Output Feedback Controller Synthesis for Descriptor Systems Satisfying Closed-Loop Dissipativity

Izumi Masubuchi [a]

[a] *Graduate School of Engineering, Hiroshima University*
1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan
E-mail: msb@hiroshima-u.ac.jp

Abstract

This paper is concerned with synthesis of output feedback controllers for descriptor systems to attain dissipativity of the closed-loop system. A necessary and sufficient condition is provided in terms of LMIs for the existence of a controller satisfying dissipativity and admissibility (an internal stability) of the closed-loop system. Unlike previous results, the condition does not depend on the choice of the descriptor realization. The derived LMI condition with a rank constraint for synthesis is a generalization of those for LMI-based H_∞ control of state-space systems to descriptor systems.

Key words: Descriptor systems; synthesis; output feedback; dissipativity; LMIs

1 Introduction

Descriptor representation of dynamical systems is more general and often more natural than state-space systems (See e.g., Lewis (1986)). The descriptor form is useful to represent and to handle systems such as mechanical systems, electric circuits, interconnected systems, and so on. Moreover, descriptor representation is utilized in some of recent results on analysis of time-delay systems (e.g., Fridman & Shaked (2002)) and gain-scheduled control based on linear parameter-varying systems (Masubuchi, Akiyama, & Saeki, 2003; Masubuchi, Kato, Saeki, & Ohara, A., 2004). These new applications exhibit further importance and usefulness of descriptor systems.

Among basic notions of state-space systems generalized to descriptor systems, dissipativity is one of the most important properties of dynamical systems and plays crucial roles in various problems of analysis and synthesis of control systems, including positive and bounded realness. For state-space systems, Kalman-Popov-Yakuvobich (KYP) Lemma and related results provide characterization of positive or bounded realness in terms of state-space realization (Anderson, 1967; Willems, 1971; Rantzer, 1996). Also for descriptor systems there have been proposed several criteria for properties related to dissipativity in such as Takaba, Morihira, & Katayama (1994); Masubuchi, Kamitane, Ohara, & Suda (1997); Wang, Yung, & Chang (1998); Uezato, & Ikeda (1999); Rehm, & Allgöwer (2000); Rehm (2000); Rehm, & Allgöwer (2002); Zhang, Lam, & Xu (2002); Freund, & Jarre (2004). However, most of existing results require a certain assumption or restriction on the realization of descriptor systems, while KYP Lemma for state-space systems is valid independently of the choice of the realization. On the other hand, a new matrix inequality condition is proposed recently in (Masubuchi, 2004; Masubuchi, 2006) that is necessary and sufficient for dissipativity of a descriptor system with any realization of a descriptor system.

In this paper, we consider output feedback controller synthesis for descriptor systems to attain dissipativity of the closed-loop system. Based on the criterion shown in Masubuchi (2006), a necessary and sufficient condition is derived in terms of LMIs with a rank constraint for existence of a controller satisfying dissipativity and admissibility [1] of the closed-loop system. Unlike previous results, the proposed condition does not depend on the choice of the descriptor realization and thus the remedy of those results are removed.

The proposed LMI condition for the synthesis problem generalizes the results of H_∞ synthesis for state-space systems in Gahinet & Apkarian (1994); Iwasaki & Skelton (1994) to synthesis for descriptor systems with dis-

[1] An internal stability of descriptor systems (See e.g. Masubuchi et al. (1997).)

sipativity specification, including a rank condition corresponding to the order of the dynamic part of a controller. The derivation of the existence criterion is based on the 'variable elimination' methodology, which is a key technique in H_∞ synthesis of state-space systems and has been applied to synthesis for descriptor systems (Rehm, 2000; Rehm, & Allgöwer, 2002). However, the LMI of the dissipativity criterion (Masubuchi, 2004) that our results are based on has structure for which output feedback synthesis has never been considered before. Hence we provide methods to handle the new LMI for synthesis of descriptor systems.

The rest of the paper is organized as follows. Section 2 is devoted to preliminaries including the disspativity criterion for descriptor systems. Section 3 provides the main result: Description of control systems and the problem statement are given in Subsections 3.1 and 3.2, respectively. The existence condition is shown in Subsection 3.3, followed by the proof in Subsections 3.4 and 3.5. The main result is illustrated via numerical examples in Section 4. Lastly Section 5 concludes the paper.

