GRAPHIC DESCRIPTIONS OF MONODROMY REPRESENTATIONS

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Dedicated to Professor Takao Matumoto on his sixtieth birthday

Abstract. Various topological objects; 2-dimensional braids, braided surfaces, Lefschetz fibrations of 4-manifolds, algebraic curves, etc., can be treated by their monodromy representations. We introduce a graphical method, called the chart description method, to describe and calculate such monodromy representations. This method is, in a sense, a generalization of the picture method.

1. Introduction

Various topological objects; 2-dimensional braids, braided surfaces, Lefschetz fibrations of 4-manifolds, algebraic curves, etc., can be treated by their monodromy representations. We introduce a graphical method, called the chart description method, to describe and calculate monodromy representations. This method is, in a sense, a generalization of the picture method due to Igusa [7, 8] and Rourke [25] (cf. [2, 24]).

In § 2 we recall the picture method, and in § 3 the notion of a chart is introduced. It is proved in § 4 that any G-monodromy representation can be described by a chart (Theorem 5 and Corollary 6). In § 5 chart moves are introduced and uniqueness of chart description modulo chart moves is proved when $\mathcal{C}$ is full (Theorem 12 and Corollary 13). In § 6 we modify chart moves so that the uniqueness theorem is valid even if $\mathcal{C}$ is not full (Theorem 16 and Corollary 17). These are main results of this paper. In § 7 and § 8 we show how chart descriptions change under conjugacy, isotopic, or homeomorphic equivalence. Our results are applicable to monodromy representations of any topological objects. Especially we demonstrate how the chart description method works for studies in braided surfaces (and 2-dimensional braids) in § 9 and studies in genus-1 Lefschetz fibrations in § 10.

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2. Pictures— traditional graphics—

Let $\langle \mathcal{X}; \mathcal{R} \rangle$ be a finite group presentation of a finitely presentable group G, where $\mathcal{X}$ is a finite set and $\mathcal{R}$ is a finite set of words in $\mathcal{X} \cup \mathcal{X}^{-1}$. The *complex associated with* $\langle \mathcal{X}; \mathcal{R} \rangle$ is a 2-complex with a single 0-cell, oriented 1-cells corresponding to the generators in $\mathcal{X}$, and oriented 2-cells corresponding to the relators in $\mathcal{R}$. We denote this complex by $K^2(\mathcal{X}; \mathcal{R})$ or simply by K^2. (0-cells, 1-cells and 2-cells are usually called vertices, edges and faces, respectively. When cells are based, we assume that the base point of an edge of K^2 is the initial point and the base point of a face is a boundary point where the relator starts.)

For example, when $\mathcal{X} = \{a, b, c\}$ and $\mathcal{R} = \{r_1, r_2, r_3\}$ with $r_1 = abab^{-1}a^{-1}b^{-1}$, $r_2 = bcbc^{-1}b^{-1}c^{-1}$, $r_3 = aca^{-1}c^{-1}$, then $K^2(\mathcal{X}; \mathcal{R})$ consists of 7 cells in Figure 1, where faces are oriented anticlockwise and base points of faces are the corners indicated by asterisks.

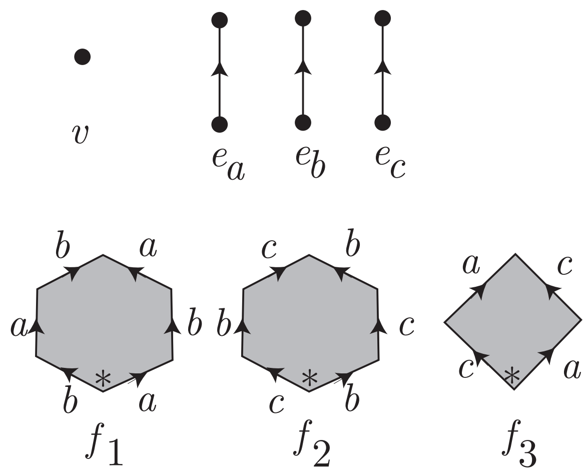

Figure 1

By the Seifert-van Kampen theorem, we have a well-known fact that the fundamental group $\pi_1(|K^2|, v)$ has the presentation $\langle \mathcal{X}; \mathcal{R} \rangle$, i.e., isomorphic to G. A word w in $\mathcal{X} \cup \mathcal{X}^{-1}$ induces a path $\eta_w : I \to |K^1|$ in the usual way, where K^1 is the 1-skeleton. The fundamental theorem in combinatorial group theory states that two words w_1 and w_2 are congruent mod $\mathcal{R}$ (i.e., they represent the same element in the group G) if and only if two paths η_{w_1} and η_{w_2} are homotopic in $|K^2|$. Then there exists a map $\varphi : D^2 \to |K^2|$ between η_{w_1} and η_{w_2}. For a generic φ, the preimages of the cells of K^2 tessellates the 2-disk D^2, for example, as in the left of Figure 2.

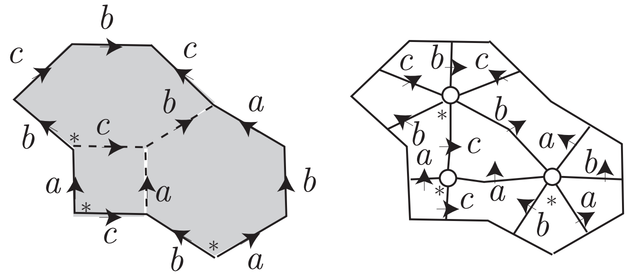

FIGURE 2

Considering the dual graph of the tessellation, we have a graph in D^2 such that each edge is co-oriented and labeled by an element of $\mathcal{X}$ and each vertex corresponds to a relator in $\mathcal{R}$ or its inverse. We asterisk a region around each vertex which indicates the starting point of the relator. (See the middle of Figure 2.) Such a graph is called a *picture* or a *co-oriented picture*. Instead of co-orientation, we may give an orientation to each edge by a rule such that the orientation vector followed by the co-orientation vector matches the orientation of the 2-disk D^2. See the right of Figure 2. Then such a graph is called an *oriented picture*.

In a (co-oriented or oriented) picture, each vertex corresponds to a relator in $\mathcal{R}$ or its inverse. In the former case, we call the vertex a *positive vertex*; in the latter case a *negative vertex*. Thus a picture has vertices with signs, unless $\mathcal{R}$ has both a relator and its inverse. (When $\mathcal{R}$ has both a relator and its inverse, we shall specify one of them as a positive relator so that we can consider pictures to have signed vertices.)

For a path $\eta : I \to D^2$ intersecting a picture Γ transversely, we have a word in $\mathcal{X} \cup \mathcal{X}^{-1}$ by reading off the labels of intersecting edges along η with exponents. The exponent is $+1$ (or -1, resp.) if the orientation of η matches the co-orientation of the edge (or does not, resp.) at the intersection. We call the word the *intersection word of* η *with respect to* Γ and denote it by $w_\Gamma(\eta)$. According to our orientation/co-orientation convention, when Γ is an oriented picture, the exponent is $+1$ (or -1, resp.) if η crosses the oriented edge from the right (or from the left, resp.). For example, $w_\Gamma(\eta) = ba^{-1}c^{-1}$ for Figure 3.

A merit of this graphical description is to enable us to transform a word in $\mathcal{X} \cup \mathcal{X}^{-1}$ to another word representing the same element of the group by using relators in $\mathcal{R}$ and their inverses, modulo trivial relators xx^{-1} and $x^{-1}x$ ($x \in \mathcal{X}$). For example, Figure 4 or Figure 5 shows

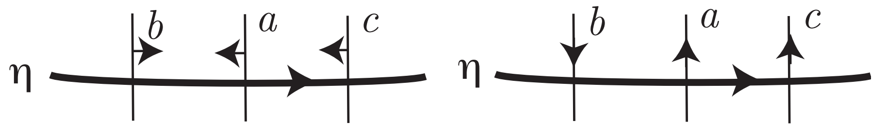

FIGURE 3

a process transforming $bc^{-1}abcb \mapsto bc^{-1}(cac^{-1}\,a^{-1})abcb \equiv bac^{-1}bcb \mapsto bac^{-1}(cbcb^{-1}c^{-1}b^{-1})bcb \equiv babc \mapsto (abab^{-1}a^{-1}b^{-1})babc \equiv abac$ with respect to the presentation used in Figure 1.

