PAPER

Modification of Cubic Bézier Spirals for Curvature Linearity

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SUMMARY We propose the extended Bézier spiral in this paper. The spiral is useful for both design purposes and improved aesthetics. This is because the spiral is one of the Bézier curves, which play an important role in interactive curve design, and because the assessment of the curve is based on the human reception of the curve. For the latter purpose we utilize the logarithmic distribution graph that quantifies the designers' preferences. This paper contributes the unification of the two different curve design objectives (the interactive operation and so called "eye pleasing" result generation); which have been independently investigated so far.
key words: *spirals, curvature variation, clothoid, Bézier curve, volume of a curve, transition curve*

1. Introduction

For CAD systems that are used for designing surfaces such as automobile bodies, it is necessary to generate smooth surfaces that satisfy the designer's task purpose. In order to do so, curvature variation of the obtained curves that compose the surface must be smooth enough. Therefore, defining smooth curves is very important in the fields of computer aided design (CAD) and computer aided geometric design (CAGD). Much research has focused on smooth curves (see [1], for example).

Generally, tools to analyze curve's shape are provided by curvature plot of the curve. A curve is said to be *fair* if its curvature plot is continuous and consists of only a few monotone pieces. According to this definition, the clothoid (its curvature is a linear function of the arc length) [2] and the logarithmic spirals (its radius of curvature is a linear function of the arc length) [3] are useful. On the other hand, the Bézier curves (a special case of well-known NURBs) are widely used in CAD applications, because their shape can be controlled easily. Unfortunately we can not guarantee in general that the obtained Bézier curves is fair. Recently the cubic Bézier spiral segment was reported [4] that has monotonically changing curvature.

Most of research on smooth curves describe definitions, continuity and controlling methods of the curves. There is relatively few research that focus on the characteristics of curves from the view point of designer's impression.

The characteristics that are considered when designers generate and modify a curve are the pattern of curvature variation and the volume [5]. Designers express these characteristics with special imaginary words, and control the curve's shape. However, these imaginary words are not easily understood in general. So, it is necessary to quantify these characteristics in order to propose better CAD systems. The logarithmic distribution graph of curvature is one of the quantities for this purpose. In [5], it is described that the pattern of curvature variation is related with the self-affinity, and it affects the smoothness of a curve.

Recently, Miura et al. [6] showed mathematical description of the logarithmic distribution graph of the curvature (they use LCH to refer to the histograms). They even propose a general formula of aesthetic curves. The definition gives more insight between the relation of human impression and the shape of curves.

In this paper, we propose the extended cubic Bézier curves (the definition is given in Sect. 4). The proposed curves are useful in CAD and CAGD because they are Bézier curves. For attaining its optimum state, we evaluate each Bézier curve by using logarithmic distribution graph of the curvature (LDGC). The judgement should match that of seasoned designers thanks to the merits of LDGC (the designers preferences are supposed to be closely related to this graph). The obtained Bézier spirals are expected to be (i) visually pleasing to human eyes, and (ii) appropriate as the design curves due to its monotone change of curvature. However, some important requirements on the curvature value for actual curve design are missing in the discussion. For designing curves, we should freely set curvature to its appropriate value. The focus of this paper is more on curvature variation, while the curvature value at the beginning or the end point is expected to be the desirable value. Higashi et al. [7] presented by the first and second derivatives the criteria that a curve's curvature changes monotonously and smoothly. Their consideration might solve the case when some curvature values are set as the boundary condition.

This paper is organized as follows: Firstly, we briefly describe the clothoid and Bézier spirals for later discussion. Secondly, we summarize the logarithmic distribution graph that plays very important role in this paper. Then, new spiral (extended cubic Bézier spiral) will be proposed for smooth curve generation; our extended Bézier spirals will be evaluated with respect to the LDGC. The existence of extended cubic Bézier spirals is investigated with respect to its dynamic range, and we conclude the merits of Bézier curves as the curve design tool.