Notation. For a matrix X, we denote by X^{-1}, X^{T}, $X^{-\mathsf{T}}$ and X^* the inverse, the transpose, the inverse of the transpose and the conjugate transpose of X, respectively. Let $\mathbf{He}X$ stand for $X+X^{\mathsf{T}}$ for square X. In addition, $X=(*)^{\mathsf{T}}$ and $X+(*)^{\mathsf{T}}$ mean $X=X^{\mathsf{T}}$ and $X+X^{\mathsf{T}}$, respectively. For a symmetric matrix represented blockwise, offdiagonal blocks can be abbreviated with '$*$', such as $\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & * \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}$. For a matrix $M \in \mathbf{R}^{m\times n}$ with $m>n$, let $M^\perp$ be a matrix satisfying $M^\perp \begin{bmatrix} (M^\perp)^{\mathsf{T}} & M \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}$. For $M \in \mathbf{R}^{m\times n}$ with $m<n$, define $M^\perp := ((M^{\mathsf{T}})^\perp)^{\mathsf{T}}$. The $n\times n$ identity matrix is represented by I_n. The zero matrix of the size $m\times n$ is $0_{m\times n}$.

2 Preliminaries

Consider the following descriptor system:

$$\begin{cases} E\dot{x} = Ax + Bw, \\ \ \ z = Cx + Dw, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^m$ is the input and $z \in \mathbf{R}^p$ is the output of the system. Let $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}E = r$.

Definition 1 *(1°) The pencil $sE-A$ is* regular *if* $\det(sE-A)$ *is not identically zero.*

(2°) Suppose that $sE-A$ is regular. The exponential modes *of $sE-A$ are the finite eigenvalues of $sE-A$, namely, $s\in\mathbf{C}$ such that* $\det(sE-A)=0$.

(3°) Let a vector v_1 satisfy $Ev_1=0$. Then the infinite eigenvalues associated with the generalized eigenvectors v_k satisfying $Ev_k=Av_{k-1}$, $k=2,3,4,\dots$ are impulsive modes *of $sE-A$.*

(4°) The descriptor system (1) is impulse-free *if the pencil $sE-A$ is regular and has no impulsive modes.*

(5°) The pencil $sE-A$ is said to be admissible *if the pencil $sE-A$ is regular, impulse-free and has no unstable exponential modes.*

Next, let $S=S^{\mathsf{T}} \in \mathbf{R}^{(m+p)\times(m+p)}$ and consider the following quadratic *supply rate*:

$$s(w,z) = \begin{bmatrix} w \\ z \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} w \\ z \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathsf{T}} & S_{22} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \tag{2}$$

where the partition of S corresponds to the sizes of w, z.

Definition 2 *The descriptor system (1) is said to be dissipative with respect to the supply rate $s(\cdot,\cdot)$ if the descriptor system (1) is impulse-free and for any $w \in L_2[0,T]$ it holds that $\int_0^T s(w(t),z(t))\mathrm{d}t \le 0$ for $\forall T\ge 0$ provided $x(0)=0$.*

The time-domain condition in Definition 2 is equivalent to the following condition in the frequency-domain:

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^* S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} \le 0, \quad \forall\omega \in \mathbf{R}\cup\{\infty\}, \tag{3}$$

where $G(s) = C(sE-A)^{-1}B + D$. The dissipativity condition can represent several performance criteria, such as the H_∞ norm condition and the extended strict positive realness (ESPR in short (Zhang et al., 2002)) with setting S appropriately. There have been proposed several matrix inequality criteria for H_∞ norm condition (Masubuchi et al., 1997; Rehm, & Allgöwer, 2000; Rehm, & Allgöwer, 2002; Rehm, 2000; Takaba et al., 1994; Uezato, & Ikeda, 1999), ESPR (Wang et al., 1998; Zhang et al., 2002) and dissipativity (Rehm, & Allgöwer, 2002; Rehm, 2000). However, matrix inequalities shown in these previous results need a certain additional condition on D-matrix to show necessity, such as $D=0$ (Masubuchi et al., 1997; Rehm, & Allgöwer, 2000; Rehm, & Allgöwer, 2002; Uezato, & Ikeda, 1999) or $\|D\|<\gamma$ (Rehm, 2000; Takaba et al., 1994) for H_∞ norm, where γ is the norm bound, $D+D^{\mathsf{T}}>0$ for ESPR (Wang et al., 1998; Zhang et al., 2002) and $D=0$, $S_{12}=0$ (Rehm, & Allgöwer, 2002; Rehm, 2000). Such drawback of existing criteria is removed in the recent result for admissibility and dissipativity shown in Masubuchi (2006):

Lemma 1 *The following two conditions are equivalent:*

(i) The descriptor system (1) is admissible and satisfies

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^{*} S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} < 0 \tag{4}$$

for any $\omega \in \mathbf{R} \cup \{\infty\}$.

(ii) There exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ satisfying

$$E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \quad E^{\mathsf{T}}W = 0, \tag{5}$$

$$M + \begin{bmatrix} X^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + (*)^{\mathsf{T}} < 0, \tag{6}$$

where

$$M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}. \tag{7}$$

In this paper we consider only strict inequalities for dissipativity as in (6).