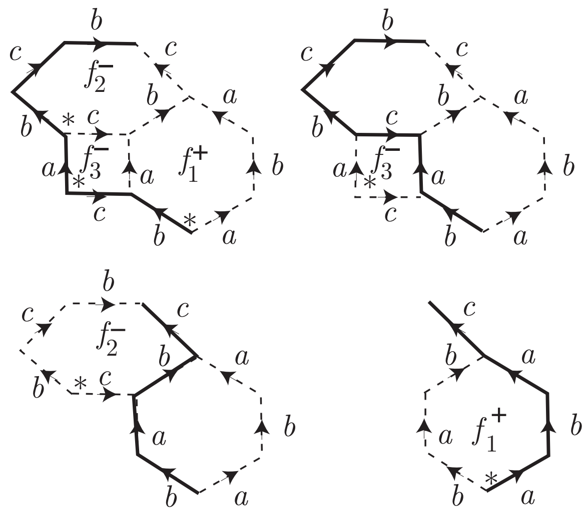

FIGURE 4

We shall generalize the notion of picture to chart.

3. CHARTS— ENHANCED GRAPHICS—

Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ be a triple consisting of a finite set $\mathcal{X}$, and two (possibly infinite) sets $\mathcal{R}$ and $\mathcal{S}$ of words in $\mathcal{X} \cup \mathcal{X}^{-1}$.

Let Σ be an oriented surface, and let Γ be a graph in Σ such that each edge is oriented (or co-oriented) and labeled by an element of $\mathcal{X}$. For a vertex v of Γ lying in $\mathrm{Int}\Sigma$ (where "Int" means interior), a small simple closed curve surrounding v in the positive direction of the surface Σ is called a *meridian loop of* v and denoted by m_v. When v is *marked*, i.e., one of the regions around v is specified by an asterisk, the intersection word $w_\Gamma(m_v)$ of the meridian loop with respect to Γ is well-defined. When v is not marked, it is determined up to cyclic permutation. We denote $w_\Gamma(m_v)$ also by $w_\Gamma(\partial v)$. Actually we usually denote a vertex

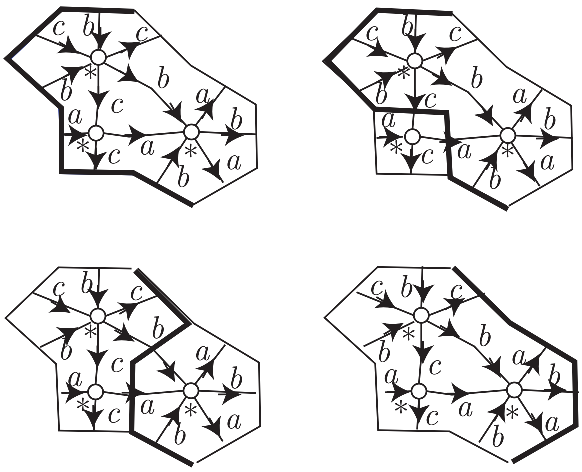

FIGURE 5

in $\mathrm{Int}\Sigma$ by a fat vertex in figures and ∂v is the boundary of this fat vertex.

Definition 1. A $\mathcal{C}$-*chart*, or simply a *chart*, is a finite graph Γ in Σ each edge of which is oriented (or co-oriented) and labeled by an element of $\mathcal{X}$ satisfying the following.

- $\Gamma \cap \partial\Sigma$ is empty or consists of some deg-1 vertices (called *boundary vertices of* Γ),
- Vertices in $\mathrm{Int}\Sigma$ are classified into two families, *white vertices* and *black vertices*,
- When v is a white vertex (or black vertex, resp.) the word $w_\Gamma(\partial v)$ is a cyclic permutation of an element of $\mathcal{R} \cup \mathcal{R}^{-1}$ (or of $\mathcal{S}$, resp.).

A $\mathcal{C}$-*chart with marked vertices* is a $\mathcal{C}$-chart such that each white vertex (or black vertex, resp.) is marked and the word $w_\Gamma(\partial v)$ is exactly an element of $\mathcal{R} \cup \mathcal{R}^{-1}$ (or of $\mathcal{S}$, resp.).

A white vertex is said to be *of type* r (or *of type* r^{-1}, resp.) if $w_\Gamma(\partial v)$ is a cyclic permutation of $r \in \mathcal{R}$ (or $r^{-1} \in \mathcal{R}^{-1}$, resp.).

When a base point y_0 of Σ is specified, we always assume that a $\mathcal{C}$-chart Γ is disjoint from y_0.

Example 2. Let $\mathcal{X} = \{\sigma_1, \dots, \sigma_{m-1}\}$, $\mathcal{R} = \{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ (for $i = 1, \dots, m-2$), $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ (for i, j with $i+1 < j$)$\}$, and $\mathcal{S} = \{\sigma_i, \sigma_i^{-1}$(for $i = 1, \dots, m-1$)$\}$. Then $\langle \mathcal{X}, \mathcal{R} \rangle$ is a presentation of the m-braid group B_m, called Artin's presentation. This $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ is used for chart descriptions of simple 2-dimensional braids and braided surfaces (§ 9).

Vertices are as in Figure 6, where labels σ_i are abbreviated to their subscripts. Vertices in the top row are boundary vertices, where vertical segments are portions of $\partial\Sigma$. White vertices in the second row are of types the relator $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$ and its inverse, respectively. White vertices in the third row correspond to the relator $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ with $i+1<j$ and its inverse, respectively. Black vertices in the bottom row correspond to σ_i and σ_i^{-1}.

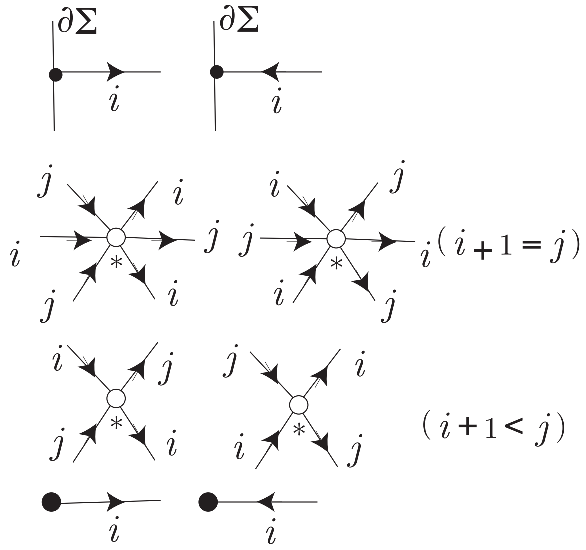

FIGURE 6

Example 3. Let $\mathcal{X} = \{1, 2\}$, $\mathcal{R} = \{1212^{-1}1^{-1}2^{-1}, (12)^6\}$, and $\mathcal{S} = \{1, 2, 1^{-1}, 2^{-1}\}$. Then $\langle \mathcal{X}, \mathcal{R}\rangle$ is a presentation of $SL(2, \mathrm{Z})$, which is isomorphic to the mapping class group of a torus. This $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ is used for chart descriptions of genus-1 Lefschetz fibrations (§ 10). Vertices are as in Figure 7.

Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$, and let Γ be a $\mathcal{C}$-chart in an oriented surface Σ with base point y_0. We denote by Δ_Γ the set of black vertices of Γ. The *homomorphism determined by* Γ means a homomorphism

$$\rho_\Gamma : \pi_1(\Sigma \setminus \Delta_\Gamma, y_0) \to \langle \mathcal{X}; \mathcal{R}\rangle$$

sending $[\eta]$ to $[w_\Gamma(\eta)]$ for a representative η which intersects Γ transversely. It is well-defined, since white vertices correspond to elements of $\mathcal{R} \cup \mathcal{R}^{-1}$.

Example 4. Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ be as in Example 3, and let Γ be a chart in a 2-disk depicted in Figure 8. When we take loops $C_1, \ldots, C_4$

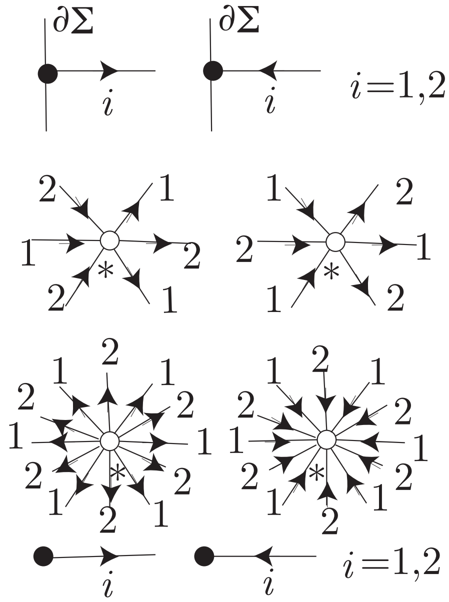

FIGURE 7

as in the figure, the homomorphism $\rho_\Gamma : \pi_1(\Sigma \setminus \Delta_\Gamma, y_0) \to \langle \mathcal{X}; \mathcal{R}\rangle$ sends them as follows.

$$\begin{array}{ll} \rho_\Gamma([C_1]) = 1, & \rho_\Gamma([C_2]) = 21^{-1}2^{-1}1^{-1}212^{-1} = 2^{-1}, \\ \rho_\Gamma([C_3]) = 21^{-1}2^{-1}, & \rho_\Gamma([C_4]) = 2. \end{array}$$