2. Clothoid and Bézier Spirals

The clothoid is a spiral defined parametrically in terms of Fresnel integrals by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \pi B \begin{pmatrix} C(t) \\ S(t) \end{pmatrix}, \tag{1}$$

where the scaling factor πB is positive, the parameter t is nonnegative, and the Fresnel integrals are

$$C(t) = \int_0^t \cos \frac{1}{2} \pi u^2 du, \tag{2}$$

$$S(t) = \int_0^t \sin \frac{1}{2} \pi u^2 du. \tag{3}$$

In the case of clothoid, the radius of curvature is given as

$$\rho = \frac{B}{t}. \tag{4}$$

Therefore, the curvature of the clothoid changes linearly with respect to the arc length. By this reason the clothoid has been widely used for highway design [8].

The curvature change along the curve is also very important to generate "visually pleasing" interpolants. Meek et al. [9] proposed a technique to utilize clothoids as the interpolants whose curvature is easily controlled. However, as Eq. (1) shows the calculation for the clothoid requires much computational effort compared to the widely used interpolation curves such as Bézier curves.

The Bézier curve is used extensively for CAD applications. Bézier curve of degree n is defined as

$$\mathbf{Q}(t) = \sum_{i=0}^n \mathbf{P}_i B_i^n(t), \qquad 0 \le t \le 1, \tag{5}$$

where $B_i^n(t)$ is the Bernstein polynomials that is defined by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

The points, $\mathbf{P}_i$, $i = 0, \cdots, n$ form the Bézier control polygon for $\mathbf{Q}(t)$. Since they are polynomial, the resulting algorithms are convenient for implementation in interactive computer graphics environments. We will investigate the case of $n = 3$ in this paper. Then, the resulting Bézier curve is cubic.

Recently it was discovered that by choosing the second control point ($\mathbf{P}_1$) of a cubic Bézier midway between the first and the third control points, the derivative of the curvature becomes more manageable [10] (*see* Fig. 1). By analyzing the derivative of the curvature of this special cubic, a cubic Bézier spiral was developed which can be used as a transition curve in a manner similar to the way in which the clothoid is used to join straight lines to circles, circles to circles, and straight lines to straight lines.

A cubic Bézier *spiral* is the curve whose curvature varies monotonically with arc length. Let Ω be the family of those cubic Bézier curves (5) with
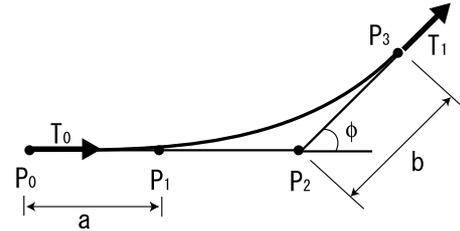
Fig. 1   A cubic Bézier spiral.

$$\mathbf{P}_1 = 0.5(\mathbf{P}_0 + \mathbf{P}_2). \tag{6}$$

Let

$$a = \|\mathbf{P}_1 - \mathbf{P}_0\| > 0, \tag{7}$$
$$\mathbf{T}_0 = (\mathbf{P}_1 - \mathbf{P}_0)/a, \tag{8}$$
$$b = \|\mathbf{P}_3 - \mathbf{P}_2\| > 0, \tag{9}$$

and

$$\mathbf{T}_1 = (\mathbf{P}_3 - \mathbf{P}_2)/b. \tag{10}$$

Then, we can describe the first and the second derivatives of $\mathbf{Q}(t)$ as

$$\mathbf{Q}'(t) = 3a\mathbf{T}_0(1-t)^2 + 6a\mathbf{T}_0(1-t)t + 3b\mathbf{T}_1 t^2, \tag{11}$$

and

$$\mathbf{Q}''(t) = 6(b\mathbf{T}_1 - a\mathbf{T}_0)t. \tag{12}$$

The curvature of a plane curve $\mathbf{Q}(t)$ is

$$\kappa(t) = \frac{\mathbf{Q}'(t) \times \mathbf{Q}''(t)}{\|\mathbf{Q}'(t)\|^3}. \tag{13}$$

By analyzing Eq. (13), we find,

- the numerator is linear in t
- the denominator is of degree t^6

These properties allow simplification in the analysis of the curvature to determine a condition for members of Ω to be spirals. By differentiating Eq. (13) we get the condition (*see* [10] for details) that $\mathbf{Q}(t)$ be a spiral if

$$b \le 1.2a \cos \phi, \tag{14}$$

where ϕ is the angle spanned by $\mathbf{T}_0$ and $\mathbf{T}_1$.