3 Main Result

3.1 Control system in the descriptor form

Based on the new criterion for admissibility and dissipativity of descriptor systems shown in the previous section, we consider synthesis of an output feedback controller to attain admissibility and dissipativity of a control system in the descriptor form. Let us represent the plant as follows:

$$\begin{cases} E\dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w, \end{cases} \tag{8}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^{m_1}$ is the external input, $u \in \mathbf{R}^{m_2}$ is the control input, $z \in \mathbf{R}^{p_1}$ is the controlled output and $y \in \mathbf{R}^{p_2}$ is the measured output. Let $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}E = r$. We consider the following output feedback controller:

$$\begin{cases} E_c\dot{x}_c = A_c x_c + B_c y, \\ u = C_c x_c + D_c y, \end{cases} \tag{9}$$

where $E_c \in \mathbf{R}^{n_c\times n_c}$ with $\mathrm{rank}E_c = r_c$ and $x_c \in \mathbf{R}^{n_c}$. Connecting this controller to the plant (8) forms the closed-loop system as follows:

$$\begin{cases} E_{cl}\dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}w, \\ z = C_{cl}x_{cl} + D_{cl}w, \end{cases} \tag{10}$$

where $x_{cl} = \begin{bmatrix} x^{\mathsf{T}} & x_c^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}} \in \mathbf{R}^{n_{cl}}$, $n_{cl} = n + n_c$ and

$$E_{cl} = \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A + B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{bmatrix},$$

$$B_{cl} = \begin{bmatrix} B_1 + B_2D_cD_{21} \\ B_cD_{21} \end{bmatrix},$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12}D_cC_2 & D_{12}C_c \end{bmatrix},$$

$$D_{cl} = D_{11} + D_{12}D_cD_{21}.$$

3.2 Problem statement

Let us consider the supply rate (2) for $S \in \mathbf{R}^{(m_1+p_1)\times(m_1+p_1)}$.

Assumption 1 *(1°) $S_{22} \geq 0$. (2°) $\begin{bmatrix} S_{12}^{\mathsf{T}} & S_{22}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ has full column rank.*

Let $T_{22} \in \mathbf{R}^{q\times p_1}$ be a matrix satisfying $S_{22} = T_{22}^{\mathsf{T}}T_{22}$.

Remark 1 *The supply rates for the two important specific dissipativity conditions of H_∞ norm and positive realness meet (1°) of Assumption 1 (See Subsection 3.3). The LMI (6) implies $A^{\mathsf{T}}X + X^{\mathsf{T}}A < 0$ and hence X is nonsingular. When the second item is not satisfied, one can always redefine the controlled output z so that the same supply rate is defined for the new z with a new S satisfying the assumption.*

Now the synthesis problem is stated as follows: *given a plant in the descriptor form (8) and a quadratic supply rate (2), find a controller (9) for which the closed-loop system (10) is admissible and satisfies dissipativity with respect to the supply rate (2).*

It is easy to see from Lemma 1 that the synthesis problem is solvable if and only if there exist matrices X_{cl} and W_{cl} satisfying

$$E_{cl}^{\mathsf{T}}X_{cl} = X_{cl}^{\mathsf{T}}E_{cl} \geq 0, \quad E_{cl}^{\mathsf{T}}W_{cl} = 0, \tag{11}$$

$$\mathbf{He}U_X + \mathrm{diag}\{0_{n_{cl}\times n_{cl}}, S_{11}, -I_q\} < 0, \tag{12}$$

where

$$U_X := \left[\begin{array}{c|c} 0 & X_{cl}^{\mathsf{T}} \\ \hline S_{12} & W_{cl}^{\mathsf{T}} \\ T_{22} & 0 \end{array}\right] \left[\begin{array}{c|c} D_{cl} & C_{cl} \\ \hline B_{cl} & A_{cl} \end{array}\right] \left[\begin{array}{c|cc} 0 & I_{m_1} & 0 \\ \hline I_{n_{cl}} & 0 & 0_{n_{cl}\times q} \end{array}\right].$$

This LMI condition is equivalent to (5)–(6) in Lemma 1 applied to the closed-loop system (10). Since X_{cl} is nonsingular if (12) holds, the condition (11)–(12) is equivalent to the following:

$$E_{cl}Y_{cl}^{\mathsf{T}} = Y_{cl}E_{cl}^{\mathsf{T}} \geq 0, \quad E_{cl}Z_{cl}^{\mathsf{T}} = 0, \tag{13}$$
$$\mathbf{He}U_Y + \mathrm{diag}\{0_{n_{cl}\times n_{cl}}, S_{11}, -I_q\} < 0, \tag{14}$$

where

$$U_Y := \begin{bmatrix} 0 & I_{n_{cl}} \\ \hline S_{12} & 0 \\ T_{22} & 0 \end{bmatrix} \begin{bmatrix} D_{cl} & C_{cl} \\ \hline B_{cl} & A_{cl} \end{bmatrix} \begin{bmatrix} 0 & I_{m_1} & 0 \\ \hline Y_{cl}^{\mathsf{T}} & Z_{cl}^{\mathsf{T}} & 0_{n_{cl}\times q} \end{bmatrix}$$

and Y_{cl}, Z_{cl} are set by

$$Y_{cl} = X_{cl}^{-\mathsf{T}}, \quad Z_{cl} = -W_{cl}^{\mathsf{T}}X_{cl}^{-\mathsf{T}}. \tag{15}$$

It is easy to see that the left hand sides of (11) and (12) are congruent to each other and that $E_{cl}Z_{cl}^{\mathsf{T}} = 0$ is equivalent to $E_{cl}^{\mathsf{T}}W_{cl} = 0$.