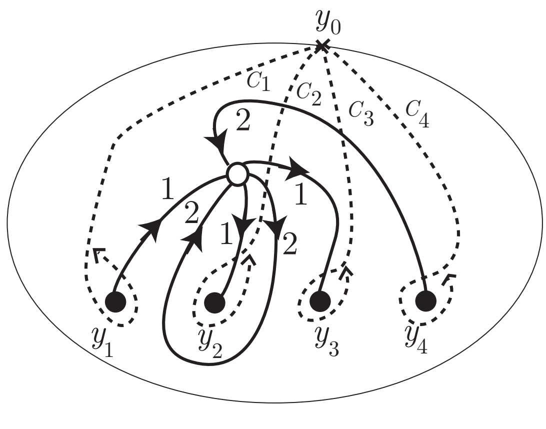

FIGURE 8

4. G-MONODROMY REPRESENTATIONS

Let Σ be a connected oriented surface, Δ a finite subset of $\mathrm{Int}\Sigma$, and y_0 a base point of $\Sigma \setminus \Delta$. When $\partial\Sigma \neq \emptyset$, we assume that the base point y_0 is in $\partial\Sigma$. Moreover, when $\partial\Sigma \cong S^1$, $\partial\Sigma$ also means a simple loop along $\partial\Sigma$ with base point y_0 whose orientation is induced from Σ. We shall denote by Σ_g a closed connected and oriented surface of genus g for $g \geq 0$, and by $\Sigma_{g,1}$ a compact surface Σ_g removed an open disk.

Let G be a group. By a *G-monodromy representation*, we mean a homomorphism $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$.

Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ be a triple as before with $G = \langle \mathcal{X}; \mathcal{R} \rangle$.

By $\mathcal{M}(\Sigma, \Delta, y_0; \mathcal{C})$, we denote the set of G-monodromy representations $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ such that for each meridional loop ℓ, the element $\rho([\ell])$ is a conjugate in G of an element of $\mathcal{S}$, where a "meridional loop" is a loop obtained from a meridian loop m_v of a point v of Δ by connecting with the base point y_0 along a path in $\Sigma \setminus \Delta$.

Let $\Sigma = \Sigma_{g,1}$ and let Λ be a finite set of points in $\partial\Sigma$ missing y_0 such that each point is co-oriented in $\partial\Sigma$ and labeled by an element of $\mathcal{X}$. We define the *intersection word* $w_\Lambda(\partial\Sigma)$ by reading off the labels of Λ with exponents determined by co-orientations of the points and the orientation of $\partial\Sigma$.

Theorem 5. *Let $\Sigma = \Sigma_{g,1}$, and Λ be as above. For any G-monodromy representation belonging to $\mathcal{M}(\Sigma, \Delta, y_0; \mathcal{C})$ such that $\rho([\partial\Sigma]) = [w_\Lambda(\partial\Sigma)]$, there exists a $\mathcal{C}$-chart Γ such that $\rho_\Gamma = \rho$ and $\Gamma \cap \partial\Sigma = \Lambda$.*

Proof. Let D be a 2-disk in Σ such that $D \cap \partial\Sigma = \{y_0\}$ and $\Delta \subset \mathrm{Int}(D)$. Let $\alpha_1, \dots, \alpha_n$ be mutually disjoint simple paths in D except at the common starting point $y_0 \in \partial D$ such that they appear in this order around y_0 and that their terminal points are the points of Δ. We denote by a_i $(1 \leq i \leq n)$ the element of $\pi_1(D \setminus \Delta, y_0)$ represented by a meridional loop starting from y_0, going along α_i toward the endpoint (say y_i) of α_i, turning around y_i in the positive direction (i.e., a meridian loop of y_i) and going back to y_0 along α_i. Then $\pi_1(D \setminus \Delta, y_0)$ is freely generated by the elements $a_1, \dots, a_n$.

Let $\beta_1, \beta_2, \dots, \beta_{2g}$ be simple closed paths in Σ which are mutually disjoint except at the common base point y_0 such that (1) we obtain a 2-disk D' by cutting Σ along these paths, (2) the 2-disk D is contained in this 2-disk D', and (3) we have

$$a_1 \cdots a_n [b_1, b_2] \cdots [b_{2g-1}, b_{2g}] = [\partial\Sigma]$$

in $\pi_1(\Sigma \setminus \Delta, y_0)$, where b_i $(1 \leq i \leq 2g)$ is the homotopy class of β_i and $[a, b]$ stands for the commutator $aba^{-1}b^{-1}$ of a and b. For example, see Figure 9, where $n = 4$ and $g = 2$.

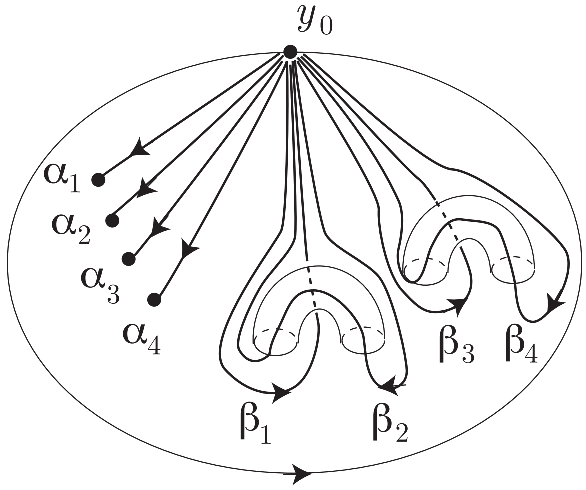

Figure 9

A G-monodromy representation $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ is completely determined by the values $\rho(a_1), \ldots, \rho(a_n)$ and $\rho(b_1), \ldots, \rho(b_{2g})$. Since $\rho \in \mathcal{M}(\Sigma, \Delta, y_0; \mathcal{C})$, $\rho(a_i)$ is a conjugate of an element of $\mathcal{S}$.

We decompose a regular neighborhood of $(\cup_{i=1}^{n} \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j) \cup \partial\Sigma$ in Σ into $n+1$ disks and $n+2g+1$ bands as follows: For each i $(1 \leq i \leq n)$, let $y_i \in \Delta$ be the terminal point of α_i. Let $N(y_i)$ be a regular neighborhood of y_i in Σ for i $(0 \leq i \leq n)$, and put $W = \mathrm{Cl}(\Sigma \setminus \cup_{k=0}^{n} N(y_k))$, where "Cl" means closure. Let $N'(\alpha_i)$ $(1 \leq i \leq n)$ be a regular neighborhood of $\alpha_i \cap W$ in W, let $N'(\beta_j)$ $(1 \leq j \leq 2g)$ be a regular neighborhood of $\beta_j \cap W$ in W, and let $N'(\partial\Sigma)$ be a regular neighborhood of $\partial\Sigma \cap W$ in W. Then the union of $n+1$ disks $N(y_0), \ldots, N(y_n)$ and $n+2g+1$ bands $N'(\alpha_1), \ldots, N'(\alpha_n)$, $N'(\beta_1), \ldots, N'(\beta_{2g})$, $N'(\partial\Sigma)$ is a regular neighborhood of $(\cup_{i=1}^{n} \alpha_i) \cup (\cup_{j=1}^{2g} \beta_j) \cup \partial\Sigma$ in Σ.

We construct a desired chart Γ piece by piece. Define $\Gamma \cap N(y_0)$ to be empty. For each i $(1 \leq i \leq n)$, the monodromy $\rho(a_i)$ is a conjugate of an element represented by a word, say s_i, of $\mathcal{S}$. So $\rho(a_i)$ is represented by a word $w_i s_i w_i^{-1}$ for some word w_i in $\mathcal{X} \cup \mathcal{X}^{-1}$. Define $\Gamma \cap N(y_i)$ to be a union of radial arcs in $N(y_i)$ connecting the center y_i and some points of $\partial N(y_i)$ missing $N'(\alpha_i)$ whose labels and co-orientations are determined by the word s_i, where we take the point $\alpha_i \cap \partial N(y_i)$ as starting point of $\partial N(y_i)$.

Define $\Gamma \cap N'(\alpha_i)$ to be a union of some parallel arcs in $N'(\alpha_i)$ missing $N(y_0)$ and $N(y_i)$ such that they are labeled and co-oriented so that the intersection word of α_i (restricted to $N'(\alpha_i)$) is equal to the word w_i. See Figure 10, where $w_i = ba^{-1}c^{-1}$ and $s_i = d^{-1}bd$.

