to appear in Journal of Algebra

Parabolic Kazhdan-Lusztig polynomials and Schubert varieties

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February 16, 2000

Abstract

We shall give a description of the intersection cohomology groups of the Schubert varieties in partial flag manifolds over symmetrizable Kac-Moody Lie algebras in terms of parabolic Kazhdan-Lusztig polynomials introduced by Deodhar.

1 Introduction

For a Coxeter system (W,S) Kazhdan-Lusztig [6], [7] introduced polynomials

$$P_{y,w}(q)=\sum_{k\in\mathbb{Z}}P_{y,w,k}q^k\in\mathbb{Z}[q],\qquad Q_{y,w}(q)=\sum_{k\in\mathbb{Z}}Q_{y,w,k}q^k\in\mathbb{Z}[q],$$

called a Kazhdan-Lusztig polynomial and an inverse Kazhdan-Lusztig polynomial respectively. Here, (y,w) is a pair of elements of W such that $y\leqq w$ with respect to the Bruhat order. These polynomials play important roles in various aspects of the representation theory of reductive algebraic groups.

In the case W is associated to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, the polynomials have the following geometric meanings. Let $X=G/B$ be the corresponding flag variety (see Kashiwara [3]), and set $X^w=B^-wB/B$ and $X_w=BwB/B$ for $w\in W$. Here B and B^- are the "Borel subgroups" corresponding to the standard Borel subalgebra $\mathfrak{b}$ and its opposite $\mathfrak{b}^-$ respectively. Then X^w (resp. X_w) is an $\ell(w)$-codimensional (resp. $\ell(w)$-dimensional) locally closed subscheme of the infinite-dimensional scheme X.

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Here $\ell(w)$ denotes the length of w as an element of the Coxeter group W. Set $X' = \bigcup_{w\in W} X_w$. Then X' coincides with the flag variety considered by Kac-Peterson [2], Tits [10], et al. Moreover we have

$$X = \bigsqcup_{w\in W} X^w, \qquad X' = \bigsqcup_{w\in W} X_w,$$

and

$$\overline{X^w} = \bigsqcup_{y\geqq w} X^y, \qquad \overline{X_w} = \bigsqcup_{y\leqq w} X_y$$

for any $w \in W$.

By Kazhdan-Lusztig [7] we have the following result (see also Kashiwara-Tanisaki [4]).

Theorem 1.1. (i) *Let $w, y \in W$ satisfying $w \leqq y$. Then we have*

$$H^{2k+1}({}^{\pi}\mathbb{Q}^H_{X^w})_{yB/B} = 0, \qquad H^{2k}({}^{\pi}\mathbb{Q}^H_{X^w})_{yB/B} = \mathbb{Q}^H(-k)^{\oplus Q_{w,y,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module ${}^{\pi}\mathbb{Q}^H_{X^y}[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}^H_{X^w}[-\ell(w)]$ coincides with $P_{w,y,k}$.*

Theorem 1.2. (i) *Let $w, y \in W$ satisfying $w \geqq y$. Then we have*

$$H^{2k+1}({}^{\pi}\mathbb{Q}^H_{X_w})_{yB/B} = 0, \qquad H^{2k}({}^{\pi}\mathbb{Q}^H_{X_w})_{yB/B} = \mathbb{Q}^H(-k)^{\oplus P_{y,w,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module ${}^{\pi}\mathbb{Q}^H_{X_y}[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}^H_{X_w}[\ell(w)]$ coincides with $Q_{y,w,k}$.*

Here ${}^{\pi}\mathbb{Q}^H_{X^w}[-\ell(w)]$ and ${}^{\pi}\mathbb{Q}^H_{X_w}[\ell(w)]$ denote the Hodge modules corresponding to the perverse sheaves ${}^{\pi}\mathbb{Q}_{X^w}[-\ell(w)]$ and ${}^{\pi}\mathbb{Q}_{X_w}[\ell(w)]$ respectively. In Theorem 1.1 we have used the convention so that ${}^{\pi}\mathbb{Q}^H_Z[-\operatorname{codim} Z]$ is a Hodge module for a locally closed finite-codimensional subvariety Z since we deal with sheaves supported on finite-codimensional subvarieties, while in Theorem 1.2 we have used another convention so that ${}^{\pi}\mathbb{Q}^H_Z[\dim Z]$ is a Hodge modules for a locally closed finite-dimensional subvariety Z since we deal with sheaves supported on finite-dimensional subvarieties.

Let J be a subset of S. Set $W_J = \langle J \rangle$ and denote by W^J the set of elements $w \in W$ whose length is minimal in the coset wW_J. In [1] Deodhar introduced two generalizations of the Kazhdan-Lusztig polynomials to this relative situation. For $(y, w) \in W^J \times W^J$ such that $y \leqq w$ we denote the parabolic Kazhdan-Lusztig polynomial for $u = -1$ by

$$P^{J,q}_{y,w}(q) = \sum_{k \in \mathbb{Z}} P^{J,q}_{y,w,k} q^k \in \mathbb{Z}[q],$$

and that for $u = q$ by

$$P^{J,-1}_{y,w}(q) = \sum_{k \in \mathbb{Z}} P^{J,-1}_{y,w,k} q^k \in \mathbb{Z}[q]$$

contrary to the original reference [1]. We can also define inverse parabolic Kazhdan-Lusztig polynomials

$$Q^{J,q}_{y,w}(q) = \sum_{k \in \mathbb{Z}} Q^{J,q}_{y,w,k} q^k \in \mathbb{Z}[q], \qquad Q^{J,-1}_{y,w}(q) = \sum_{k \in \mathbb{Z}} Q^{J,-1}_{y,w,k} q^k \in \mathbb{Z}[q]$$

(see § 2 below)

The aim of this paper is to extend Theorem 1.1 and Theorem 1.2 to this relative situation using the partial flag variety corresponding to J.

Let Y be the partial flag variety corresponding to J. Let 1_Y be the origin of Y and set $Y^w = B^- w 1_Y$ and $Y_w = Bw1_Y$ for $w \in W^J$. Then Y^w (resp. Y_w) is an $\ell(w)$-codimensional (resp. $\ell(w)$-dimensional) locally closed subscheme of the infinite-dimensional scheme Y. Set $Y' = \bigcup_{w \in W^J} Y_w$. Then we have

$$Y = \bigsqcup_{w \in W^J} Y^w, \qquad Y' = \bigsqcup_{w \in W^J} Y_w,$$

and

$$\overline{Y^w} = \bigsqcup_{y \geqq w} Y^y, \qquad \overline{Y_w} = \bigsqcup_{y \leqq w} Y_y$$

for any $w \in W^J$.

We note that the construction of the partial flag variety similar to the ordinary flag variety in Kashiwara [3] has not yet appeared in the literature. In the case where W_J is a finite group (especially when W is an affine Weyl group), we can construct the partial flag variety $Y = G/P$ and the properties of Schubert varieties in Y stated above are established in exactly the same manner as in Kashiwara [3] and Kashiwara-Tanisaki [5]. In the case W_J is an infinite group we can not define the "parabolic subgroup" P corresponding to J as a group scheme and hence the arguments in Kashiwara [3] are not directly generalized. We leave the necesary modification in the case W_J is an infinite group to the future work.