Walton et al. [4] applies the cubic Bézier spiral that meets the condition of Eq. (14) as the substitute for clothoid. However, there is major difference in their curvature change pattern (the curvature change of the cubic Bézier spiral is monotone with respect to the arc length, while the change is linear in the case of clothoid).

It is not clear in what extent the resemblance in the curvature characteristics is required in actual applications. One of the interesting reports regarding this matter is the human perception to the curvature variation. The topic is described in the next chapter.

3. Curvature Change Pattern

Smooth curves are important in CAD applications. Definitions and controlling methods for smooth curves have been proposed by many researchers, but there is few research that describes the relations between the features and the impression of the curve.

Designers create a curve by considering some features of the curve. They control features of the curve by the use of some imaginary words. In [5], these imaginary words are classified into two groups:

- The pattern of curvature variation
- The volume

The pattern of curvature variation means that how the curve's curvature vary, or whether the curve has the point where the curvature is not continuous. Curve's volume is defined as the inside area that is enclosed by the curve and its chord. If the curve is smooth, it has "rhythm" regarding the pattern of curvature variation.

Let the logarithm of the radius of curvature that has constant breadth of interval be $\bar{\rho}_j$, and let the logarithm of the partial arc length of the curve that is included in the interval $\bar{\rho}_j$ be $\bar{s}_j$. The curves that has "rhythm" regarding the pattern of curvature variation if they satisfy the following conditions (See [5] for details).

$$\frac{d\bar{s}}{d\bar{\rho}} = \lim \frac{\bar{s}_1 - \bar{s}_2}{\bar{\rho}_1 - \bar{\rho}_2} = \cdots = \lim \frac{\bar{s}_{j-1} - \bar{s}_j}{\bar{\rho}_{j-1} - \bar{\rho}_j} = const. \quad (15)$$

Curve's features, the pattern of curvature variation and the volume, can be expressed by the Logarithmic Distribution Graph of Curvature (LDGC).

In Fig. 2, an example of LDGC is given. In this graph, pattern of curvature variation is expressed as the locus of the curve C, and the volume is expressed as the position of both endpoints of locus. Let the right point be A, and the left point be B. If point A goes to the right or the interval between points A and B become greater, the volume becomes smaller. In this graph, the volume is expressed as relative value. This graph can be used for investigating how to change the volume of some cross-sections of a surface.

If the locus C on LDGC is a straight line and it has constant slope as the graph in X-Y orthogonal coordinate
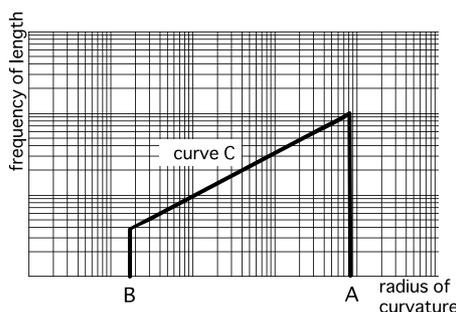
system (in this case, the curve satisfies condition (15)), the curve is considered to be smooth. We can classify the curve by the slope of the locus C. If two curves have the same slope on LDGC, they are in the same category, these two curves share the same impression.

Figure 3 shows the LDGC of a clothoid (the case of $B = 200$, $t = 0.5$ in Eq. (1)). We observe the linear change in LDGC that matches the calculation result in Appendix. Miura et al. [6] presented the analysis of clothoid curves to LDGC (LCH is their terminology). They modified the equation that describes the derivative of the arc length with respect to the logarithm of the radius of curvature, and then obtained a more informative equation (Eq. (2) in the reference).