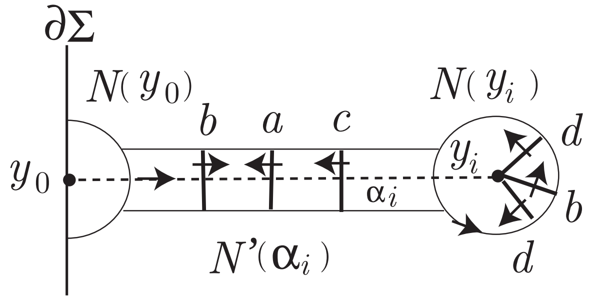

FIGURE 10

For each j $(1 \leq j \leq 2g)$, we define $\Gamma \cap N'(\beta_j)$ to be the union of some parallel arcs in $N'(\beta_j)$ which are labeled and co-oriented such that the intersection word of β_j is a word representing the monodromy $\rho(b_j)$.

For $N'(\partial\Sigma)$, we construct $\Gamma \cap N'(\partial\Sigma)$ by using parallel arcs which are labeled and co-oriented such that $\Gamma \cap \partial\Sigma = \Lambda$.

We have constructed Γ on the neighborhood $N((\cup_{i=1}^{n}\alpha_i) \cup (\cup_{j=1}^{2g}\beta_j) \cup \partial\Sigma)$ of $(\cup_{i=1}^{n}\alpha_i) \cup (\cup_{j=1}^{2g}\beta_j) \cup \partial\Sigma$. Let E be $\mathrm{Cl}(\Sigma \setminus N((\cup_{i=1}^{n}\alpha_i) \cup (\cup_{j=1}^{2g}\beta_j) \cup \partial\Sigma))$, which is a 2-disk in Σ. By the construction, the intersection word $w_\Gamma(\partial E)^{-1}$ is a word representing

$$\rho(a_1) \cdots \rho(a_n)[\rho(b_1), \rho(b_2)] \cdots [\rho(b_{2g-1}), \rho(b_{2g})]\rho([\partial\Sigma])^{-1}$$

in G. Since ∂E is null-homotopic in $\Sigma \setminus \Delta$, this word $w_\Gamma(\partial E)^{-1}$ represents the identity element of $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$. Thus there exists a finite sequence of words in $\mathcal{X} \cup \mathcal{X}^{-1}$ starting from the word $w_\Gamma(\partial E)^{-1}$ to the empty word such that each word is related to the previous one by one of the following transformations;

- insertion/deletion of a trivial relator $x^\varepsilon x^{-\varepsilon}$ for $x \in \mathcal{X}$ and $\varepsilon \in \{+1, -1\}$,
- insertion of r^ε or $r^{-\varepsilon}$ for $r \in \mathcal{R}$ and $\varepsilon \in \{+1, -1\}$.

(Deletion of r^ε is obtained from insertion of $r^{-\varepsilon}$ and deletion of trivial relators.) Therefore, by the same argument as in p. 147 of [10] or in Chapter 18 of [13], we can extend the chart Γ defined on $N((\cup_{i=1}^{n}\alpha_i) \cup (\cup_{j=1}^{2g}\beta_j) \cup \partial\Sigma)$ to a chart in Σ. This is a desired chart Γ in Σ, since by construction, we have $\rho_\Gamma(a_i) = \rho(a_i)$ for i $(1 \leq i \leq n)$, $\rho_\Gamma(b_j) = \rho(b_j)$ for j $(1 \leq j \leq 2g)$ and $\Gamma \cap \partial\Sigma = \Lambda$. $\square$

Corollary 6. *Let $\Sigma = \Sigma_g$. For any G-monodromy representation belonging to $\mathcal{M}(\Sigma, \Delta, y_0; \mathcal{C})$, there exists a $\mathcal{C}$-chart Γ such that $\rho_\Gamma = \rho$.*

Proof. This follows from Theorem 5 with $\Lambda = \emptyset$. □

5. Moves for charts

Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$, and let Γ and Γ' be $\mathcal{C}$-charts on a surface Σ.

Let E be a disk region in $\mathrm{Int}\Sigma$ which misses the base point y_0 of Σ. Suppose that ∂E intersects Γ and Γ' transversely (or ∂E is disjoint from them).

Definition 7. Suppose that $\Gamma \cap \mathrm{Cl}(\Sigma \setminus E) = \Gamma' \cap \mathrm{Cl}(\Sigma \setminus E)$ and that $\Gamma \cap E$ and $\Gamma' \cap E$ have no black vertices. Then we say that Γ' is obtained from Γ by a *chart move of type W*.

We call moves in Figure 11 *basic chart moves of type W*. A move in the top row is called a *channel change*. In the second row, the simple loop may be oriented (or co-oriented) in any direction. We call the move a *birth/death of a hoop*. In the third or fourth row, two white vertices are of type r and of type r^{-1}. We call this move a *birth/death of a pair of white vertices*. The fourth move is equivalent to the third one modulo the first and the second ones.

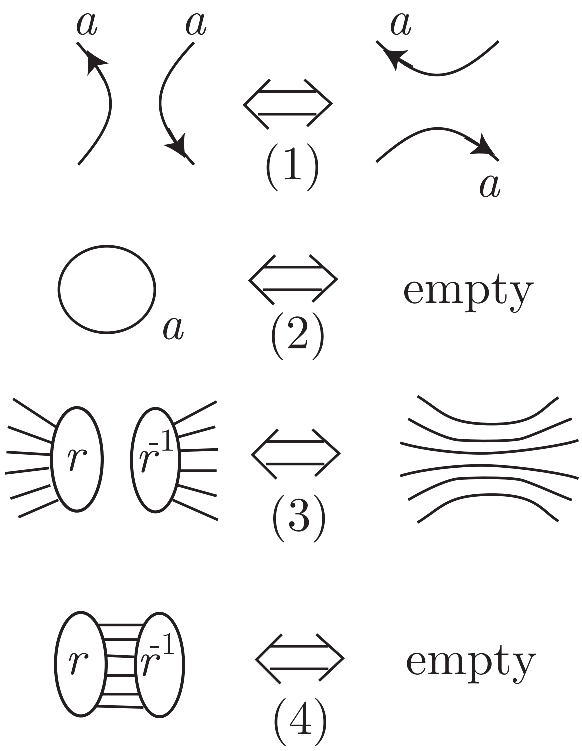

FIGURE 11. Basic chart moves of type W

Lemma 8. *Any chart move of type W does not change the homomorphism ρ_Γ determined by a chart Γ.*

Proof. Since E is a 2-disk in $\mathrm{Int}\Sigma$ missing $\Delta_\Gamma \cup \{y_0\}$, any element of $\pi_1(\Sigma \setminus \Delta_\Gamma, y_0)$ has a representative loop ℓ which is disjoint from E and intersects Γ transversely. Then the intersection word $w_\Gamma(\ell)$ does not change by any chart move on E. Thus $\rho_\Gamma([\ell])$ is preserved. $\square$

Let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$, Γ, Γ' and E be as above.

Definition 9. Suppose that $\Gamma \cap \mathrm{Cl}(\Sigma \setminus E) = \Gamma' \cap \mathrm{Cl}(\Sigma \setminus E)$ and that Γ' differs from Γ in E as one of Figure 12. Then we say that Γ' is obtained from Γ by a *chart move of type B*.

Now let E be a 2-disk in Σ missing y_0 such that $E \cap \partial\Sigma$ is an arc.

Definition 10. Suppose that $\Gamma \cap \mathrm{Cl}(\Sigma \setminus E) = \Gamma' \cap \mathrm{Cl}(\Sigma \setminus E)$ and that Γ' differs from Γ in E as one of Figure 13, where the vertical arcs are $E \cap \partial\Sigma$. Then we say that Γ' is obtained from Γ by a *chart move of type ∂*.

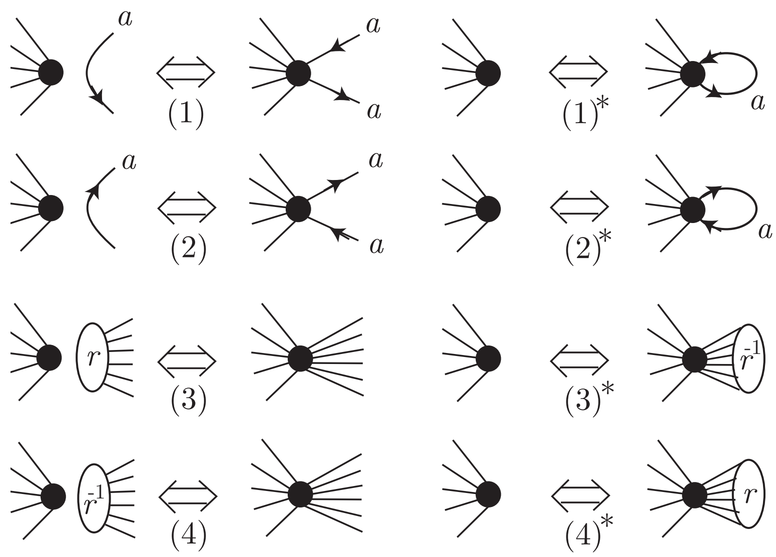

FIGURE 12. Chart moves of type B

For $i = 1, \ldots, 4$, the move (i) in Figure 12 (or in Figure 13) is equivalent to the move $(i)^*$ modulo basic chart moves of type W.

