters ($\lambda_{21}, \lambda_{22}$), except the first replacement with exponential lifetime. Table 5 shows the characteristics of the components.

The cost unit is in hundred dollars and the time unit is in year. The mission time, u, is assumed to be equal 1.8. It is also assumed that $S_{ij}=\{0.5,0.6,\ldots,1\}$ for $i=1,2,\ldots,10,$, $j=1,2,\ldots,n_i$. We set the goals as $b_1=100$, $b_2=3.5$, $b_3=0.5$ and $b_4=0.8$. Under-attainment of the goals are assumed to be $c_1=0.7547$, $c_2=0.0377$, $c_3=0.0189$, $c_4=0.1887$, like the previous case.

The length of the time interval is approximated as $\hat{T} = 6$. Therefore, we consider *K=10* and Δt *=0.6*, at the beginning. Then, we formulate and solve the proper multi-objective reliability optimization problem for both combinations of n_2 (1 and 2), according to (24).

Table 5. Characteristics of the components of Case II.

i	Distribution	Parameters	Purchase cost
1	Exponential	λ_{11}	$7\left(\frac{1}{\lambda_{11}}\right)^2 + 5$
2_1	Exponential	λ_{21} (n_2 =1)	$2\left(\frac{1}{\lambda_{21}}\right) + 2$
2_2	Generalized Erlang	($\lambda_{21}, \lambda_{22}$) ($n_2$ =2)	$\left(\frac{1}{\lambda_{21}}\right) + \left(\frac{1}{\lambda_{22}}\right) + 2$
3	Exponential	λ_{31}	$8\left(\frac{1}{\lambda_{31}}\right) + 6$
4	Exponential	λ_{41}	$4\left(\frac{1}{\lambda_{41}}\right) + 3$
5	Exponential	λ_{51}	$4\left(\frac{1}{\lambda_{51}}\right) + 3$
6	Exponential	λ_{61}	$10\left(\frac{1}{\lambda_{61}}\right) + 4$
7	Exponential	λ_{71}	$3\left(\frac{1}{\lambda_{71}}\right)^2 + 7$
8	Exponential	λ_{81}	$5\left(\frac{1}{\lambda_{81}}\right) + 2$
9	Exponential	λ_{91}	$5\left(\frac{1}{\lambda_{91}}\right) + 2$
10	Exponential	λ_{101}	$5\left(\frac{1}{\lambda_{101}}\right) + 2$

Table 6 shows the trade-off between the accuracy and the computational time for the different pairs of K and Δt. Considering *K=500* and Δt *=0.012*, we obtain λ^*_{ij} =1 for all *i=1,2,…,10* and *j=1,2*. Moreover, the optimal replacement for component 2 (power supply) would be a hardware with Generalized Erlang distribution lifetime of order n^*_2 =2 and the parameters ($\lambda^*_{21} = 1, \lambda^*_{22} = 1$), and we should pay 400 dollars for purchasing this kind of hardware.

Table 6. Trade-off results in Case II.

No.	*C*	$MTTF_a$	$VTTF_a$	R_a	PD.M %	PD.V %	*K*	Δt	C.T.
1	*111.714*	*2.937*	*0.71*	*0.849*	*0.71*	*71.62*	*10*	*0.6*	*7:39*
2	*112.656*	*2.59*	*0.943*	*0.724*	*0.77*	*46.36*	*20*	*0.3*	*9:23*
3	*110.937*	*2.46*	*1.053*	*0.677*	*1.16*	*34.31*	*30*	*0.2*	*10:56*
4	*107*	*2.347*	*1.072*	*0.637*	*1.18*	*27.76*	*40*	*0.15*	*20:34*
5	*107*	*2.341*	*1.13*	*0.636*	*1.43*	*23.85*	*50*	*0.12*	*22:41*
6	*89*	*2.043*	*1.125*	*0.532*	*1.25*	*7.02*	*500*	*0.012*	*56:38*

Figures 6, 7 and 8 show the computational time, PD.M and PD.V, respectively, according to the different pairs of K and Δt in two indicated cases. According to Figure 6, computational time grows with K. Computational time is also strongly dependent on n_i, the network size and the number of different shape parameters (n_i) for each component. According to Figure 8, when K is increased, the percentage difference between the approximated and the exact variance of the system lifetime, which is one of the most important criteria for the accuracy of the solution, will be decreased.