3.3 Existence condition

In this subsection, we show the existence condition of a controller satisfying admissibility and dissipativity of the closed-loop system. Under Assumption 1, define the following matrices from S:

$$M := \begin{bmatrix} \begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix} & \left(\begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix}^{\perp} \right)^{\mathsf{T}} \end{bmatrix}, \tag{16}$$
$$\begin{bmatrix} N_1 & N_2 \end{bmatrix} := \begin{bmatrix} I & 0 \end{bmatrix} M^{-\mathsf{T}}, \tag{17}$$
$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^{\mathsf{T}} & H_{22} \end{bmatrix} := M^{-1} \begin{bmatrix} S_{11} & 0 \\ 0 & -I \end{bmatrix} M^{-\mathsf{T}}. \tag{18}$$

Note that $\begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix}$ has full column rank iff so does $\begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$.

Now we show the main theorem, which provides an existence condition of a controller that solves the synthesis problem stated in the previous subsection. The criterion is given in terms of LMIs and a rank condition.

Theorem 1 *The following statements (I) and (II) are equivalent:*

(I) There exists a controller (9) for which the closed-loop system (10) is admissible and satisfies dissipativity for the supply rate (2).

(II) There exist matrices X, Y, W, Z with appropriate sizes satisfying the following LMIs and LMEs:

$$\begin{bmatrix} E^{\mathsf{T}} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} X & I \\ I & Y^{\mathsf{T}} \end{bmatrix} = (*)^{\mathsf{T}} \geq 0, \tag{19}$$
$$E^{\mathsf{T}}W = 0, \quad EZ^{\mathsf{T}} = 0, \tag{20}$$
$$N_B(L_B + L_B^{\mathsf{T}} + H_B)N_B^{\mathsf{T}} < 0, \tag{21}$$
$$N_C^{\mathsf{T}}(L_C + L_C^{\mathsf{T}} + H_C)N_C < 0 \tag{22}$$

and the rank condition:

$$\mathrm{rank} \begin{bmatrix} E^{\mathsf{T}} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} X & I \\ I & Y^{\mathsf{T}} \end{bmatrix} \leq r + r_c, \tag{23}$$

where

$$N_B := \begin{bmatrix} N_{B0} & 0 \\ \hline 0 & I \end{bmatrix}, \ N_{B0} := \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}^{\perp},$$
$$N_C := \begin{bmatrix} N_{C0} & 0 \\ \hline 0 & I \end{bmatrix}, \ N_{C0} := \begin{bmatrix} C_2 & D_{21} \end{bmatrix}^{\perp},$$
$$L_B := \begin{bmatrix} \begin{bmatrix} A \\ C_1 \end{bmatrix} Y^{\mathsf{T}} & \begin{bmatrix} B_1 + AZ^{\mathsf{T}} \\ D_{11} + C_1 Z^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \\ \hline 0 & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix},$$
$$L_C := \begin{bmatrix} X^{\mathsf{T}}A & X^{\mathsf{T}}B_1 & 0 \\ S_{12}C_1 + W^{\mathsf{T}}A & S_{12}D_{11} + W^{\mathsf{T}}B_1 & 0 \\ \hline T_{22}C_1 & T_{22}D_{11} & 0 \end{bmatrix},$$
$$H_B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{11} & H_{12} \\ \hline 0 & H_{12}^{\mathsf{T}} & H_{22} \end{bmatrix}, \quad H_C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{11} & 0 \\ \hline 0 & 0 & -I \end{bmatrix}.$$

If the conditions (19)–(22) are fulfilled, a controller in the descriptor form (9) with $\mathrm{rank}E_c \leq r_c$ *solving the synthesis problem is constructed from a solution X, Y, W, Z. Furthermore, also a controller in the state-space form with order no more than r_c is derived from the solution.*

We provide the procedure to obtain a controller satisfying the condition (I) in Subsection 3.5 with proving the sufficiency, while the proof of the necessity is shown in Subsection 3.4. The proof is based on application of the matrix elimination lemma (Gahinet & Apkarian, 1994; Iwasaki & Skelton, 1994) to LMIs (11)–(12) and (13)–(14).