Definition 11. A triple $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ (or $\mathcal{S}$) is *full* when it has the following property:

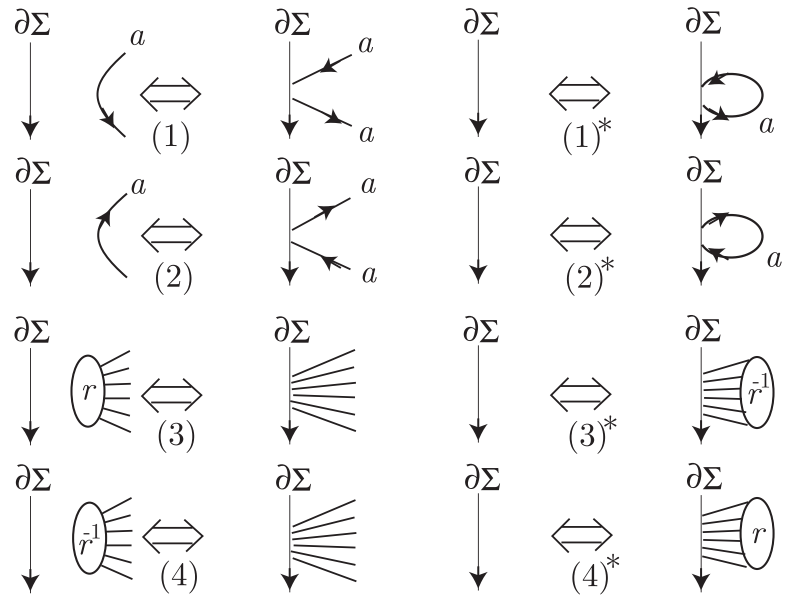

FIGURE 13. Chart moves of type ∂

- For any two words w and w' in $\mathcal{X} \cup \mathcal{X}^{-1}$ such that $[w]$ is a conjugate of $[w']$ in the group $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$, if $w \in \mathcal{S}$ then $w' \in \mathcal{S}$.

Note that, when we apply a move illustrated in Figure 12 to a $\mathcal{C}$-chart Γ, the result, say Γ', may not be a $\mathcal{C}$-chart. However if $\mathcal{C}$ is full, then Γ' is a $\mathcal{C}$-chart.

Theorem 12. *Let $\mathcal{C}$ be full, and let $\Sigma = \Sigma_{g,1}$. If two $\mathcal{C}$-charts Γ and Γ' determine the same homomorphism $\rho_\Gamma = \rho_{\Gamma'} : \pi_1(\Sigma \setminus \Delta, y_0) \to G$, then Γ is transformed to Γ' by a finite sequence of chart moves of type W, chart moves of type B, chart moves of type ∂ and isotopies of Σ rel $\Delta \cup \{y_0\}$. Moreover, when $\Gamma \cap \partial\Sigma = \Gamma' \cap \partial\Sigma$, we do not need chart moves of type ∂ in the sequence.*

Proof. Let $\alpha_i, \beta_j, a_i, b_j, y_i, N(y_i), N'(\alpha_i), N'(\beta_j), N'(\partial\Sigma), W = \mathrm{Cl}(\Sigma \setminus \cup_{k=0}^{n} N(y_k))$, $E = \mathrm{Cl}(\Sigma \setminus N((\cup_{i=1}^{n}\alpha_i) \cup (\cup_{j=1}^{2g}\beta_j) \cup \partial\Sigma))$ be as in the proof of Theorem 5. Let $N(\alpha_i)$ be $N(y_0) \cup N'(\alpha_i) \cup N(y_i)$ $(1 \le i \le n)$.

By an isotopy of Σ rel $\Delta \cup \partial\Sigma$, we deform Γ so that, for each $i = 1, \dots, n$, $\Gamma \cap N(\alpha_i)$ is as in Figure 10, i.e., $\Gamma \cap N(y_0) = \emptyset$, $\Gamma \cap N'(\alpha_i)$ is a union of some parallel arcs, and $\Gamma \cap N(y_i)$ is a union of radial arcs. Using chart moves (1) and (2) in Figure 12, we move all arcs in $\Gamma \cap N'(\alpha_i)$ toward the black vertex y_i and hence we may assume that $\Gamma \cap N'(\alpha_i) = \emptyset$. Then $\rho_\Gamma(a_i)$ is represented by $w_\Gamma(\partial y_i) = w_\Gamma(\partial N(y_i))$, where we assume the starting point of $\partial N(y_i)$ is the intersection $\alpha_i \cap \partial N(y_i)$. Do the same for Γ'. Since $\rho_\Gamma(a_i) = \rho_{\Gamma'}(a_i)$, the word $w_\Gamma(\partial y_i)$ is transformed to $w_{\Gamma'}(\partial y_i)$ by a finite number of insertion/deletion of trivial relators and relators in $\mathcal{R}$. According to it, we transform Γ by chart moves of type B so that $\Gamma \cap N(\alpha_i) = \Gamma' \cap N(\alpha_i)$.

By an isotopy of Σ whose support is a neighborhood of $N'(\beta_j)$ ($j = 1, \dots, 2g$), we deform Γ so that $\Gamma \cap N'(\beta_j)$ is a union of parallel arcs. Identify the neighborhood $N'(\beta_j)$ of $\beta_j \cap W$ in W with $(\beta_j \cap W) \times [-1, 1]$ such that $(\beta_j \cap W) \times \{0\} = \beta_j \cap W$. Consider a chart in $(\beta_j \cap W) \times [0, 1]$ whose restriction to $(\beta_j \cap W) \times \{1\}$ is equal to that of Γ and the restriction to $(\beta_j \cap W) \times \{0\} = \beta_j \cap W$ is equal to that of Γ' such that there are no black vertices. Since $[w_\Gamma(\beta_j)] = \rho_\Gamma(b_j) = \rho_{\Gamma'}(b_j) = [w_{\Gamma'}(\beta_j)]$, such a chart in $(\beta_j \cap W) \times [0, 1]$ exists. The union of this chart and its mirror image in $(\beta_j \cap W) \times [-1, 0]$ forms a chart in $N'(\beta_j) = (\beta_j \cap W) \times [-1, 1]$ without black vertices. Replacement of $\Gamma \cap N'(\beta_j)$ by this chart is a chart move of type W. Then the new chart, say Γ again, satisfies that $\Gamma \cap \beta_j = \Gamma' \cap \beta_j$. By an isotopy of Σ, we may assume that $\Gamma \cap N'(\beta_j) = \Gamma' \cap N'(\beta_j)$ for each $j = 1, \dots, 2g$.

Since $[w_\Gamma(\partial\Sigma)] = \rho_\Gamma([\partial\Sigma]) = \rho_{\Gamma'}([\partial\Sigma]) = [w_{\Gamma'}(\partial\Sigma)]$, we can transform Γ in $N'(\partial\Sigma)$, by chart moves of type ∂ and an isotopy of Σ rel $\Delta \cup \{y_0\}$, so that $\Gamma \cap N'(\partial\Sigma) = \Gamma' \cap N'(\partial\Sigma)$. (When $\Gamma \cap \partial\Sigma = \Gamma' \cap \partial\Sigma$, this is done by an isotopy of Σ rel $\Delta \cup \partial\Sigma$.)

Now Γ and Γ' are identical except E. Apply a chart move of type W in E, and we change Γ to Γ'. $\square$

Corollary 13. *Let $\mathcal{C}$ be full, and let $\Sigma = \Sigma_g$. If two $\mathcal{C}$-charts Γ and Γ' determine the same homomorphism $\rho_\Gamma = \rho_{\Gamma'} : \pi_1(\Sigma \setminus \Delta, y_0) \to G$, then Γ is transformed to Γ' by a finite sequence of chart moves of type W, chart moves of type B, and isotopies of Σ rel $\Delta \cup \{y_0\}$.*

For $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$, we denote by $\overline{\mathcal{S}}$ the set of all words in $\mathcal{X} \cup \mathcal{X}^{-1}$ each of which represents a conjugate of an element of $\mathcal{S}$ in $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$, and we denote by $\overline{\mathcal{C}}$ the triple $(\mathcal{X}, \mathcal{R}, \overline{\mathcal{S}})$. Then the triple $\overline{\mathcal{C}}$ is full.

A $\mathcal{C}$-chart is a $\overline{\mathcal{C}}$-chart. If two $\mathcal{C}$-charts Γ and Γ' in $\Sigma = \Sigma_{g,1}$ or in $\Sigma = \Sigma_g$ determine the same homomorphism, by Theorem 12 or Corollary 13, there exists a sequence of $\overline{\mathcal{C}}$-charts from Γ to Γ' related by chart moves (as $\overline{\mathcal{C}}$-charts) and isotopies of Σ. However these $\overline{\mathcal{C}}$-charts are not $\mathcal{C}$-charts in general. In the next section, we shall study about existence of a sequence of $\mathcal{C}$-charts by modifying the definition of chart moves.