Figure 4 shows the LDGC of a cubic Bézier curve. If we slightly move a control point, the LDGC changes to Fig. 5. These figures show that:

- Bézier curves' LDGC is in general quite different from that of clothoid where plot is linear as in Fig. 3.
- A slight change of a control point might cause abrupt change in LDGC.
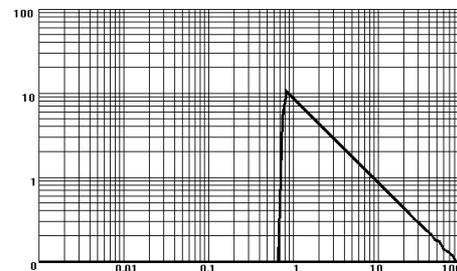
Fig. 3　LDGC of a clothoid.
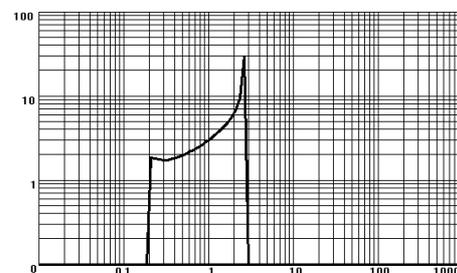
Fig. 4　LDGC of a Bézier curve.
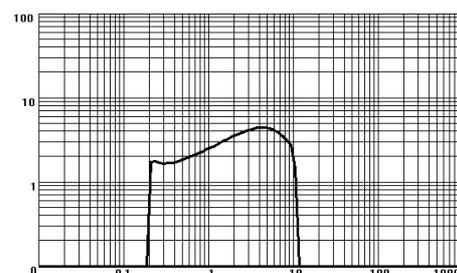
Fig. 5　LDGC of a Bézier curve. (a control point is slightly moved from the case of Fig. 4)

Fig. 2　The logarithmic distribution graph of curvature. (LDGC)

The next section gives more precise analysis between Bézier curves and its LDGC which is the main topic of this paper.

4. Extended Cubic Bézier Spirals

Even if the pattern of curvature variation of a cubic Bézier spiral is the same as the pattern of a clothoid at the start point, they are quite different at the end point in general. In the case of the cubic Bézier spiral, the second control point $\mathbf{P}_1$ is chosen on the midpoint between the first control point $\mathbf{P}_0$ and the third control point $\mathbf{P}_2$ (given as (6)). Figure 6 indicates a Bézier spiral and its LDGC. Figure 6 (b) shows that a Bézier spiral, which has been introduced as an appropriate Bézier curve based on its monotone curvature change, does not necessarily emulate the clothoid curve from the view point of LDGC evaluation.

Now, the second control point is chosen arbitrarily between the first and the third control points, i.e. the second control point is given as

$$\mathbf{P}_1 = m\mathbf{P}_0 + (1 - m)\mathbf{P}_2, \qquad 0 < m < 1. \qquad (16)$$

Let Ω be the family of cubic Bézier curve with (16), and a, $\mathbf{T}_0$, b, and $\mathbf{T}_1$ is defined as (7), (8), (9), and (10), respectively (same as the cubic Bézier spiral case). A member of Ω and its first two parametric derivative can thus be expressed as

$$
\begin{aligned}
\mathbf{Q}(t) = {} & \mathbf{P}_0(1 - t)^3 + 3(\mathbf{P}_0 + a\mathbf{T}_0)(1 - t)^2 t \\
& + 3(\mathbf{P}_0 + \frac{a}{m}\mathbf{T}_0)(1 - t)t^2 \\
& + (\mathbf{P}_0 + \frac{a}{m}\mathbf{T}_0 + b\mathbf{T}_1)t^3, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}'(t) = {} & 3a\mathbf{T}_0(1 - t)(1 - 3t) \\
& + \frac{6}{m}a\mathbf{T}_0(1 - t)t + 3b\mathbf{T}_1 t^2, \qquad (18)
\end{aligned}
$$

and

$$\mathbf{Q}''(t) = -6a\mathbf{T}_0(2 - 3t) + \frac{6}{m}a\mathbf{T}_0(1 - 2t) + 6b\mathbf{T}_1 t. \quad (19)$$

It follows from (13), (18) and (19) that the curvature of the member of Ω is given by

(a)

(b)
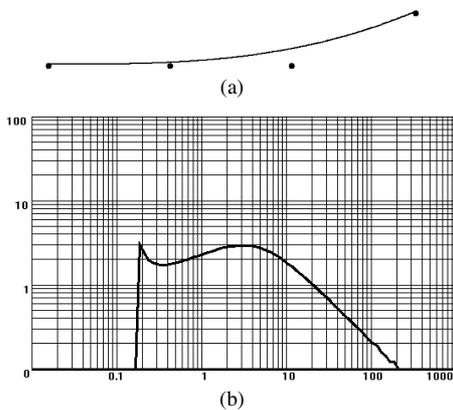
Fig. 6   (a) A Bézier spiral and (b) its LDGC.