Below we show the LMIs (21) and (22) for specific supply rates. Setting $S = \mathrm{diag}\{-\gamma^2 I, I\}$ and matrices M, N_i, H_{ij} appropriately, we see that the inequalities (21) and

(22) for H_∞ synthesis are

$$N_B \left[\begin{array}{cc|c} \mathbf{He}(AY^\top) & YC_1^\top & B_1 + AZ^\top \\ * & -I & D_{11} + C_1 Z^\top \\ \hline * & * & -\gamma^2 I \end{array}\right] N_B^\top < 0, \quad (24)$$

$$N_C^\top \left[\begin{array}{cc|c} \mathbf{He}(X^\top A) & X^\top B_1 + W^\top A & C_1^\top \\ * & \mathbf{He}(W^\top B_1) - \gamma^2 I & D_1^\top \\ \hline * & * & -I \end{array}\right] N_C < 0, \quad (25)$$

respectively. For ESPR, it is easy to see that the conditions (21) and (22) are equivalent to

$$N_{B0}\left(\mathbf{He}\begin{bmatrix} AY^\top & -(B_1 + AZ^\top) \\ C_1 Y^\top & -(D_{11} + C_1 Z^\top) \end{bmatrix}\right) N_{B0}^\top < 0,$$

$$N_{C0}^\top\left(\mathbf{He}\begin{bmatrix} X^\top A & X^\top B_1 \\ W^\top A - C_1 & W^\top B_1 - D_{11} \end{bmatrix}\right) N_{C0} < 0,$$

respectively, with $S = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$.

The inequalities of the existence condition for H_∞ synthesis are similar to those in previous papers for descriptor systems. The crucial difference is that the new inequality condition has variables W and Z. With these variables the LMI condition (19)–(22) is equivalent to the solvability of the synthesis problem without depending on the realization of the plant. Actually, denoting by γ^* the infimum of H_∞ norm from w to z that the closed-loop system can attain via feedback (9), we see that (24) and (25) never hold if W and Z are set zero and the realization of (8) is such that $\|D_{11}\|_\infty > \gamma^*$; see Section 4.

As in the state-space case, the order of the dynamic part of the controller corresponds to the rank condition (23). This makes the whole inequality condition nonconvex. When the order r_c is not required to be less than r, the rank condition (23) is removed and the other inequalities are convex.

Also similarly to the H_∞ control of state-space systems, an appropriate subset of the condition (II) of Theorem 1 gives a feasibility condition of the full-information problem; namely, consider the control input $u = Fx + Gw$ with constant gains F and G. Then it is easy to see that there exists a pair of gains (F, G) with which the disspativity of the closed-loop system holds iff there exist X and W satisfying $E^\top X = X^\top E \geq 0$, $E^\top W = 0$ and (21). An LMI condition derived via the change-of-variables method has been shown for the full-information problem in Masubuchi (2006). A dual result holds for an observer problem with disspativity specification on the observation error.

Lastly in this subsection, we note that when $E = I$, which means that the descriptor plant is reduced to a state-space one, the existence condition (19)–(22) coincides with that for state-space systems (Gahinet & Apkarian, 1994; Iwasaki & Skelton, 1994). The equalities in (20) imply that W and Z vanish for $E = I$.

3.4 Proof of the necessity

Suppose that there exists a controller (9) for which the closed-loop system is admissible and dissipative. Then LMI conditions (11)–(12) and (13)–(14) hold for some (X_{cl}, W_{cl}) and (Y_{cl}, Z_{cl}), respectively, with satisfying the identities in (15). For the purpose of the proof, without loss of generality we can assume $E = \mathrm{diag}\{I_r, 0_{s\times s}\}$ and $E_c = \mathrm{diag}\{I_{r_c}, 0_{s_c\times s_c}\}$, where $r+s = n$ and $r_c+s_c = n_c$. In accordance with this form and the equality constraints in (11) and (13), represent the variables X_{cl}, W_{cl}, Y_{cl}, Z_{cl} with subblocks as follows:

$$X_{cl} = \left[\begin{array}{cc|cc} X_{p11} & 0 & X_{pc11} & 0 \\ X_{p21} & X_{p22} & X_{pc21} & X_{pc22} \\ \hline X_{cp11} & 0 & X_{c11} & 0 \\ X_{cp21} & X_{cp22} & X_{c21} & X_{c22} \end{array}\right], \quad (26)$$

$$W_{cl} = \left[\begin{array}{cc|cc} 0 & W_{p2}^\top & 0 & W_{c2}^\top \end{array}\right]^\top, \quad (27)$$

$$Y_{cl} = \left[\begin{array}{cc|cc} Y_{p11} & Y_{p12} & Y_{pc11} & Y_{pc12} \\ 0 & Y_{p22} & 0 & Y_{pc22} \\ \hline Y_{cp11} & Y_{cp12} & Y_{c11} & Y_{c12} \\ 0 & Y_{cp22} & 0 & Y_{c22} \end{array}\right], \quad (28)$$

$$Z_{cl} = \left[\begin{array}{cc|cc} 0 & Z_{p2} & 0 & Z_{c2} \end{array}\right]. \quad (29)$$