Our main result is the following.

Theorem 1.3. (i) *Let $w, y \in W^J$ satisfying $w \leqq y$. Then we have*

$$H^{2k+1}({}^{\pi}\mathbb{Q}^H_{Y^w})_{y1_Y} = 0, \qquad H^{2k}({}^{\pi}\mathbb{Q}^H_{Y^w})_{y1_Y} = \mathbb{Q}^H(-k)^{\oplus Q^{J,-1}_{w,y,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module ${}^{\pi}\mathbb{Q}^H_{Y^y}[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}^H_{Y^w}[-\ell(w)]$ coincides with $P^{J,-1}_{w,y,k}$.*

Theorem 1.4. (i) *Let $w, y \in W^J$ satisfying $w \geqq y$. Then we have*

$$H^{2k+1}({}^{\pi}\mathbb{Q}^H_{Y_w})_{y1_Y} = 0, \qquad H^{2k}({}^{\pi}\mathbb{Q}^H_{Y_w})_{y1_Y} = \mathbb{Q}^H(-k)^{\oplus P^{J,q}_{y,w,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module ${}^{\pi}\mathbb{Q}^H_{Y_y}[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}^H_{Y_w}[\ell(w)]$ coincides with $Q^{J,-1}_{y,w,k}$.*

In Theorem 1.3 we have used the convention so that ${}^{\pi}\mathbb{Q}^H_Z[-\operatorname{codim} Z]$ is a Hodge module for a locally closed finite-codimensional subvariety Z, and in Theorem 1.4 we have used another convention so that ${}^{\pi}\mathbb{Q}^H_Z[\dim Z]$ is a Hodge modules for a locally closed finite-dimensional subvariety Z.

We note that a result closely related to Theorem 1.4 was already obtained by Deodhar [1].

The above results imply that the coefficients of the four (oridnary or inverse) parabolic Kazhdan-Lusztig polynomials are all non-negative in the case W is the Weyl group of a symmetrizable Kac-Moody Lie algebra.

We would like to thank B. Leclerc for leading our attention to this problem. We also thank H. Tagawa for some helpful comments on the manuscript.

2 Kazhdan-Lusztig polynomials

Let R be a commutative ring containing $\mathbb{Z}[q, q^{-1}]$ equipped with a direct sum decomposition $R = \bigoplus_{k \in \mathbb{Z}} R_k$ into $\mathbb{Z}$-submodules and an involutive ring endomorphism $R \ni r \mapsto \overline{r} \in R$ satisfying the following conditions:

$$(2.1) \qquad R_i R_j \subset R_{i+j}, \quad \overline{R_i} = R_{-i}, \quad 1 \in R_0, \quad q \in R_2, \quad \overline{q} = q^{-1}.$$

Let (W, S) be a Coxeter system. We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ and $\geqq$ the length function and the Bruhat order respectively. The Hecke algebra $H = H(W)$ over R is an R-algebra with free R-basis $\{T_w\}_{w \in W}$ whose multiplication is determined by the following:

$$(2.2) \qquad T_{w_1} T_{w_2} = T_{w_1 w_2} \qquad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2),$$

$$(2.3) \qquad (T_s + 1)(T_s - q) = 0 \qquad \text{for } s \in S.$$

Note that $T_e = 1$ by (2.2).

We define involutive ring endomorphisms $H \ni h \mapsto \overline{h} \in H$ and $j : H \to H$ by

$$(2.4) \quad \overline{\sum_{w \in W} r_w T_w} = \sum_{w \in W} \overline{r}_w T_{w^{-1}}^{-1}, \qquad j(\sum_{w \in W} r_w T_w) = \sum_{w \in W} r_w (-q)^{\ell(w)} T_{w^{-1}}^{-1}.$$

Note that j is an endomorphism of an R-algebra.

Proposition 2.1 (Kazhdan-Lusztig [6]). *For any $w \in W$ there exists a unique $C_w \in H$ satisfying the following conditions:*

$$(2.5) \qquad C_w = \textstyle\sum_{y \leqq w} P_{y,w} T_y \text{ with } P_{w,w} = 1 \text{ and } P_{y,w} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i \text{ for } y < w,$$

$$(2.6) \qquad \overline{C}_w = q^{-\ell(w)} C_w.$$

Moreover we have $P_{y,w} \in \mathbb{Z}[q]$ for any $y \leqq w$.

Note that $\{C_w\}_{w \in W}$ is a basis of the R-module H. The polynomials $P_{y,w}$ for $y \leqq w$ are called Kazhdan-Lusztig polynomials. We write

$$(2.7) \qquad P_{y,w} = \sum_{k \in \mathbb{Z}} P_{y,w,k} q^k.$$

Set $H^* = H^*(W) = \mathrm{Hom}_R(H, R)$. We denote by $\langle\ ,\ \rangle$ the coupling between H^* and H. We define involutions $H^* \ni m \mapsto \overline{m} \in H^*$ and $j : H^* \to H^*$ by

$$(2.8) \quad \langle \overline{m}, h \rangle = \overline{\langle m, \overline{h} \rangle}, \quad \langle j(m), h \rangle = \langle m, j(h) \rangle \qquad \text{for } m \in H^* \text{ and } h \in H.$$

Note that j is an endomorphism of an R-module. For $w \in W$ we define elements $S_w, D_w \in H^*$ by

$$(2.9) \qquad \langle S_w, T_x \rangle = (-1)^{\ell(w)} \delta_{w,x}, \qquad \langle D_w, C_x \rangle = (-1)^{\ell(w)} \delta_{w,x}.$$

Then any element of H^* is uniquely written as an infinite sum in two ways $\sum_{w \in W} r_w S_w$ and $\sum_{w \in W} r'_w D_w$ with $r_w, r'_w \in R$. Note that we have

$$(2.10) \qquad S_w = \sum_{y \geqq w} (-1)^{\ell(w)-\ell(y)} P_{w,y} D_y$$

by $C_w = \sum_{y \leqq w} P_{y,w} T_y$. By (2.6), we have

$$(2.11) \qquad \overline{D}_w = q^{\ell(w)} D_w,$$

and we can write

$$(2.12) \qquad D_w = \sum_{y \geqq w} Q_{w,y} S_y,$$

where $Q_{w,y}$ are determined by

$$(2.13) \qquad \sum_{w \leqq y \leqq z} (-1)^{\ell(y)-\ell(w)} Q_{w,y} P_{y,z} = \delta_{w,z}.$$

Note that (2.12) is equivalent to

$$(2.14) \qquad T_w = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} Q_{y,w} C_y.$$

By (2.13) we see easily that

(2.15) $\qquad Q_{w,y} \in \mathbb{Z}[q]$,

(2.16) $\qquad Q_{w,w} = 1$ and $\deg Q_{w,y} \leqq (\ell(y) - \ell(w) - 1)/2$ for $w < y$.

The polynomials $Q_{w,y}$ for $w \leqq y$ are called inverse Kazhdan-Lusztig polynomials (see Kazhdan-Lusztig [7]). We write

$$(2.17) \qquad Q_{w,y} = \sum_{k \in \mathbb{Z}} Q_{w,y,k} q^k.$$

The following result is proved similarly to Proposition 2.1 (see Kashiwara-Tanisaki [4]).