$$\kappa(t) = \frac{2abt \sin\phi \left\{ 1 + (\frac{1}{m} - 2)t \right\}}{3\{f(t)\}^{3/2}}, \qquad (20)$$

where

$$
\begin{aligned}
f(t) = {} & a^2 - 4a^2\left(2 - \frac{1}{m}\right)t + 2a\left(11a - \frac{10}{m}a \right. \\
& \left. + \frac{2}{m^2}a + b\cos\phi\right)t^2 \\
& - 4a\left(6a - \frac{7}{m}a + \frac{2}{m^2}a + 2b\cos\phi - \frac{1}{m}b\cos\phi\right)t^3 \\
& + \left(9a^2 - \frac{12}{m}a^2 + \frac{4}{m^2}a^2 \right. \\
& \left. + 6ab\cos\phi - \frac{4}{m}ab\cos\phi + b^2\right)t^4, \qquad (21)
\end{aligned}
$$

and ϕ is the angle from $\mathbf{T}_0$ to $\mathbf{T}_1$. Hence the curvature at both endpoints of member of Ω are

$$\kappa(0) = 0, \qquad \kappa(1) = \frac{2a\sin\phi\left(\frac{1}{m} - 1\right)}{3b^2}. \qquad (22)$$

If the three control point $\mathbf{P}_0$, $\mathbf{P}_2$ and $\mathbf{P}_3$ are given, the curvature $\kappa(1)$ at the last endpoint $\mathbf{P}_3$ varies according to the position of the control point $\mathbf{P}_1$ (it changes according to m in turn).

In order to approximate the clothoid, the cubic Bézier curve is obtained by placing $\mathbf{P}_0$ and $\mathbf{P}_3$ at the first ($\mathbf{Q}_0$) and the last ($\mathbf{Q}_1$) end point of the clothoid segment, respectively (see Fig. 7). The direction of the unit tangent vector $\mathbf{T}_0$ and $\mathbf{T}_1$ is chosen as the tangent direction at the first and the last endpoints of the clothoid, respectively. The third control point $\mathbf{P}_2$ is chosen as the intersection point of tangents at both endpoints.

We can choose the second control point $\mathbf{P}_1$ such that the curvature of the member of Ω is equal to the curvature at the last endpoint of the clothoid. To determine the position of the second control point $\mathbf{P}_1$, the coefficient m is calculated as

$$m = 1 - \frac{3b^2\kappa_1}{2c\sin\phi}, \qquad (23)$$

where

c
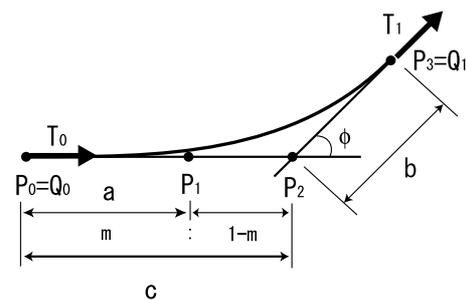
Fig. 7   The cubic Bézier curve that approximates the clothoid.
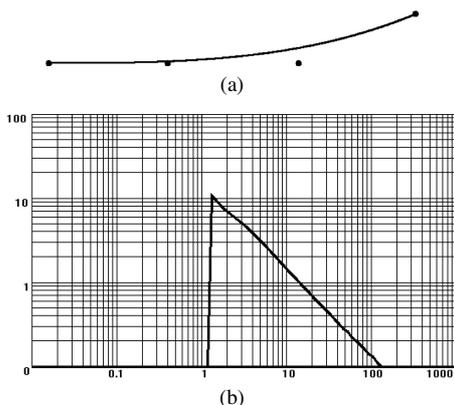
Fig. 8 (a) The Bézier curve that approximates to the clothoid and (b) its LDGC.

$$c = \|\mathbf{P}_2 - \mathbf{P}_0\|.$$

If a cubic Bézier spiral's second control point $\mathbf{P}_1$ is chosen based on Eq. (23), then we can expect much closer similarity to a clothoid curve. Let us call this cubic Bézier spirals as extended cubic Bézier spiral.

