An algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras

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Abstract

We give an algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras from the representation theoretic viewpoint. For this purpose, we give the $GL(V)$-irreducible decomposition of the polynomial ring of the space $\wedge^2 V^* \otimes V$ ($V = \boldsymbol{C}^4$) up to degree three, and show that intrinsic concepts defined by the vanishing of these covariants are sufficient to distinguish the isomorphism classes. As an application, we describe the variety of 4-dimensional Lie algebras and their degenerations in a comparatively simple form, by introducing a new family of normal forms of 4-dimensional Lie algebras that are just fitted for these purposes.

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Introduction

In this paper, we study the set of 4-dimensional complex Lie algebras from the representation theoretic viewpoint. In particular, we give an algorithm to determine the isomorphism class of a given 4-dimensional Lie algebra, in terms of a finite number of covariants and invariants of the group $GL(V)$.

Many results are already known for 4-dimensional (real or complex) Lie algebras, such as the classification, degeneration and deformation, and the number of varieties consisting of Lie algebras, etc. (cf. [7], [9], [12], [19], [20], [24], [27], [31].) But, in spite of these results, several important problems are still left unsolved. For example, it is in general a hard algebraic problem to determine the explicit isomorphism class of a given Lie algebra $\mathfrak{g}$, i.e., which normal form in the classification table is isomorphic to a given $\mathfrak{g}$. Of course, the dimensions of $[\mathfrak{g}, \mathfrak{g}]$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]$, etc. give some necessary conditions to determine this isomorphism class. But, these conditions are not in general enough to determine it. In this paper, we give a finite number of intrinsic concepts of 4-dimensional complex Lie algebras,

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by which we can determine their isomorphism classes uniquely without constructing the explicit isomorphisms. (See the examples at the end of §6.)

For this purpose, we consider the set of all Lie algebra structures on a fixed 4-dimensional complex vector space V. By fixing a basis of V, the structure constant $\{c_{ij}^k\}$ of $\mathfrak{g}$ can be naturally considered as an element of $\wedge^2 V^* \otimes V$, and we may identify it with the Lie algebra $\mathfrak{g}$ itself. Since $\{c_{ij}^k\}$ satisfies the Jacobi identities that are the quadratic polynomial relations of $\{c_{ij}^k\}$, the set of all Lie algebra structures on V constitutes an algebraic set of $\wedge^2 V^* \otimes V$, which splits into four irreducible varieties. (cf. [9], [12], [19], [27]. See also Proposition 8.) The group $GL(V)$ naturally acts on this space, and it is clear that the orbit decomposition of this algebraic set is equivalent to the classification of Lie algebras.

In order to distinguish these $GL(V)$-orbits, we consider the polynomial ring of $\wedge^2 V^* \otimes V$ and its $GL(V)$-irreducible decomposition as a main tool. The above mentioned "intrinsic concepts" on $\mathfrak{g}$ can be expressed as the vanishing of some irreducible components of the polynomial ring (= covariants). For example, the space of linear polynomials of $\{c_{ij}^k\}$ splits into two $GL(V)$-irreducible components, and the vanishing of one component is equivalent to the "unimodularity" of $\mathfrak{g}$, the vanishing of the other component is equivalent to the condition that the Lie bracket is expressed as $[X, Y] = f(X)Y - f(Y)X$ for some $f \in \mathfrak{g}^*$. Both concepts play fundamental roles in determining the isomorphism classes. Two fundamental values $\dim [\mathfrak{g}, \mathfrak{g}]$ and $\dim [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]$ also can be characterized by the vanishing or non-vanishing of some covariants. (For details, see Proposition 4.)

But to distinguish the individual $GL(V)$-orbits, we need more delicate additional device. Since there exists a family of continuously deformable Lie algebras depending on two parameters (see Table 1), we must introduce at least two independent invariants of $\mathfrak{g}$ in order to distinguish them. These invariants are a natural generalization of the invariants of 3-dimensional Lie algebras introduced by [19], [35], etc., and they can be defined as a ratio of some covariants. As a result, we can show that "intrinsic concepts" that are needed in determining the isomorphism classes of $\mathfrak{g}$ are all expressible in terms of polynomials of $\{c_{ij}^k\}$ up to degree three. And by these concepts, we can explicitly give an algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras, which is the main result of this paper (Theorem 10 and Figure 2).