Proposition 2.2. *Let $w \in W$. Assume that $D \in H^*$ satisfies the following conditions:*

$$D = \textstyle\sum_{y \geqq w} r_y S_y \text{ with } r_w = 1 \text{ and } r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i \text{ for } w < y, \tag{2.18}$$

$$\overline{D} = q^{\ell(w)} D. \tag{2.19}$$

Then we have $D = D_w$.

We fix a subset J of S and set

$$W_J = \langle J \rangle, \qquad W^J = \{w \in W\,;\, ws > w \quad \text{for any } s \in J\}. \tag{2.20}$$

Then we have

$$W = \bigsqcup_{w \in W^J} wW_J, \tag{2.21}$$

$$\ell(wx) = \ell(w) + \ell(x) \text{ for any } w \in W^J \text{ and } x \in W_J. \tag{2.22}$$

When W_J is a finite group, we denote the longest element of W_J by w_J.

Let $a \in \{q, -1\}$ and define $a^\dagger \in \{q, -1\}$ by $aa^\dagger = -q$. Define an algebra homomorphism $\chi^a : H(W_J) \to R$ by $\chi^a(T_w) = a^{\ell(w)}$, and denote the corresponding one-dimensional $H(W_J)$-module by $R^a = R1^a$. We define the induced module $H^{J,a}$ by

$$H^{J,a} = H \otimes_{H(W_J)} R^a, \tag{2.23}$$

and define $\varphi^{J,a} : H \to H^{J,a}$ by $\varphi^{J,a}(h) = h \otimes 1^a$.

It is easily checked that $H^{J,a} \ni k \mapsto \overline{k} \in H^{J,a}$ and $j^a : H^{J,a} \to H^{J,a^\dagger}$ are well defined by

$$\overline{\varphi^{J,a}(h)} = \varphi^{J,a}(\overline{h}), \quad j^a(\varphi^{J,a}(h)) = \varphi^{J,a^\dagger}(j(h)) \qquad \text{for } h \in H. \tag{2.24}$$

Note that j^a is a homomorphism of R-modules and that

$$\overline{rk} = \overline{r}\,\overline{k} \qquad \text{for } r \in R \text{ and } k \in H^{J,a}, \tag{2.25}$$

$$\overline{\overline{k}} = k \qquad \text{for } k \in H^{J,a}, \tag{2.26}$$

$$j^{a^\dagger} \circ j^a = \mathrm{id}_{H^{J,a}}\,. \tag{2.27}$$

For $w \in W^J$ set $T_w^{J,a} = \varphi^{J,a}(T_w)$. It is easily seen that $H^{J,a}$ is a free R-module with basis $\{T_w^{J,a}\}_{w \in W^J}$. Note that we have

$$\varphi^{J,a}(T_{wx}) = a^{\ell(x)} T_w^{J,a} \qquad \text{for } w \in W^J \text{ and } x \in W_J. \tag{2.28}$$

Proposition 2.3 (Deodhar [1]). *For any $w \in W^J$ there exists a unique $C_w^{J,a} \in H^{J,a}$ satisfying the following conditions.*

(2.29) $C_w^{J,a} = \sum_{y \leqq w} P_{y,w}^{J,a} T_y$ *with* $P_{w,w}^{J,a} = 1$ *and* $P_{y,w}^{J,a} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i$ *for* $y < w$.

(2.30) $\overline{C_w^{J,a}} = q^{-\ell(w)} C_w^{J,a}$.

Moreover we have $P_{y,w}^{J,a} \in \mathbb{Z}[q]$ *for any* $y \leqq w$.

The polynomials $P_{y,w}^{J,a}$ for $y, w \in W^J$ with $y \leqq w$ are called parabolic Kazhdan-Lusztig polynomials. We write

$$P_{y,w}^{J,a} = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,a} q^k. \tag{2.31}$$

Remark 2.4. In the original reference [1] Deodhar uses

$$(-1)^{\ell(w)} j^{a^\dagger}(C_w^{J,a^\dagger}) = \sum_{y \leqq w} (-q)^{\ell(w)-\ell(y)} \overline{P_{y,w}^{J,a^\dagger}} T_y^{J,a}$$

instead of $C_w^{J,a}$ to define the parabolic Kazhdan-Lusztig polynomials. Hence our $P_{y,w}^{J,a}$ is actually the parabolic Kazhdan-Lusztig polynomial $P_{y,w}^J$ for $u = a^\dagger$ in the terminology of [1].

Proposition 2.5 (Deodhar [1]). *Let $w, y \in W^J$ such that $w \geqq y$.*

(i) *We have*

$$P_{y,w}^{J,-1} = \sum_{x \in W_J, yx \leqq w} (-1)^{\ell(x)} P_{yx,w}.$$

(ii) *If W_J is a finite group, then we have $P_{y,w}^{J,q} = P_{yw_J, ww_J}$.*

Set

$$H^{J,a,*} = \mathrm{Hom}_R(H^{J,a}, R), \tag{2.32}$$

and define ${}^t\varphi^{J,a} : H^{J,a,*} \to H^*$ by

$$\langle {}^t\varphi^{J,a}(n), h \rangle = \langle n, \varphi^{J,a}(h) \rangle \qquad \text{for } n \in H^{J,a,*} \text{ and } h \in H.$$

Then ${}^t\varphi^{J,a}$ is an injective homomorphism of R-modules. We define an involution $-$ of $H^{J,a,*}$ similarly to (2.8). We can easily check that

$$\overline{{}^t\varphi^{J,a}(n)} = {}^t\varphi^{J,a}(\overline{n}) \qquad \text{for any } n \in H^{J,a,*}. \tag{2.33}$$

For $w \in W^J$ we define $S_w^{J,a}, D_w^{J,a} \in H^{J,a,*}$ by

$$\langle S_w^{J,a}, T_x^{J,a}\rangle = (-1)^{\ell(w)}\delta_{w,x}, \qquad \langle D_w^{J,a}, C_x^{J,a}\rangle = (-1)^{\ell(w)}\delta_{w,x}. \tag{2.34}$$

Then any element of $H^{J,a,*}$ is written uniquely as an infinite sum in two ways $\sum_{w\in W^J} r_w S_w^{J,a}$ and $\sum_{w\in W^J} r'_w D_w^{J,a}$ with $r_w, r'_w \in R$. Note that we have

$$S_w^{J,a} = \sum_{y\in W^J, y\geqq w} (-1)^{\ell(w)-\ell(y)} P_{w,y}^{J,a} D_y^{J,a} \tag{2.35}$$

by $C_w^{J,a} = \sum_{y\leqq w} P_{y,w}^{J,a} T_y$. We see easily by (2.28) that

$${}^t\varphi^{J,a}(S_w^{J,a}) = \sum_{x\in W_J} (-a)^{\ell(x)} S_{wx} \qquad \text{for } w \in W^J. \tag{2.36}$$

By the definition we have

$$\overline{D_w^{J,a}} = q^{\ell(w)} D_w^{J,a}, \tag{2.37}$$

and we can write

$$D_w^{J,a} = \sum_{y\in W_J, y\geqq w} Q_{w,y}^{J,a} S_y^{J,a} \tag{2.38}$$

where $Q_{w,y}^{J,a} \in R$ are determined by

$$\sum_{y\in W^J, w\leqq y\leqq z} (-1)^{\ell(y)-\ell(w)} Q_{w,y}^{J,a} P_{y,z}^{J,a} = \delta_{w,z} \tag{2.39}$$

for $w, z \in W^J$ satisfying $w \leqq z$.