We give an example of the extended Bézier spiral. The target clothoid curve was generated by setting $B = 200$ and $t = 0.5$ (see Eqns. (1) through (4)). Then $\mathbf{P}_0$ and $\mathbf{P}_3$ are $(0.0, 0.0)$ and $(309.35, 40.67)$, respectively. Two curvature values at the two end points are 0.0 and 0.0025. These data determine in turn $b = 106.28$, $c = 211.17$, $\mathbf{P}_1 = (100.46, 0.0)$ and $\mathbf{P}_2 = (211.17, 0.0)$. The position of $\mathbf{P}_1$ shows $m = 0.476$ which is not in the category of Walton's Bézier spiral. Figure 8 (a) depicts the cubic Bézier curve, and Fig. 8 (b) is its LDGC. Because the slope of the locus on LDGC is -1 (same as the case of clothoid), we can say that this approximation of the clothoid is of high quality.

5. Dynamic Range for Extended Cubic Bézier Spirals

Walton's main contribution [10] regarding the Bézier spiral is the Eq. (14). This equation shows the condition to obtain Bézier spiral when a, ϕ are fixed. If $\mathbf{P}_3$ locates inside the circle derived by Eq. (14) the resulting Bézier curve is a spiral. This is the special case analysis of $m = 0.5$. Unfortunately, Walton's restriction on the position of $\mathbf{P}_1$ does not bring *clothoid-like* curve if we adopt LDGC for evaluation, as our simple example shows.

LDGC was first introduced to explain the human factor that brings designers' preferences on curve design, and then mathematical theory for it has been proposed that derives a general formula of aesthetic curves [6]. Regarding the relation between LDGC and Bézier curves, what we have found so far are: (1) if a curve (more specifically, a spiral) is similar to a clothoid, then it is of high quality regarding the LDGC measure, and (2) the high quality curves are derived by moving the second control point.

The location of the second control point should be selected so that the resulting curve is at least a spiral. This consideration suggests the need investigating the allowable

region of m (a dynamic range of m) that guarantees the curve to be a spiral. Analysing the relation between m and the shape of the curve's curvature profile is thus very important for the later discussion.

Since the equations of Bézier spirals are too complicated, the full analysis of Eqs. (17) through (21) are almost impossible. We focus on a necessary condition instead that guarantees the curve to be a cubic Bézier spiral.

By investigating the derivative of $\kappa(t)$ of Eq. (20), we find that the critical value of m (indicated with m^*) is obtained by solving the equation $\kappa'(1) = 0$. The equation is,

$$m^* = \frac{2b + 6a\cos\phi \pm \sqrt{2b}\sqrt{2b + 3a\cos\phi}}{3(b + 2a\cos\phi)}. \tag{24}$$

As is clear by Fig. 7, a is affected by the position of $\mathbf{P}_1$. For further investigation of the general case we use c instead of a which enables easier observation of the position of $\mathbf{P}_1$. By replacing a with cm^* in Eq. (24), we obtain the following nonlinear equation,

$$m^* = \frac{2b + 6cm^*\cos\phi \pm \sqrt{2b}\sqrt{2b + 3cm^*\cos\phi}}{3(b + 2cm^*\cos\phi)}. \tag{25}$$

With the aid of equation processing software, we can derive the following two roots (suppose $F(m^*) = 169b^2 - 336bc\cos\phi + 144c^2\cos^2\phi$).

$$m^* = \frac{-13b + 24c\cos\phi \pm \sqrt{F(m^*)}}{36c\cos\phi}. \tag{26}$$

An obvious condition for the existence of valid m^* is,

$$169b^2 - 336bc\cos\phi + 144c^2\cos^2\phi \geq 0. \tag{27}$$

We find that this condition is automatically met for realistic situations ($0 < b, c; 0 \leq \phi < \pi/2$). For the further analysis of Eq. (26), we suppose the following relation,

$$b = \alpha c \cos\phi. \tag{28}$$

By inserting this relation into Eq. (26), and solve for α, we obtain the following simple relation between m^* and α.

$$\alpha = \frac{6(1 - 4m^* + 3m^{*2})}{4 - 13m^*}. \tag{29}$$

As is clear from this equation, the valid span for α is split into two parts by the singular value of m^* that makes the denominator of Eq. (29) equal to zero. The singular value is $4/13$, that is approximately 0.3. The plot of Eq. (29) shows that α value is positive except at the singular point and its vicinity for the range $0 \leq m^* \leq 1$. We omit the range below this singular point for m^* because it is too far from the Walton's case. Figure 9 shows the plot of Eq. (29) for further consideration.