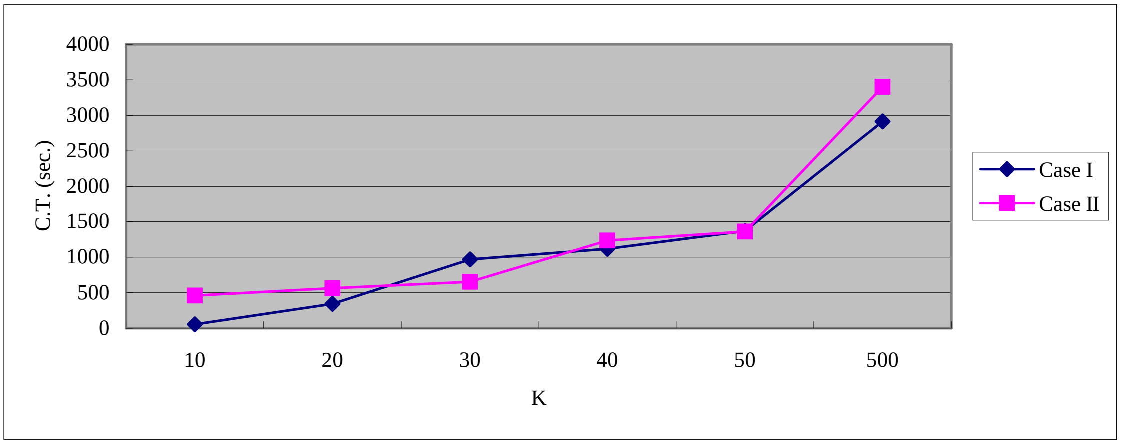

Fig. 6. Computational time (sec.) versus *K*.

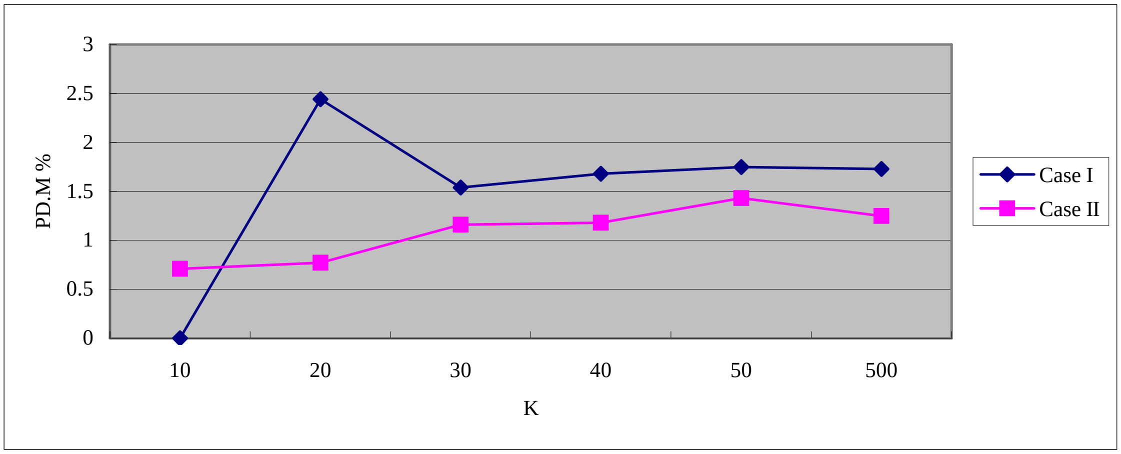

Fig. 7. PD.M versus *K*.