Note that (2.38) is equivalent to

$$T_w^{J,a} = \sum_{y\in W^J, y\leqq w} (-1)^{\ell(w)-\ell(y)} Q_{y,w}^{J,a} C_y^{J,a}. \tag{2.40}$$

By (2.39) we have for $w, y \in W_J$

(2.41) $\quad Q_{w,y}^{J,a} \in \mathbb{Z}[q]$,

(2.42) $\quad Q_{w,w}^{J,a} = 1$ and $\deg Q_{w,y}^{J,a} \leqq (\ell(y) - \ell(w) - 1)/2$ for $w < y$.

We call the polynomials $Q^{J,a}_{w,y}$ for $w \leqq y$ inverse parabolic Kazhdan-Lusztig polynomials. We write

$$Q^{J,a}_{w,y} = \sum_{k\in\mathbb{Z}} Q^{J,a}_{w,y,k} q^k. \tag{2.43}$$

Similarly to Propositions 2.1, 2.2, 2.3, we can prove the following.

Proposition 2.6. *Let $w \in W^J$. Assume that $D \in H^{J,a,*}$ satisfies the following conditions:*

(2.44) $D = \sum_{y\in W^J, y\geq w} r_y S^{J,a}_y$ *with* $r_w = 1$ *and* $r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i$ *for* $y \in W^J$ *satisfying* $w < y$.

(2.45) $\overline{D} = q^{\ell(w)} D$.

Then we have $D = D^{J,a}_w$.

Proposition 2.7 (Soergel [9]). *Let $w, y \in W^J$ such that $w \leqq y$.*

(i) *We have* $Q^{J,-1}_{w,y} = Q_{w,y}$.

(ii) *If W_J is a finite group, then we have*

$$Q^{J,q}_{w,y} = \sum_{x\in W_J, ww_J\leqq yx} (-1)^{\ell(x)+\ell(w_J)} Q_{ww_J,yx}.$$

3 Hodge modules

In this section we briefly recall the notation from the theory of Hodge modules due to M. Saito [8].

We denote by HS the category of mixed Hodge structures and by HS_k the category of pure Hodge structures with weight $k \in \mathbb{Z}$. Let R and R_k be the Grothendieck groups of HS and HS_k respectively. Then we have $R = \bigoplus_{k\in\mathbb{Z}} R_k$ and R is endowed with a structure of a commutative ring via the tensor product of mixed Hodge structures. The identity element of R is given by $[\mathbb{Q}^H]$, where $\mathbb{Q}^H$ is the trivial Hodge structure. We denote by $R \ni r \mapsto \overline{r} \in R$ the involutive ring endomorphism induced by the duality functor $\mathbb{D} : \mathrm{HS} \to \mathrm{HS}^{\mathrm{op}}$. Here $\mathrm{HS}^{\mathrm{op}}$ denotes the opposite category of HS. Let $\mathbb{Q}^H(1)$ and $\mathbb{Q}^H(-1)$ be the Hodge structure of Tate and its dual respectively, and set $\mathbb{Q}^H(\pm n) = \mathbb{Q}^H(\pm 1)^{\otimes n}$ for $n \in \mathbb{Z}_{\geq 0}$. We can regard $\mathbb{Z}[q, q^{-1}]$ as a subring of R by $q^n = [\mathbb{Q}^H(-n)]$. Then the condition (2.1) is satisfied for this R.

Let Z be a finite-dimensional algebraic variety over $\mathbb{C}$. There are two conventions for perverse sheaves on Z according to whether $\mathbb{Q}_U[\dim U]$ is a perverse sheaf or $\mathbb{Q}_U[-\operatorname{codim} U]$ is a perverse sheaf for a closed smooth subvariety U of Z. Correspondingly, we have two conventions for Hodge modules. When we use the convention so that $\mathbb{Q}_U[\dim U]$ is a perverse sheaf, we denote the category of Hodge modules on Z by $\mathrm{HM}_{\mathrm{d}}(Z)$, and when we use the other one we denote it by $\mathrm{HM}_{\mathrm{c}}(Z)$. Let $D^b(\mathrm{HM}_{\mathrm{d}}(Z))$ and $D^b(\mathrm{HM}_{\mathrm{c}}(Z))$ denote the bounded derived categories of $\mathrm{HM}_{\mathrm{d}}(Z)$ and $\mathrm{HM}_{\mathrm{c}}(Z)$ respectively. Note that d is for dimension and c for codimension. Then the functor $\mathrm{HM}_{\mathrm{d}}(Z) \to \mathrm{HM}_{\mathrm{c}}(Z)$ given by $M \mapsto M[-\dim Z]$ gives the category equivalences

$$\mathrm{HM}_{\mathrm{d}}(Z) \cong \mathrm{HM}_{\mathrm{c}}(Z), \qquad D^b(\mathrm{HM}_{\mathrm{d}}(Z)) \cong D^b(\mathrm{HM}_{\mathrm{c}}(Z)).$$

We shall identify $D^b(\mathrm{HM}_{\mathrm{d}}(Z))$ with $D^b(\mathrm{HM}_{\mathrm{c}}(Z))$ via this equivalence, and then we have

$$\mathrm{HM}_{\mathrm{c}}(Z) = \mathrm{HM}_{\mathrm{d}}(Z))[-\dim Z]. \tag{3.1}$$

Although there are no essential differences between $\mathrm{HM}_{\mathrm{d}}(Z)$ and $\mathrm{HM}_{\mathrm{c}}(Z)$, we have to be careful in extending the theory of Hodge modules to the infinite-dimensional situation. In dealing with sheaves supported on finite-dimensional subvarieties embedded into an infinite-dimensional manifold we have to use HM_{d}, while we need to use HM_{c} when we treat sheaves supported on finite-codimensional subvariety of an infinite-dimensional manifold. In fact what we really need in the sequel is the results for infinite-dimensional situation; however, we shall only give below a brief explanation for the finite-dimensional case. The extension of HM_{d} to the infinite-dimensional situation dealing with sheaves supported on finite-dimensional subvarieties is easy, and as for the extension of HM_{c} to the infinite-dimensional situation dealing with sheaves supported on finite-codimensional subvarieties we refer the readers to Kashiwara-Tanisaki [4].