From the inequality conditions in (11) and (13), matrices

$$X_{11} = \begin{bmatrix} X_{p11} & X_{pc11} \\ X_{cp11} & X_{c11} \end{bmatrix}, \quad Y_{11} = \begin{bmatrix} Y_{p11} & Y_{pc11} \\ Y_{cp11} & Y_{c11} \end{bmatrix} \quad (30)$$

are positive definite and satisfy $X_{11}Y_{11} = I$, from which we see (19) and (23) for

$$\begin{cases} X = \begin{bmatrix} X_{p11} & 0 \\ X_{p21} & X_{p22} \end{bmatrix}, & W = \begin{bmatrix} 0 \\ W_{p2} \end{bmatrix}, \\ Y = \begin{bmatrix} Y_{p11} & Y_{p12} \\ 0 & Y_{p22} \end{bmatrix}, & Z = \begin{bmatrix} 0 & Z_{p2} \end{bmatrix}. \end{cases} \quad (31)$$

The equality conditions in (20) are obvious from the definition of W and Z in (31) and the equalities on W_{cl} and Z_{cl} in (11) and (13). Lastly, to prove (21) and (22), let us recall Gahinet & Apkarian (1994); Iwasaki & Skelton (1994) for the matrix elimination lemma:

Lemma 2 *Let $\bar{Q} \in \mathbf{R}^{N\times N}$, $\bar{B} \in \mathbf{R}^{N\times M}$, $\bar{C} \in \mathbf{R}^{P\times N}$ and $\tilde{Y} \in \mathbf{R}^{N\times N}$ be given, where $\bar{Q}$ is symmetric, $\tilde{Y}$ is nonsingular and $\bar{B}$ and $\bar{C}$ are of full column and row rank, respectively. Suppose that*

$$\tilde{Q} = \tilde{Y}\bar{Q}\tilde{Y}^{\mathsf{T}}, \quad \tilde{B} = \tilde{Y}\bar{B}, \quad \tilde{C} = \bar{C}\tilde{Y}^{\mathsf{T}}. \tag{32}$$

Then there exists a matrix $K \in \mathbf{R}^{M\times P}$ satisfying $\bar{Q} + \mathbf{He}\bar{B}K\bar{C} < 0$ if and only if

$$(\bar{C}^{\perp})^{\mathsf{T}}\bar{Q}\bar{C}^{\perp} < 0, \quad \tilde{B}^{\perp}\tilde{Q}(\tilde{B}^{\perp})^{\mathsf{T}} < 0. \tag{33}$$

The proof is completed by identifying the matrices in Lemma 2 for our problem. Define

$$K = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \tag{34}$$

and determine matrices $(\bar{Q}, \bar{B}, \bar{C})$ and $(\tilde{Q}, \tilde{B}, \tilde{C})$ so that $\bar{Q} + \mathbf{He}\bar{B}K\bar{C}$ and $\tilde{Q} + \mathbf{He}\tilde{B}K\tilde{C}$ coincide with the left-hand sides of (14) and (12), respectively. Then it is easy to see that (32) holds for

$$\tilde{Y} = \left[\begin{array}{c|cc} Y_{cl} & 0 & 0 \\ \hline Z_{cl} & I_{m_1} & 0 \\ 0 & 0 & I_q \end{array}\right] \tag{35}$$

and we derive

$$N_B(L_B + L_B^{\mathsf{T}} + H_B)N_B^{\mathsf{T}} = \tilde{B}^{\perp}\tilde{Q}(\tilde{B}^{\perp})^{\mathsf{T}}, \tag{36}$$
$$N_C^{\mathsf{T}}(L_C + L_C^{\mathsf{T}} + H_C)N_C = (\bar{C}^{\perp})^{\mathsf{T}}\bar{Q}\bar{C}^{\perp}, \tag{37}$$

which prove (21) and (22) via Lemma 2.

3.5 Construction of a controller with proof of the sufficiency

Here we show a procedure to derive a controller satisfying admissibility and dissipativity of the closed-loop system when the conditions (19)–(22) hold, which provides a constructive proof for the sufficiency. Again without loss of generality, we assume $E = \mathrm{diag}\{I_r, 0_{s\times s}\}$ and seek a controller with setting $E_c = \mathrm{diag}\{I_{r_c}, 0_{s_c\times s_c}\}$. According to the block form of E and E_c, the variables X, Y, W, Z satisfying the equality conditions (19) and (20) are represented as in (31), where we can assume the submatrices X_{p22} and Y_{p22} are nonsingular; see Masubuchi et al. (1997). Let us consider the condition (19) and (23). The block structure of E implies that the condition (19)–(23) is equivalent to:

$$\begin{bmatrix} X_{p11} & I \\ I & Y_{p11} \end{bmatrix} \geq 0, \ \mathrm{rank}\begin{bmatrix} X_{p11} & I \\ I & Y_{p11} \end{bmatrix} \leq r + r_c. \tag{38}$$