As a by-product of this method, we can also show several facts on the set of 4-dimensional Lie algebras, in particular, degenerations and the variety of Lie algebras. (For the definition of degeneration, see §5.) Degenerations of 4-dimensional Lie algebras are already completely determined in [7]. But the final results given in [7] are quite complicated. In this paper, we introduce a new family of normal forms of 4-dimensional complex Lie algebras that are just fitted to describe the varieties and the degenerations. And in terms of these normal forms, we summarize the results on these subjects in a comparatively simple form. To give such nice normal forms is another main result of the present paper.

Now, we state the contents of this paper. In §1, we give a new classification table of 4-dimensional complex Lie algebras, consisting of ten normal forms (Table 1). These normal forms possess several nice properties, and after explaining these features, we next summarize the fundamental quantities of these Lie algebras (Table 2). Among others, we state a remarkable property that except for one Lie algebra, the ratio of the eigenvalues of ad X does not depend on the choice of $X \in \mathfrak{g}$ if X is sufficiently generic (Proposition 2). This property leads us to define three fundamental invariants of 4-dimensional Lie algebras, which play a crucial role in determining the isomorphism classes. In §2, we give the $GL(V)$-irreducible decomposition of the polynomial ring of the space $\wedge^2 V^* \otimes V$ up to degree three. We also explicitly give the generators of these irreducible components which we use in this paper. And we evaluate them for each normal form by using computers (Table 3). To know the vanishing or non-vanishing of each generator for a given Lie algebra is one crucial step in the actual determination of the isomorphism classes. In §3, we characterize "intrinsic concepts" on $\mathfrak{g}$ determined by the vanishing of these covariants. Some unfamiliar but important properties naturally appear. In §4, by using the ratio of the eigenvalues of ad X stated above, we define three fundamental invariants $\chi_1 \sim \chi_3$ of 4-dimensional Lie algebras. Roughly speaking, these invariants serve as the coordinate of the moduli space of the variety of Lie algebras, because the parameters appearing in the normal forms in Table 1 are uniquely determined by these invariants $\chi_1 \sim \chi_3$ (Proposition 5).

In §5, by using the above results, we summarize some known facts and some new results on degenerations and the varieties of 4-dimensional Lie algebras. On account of the nice properties of our normal forms, these results are expressed in a comparatively simple form. In particular, we give the defining equations of four irreducible varieties of Lie algebras, and the explicit orbit decompositions of them, including their degenerations (Proposition 9, Figure 1). We know that there exists one principal line of degenerations in each variety. But, several "singular" Lie algebras make the situation a little complicated. In the final section (§6), we give an algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras, which is the main subject of this paper. The results are summarized in Theorem 10 and Figure 2. Roughly speaking, dimensions of $[\mathfrak{g}, \mathfrak{g}]$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]$, three invariants $\chi_1 \sim \chi_3$, and three kinds of covariants are enough to determine the isomorphism classes.

In Appendix, we construct explicit isomorphisms between normal forms in Table 1 and those of [7] for our reference. In view of this correspondence, the readers can easily see that the list of degenerations given in Proposition 6 just coincides with the result in [7; p.736].

Finally, we add some comments for higher dimensional case. Theoretically, we can continue to develop our method to higher dimensional Lie algebras. But unfortunately, calculations of covariants and invariants, and the irreducible decomposition of the polynomial ring become complicated as the dimension becomes large, and we do not know what kind of concepts will be required in order to distinguish the isomorphism classes. This is mainly due to the lack of our knowledge on the $GL(V)$-irreducible decomposition of the space $S^p(\wedge^2 V^* \otimes V)$, especially due to the lack of decomposition formulas of the "plethysm" $\{1^2\} \otimes \{\lambda\}$ (cf. §2, [23], [4]).

We can apply the results of this paper to other geometric problems. For example, we can describe the existence (or non-existence) of left invariant symplectic structures on 4-dimensional complex Lie groups in terms of several intrinsic concepts introduced in this paper. (For details, see [6].) In addition, we can apply the representation theoretic method of this paper to other kind of geometric problems on multi-tensor spaces, such as the exterior space $\wedge^p V^*$ ($p \geq 3$), the space of curvature like tensors on V, etc. We will treat these problems in the forthcoming papers.