Let Z be a finite-dimensional algebraic variety over $\mathbb{C}$. When Z is smooth, one has a Hodge module $\mathbb{Q}_Z^H[\dim Z] \in \mathrm{Ob}(\mathrm{HM}_{\mathrm{d}}(Z))$ corresponding to the perverse sheaf $\mathbb{Q}_Z[\dim Z]$. More generally, for a locally closed smooth subvariety U of Z one has a Hodge module ${}^{\pi}\mathbb{Q}_U^H[\dim U] \in \mathrm{Ob}(\mathrm{HM}_{\mathrm{d}}(Z))$ corresponding to the perverse sheaf ${}^{\pi}\mathbb{Q}_U[\dim U]$. For $M \in \mathrm{Ob}(D^b(\mathrm{HM}_{\mathrm{d}}(Z)))$ and $n \in \mathbb{Z}$ we set $M(n) = M \otimes \mathbb{Q}^H(n)$. One has the duality functor

$$\mathbb{D}_{\mathrm{d}} : \mathrm{HM}_{\mathrm{d}}(Z) \to \mathrm{HM}_{\mathrm{d}}(Z)^{\mathrm{op}}, \qquad \mathbb{D}_{\mathrm{d}} : D^b(\mathrm{HM}_{\mathrm{d}}(Z)) \to D^b(\mathrm{HM}_{\mathrm{d}}(Z))^{\mathrm{op}} \tag{3.2}$$

satisfying $\mathbb{D}_{\mathrm{d}} \circ \mathbb{D}_{\mathrm{d}} = \mathrm{Id}$, and we have

$$\mathbb{D}_{\mathrm{d}}({}^{\pi}\mathbb{Q}_U^H[\dim U]) = {}^{\pi}\mathbb{Q}_U^H[\dim U](\dim U) \tag{3.3}$$

for a locally closed smooth subvariety U of Z.

Let $f : Z \to Z'$ be a morphism of finite-dimensional algebraic varieties. Then one has the functors:

$$f^* : D^b(\mathrm{HM_d}(Z')) \to D^b(\mathrm{HM_d}(Z)), \qquad f^! : D^b(\mathrm{HM_d}(Z')) \to D^b(\mathrm{HM_d}(Z)),$$
$$f_* : D^b(\mathrm{HM_d}(Z)) \to D^b(\mathrm{HM_d}(Z')), \qquad f_! : D^b(\mathrm{HM_d}(Z)) \to D^b(\mathrm{HM_d}(Z')),$$

satisfying

$$f^* \circ \mathbb{D}_\mathrm{d} = \mathbb{D}_\mathrm{d} \circ f^!, \qquad f_* \circ \mathbb{D}_\mathrm{d} = \mathbb{D}_\mathrm{d} \circ f_!.$$

We define the functors $f^*, f^!, f_*, f_!$ for $D^b(\mathrm{HM_c})$ by identifying $D^b(\mathrm{HM_c})$ with $D^b(\mathrm{HM_d})$. For $\mathrm{HM_c}$ we use the modified duality functor

(3.4) $\mathbb{D}_\mathrm{c} : \mathrm{HM_c}(Z) \to \mathrm{HM_c}(Z)^{\mathrm{op}}, \qquad \mathbb{D}_\mathrm{c} : D^b(\mathrm{HM_d}(Z)) \to D^b(\mathrm{HM_d}(Z))^{\mathrm{op}}$

given by

$$\mathbb{D}_\mathrm{c}(M) = (\mathbb{D}_\mathrm{d}(M))[-2\dim Z](-\dim Z).$$

It also satisfies $\mathbb{D}_\mathrm{c} \circ \mathbb{D}_\mathrm{c} = \mathrm{Id}$. For a locally closed smooth subvariety U of Z we have ${}^\pi\mathbb{Q}_U^H[-\operatorname{codim} U] \in \mathrm{Ob}(\mathrm{HM_c}(Z))$ and

(3.5) $$\mathbb{D}_\mathrm{c}({}^\pi\mathbb{Q}_U^H[-\operatorname{codim} U]) = {}^\pi\mathbb{Q}_U^H[-\operatorname{codim} U](-\operatorname{codim} U).$$

When $f : Z \to Z'$ is a proper morphism, we have $f_* = f_!$ and hence $f_! \circ \mathbb{D}_\mathrm{d} = \mathbb{D}_\mathrm{d} \circ f_!$. When f is a smooth morphism, we have $f^! = f^*[2(\dim Z - \dim Z')](\dim Z - \dim Z')$ and hence $f^* \circ \mathbb{D}_\mathrm{c} = \mathbb{D}_\mathrm{c} \circ f^*$.

4 Finite-codimensional Schubert varieties

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbb{C}$. We denote by W its Weyl group and by S the set of simple roots. Then (W, S) is a Coxeter system. We shall consider the Hecke algebra $H = H(W)$ over the Grothendieck ring R of the category HS (see § 3), and use the notation in § 2

Let $X = G/B$ be the flag manifold for $\mathfrak{g}$ constructed in Kashiwara [3]. Here B is the "Borel subgroup" corresponding to the standard Borel subalgebra of $\mathfrak{g}$. Then X is a scheme over $\mathbb{C}$ covered by open subsets isomorphic to

$$\mathbb{A}^\infty = \operatorname{Spec} \mathbb{C}[x_k; k \in \mathbb{N}]$$

(unless $\dim \mathfrak{g} < \infty$).

Let $1_X = eB \in X$ denote the origin of X. For $w \in W$ we have a point $w1_X = wB/B \in X$. Let B^- be the "Borel subgroup" opposite to B, and set $X^w = B^- w1_X = B^- wB/B$ for $w \in W$. Then we have the following result.

Proposition 4.1 (Kashiwara [3]). (i) *We have* $X = \bigsqcup_{w \in W} X^w$.

(ii) *For* $w \in W$, X^w *is a locally closed subscheme of* X *isomorphic to* $\mathbb{A}^\infty$ (*unless* $\dim \mathfrak{g} < \infty$) *with codimension* $\ell(w)$.

(iii) *For* $w \in W$, *we have* $\overline{X^w} = \bigsqcup_{y \in W, y \geqq w} X^y$.

We call X^w for $w \in W$ a finite-codimensional Schubert cell, and $\overline{X^w}$ a finite-codimensional Schubert variety.

Let J be a subset of S. We denote by Y the partial flag manifold corresponding to J. Let $\pi : X \to Y$ be the canonical projection and set $1_Y = \pi(1_X)$. We have $\pi(w1_X) = 1_Y$ for any $w \in W_J$. For $w \in W^J$ we set $Y^w = B^- w 1_Y = \pi(X^w)$. When W_J is a finite group, we have $Y = G/P_J$ and $Y^w = B^- w P_J / P_J$, where P_J is the "parabolic subgroup" corresponding to J (we cannot define P_J as a group scheme unless W_J is a finite group).

Similarly to Proposition 4.1 we have the following.

Proposition 4.2. (i) *We have* $Y = \bigsqcup_{w \in W^J} Y^w$.

(ii) *For* $w \in W^J$, Y^w *is a locally closed subscheme of* Y *isomorphic to* $\mathbb{A}^\infty$ (*unless* $\dim Y < \infty$) *with codimension* $\ell(w)$.

(iii) *For* $w \in W^J$, *we have* $\overline{Y^w} = \bigsqcup_{y \in W^J, y \geqq w} Y^y$.

(iv) *For* $w \in W^J$, *we have* $\pi^{-1}(Y^w) = \bigsqcup_{x \in W_J} X^{wx}$.