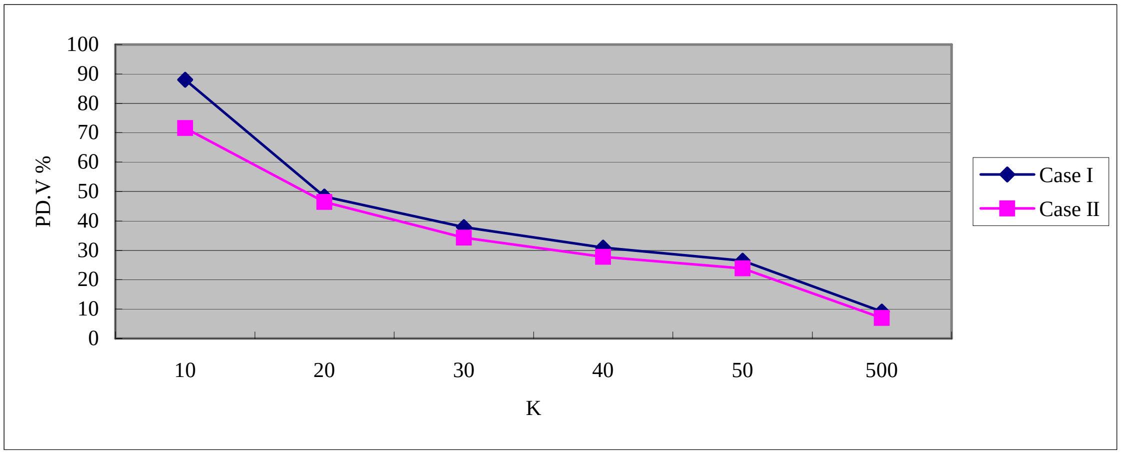

Fig. 8. PD.V versus *K*.

5. Conclusion

In this paper, we introduced a new methodology for the reliability optimization of dissimilar-unit non-repairable cold-standby redundant systems, in which each unit is composed of a number of independent components arranged in a series-parallel configuration. The system components work independently and the lifetime of each component is a random variable with generalized Erlang distribution, in which the decision variables are both scale and shape parameters. We assumed for each component, the distribution parameters are to be determined from among some discrete sets. The purchase cost of each component was also assumed to be an increasing function of its expected lifetime.

To select the desired components, we developed a goal attainment model with four conflicting objectives, minimization of the total purchase costs, maximization of the mean time to failure of the system, minimization of the variance of time to failure of the system and also maximization of the system reliability at the given mission time. Then, in order to solve the resulting optimal control problem, it was transformed into a mixed integer nonlinear programming problem.

Although, at the first glance, it seems that the proposed mixed integer nonlinear programming problem has many continuous and 0-1 decision variables, but using the shortest path technique for reliability optimization has many other advantages over the classical approaches. Computing the reliability function of these standby systems using classical approaches, which is essential for reliability optimization, if is not impossible, is at least so complicated for most real case problems, because either the convolution integrals are intractable or the size of the state space would be enormous. For example, this problem could be solved by using clever complete enumeration of network states, see [26]. According to our methodology, for a complete directed network with *l* nodes and *l(l-1)* arcs representing the components of the system (the worst case example), the size of the state space would be $2^{l-2}+1$, but the size of the state space in Shooman's method would be equal to $3^{l(l-1)}$, because each component can be in one of these three states: work, fail and standby. In the numerical example of section 4.2, the size of the state space for n_2 =1 is equal to 7 and for n_2 =2 is equal to 8 (totally 15), but according to the Shooman's method, the size of the state space would be $2*3^{10}$, which is much larger than our proposed methodology.

According to the computational experiments, when K grows and Δt goes to zero, the percentage difference between the approximated and the exact mean would be about zero, and the percentage difference between the approximated and the exact variance approaches zero. Therefore, the approximated discrete lifetime distribution approaches to the continuous distribution, because the first two moments of the approximated distribution and the exact distribution are matched with each other. In this case, the optimal solution of the discrete-time problem also approaches to the optimal solution of the original continuous-time problem, accordingly. In more realistic sized problems, the values of K and Δt should be selected, such that we can solve the problem in an acceptable level of accuracy with reasonable computational time. According to these experiments, there is no significant relation between PD.M and the network size and also between PD.V and the network size, but the computational time grows with the size of the network, the value of n_i and also the number of different shape parameters for each component.

The proposed model can be easily extended to the general types of non-constant hazard functions. In the case of general distribution of lifetime, the lifetime distribution can be approximated by a generalized Erlang distribution, by matching the first three moments, because the generalized Erlang distributions are a special class of Coxian distributions and each general distribution can be easily approximated by a Coxian distribution.

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