Preliminary Remark

A Lie algebra is by definition a pair $(V, [\ ,\ ])$, where V is a vector space and $[\ ,\ ]$ is an element of $\wedge^2 V^* \otimes V$ satisfying the Jacobi identity. In this paper, we often express the vector space V as $\mathfrak{g}$ if a Lie algebra structure $[\ ,\ ]$ on V is explicitly or implicitly given. But sometimes, in case the underlying vector space V is fixed, we often identify the Lie bracket $[\ ,\ ]$ with $\mathfrak{g}$, and we consider $\mathfrak{g}$ as an element of $\wedge^2 V^* \otimes V$. This is clearly an abuse of notation. But, the author believes that the readers can correctly understand the situation without any confusion. (In this paper, V always means $\boldsymbol{C}^4$.)

1. Normal forms

In this section, we give a classification table of 4-dimensional complex Lie algebras, fitted to describe an algorithm to determine the isomorphism classes in addition to several properties on the varieties of Lie algebras, such as deformations, degenerations, orbit decompositions, etc.

There are already several classifications of 4-dimensional Lie algebras. But unfortunately, it seems that all of them are not fitted to describe the above subjects. In fact, for these classifications, several normal forms depending on parameters often degenerate to singular Lie algebras at some special values of parameters. (An example is given in Remark (6) below.) And this phenomenon makes a description of the above subjects quite complicated. Here, based on previously known classifications, we give a new classification table of 4-dimensional complex Lie algebras fitted for our purposes. Explicit correspondences to other normal forms are given in Appendix.

Proposition 1. *Any 4-dimensional complex Lie algebra is isomorphic to one of the following Lie algebras in Table 1 (α and β are complex parameters):*

Table 1

	non-trivial bracket operations
L_0	
L_1	$[X_1, X_2] = X_3$
L_2	$[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$
L_3	$[X_1, X_2] = X_2$, $[X_1, X_3] = X_3$, $[X_1, X_4] = X_4$
$L_4(\alpha)$	$[X_1, X_2] = X_2$, $[X_1, X_3] = X_3$, $[X_1, X_4] = X_3 + \alpha X_4$
$L_4(\infty)$	$[X_1, X_2] = X_2$
L_5	$[X_1, X_2] = X_2$, $[X_1, X_3] = X_3$, $[X_1, X_4] = 2X_4$, $[X_2, X_3] = X_4$
L_6	$[X_1, X_2] = X_2$, $[X_1, X_3] = -X_3$, $[X_2, X_3] = X_1$
$L_7(\alpha, \beta)$	$[X_1, X_2] = X_2$, $[X_1, X_3] = X_2 + \alpha X_3$, $[X_1, X_4] = X_3 + \beta X_4$
$L_8(\alpha)$	$[X_1, X_2] = X_2$, $[X_1, X_3] = X_2 + \alpha X_3$, $[X_1, X_4] = (\alpha + 1) X_4$, $[X_2, X_3] = X_4$
L_9	$[X_1, X_2] = X_2$, $[X_3, X_4] = X_4$

These Lie algebras are not isomorphic to each other except for the following cases:

- $L_7(\alpha,\beta) \cong L_7(\alpha',\beta')$ *if and only if two ratios* $1:\alpha:\beta$ *and* $1:\alpha':\beta'$ *coincide after a suitable change of ordering.*
- $L_8(\alpha) \cong L_8(\alpha')$ *if and only if* $\alpha = \alpha'$ *or* $\alpha\alpha' = 1$.

(*As for the Lie algebra* $L_4(\alpha)$ ($\alpha \in \boldsymbol{C} \cup \{\infty\}$), $L_4(\alpha)$ *is isomorphic to* $L_4(\alpha')$ *if and only if* $\alpha = \alpha'$.)

Outline of the proof. We can show that the above table exhausts all 4-dimensional complex Lie algebras by constructing the isomorphisms to other known normal forms. (For details, see Appendix.) By calculating the dimensions of $[L_i, L_i]$ and the $GL(V)$-orbit of L_i, we have immediately $L_i \not\cong L_j$ for $i \neq j$. (See Table 2 below.) The remaining special isomorphisms for L_4, L_7 and L_8 can be checked directly. □

We must state some remarks on the typical features of these normal forms in order to understand the arguments in this paper.