We call a subset Ω of W^J (resp. W) admissible if it satisfies

$$w, y \in W^J (\text{resp. } W), w \leqq y, y \in \Omega \Rightarrow w \in \Omega. \tag{4.1}$$

For a finite admissible subset Ω of W^J we set $Y^\Omega = \bigcup_{w \in \Omega} Y^w$. It is a quasi-compact open subset of Y. Let $\mathrm{HM}_c^{B^-}(Y^\Omega)$ be the category of B^--equivariant Hodge modules on Y^Ω (see Kashiwara-Tanisaki [4] for the equivariant Hodge modules on infinite-dimensional manifolds), and denote its Grothendieck group by $K(\mathrm{HM}_c^{B^-}(Y^\Omega))$. For $w \in W^J$ the Hodge modules $\mathbb{Q}^H_{Y^w}[-\ell(w)]$ and ${}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)]$ are objects of $K(\mathrm{HM}_c^{B^-}(Y^\Omega))$. Note that $\mathbb{Q}_{Y^w}[-\ell(w)]$ is a perverse sheaf on Y because Y^w is affine. Set

$$\mathrm{HM}_c^{B^-}(Y) = \varprojlim_\Omega \mathrm{HM}_c^{B^-}(Y^\Omega),\ K(\mathrm{HM}_c^{B^-}(Y)) = \varprojlim_\Omega K(\mathrm{HM}_c^{B^-}(Y^\Omega)), \tag{4.2}$$

where Ω runs through finite admissible subsets of W^J. By the tensor product, $K(\mathrm{HM}_c^{B^-}(Y))$ is endowed with a structure of an R-module. Then any element of $K(\mathrm{HM}_c^{B^-}(Y))$ is uniquely written as an infinite sum

$$\sum_{w \in W^J} r_w [\mathbb{Q}^H_{Y^w}[-\ell(w)]] \text{ with } r_w \in R.$$

Denote by $K(\mathrm{HM}_c^{B^-}(Y)) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_c^{B^-}(Y))$ the involution induced by the duality functor $\mathbb{D}_c$. Then we have $\overline{rm} = \overline{r}\,\overline{m}$ for any $r \in R$ and $m \in K(\mathrm{HM}_c^{B^-}(Y))$.

We can similarly define $\mathrm{HM}_c^{B^-}(X)$, $\mathbb{Q}^H_{X^w}[-\ell(w)]$ and ${}^{\pi}\mathbb{Q}^H_{X^w}[-\ell(w)]$ for $w \in W$, $K(\mathrm{HM}_c^{B^-}(X))$, and $K(\mathrm{HM}_c^{B^-}(X)) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_c^{B^-}(X))$ (for $J = \emptyset$).

Let pt denote the algebraic variety consisting of a single point. For $w \in W$ (resp. $w \in W^J$) we denote by $i_{X,w} : \mathrm{pt} \to X$ (resp. $i_{Y,w} : \mathrm{pt} \to Y$) denote the morphism with image $\{w1_X\}$ (resp. $\{w1_Y\}$). We define homomorphisms

$$(4.3) \qquad \Phi : K(\mathrm{HM}_c^{B^-}(X)) \to H^*, \qquad \Phi^J : K(\mathrm{HM}_c^{B^-}(Y)) \to H^{J,-1,*}$$

of R-modules by

$$(4.4) \qquad \Phi([M]) = \sum_{w \in W} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i^*_{X,w}(M)] \right) S_w,$$

$$(4.5) \qquad \Phi^J([M]) = \sum_{w \in W^J} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i^*_{Y,w}(M)] \right) S^{J,-1}_w.$$

By the definition we have

$$(4.6) \qquad \Phi([\mathbb{Q}^H_{X^w}[-\ell(w)]]) = (-1)^{\ell(w)} S_w \quad \text{for } w \in W,$$

$$(4.7) \qquad \Phi^J([\mathbb{Q}^H_{Y^w}[-\ell(w)]]) = (-1)^{\ell(w)} S^{J,-1}_w \quad \text{for } w \in W^J,$$

and hence Φ and Φ^J are isomorphisms of R-modules.

The projection $\pi : X \to Y$ induces a homomorphism

$$\pi^* : K(\mathrm{HM}_c^{B^-}(Y)) \to K(\mathrm{HM}_c^{B^-}(X))$$

of R-modules.

Lemma 4.3. (i) *The following diagram is commutative.*

$$\begin{array}{ccc} K(\mathrm{HM}_c^{B^-}(Y)) & \xrightarrow{\Phi^J} & H^{J,-1,*} \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle {}^t\varphi^{J,-1}} \\ K(\mathrm{HM}_c^{B^-}(X)) & \xrightarrow[\Phi]{} & H^* \end{array}$$

(ii) $\overline{\pi^*(m)} = \pi^*(\overline{m})$ *for any* $m \in K(\mathrm{HM}_{\mathrm{c}}^{B^-}(Y))$.

(iii) $\overline{\Phi(m)} = \Phi(\overline{m})$ *for any* $m \in K(\mathrm{HM}_{\mathrm{c}}^{B^-}(X))$.

(iv) $\overline{\Phi^J(m)} = \Phi^J(\overline{m})$ *for any* $m \in K(\mathrm{HM}_{\mathrm{c}}^{B^-}(Y))$.

Proof. For $w \in W^J$ we have $\pi^*(\mathbb{Q}^H_{Y^w}) = \mathbb{Q}^H_{\pi^{-1}Y_w}$, and hence Proposition 4.2 (iv) implies

$$\pi^*([\mathbb{Q}^H_{Y^w}]) = \sum_{x \in W_J} [\mathbb{Q}^H_{X^{wx}}].$$

Thus (i) follows from (4.6), (4.7) and (2.36)

Locally on X the morphism π is a projection of the form $Z \times \mathbb{A}^\infty \to Z$, and thus $\pi^* \circ \mathbb{D}_{\mathrm{c}} = \mathbb{D}_{\mathrm{c}} \circ \pi^*$. Hence the statement (ii) holds.

The statement (iii) is already known (see Kashiwara-Tanisaki [4]).

Then the statement (iv) follows from (i), (ii), (iii), (2.33) and the injectivity of ${}^t\varphi^{J,-1}$. □

Theorem 4.4. *Let* $w, y \in W^J$ *satisfying* $w \leqq y$. *Then we have*

$$H^{2k+1} i^*_{Y,y}({}^\pi\mathbb{Q}^H_{Y^w}) = 0, \qquad H^{2k} i^*_{Y,y}({}^\pi\mathbb{Q}^H_{Y^w}) = \mathbb{Q}^H(-k)^{\oplus Q^{J,-1}_{w,y,k}}$$

for any $k \in \mathbb{Z}$. *In particular, we have*

$$\Phi^J([{}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)]]) = (-1)^{\ell(w)} D^{J,-1}_w.$$

Proof. Let $w \in W^J$ and set

$$(-1)^{\ell(w)}\Phi^J([{}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)]]) = D = \sum_{y \in W^J, y \geqq w} r_y S^{J,-1}_y.$$

By the definition of ${}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)]$ we have

$$\mathbb{D}_{\mathrm{c}}({}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)]) = {}^\pi\mathbb{Q}^H_{Y^w}[-\ell(w)](-\ell(w)),$$

and hence we obtain

$$\overline{D} = q^{\ell(w)} D \tag{4.8}$$

by Lemma 4.3 (iv). By the definition of Φ^J we have

$$r_y = \sum_{k \in \mathbb{Z}} (-1)^k [H^k i^*_{Y,y}({}^\pi\mathbb{Q}^H_{Y^w})], \tag{4.9}$$

and by the definition of ${}^{\pi}\mathbb{Q}^H_{Y^w}[-\ell(w)]$ we have

(4.10) $\qquad r_w = 1,$

(4.11) $\qquad$ for $y > w$ we have $H^k i^*_{Y,y}({}^{\pi}\mathbb{Q}^H_{Y^w}) = 0$ unless $0 \leqq k \leqq (\ell(y) - \ell(w) - 1)$.