Note that Walton's case ($c = 2a$) is equivalent to the case $\alpha = 0.5$ ($m^* = 0.6$). This means if we set the distance between $\mathbf{P}_2$ and $\mathbf{P}_3$ be half of the length of line $\mathbf{P}_0^*\mathbf{P}_2$ ($\mathbf{P}_0^*$ indicates the closest point from $\mathbf{P}_0$ on the extension of line $\mathbf{P}_2\mathbf{P}_3$), then we can move the position of $\mathbf{P}_1$ rightward until

Fig. 9 The plot of α value of Eq. (29) with respect to m^*.

Fig. 10 Dynamic range for m.

design also offer high quality with respect to LDGC. Mathematical treatment of LDGC [6] also supports this result.

- The Bézier spirals can be elaborated to *clothoid-like* curves by adjusting the positions of their control points.
- LDGC is very sensitive to the positions of control points.
- The dynamic range that describes the possible area where Bézier curves become Bézier spirals can be widened by extending Walton's case. More specifically, if the position of the last control point $\mathbf{P}_3$ is relatively closer to the previous control point $\mathbf{P}_2$, then we can expect wider dynamic range.

All the results of this paper have been derived by extending Walton's result, and hence only valid in one case (the number of control points is four). However, these basic results suggest the importance of Bézier spirals and are expected to be extended to more general cases (for multiple control point curves and surfaces).

As is described in Introduction, curve design is not a simple process. When a designer creates a high quality curve, he or she specifies first the key curvature values at various points. This means that analysis for the desirable curves should incorporate the free setting of the curvature value. A systematic approach that includes various boundary conditions (pre-setting of the curvature) should be conducted in the future work for utilizing the results of this paper in actual curve design work.

Acknowledgement

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m increases to 0.6, while maintaining the resulting curve be a spiral.

We can define an allowable span (dynamic range) for m that can be visualized on the plot in Fig. 9. Our example is shown in Fig. 10. R_l and R_u show the lower and upper bounds for the allowable span, respectively. The length between R_l and R_u is the size of dynamic range. The vertical dotted line indicates the case $m = 0.5$ that is the base for the Walton's case. The horizontal part of the dotted line shows the critical case for Eq. (14). We can realize that Walton's condition yields very limited (narrow dynamic range) case for discussing spirals. The dynamic range of our example (shown with horizontal solid line) is much wider and enables wide selection for cubic Bézier spirals. An interesting observation is that the *clothoid-like* case (of high quality in the sense of LDGC) shown by R_c in Fig. 10 is very close to the middle of the dynamic range.

6. Conclusion

We evaluated the widely used Bézier curves from the view point of human's impression to curves. The quantitative evaluation is based on LDGC and found the following:

- The clothoid curves which are widely used for road

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Appendix: Slope of Clothoid Curve in LDGC

An element of arc length is

$$ds = \pi B dt, \tag{A·1}$$

hence

$$s = \int ds = \int \pi B dt = \pi B t. \tag{A·2}$$

Substitution of (A·2) into (4) with arrangement gives

$$\rho = \frac{\pi B^2}{s}. \tag{A·3}$$

Taking logarithm of (A·3) yields

$$\log \rho = \log \pi B^2 - \log s. \tag{A·4}$$

Let $X = \log \rho$, and $Y = \log s$. (A·4) can be written as

$$Y = -X + C. \qquad (C : const.) \tag{A·5}$$
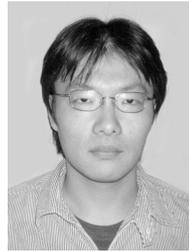
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