6. Modified version of chart moves

In this section, we study chart moves for $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ that is possibly not full.

Let s and s' be words in $\mathcal{S}$ and suppose that $[s'] = [wsw^{-1}]$ in $G = \langle \mathcal{X} \mid \mathcal{R} \rangle$ for some word w in $\mathcal{X} \cup \mathcal{X}^{-1}$. Then there exists a chart in a rectangle R without black vertices such that the intersection words of the four edges are w, s, w^{-1} and $(s')^{-1}$, respectively. Let $T(s \to s', w)$ be such a chart. A local replacement of a chart depicted in Figure 14 changes a black vertex v of a chart Γ to a vertex v' of a new chart Γ' such that $w_\Gamma(\partial v) = s$ and $w_{\Gamma'}(\partial v') = s'$. (In the figure, T stands for $T(s \to s', w)$ and the symbols s, s', w nearby dashed arrows mean that the intersection words of the arrows are these words.)

Definition 14. A *chart move of transition*, or a *transition of a black vertex*, is a local replacement depicted in Figure 14.

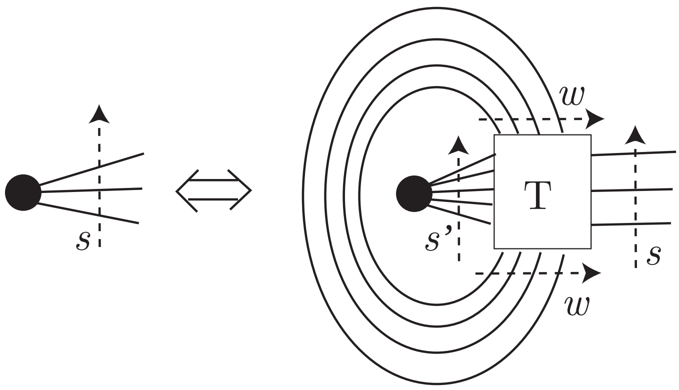

Figure 14. Chart move of transition

Remark 15. A chart move of type B is a consequence of a chart move of transition (using $T = T(s \to s', w)$ with $w = \emptyset$) modulo basic chart moves of type W. For example, see Figure 15 for the move (3) in Figure 12.

Conversely, a chart move of transition is a consequence of chart moves of type B. This is seen as follows. Let Γ' be obtained from Γ by a transition using $T = T(s \to s', w)$ in a rectangle R. Modify Γ' by an isotopy of R rel ∂R and split R into strips such that strips containing white vertices of Γ' are as in the left of Figure 16. (For a strip containing a white vertex, all arcs attached to the white vertex are connected to the top side of the strip, and the other arcs are vertical.) Apply chart moves of type B and transform the chart as in the right of Figure 16.

Then there exist no white vertices in R. If necessary applying chart moves (1) and (2) in Figure 12, we may assume that the chart is a union of Γ and a bouquet. The bouquet can be removed by moves $(1)^*$ and $(2)^*$ and we get Γ.

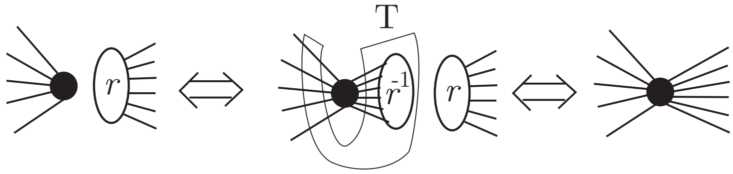

Figure 15

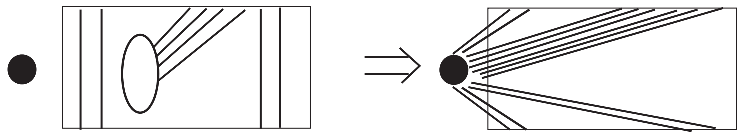

Figure 16

Theorem 16. *Let $\Sigma = \Sigma_{g,1}$. If two $\mathcal{C}$-charts Γ and Γ' determine the same homomorphism $\rho_\Gamma = \rho_{\Gamma'} : \pi_1(\Sigma \setminus \Delta, y_0) \to G$, then Γ is transformed to Γ' by a finite sequence of chart moves of type W, chart moves of transition, chart moves of type ∂ and isotopies of Σ rel $\Delta \cup \{y_0\}$. Moreover, when $\Gamma \cap \partial\Sigma = \Gamma' \cap \partial\Sigma$, we do not need chart moves of type ∂ in the sequence.*

Proof. In the proof of Theorem 12, when we transform Γ so that $\Gamma \cap N(\alpha_i) = \Gamma' \cap N(\alpha_i)$, we used chart moves of type B. Thus it is sufficient to show that we can transform Γ so that $\Gamma \cap N(\alpha_i) = \Gamma' \cap N(\alpha_i)$ by chart moves of transition and chart moves of type W. We may suppose that $\Gamma \cap N(\alpha_i)$ is as in Figure 10, i.e., $\Gamma \cap N(y_0) = \emptyset$, $\Gamma \cap N'(\alpha_i)$ is a union of some parallel arcs, and $\Gamma \cap N(y_i)$ is a union of radial arcs. Suppose that Γ' is so. Put $w_i = w_\Gamma(\alpha_i \cap W)$, $s_i = w_\Gamma(\partial N(y_i))$, $w'_i = w_{\Gamma'}(\alpha_i \cap W)$, $s'_i = w_{\Gamma'}(\partial N(y_i))$. Since $[w_i s_i w_i^{-1}] = [w'_i s'_i (w'_i)^{-1}]$, we can apply a chart move of transition with $T = T(s_i \to s'_i, (w'_i)^{-1} w_i)$ to the black vertex of Γ, and let $\tilde{\Gamma}$ be the result. Here we assume that α_i is disjoint from the rectangle containing T. Then $\tilde{w}_i = w_{\tilde{\Gamma}}(\alpha_i \cap W)$ and $\tilde{s}_i = w_{\tilde{\Gamma}}(\partial N(y_i))$ for this chart $\tilde{\Gamma}$ are $w_i w_i^{-1} w'_i$ and s'_i, respectively.

Since $\tilde{s}_i = s'_i$, by an isotopy of Σ rel $\Delta \cup \{y_0\}$, we can deform $\tilde{\Gamma}$ so that $\tilde{\Gamma}$ and Γ' are identical on $N(y_i)$.

Since $\tilde{w}_i = w_i w_i^{-1} w_i'$ and w_i' represent the same element of G, we can transform $\tilde{\Gamma}$ by chart moves of type W so that $\tilde{\Gamma}$ and Γ' are identical on $N'(\alpha_i \cap W)$. (Recall the argument in the proof of Theorem 12 which is used in order to change the chart on $N'(\beta_j)$.)

Therefore $\tilde{\Gamma}$ and Γ' are identical on $N(\alpha_i)$. Continue the remainder of the proof of Theorem 12, and we see that Γ and Γ' are related by moves stated in the theorem. □

Corollary 17. *Let $\Sigma = \Sigma_g$. If two $\mathcal{C}$-charts Γ and Γ' determine the same homomorphism $\rho_\Gamma = \rho_{\Gamma'} : \pi_1(\Sigma \setminus \Delta, y_0) \to G$, then Γ is transformed to Γ' by a finite sequence of chart moves of type W, chart moves of transition, and isotopies of Σ rel $\Delta \cup \{y_0\}$.*

7. Conjugacy equivalence on G-monodromy

Two G-monodromy representations $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ and $\rho' : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ are *conjugacy equivalent* if there is an inner-automorphism ι of G such that $\rho' = \iota \circ \rho$.

Definition 18. *Chart moves of conjugacy type* are local moves depicted in Figure 17.

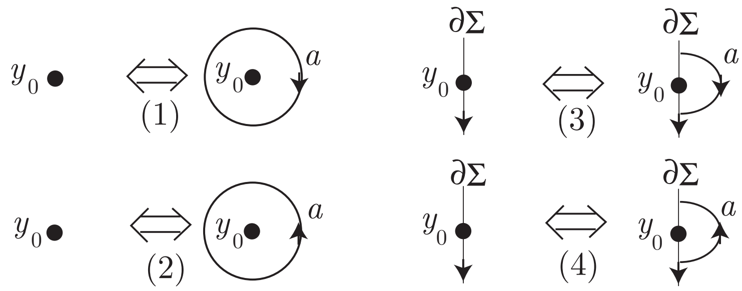

Figure 17. Chart moves of conjugacy

Chart moves of conjugacy type change ρ_Γ to $\iota_a \circ \rho_\Gamma$ or $\iota_a^{-1} \circ \rho_\Gamma$, where ι_a is the inner-automorphism of G by a. For a generic inner-automorphism of G, iterate such moves suitably.