By the argument similar to Kashiwara-Tanisaki [4] (see also Kazhdan-Lusztig [7]) we have

$$[H^k i^*_{Y,y}({}^{\pi}\mathbb{Q}^H_{Y^w})] \in R_k. \tag{4.12}$$

In particular, we have

(4.13) $\qquad$ for $y > w$ we have $r_y \in \bigoplus_{k=0}^{\ell(y)-\ell(w)-1} R_k$.

Thus we obtain $D = D^{J,-1}_w$ by (4.8), (4.10), (4.13) and Proposition 2.6. Hence $r_y = Q^{J,-1}_{y,w}$. By (4.9) and (4.12) we have $[H^{2k+1} i^*_{Y,y}({}^{\pi}\mathbb{Q}^H_{Y^w})] = 0$ and $[H^{2k} i^*_{Y,y}({}^{\pi}\mathbb{Q}^H_{Y^w})] = q^k Q_{w,y,k}$ for any $k \in \mathbb{Z}$. The proof is complete. □

By (2.35) and Theorem 4.4 we obtain the following.

Corollary 4.5. *We have*

$$[\mathbb{Q}^H_{Y^w}[-\ell(w)]] = \sum_{y \geqq w} P^{J,-1}_{w,y}[{}^{\pi}\mathbb{Q}^H_{Y^y}[-\ell(y)]]$$

in the Grothendieck group $K(\mathrm{HM}^{B^-}_c(Y))$. In particular, the coefficient $P^{J,-1}_{w,y,k}$ of the parabolic Kazhdan-Lusztig polynomial $P^{J,-1}_{w,y}$ is non-negative and equal to the multiplicity of the irreducible Hodge module ${}^{\pi}\mathbb{Q}^H_{Y^y}[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}^H_{Y^w}[-\ell(w)]$.

5 Finite-dimensional Schubert varieties

Set

$$X_w = Bw1_X = BwB/B \quad \text{for } w \in W. \tag{5.1}$$

Then we have the following result.

Proposition 5.1 (Kashiwara-Tanisaki [5]). *Set $X' = \bigcup_{w \in W} X_w$. Then X' is the flag manifold considered by Kac-Peterson [2], Tits [10], et al. In particular, we have the following.*

(i) *We have* $X' = \bigsqcup_{w \in W} X_w$.

(ii) *For* $w \in W$ X_w *is a locally closed subscheme of* X *isomorphic to* $\mathbb{A}^{\ell(w)}$.

(iii) *For* $w \in W$ *we have* $\overline{X}_w = \bigsqcup_{y \in W, y \leqq w} X_y$.

We call X_w for $w \in W$ a finite-dimensional Schubert cell and $\overline{X}_w$ a finite-dimensional Schubert variety. Note that X' is not a scheme but an inductive limit of finite-dimensional projective schemes (an ind-scheme).

For $w \in W^J$, we set $Y_w = Bw1_Y = \pi(X_w)$. Similarly to Proposition 5.1 we have the following.

Proposition 5.2. *Set* $Y' = \bigcup_{w \in W^J} Y_w$. *Then we have the following.*

(i) *We have* $Y' = \bigsqcup_{w \in W^J} Y_w$.

(ii) *For* $w \in W^J$, Y_w *is a locally closed subscheme of* Y *isomorphic to* $\mathbb{A}^{\ell(w)}$.

(iii) *For* $w \in W^J$, *we have* $\overline{Y}_w = \bigsqcup_{y \in W^J, y \leqq w} Y_y$.

(iv) *For* $w \in W^J$, *we have* $\pi^{-1}(Y_w) = \bigsqcup_{x \in W_J} X_{wx}$.

For a finite admissible subset Ω of W^J we set $Y'_\Omega = \bigcup_{w \in \Omega} Y'_w$. It is a finite dimensional projective scheme.

Let $\mathrm{HM}_{\mathrm{d}}^B(Y'_\Omega)$ be the category of B-equivariant Hodge modules on Y'_Ω. For $w \in W^J$ the Hodge modules $\mathbb{Q}^H_{Y_w}[\ell(w)]$ and ${}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]$ are objects of $\mathrm{HM}_{\mathrm{d}}^B(Y'_\Omega)$. Note that $\mathbb{Q}_{Y_w}[\ell(w)]$ is a perverse sheaf because Y_w is affine. Set

$$\mathrm{HM}_{\mathrm{d}}^B(Y') = \varinjlim_{\Omega} \mathrm{HM}_{\mathrm{d}}^B(Y'_\Omega),\ K(\mathrm{HM}_{\mathrm{d}}^B(Y')) = \varinjlim_{\Omega} K(\mathrm{HM}_{\mathrm{d}}^B(Y'_\Omega)), \tag{5.2}$$

where Ω runs through finite admissible subsets of W^J. By the tensor product $K(\mathrm{HM}_{\mathrm{d}}^B(Y'))$ is endowed with a structure of an R-module. Then any element of $K(\mathrm{HM}_{\mathrm{d}}^B(Y'))$ is uniquely written as a finite sum in two ways

$$\sum_{w \in W^J} r_w[\mathbb{Q}^H_{Y_w}[\ell(w)]] \ \text{ and } \ \sum_{w \in W^J} r_w[{}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]] \ \text{ with } r_w,\ r'_w \in R.$$

Denote by $K(\mathrm{HM}_{\mathrm{d}}^B(Y')) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_{\mathrm{d}}^B(Y'))$ the involution of an abelian group induced by the duality functor $\mathbb{D}_{\mathrm{d}}$. Then we have $\overline{rm} = \overline{r}\,\overline{m}$ for any $r \in R$ and $m \in K(\mathrm{HM}_{\mathrm{d}}^B(Y'))$.

We can similarly define $\mathrm{HM}_{\mathrm{d}}^B(X')$, $\mathbb{Q}^H_{X_w}[\ell(w)]$ and ${}^\pi\mathbb{Q}^H_{X_w}[\ell(w)]$ for $w \in W$, $K(\mathrm{HM}_{\mathrm{d}}^B(X'))$, and $K(\mathrm{HM}_{\mathrm{d}}^B(X')) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_{\mathrm{d}}^B(X'))$ (for $J = \emptyset$).