As in the results for state-space systems of Gahinet & Apkarian (1994); Iwasaki & Skelton (1994), by setting subblocks X_{*11} and Y_{*11} for '$*$'='pc', 'cp', 'c' appropriately, we derive positive definite matrices X_{11} and Y_{11} in (30) so that $X_{11}Y_{11} = I_{r+r_c}$. Set the rest of the subblocks below to define X_{cl}, W_{cl}, Y_{cl} and Z_{cl} as (26)–(29):

$$\begin{aligned}
&X_{pc21} = 0, \quad X_{pc22} = I_s, \\
&X_{cp21} = -(Y_{p22}^{\mathsf{T}}X_{p21} + Y_{p12}^{\mathsf{T}}\Delta_Y^{-\mathsf{T}}), \\
&X_{cp22} = I_s - Y_{p22}^{\mathsf{T}}X_{p22}, \\
&X_{c21} = Y_{p12}^{\mathsf{T}}\Delta_Y^{-\mathsf{T}}Y_{cp11}^{\mathsf{T}}Y_{c11}^{-\mathsf{T}}, \quad X_{c22} = -Y_{p22}^{\mathsf{T}}, \\
&Y_{cp12}^{\mathsf{T}} = 0, \quad Y_{cp22}^{\mathsf{T}} = I_s, \\
&Y_{pc12}^{\mathsf{T}} = -(X_{p21}Y_{p11}^{\mathsf{T}} + X_{p22}Y_{p12}^{\mathsf{T}}), \\
&Y_{pc22}^{\mathsf{T}} = I_s - X_{p22}Y_{p22}^{\mathsf{T}}, \\
&Y_{c12}^{\mathsf{T}} = -X_{p21}Y_{cp11}^{\mathsf{T}}, \quad Y_{c22}^{\mathsf{T}} = -X_{p22}, \\
&W_{c2} = -(Z_{p2} + Y_{p22}^{\mathsf{T}}W_{p2}), \\
&Z_{c2}^{\mathsf{T}} = -(X_{p22}Z_{p2}^{\mathsf{T}} + W_{p2}),
\end{aligned}$$

where

$$\Delta_Y = Y_{p11} - Y_{pc11}Y_{c11}^{-1}Y_{cp11}(> 0).$$

Then matrices X_{cl}, Y_{cl}, W_{cl}, Z_{cl} satisfy (15). Moreover, setting K by (34), $\tilde{Y}$ by (35) and $\bar{Q}$, $\bar{B}$, $\bar{C}$, $\tilde{Q}$, $\tilde{B}$, $\tilde{C}$, as stated in the last part of the proof of the necessity, we see that the equality in (32) and the inequalities in (33) hold in Lemma 2. Thus we derive A_c, B_c, C_c, D_c. Lastly, eliminating the static part of the descriptor variable x_c with perturbing A_c if necessary (Masubuchi et al., 1997), we obtain a proper (impulse-free) controller satisfying the closed-loop admissibility and dissipativity.

4 Numerical examples

Let us consider the plant (8) with $E = \mathrm{diag}\{I_5, 0_{2\times 2}\}$ and the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^{\mathsf{T}}, \ B_2 = \begin{bmatrix} 0\ 1\ 0\ 0\ 1\ 0\ 1 \end{bmatrix}^{\mathsf{T}},$$

$$C_1 = \begin{bmatrix} 0\ 0\ 0\ 2\ 1\ 1-\kappa\ 0 \\ 0\ 0\ 0\ 0\ 0\ \ \ 0\ \ \ 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix},$$

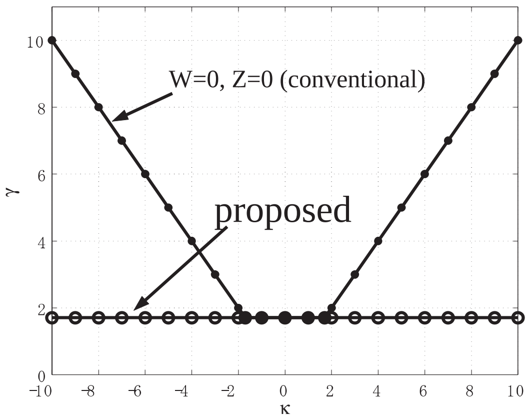

Fig. 1. Optimal γ v.s. κ

$$D_{11} = \begin{bmatrix} \kappa & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathsf{T}},$$

where κ is a scalar. This system has an identical transfer function independent of κ as shown below:

$$\begin{bmatrix} z \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|c} \frac{s^2+3s+5}{s^2+2s+3} & 0 & \frac{s^3+8s^2+14s+4}{s^4+7s^3+15s^2+19s+6} \\ 0 & 0 & 1 \\ \hline \frac{s^2}{s^2+2s+3} & 1 & \frac{s(s^3+8s^3+21s^2+21s+6)}{s^4+7s^3+15s^2+19s+6} \end{array}\right] \begin{bmatrix} w \\ \hline u \end{bmatrix}.$$

Thus the optimal H_∞ norm of the controlled system should be the same in spite of different values of κ.