When we replace a condition that $\rho_\Gamma = \rho_{\Gamma'}$ by a condition that ρ_Γ and $\rho_{\Gamma'}$ are conjugacy equivalent, Theorems 12, 16 and Corollaries 13, 17 are valid by adding chart moves of conjugacy type.

8. Isotopic and homeomorphic equivalence on G-monodromy

Two G-monodromy representations $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ and $\rho' : \pi_1(\Sigma \setminus \Delta', y_0) \to G$ are *isotopically equivalent* if there is an isotopy of

Σ rel $\partial\Sigma \cup \{y_0\}$ such that the initial map is the identity map of Σ and the terminal map, say h, sends Δ to Δ' and induces an isomorphism $h_* : \pi_1(\Sigma \setminus \Delta, y_0) \to \pi_1(\Sigma \setminus \Delta', y_0)$ with $\rho' = \rho \circ h_*^{-1}$.

In the above situation, if $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ is described by a chart Γ (i.e., $\rho = \rho_\Gamma$), then $h(\Gamma)$ is a chart description of ρ'.

When we replace a condition that $\rho_\Gamma = \rho_{\Gamma'}$ by a condition that $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ and $\rho' : \pi_1(\Sigma \setminus \Delta', y_0) \to G$ are isotopically equivalent, Theorems 12, 16 and Corollaries 13, 17 are valid by adding isotopies of Σ rel $\partial\Sigma \cup \{y_0\}$.

Two G-monodromy representations $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ and $\rho' : \pi_1(\Sigma' \setminus \Delta', y_0') \to G$ are *homeomorphically equivalent* if there is an orientation-preserving homeomorphism $h : \Sigma \to \Sigma'$ with $h(\Delta) = \Delta'$ and $h(y_0) = y_0'$ such that h induces an isomorphism $h_* : \pi_1(\Sigma\setminus\Delta, y_0) \to \pi_1(\Sigma' \setminus \Delta', y_0')$ with $\rho' = \rho \circ h_*^{-1}$.

In the above situation, if $\rho : \pi_1(\Sigma \setminus \Delta, y_0) \to G$ is described by a chart Γ, then $h(\Gamma)$ is a chart description of ρ'.

When we replace a condition that $\rho_\Gamma = \rho_{\Gamma'}$ by a condition that $\rho : \pi_1(\Sigma\setminus\Delta, y_0) \to G$ and $\rho' : \pi_1(\Sigma\setminus\Delta', y_0) \to G$ are homeomorphically equivalent, Theorems 12, 16 and Corollaries 13, 17 are valid by adding to send a chart by an orientation-preserving homeomorphism $h : \Sigma \to \Sigma'$ with $h(\Delta) = \Delta'$ and $h(y_0) = y_0'$.

9. Chart descriptions of braided surfaces

Throughout this section, let $\mathcal{C} = (\mathcal{X}, \mathcal{R}, \mathcal{S})$ be as in Example 2, and let Σ be $\Sigma_{g,1}$ or Σ_0 with base point y_0. (When $\Sigma = \Sigma_{g,1}$, we assume that $y_0 \in \partial\Sigma$.)

Let D^2 be a 2-disk, let $pr_1 : D^2 \times \Sigma \to D^2$ and $pr_2 : D^2 \times \Sigma \to \Sigma$ be the projections, and let Q_m be a fixed m interior points of D^2. We identify the m-braid group B_m with the fundamental group $\pi_1(C_m, Q_m)$ of the configuration space C_m of m points of $\mathrm{Int}D^2$ (cf. [1]).

A *braided surface* over Σ of degree m (cf. [13, 26, 27]) is a compact oriented surface F embedded in $D^2 \times \Sigma$ such that $pr_2|_F : F \to \Sigma$ is a branched covering map of degree m and ∂F is a closed m-braid in the solid torus $D^2 \times \partial\Sigma$. We assume that $pr_1((pr_2|_F)^{-1}(y_0)) = Q_m$.

A 2-*dimensional* m-*braid* is a braided surface over a 2-disk $\Sigma = \Sigma_{0,1}$ such that ∂F is a trivial closed m-braid $Q_m \times \partial\Sigma$ in $D^2 \times \partial\Sigma$.

A braided surface is said to be *simple* if $pr_2|_F : F \to \Sigma$ is a simple branched covering map of degree m (i.e., $\#((pr_2|_F)^{-1}(y)) \geq m - 1$ for every $y \in \Sigma$).

Let Δ_F be the set of branch values of $pr_2|_F$. For a path $c : [0, 1] \to \Sigma \setminus \Delta_F$, we denote by $\ell_F(c)$ a path in the configuration space C_m with

$\ell_F(c)(t) = pr_1((pr_2|_F)^{-1}(c(t)))$. Then the B_m-monodromy representation

$$\rho_F : \pi_1(\Sigma \setminus \Delta_F, y_0) \to \pi_1(C_m, Q_m) = B_m$$

of F is defined by

$$\rho_F([c]) = [\ell_F(c)].$$

It is known that braided surfaces and 2-dimensional m-braids are determined by their B_m-monodromy representations (cf. [10, 11, 13, 27]). Applying Theorem 5 and Corollary 6, we have the following theorem.

Theorem 19 (Chart description of a simple braided surface, cf. [10])**.** *A $\mathcal{C}$-chart Γ in Σ determines a monodromy representation ρ_F of a simple braided surface F. Conversely the B_m-monodromy representation ρ_F of a simple braided surface F can be described as ρ_Γ for some $\mathcal{C}$-chart Γ.*

This theorem is valid for simple 2-dimensional braids ([10]), when we restrict charts to charts Γ with $\Gamma \cap \partial\Sigma = \emptyset$.

Example 20. A $\mathcal{C}$-chart in a 2-disk $\Sigma = \Sigma_{0,1}$ is depicted in Figure 18, where $m = 4$ and $\Sigma = [0,1] \times [0,1]$. By this bi-parameterization of Σ, we have a motion picture of the corresponding 2-dimensional 4-braid F. The vertical dashed lines in Figure 19 yield the intersection words, $\sigma_1\sigma_1^{-1}$, $\sigma_3^{-1}\sigma_1$, $\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1$, $\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}$, $\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$, $\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$, σ_3^{-1}. The motion picture is illustrated in Figure 20. Refer to [4, 13] for details and further topics.

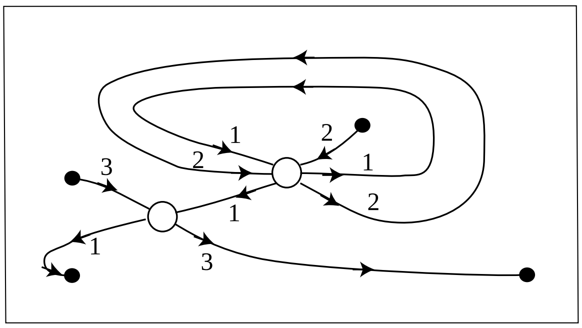

FIGURE 18

Some chart moves of type W are depicted in Figure 21. Moves (1), ..., (4) are basic chart moves of type W. Moves (5), (6), (7) are typical for our $\mathcal{C}$. It is known that any chart move of type W is a consequence of the moves listed in Figure 21, [3].

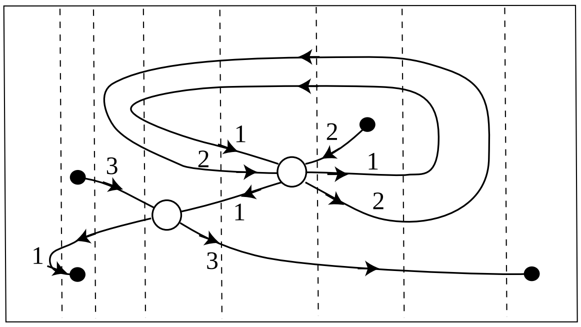

FIGURE 19

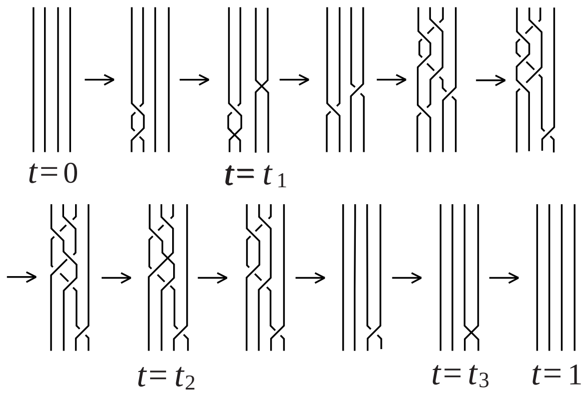

FIGURE 20

Since $\mathcal{C}$ is not full, we cannot apply the moves in Figure 12 in general. In order to connect two $\mathcal{C}$-charts determining the same B_m-monodromy representation, we need chart moves of transition. Moves in Figure 22 are chart moves of transition. Note that moves in Figure 22 are equivalent to the moves in Figure 23 modulo basic chart moves of type W. It is proved in [12] that any chart moves of transition is a consequence of the moves in Figure 23 (or equivalently, the moves in Figure 22) and chart moves of type W (cf. [13]).