For $w \in W$ (resp. $w \in W^J$) we denote by $i_{X',w} : \mathrm{pt} \to X'$ (resp. $i_{Y',w} : \mathrm{pt} \to Y'$) denote the morphism with image $\{w1_X\}$ (resp. $\{w1_Y\}$). We define homomorphisms

$$\Psi : K(\mathrm{HM}^B_{\mathrm{d}}(X')) \to H, \qquad \Psi^J : K(\mathrm{HM}^B_{\mathrm{d}}(Y')) \to H^{J,q} \tag{5.3}$$

of R-modules by

$$\Psi([M]) = \sum_{w \in W} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i^*_{X',w}(M)] \right) T_w, \tag{5.4}$$

$$\Psi^J([M]) = \sum_{w \in W^J} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i^*_{Y',w}(M)] \right) T^{J,q}_w. \tag{5.5}$$

By the definition we have

$$\Psi([\mathbb{Q}^H_{X_w}[\ell(w)]]) = (-1)^{\ell(w)} T_w \quad \text{for } w \in W, \tag{5.6}$$

$$\Psi^J([\mathbb{Q}^H_{Y_w}[\ell(w)]]) = (-1)^{\ell(w)} T^{J,q}_w \quad \text{for } w \in W^J, \tag{5.7}$$

and hence Ψ and Ψ^J are isomorphisms.

Let $\pi' : X' \to Y'$ denote the projection. Let Ω be a finite admissible subset of W and set $\Omega' = \{w \in W^J \,;\, wW_J \cap \Omega \neq \emptyset\}$. Then Ω' is a finite admissible subset of W^J and π' induces a surjective projective morphism $X'_\Omega \to Y'_{\Omega'}$. Hence we can define a homomorphism $\pi'_! : K(HM^B(X')) \to K(HM^B(Y'))$ of R-modules by

$$\pi'_!([M]) = \sum_{k \in \mathbb{Z}} (-1)^k [H^k \pi'_!(M)]. \tag{5.8}$$

Lemma 5.3. (i) *The following diagram is commutative.*

$$\begin{array}{ccc} K(\mathrm{HM}^B_{\mathrm{d}}(X')) & \xrightarrow{\Psi} & H \\ {\scriptstyle \pi'_!}\downarrow & & \downarrow{\scriptstyle \varphi^{J,q}} \\ K(\mathrm{HM}^B_{\mathrm{d}}(Y')) & \xrightarrow[\Psi^J]{} & H^{J,q} \end{array}$$

(ii) $\overline{\pi'_!(m)} = \pi'_!(\overline{m})$ *for any* $m \in K(\mathrm{HM}^B_{\mathrm{d}}(X'))$.

(iii) $\overline{\Psi(m)} = \Psi(\overline{m})$ *for any* $m \in K(\mathrm{HM}^B_{\mathrm{d}}(X'))$.

(iv) $\overline{\Psi^J(m)} = \Psi^J(\overline{m})$ *for any* $m \in K(\mathrm{HM}^B_{\mathrm{d}}(Y'))$.

Proof. Let $w \in W^J$ and $x \in W_J$. Since $X_{wx} \to Y_w$ is an $\mathbb{A}^{\ell(x)}$-bundle, we have $\pi'_!(\mathbb{Q}^H_{X_{wx}}) = \mathbb{Q}^H_{Y_w}[-2\ell(x)](-\ell(x))$, and hence

$$\pi'_!([\mathbb{Q}^H_{X_{wx}}[\ell(wx)]]) = (-q)^{\ell(x)}[\mathbb{Q}^H_{Y_w}[\ell(w)]].$$

Thus (i) follows from (5.6), (5.7) and (2.28).

The statement (ii) follows from the fact that π' is an inductive limit of projective morphisms and hence $\pi'_!$ commutes with the duality functor $\mathbb{D}_{\mathrm{d}}$.

The statement (iii) is proved similarly to Kashiwara-Tanisaki [4], and we omit the details (see also Kazhdan-Lusztig [7]). Then the statement (iv) follows from (i), (ii), (iii), (2.24) and surjectivity of $\varphi^{J,q}$. □

Theorem 5.4. *Let $w, y \in W^J$ such that $w \geqq y$. Then we have*

$$H^{2k+1}i^*_{Y',y}({}^\pi\mathbb{Q}^H_{Y_w}) = 0, \qquad H^{2k}i^*_{Y',y}({}^\pi\mathbb{Q}^H_{Y_w}) = \mathbb{Q}^H(-k)^{\oplus P^{J,q}_{y,w,k}}$$

for any $k \in \mathbb{Z}$. In particular, we have

$$\Psi^J([{}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]]) = (-1)^{\ell(w)}C^{J,q}_w.$$

Proof. Let $w \in W^J$ and set

$$(-1)^{\ell(w)}\Psi^J([{}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]]) = C = \sum_{y \in W^J, y \leqq w} r_y T^{J,q}.$$

By the definition of ${}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]$ we have $\mathbb{D}_{\mathrm{d}}({}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]) = {}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)](\ell(w))$. Hence we obtain

$$\overline{C} = q^{-\ell(w)}C \tag{5.9}$$

by Lemma 5.3 (iv). By the definition of Ψ^J we have

$$r_y = \sum_{k \in \mathbb{Z}}(-1)^k[H^k i^*_{Y',y}({}^\pi\mathbb{Q}^H_{Y_w})], \tag{5.10}$$

and by the definition of ${}^\pi\mathbb{Q}^H_{Y_w}[\ell(w)]$ we have

(5.11) $r_w = 1$,

(5.12) for $y < w$ we have $H^k i^*_{Y',y}({}^\pi\mathbb{Q}^H_{Y_w}) = 0$ unless $0 \leqq k \leqq (\ell(w) - \ell(y) - 1)$.

Moreover, by the argument similar to Kazhdan-Lusztig [7] and Kashiwara-Tanisaki [4] we have

$$[H^k i^*_{Y',y}({}^\pi\mathbb{Q}^H_{Y_w})] \in R_k. \tag{5.13}$$

In particular, we have

(5.14) for $y < w$ we have $r_y \in \bigoplus_{k=0}^{\ell(w)-\ell(y)-1} R_k$.

Thus we obtain $C = C_w^{J,q}$ by (5.9), (5.11), (5.14) and Proposition 2.3. Hence $r_y = P_{y,w}^{J,q}$. By (5.10) and (5.13) we have $[H^{2k+1} i_{Y',y}^* ({}^\pi\mathbb{Q}_{Y_w}^H)] = 0$ and $[H^{2k} i_{Y',y}^* ({}^\pi\mathbb{Q}_{Y_w}^H)] = q^k P_{y,w,k}$ for any $k \in \mathbb{Z}$. The proof is complete. □

We note that a result closely related to Theorem 5.4 above is already given in Deodhar [1].

By (2.40) and Theorem 5.4 we obtain the following.

Corollary 5.5. *We have*

$$[\mathbb{Q}_{Y'_w}^H[\ell(w)]] = \sum_{y \leqq w} Q_{y,w}^{J,q} [{}^\pi\mathbb{Q}_{Y'_y}^H[\ell(y)]]$$

in $K(\mathrm{HM}_{\mathrm{d}}^B(Y'))$. *In particular, the coefficient* $Q_{y,w,k}^{J,q}$ *of the inverse parabolic Kazhdan-Lusztig polynomial* $Q_{y,w}^{J,q}$ *is non-negative and equal to the multiplicity of the irreducible Hodge module* ${}^\pi\mathbb{Q}_{Y'_y}^H[\ell(y)](-k)$ *in the Jordan Hölder series of the Hodge module* $\mathbb{Q}_{Y'_w}^H[\ell(w)]$.

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