We solved the LMI-LME condition of Theorem 1 for H_∞ norm condition: (19), (20), (24) and (25) without the rank constraint for integer κ from -10 to 10 and obtained the same optimal value $\gamma^* = 1.7064$ along with state-space controllers satisfying the H_∞ norm condition with closed-loop admissibility. On the other hand, solving LMIs with $W = 0$ and $Z = 0$, which corresponds to the case of using conventional H_∞ norm conditions, yields values of optimal γ larger than γ^* when $\|D\|_\infty > \gamma^*$. These results of optimal γ for each κ via the conventional and proposed methods are plotted in Fig. 1.

5 Conclusions

In this paper, we considered a synthesis problem of output feedback controllers for descriptor systems to attain closed-loop dissipativity and admissibility. We provided a necessary and sufficient condition for the existence of such a controller, based on the recent result on the dissipativity analysis of descriptor systems (Masubuchi, 2006). The proposed LMI condition is a generalization of the widely-known results for state-space systems of Gahinet & Apkarian (1994) and Iwasaki & Skelton (1994). It is inherit from the criterion for analysis that the LMI condition does not depend on the choice of realization of the plant in the descriptor form.

References

Anderson, B. D. O. (1967). A system theory criterion for positive real matrices. *SIAM Journal of Control 5*, 171-182.

Freund, R. W., & Jarre, F. (2004). An extension of the positive real lemma to descriptor systems. *Optimization Methods and Software 18*, 69-87.

Fridman, E., & Shaked, U. (2002). A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control 47*(2), 253-270.

Gahinet, P., & Apkarian, P. (1994). A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control 4*, 421-448.

Iwasaki, T., & Skelton, R. E. (1994). All controllers for the general H_∞ control problem: LMI existence conditions and sate space formulas. *Automatica 30*(8), 1307-1317.

Lewis, F. L. (1986). A survey of linear singular systems. *Circuits, Systems and Signal Processing 5*(1), 3-36.

Masubuchi, I., Kamitane, Y., Ohara, A., & Suda, N. (1997). H_∞ control for descriptor systems: A matrix inequalities approach. *Automatica 33*(4), 669-673.

Masubuchi, I., Akiyama, T., & Saeki, M. (2003). Synthesis of output feedback gain-scheduling controllers based on the descriptor LPV system representation. In *Proceedings of the 42nd IEEE Conference on Decision and Control* (pp. 6115-6120), Maui.

Masubuchi, I., Kato, J., Saeki, M., & Ohara, A. (2004). Gain-scheduled controller design based on descriptor representation of LPV systems: application to flight vehicle control. In *Proceedings of the 43rd IEEE Conference on Decision and Control* (pp. 815-820), The Bahamas.

Masubuchi, I. (2004). Dissipativity inequality for continuous-time descriptor systems: A realization-independent condition. In *Proceedings of the 10th IFAC Symposium on Large Scale Systems* (pp. 417-420), Osaka.

Masubuchi, I. (2006). Dissipativity inequalities for continuous-time descriptor systems with applications to synthesis of control gains. *Systems and Control Letters 55*(2), 158-164.

Rantzer, A. (1996) On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters 28*, 7-10.

Rehm, A. (2000). *Control of linear descriptor systems: a matrix inequalities approach.* Dsseldorf: VDI Verlag.

Rehm, A., & Allgöwer, F. (2000). Self-scheduled H_∞ output feedback control of descriptor systems. *Computers and Chemical Engineering 24*, 279–284.

Rehm, A., & Allgöwer, F. (2002). General quadratic performance analysis and synthesis of differential algebraic equation (DAE) systems. *Journal of Process Control 12*(4), 467–474.

Takaba, K., Morihira, N., & Katayama, T. (1994). H_∞ control for descriptor systems – A J-spectral factorization approach –. In *Proceedings of the 33rd Conference on Decision and Control* (pp. 2251-2256), Lake Buena Vista.

Uezato, E., & Ikeda, M. (1999). Strict condition for stability, robust stabilization, H_∞ control of descriptor systems. In *Proceedings of the 38th Conference on Decision and Control* (pp. 4092-4097), Phoenix (1999)

Wang, H.-S., Yung, C.-F., & Chang, F.-R. (1998). Bounded real lemma and H_∞ control for descriptor systems. *IEE Proceedings D: Control Theory and Applications 145*, 316-322.

Willems, J. C. (1971). Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control 16*(6), 621-634.

Zhang, L., Lam, J., & Xu, S. (2002). On positive realness of descriptor systems. *IEEE Transactions on Circuits and Systems I 49*, 401-407.