We have the following theorem from Theorem 16 and Corollary 17, since chart moves of type W and chart moves of transition are consequence of particular ones in Figures 21 and 22. (This is proved in [12] for charts of 2-dimensional braids.)

Theorem 21. *Let $\Sigma = \Sigma_{g,1}$. If two $\mathcal{C}$-charts Γ and Γ' determine the same homomorphism $\rho_\Gamma = \rho_{\Gamma'} : \pi_1(\Sigma \setminus \Delta, y_0) \to B_m$, then Γ is transformed to Γ' by a finite sequence of chart moves of type W in Figure 21, chart moves of transition in Figure 22, chart moves of type*

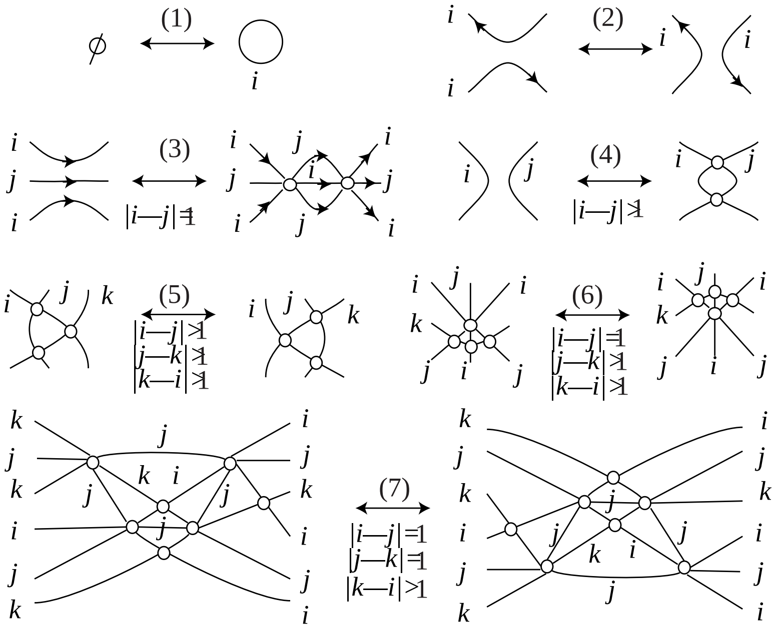

Figure 21

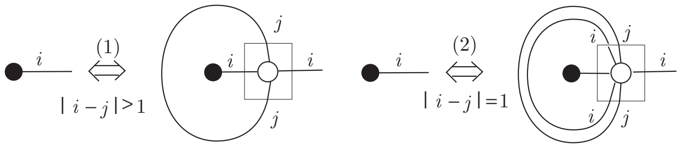

Figure 22

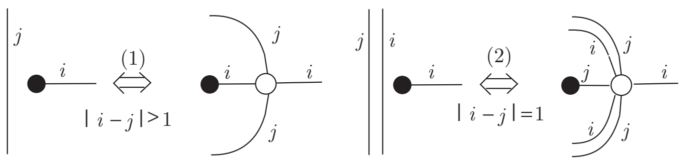

Figure 23

∂ and isotopies of Σ rel $\Delta \cup \{y_0\}$. Moreover, when $\Gamma \cap \partial\Sigma = \Gamma' \cap \partial\Sigma$, we do not need chart moves of type ∂ in the sequence.

For example, a chart depicted in Figure 18 is transformed to a simpler chart as illustrated in Figure 24.

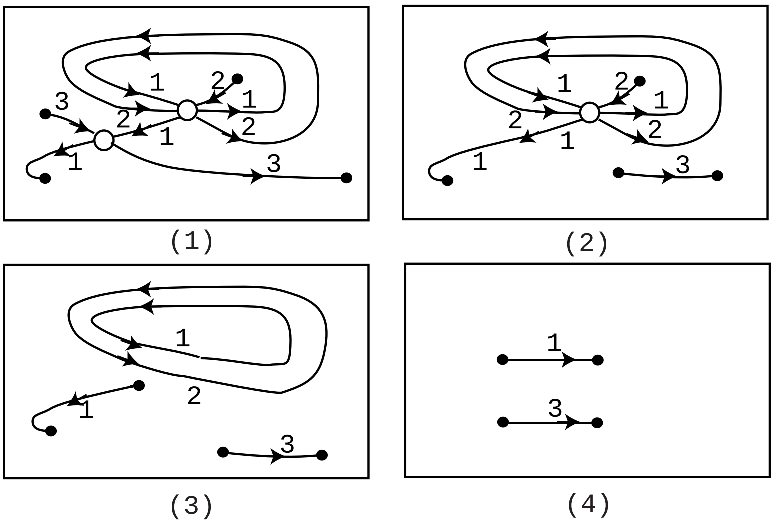

FIGURE 24

10. Genus-1 Lefschetz fibrations

Let M and Σ be closed connected and oriented 4- and 2-manifolds. Let $f : M \to \Sigma$ be a genus-1 Lefschetz fibration (in the sense of [14]) and let Δ_f be the set of singular values. It induces a G-monodromy representation

$$\rho_f : \pi_1(\Sigma \setminus \Delta_f, y_0) \to G,$$

where G is the mapping class group of $T^2 = f^{-1}(y_0)$.

It is known that genus-1 Lefschetz fibrations are determined by their monodromies if $n_+ \neq n_-$, where n_+ and n_- stand for the numbers of positive singular fibers and negative singular fibers, respectively (cf. [22]). (The condition $n_+ \neq n_-$ is equivalent to $\sigma(M) \neq 0$, where $\sigma(M)$ is the signature of M.) Refer to [9, 14, 15, 16, 17, 18, 19, 20, 21, 28], etc. for details and related topics on genus-1 Lefschetz fibrations.

Let $\mathcal{C}$ be as in Example 3. Then $G = \langle \mathcal{X} \,|\, \mathcal{R} \rangle$ is isomorphic to the mapping class group of a torus. Then we have the following.

Theorem 22 ([14])**.** *The set of (isomorphism classes of) genus-1 Lefschetz fibrations over Σ with $n_+ \neq n_-$, the set of (equivalence classes of) G-monodromy representations of genus-1 Lefschetz fibration with $n_+ \neq n_-$, and the set of chart move equivalences classes of $\mathcal{C}$-charts $n_+ \neq n_-$ are in one-to-one correspondence.*

Since we are assuming Σ to be closed, we do not need chart moves of type ∂. Since $\mathcal{C}$ is not full, we need chart moves of transition. However we do not need all chart moves of type W and chart moves of transition for chart moves in the above theorem. Actually, all moves used in [14] are basic chart moves of type W and chart moves of transition depicted in (2) of Figure 23 for $\{i, j\} = \{1, 2\}$. (The former moves are called *CI-moves* and the latter *CII-moves* in [14].)

Using the chart description method, an elementary proof to the following theorem is given in [14].

Theorem 23 (Classification theorem ([14, 20])). *Let $f : M \to \Sigma$ and $f' : M' \to \Sigma'$ be genus-1 Lefschetz fibrations with $n_+ \neq n_-$. The following conditions are mutually equivalent.*

(1) *f and f' are isomorphic as Lefschetz fibrations.*
(2) *$\sigma(M) = \sigma(M')$, $e(M) = e(M')$, and $g(\Sigma) = g(\Sigma')$.*
(3) *$M \cong M'$ and $B \cong B'$.*

Here $\sigma(M)$ and $e(M)$ are the signature and the Euler number of M. It is known ([6, 18]) that $\sigma(M) = -\frac{2}{3}(n_+ - n_-)$ and $e(M) = n_+ + n_-$.

Remark 24. A Lefschetz fibration with $n_- = 0$ is called a *chiral* Lefschetz fibration ([21]) or a *symplectic* Lefschetz fibration ([28]). For chart descriptions of genus-1 chiral Lefschetz fibrations, we use another $\mathcal{C}$ such that $\mathcal{X}$ and $\mathcal{R}$ are the same as in Example 3 and $\mathcal{S} = \{1, 2\}$.

Chart descriptions of monodromy representations of higher genus Lefschetz fibrations (cf. [5, 21, 28]) and chart descriptions of monodromy representations of algebraic curves (in the sense of [22, 23]) can be defined by Theorem 5 (Corollary 6) when we fix a suitable $\mathcal{C}$. Then we can use Theorem 12 (Corollary 13) and Theorem 16 (Corollary 17) to calculate the monodromy representations.

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