Remark. (1) Among these Lie algebras, the following ones are expressed as sums of lower dimensional Lie algebras:

$$\begin{array}{ll} L_0 \cong \boldsymbol{C}^4, & L_1 \cong M_1 \oplus \boldsymbol{C}, \\ L_4(0) \cong M_2 \oplus \boldsymbol{C}, & L_4(\infty) \cong M_3(0) \oplus \boldsymbol{C} \cong \mathfrak{aff}(1,\boldsymbol{C}) \oplus \boldsymbol{C}^2, \\ L_6 \cong \mathfrak{gl}(2,\boldsymbol{C}) \cong \mathfrak{sl}(2,\boldsymbol{C}) \oplus \boldsymbol{C}, & L_7(\alpha,0) \cong M_3(\alpha) \oplus \boldsymbol{C}, \quad (\alpha \neq 0), \\ L_9 \cong \mathfrak{aff}(1,\boldsymbol{C}) \oplus \mathfrak{aff}(1,\boldsymbol{C}). & \end{array}$$

Here, the Lie algebra $\mathfrak{aff}(1,\boldsymbol{C})$ means the non-abelian 2-dimensional Lie algebra, and M_i are 3-dimensional complex Lie algebras defined by

$$\begin{array}{lcl} M_1 & : & [Y_1,Y_2] = Y_3, \\ M_2 & : & [Y_1,Y_2] = Y_2, \ [Y_1,Y_3] = Y_3, \\ M_3(\alpha) & : & [Y_1,Y_2] = Y_2, \ [Y_1,Y_3] = Y_2 + \alpha Y_3. \end{array}$$

It should be remarked that the set of these decomposable Lie algebras does not form a "closed" subset of $\wedge^2 V^* \otimes V$ in the usual topology because the limitting Lie algebra $\lim_{\alpha\to 0} L_7(\alpha,0) = L_7(0,0)$ is not decomposable. See also Remark (1) after Proposition 6 in §5.

(2) Nilpotent Lie algebras are exhausted by L_0, L_1, L_2. All 4-dimensional Lie algebras are solvable except for the unimodular Lie algebra $L_6 \cong \mathfrak{gl}(2,\boldsymbol{C})$. As we see later, four Lie algebras $L_6 \sim L_9$ constitute the "principal part" of the set of 4-dimensional Lie algebras (cf. Proposition 8), and the remaining Lie algebras L_3, L_4, L_5 are intermediate degenerate Lie algebras.

(3) For the Lie algebra $L_7(\alpha,\beta)$, we often say that two "unordered ratios" $1:\alpha:\beta$ and $1:\alpha':\beta'$ coincide in case these ratios coincide after a suitable change of ordering.

In this case, two Lie algebras $L_7(\alpha,\beta)$ and $L_7(\alpha',\beta')$ are isomorphic, as stated above. For example, it is easy to see that $L_7(\alpha,\beta)$ with $\alpha+\beta=1$ is isomorphic to $L_7(\gamma,\gamma+1)$ for some $\gamma \in \boldsymbol{C}$.

(4) It is convenient to use the symbolical notation $L_7(\infty,1)=L_7(0,0)$ in considering the degeneration of Lie algebras (cf. §5, Figure 1). In fact, from the above remark, we have $L_7(\alpha,1)\cong L_7(\frac{1}{\alpha},\frac{1}{\alpha})$ for $\alpha\neq 0$, and hence $\lim_{\alpha\to\infty} L_7(\alpha,1)\cong L_7(0,0)$. By the same reason, we may consider $L_7(\infty,\infty)=L_7(1,0)$ and $L_7(\infty,-\infty)=L_7(-1,0)$.

(5) We may say that the Lie algebra $L_4(\alpha)$ converges to $L_4(\infty)$ as $\alpha\to\infty$ in spite of its appearance. To check this fact, we consider the family of Lie algebras $L'_4(k,l)$ ($(k,l)\neq(0,0)$) defined by

$$[Y_1,Y_2]=kY_2,\quad [Y_1,Y_3]=kY_3,\quad [Y_1,Y_4]=kY_3+lY_4.$$

Then, we have

$$L'_4(k,l)\cong\begin{cases} L_4(\infty) & k=0,\\ L_4(\frac{l}{k}) & k\neq 0.\end{cases}$$

In particular, $L'_4(k,l)\cong L'_4(k',l')$ if and only if $(k',l')=(ck,cl)$ for some $c\neq 0$, and we see that the parameter space of $L_4(\alpha)$ can be naturally identified with the 1-dimensional complex projective space $P^1(\boldsymbol{C})$. From these facts, we have $\lim_{\alpha\to\infty} L_4(\alpha)\cong \lim_{\alpha\to\infty} L'_4(1,\alpha)$ $\cong \lim_{\alpha\to\infty} L'_4(\frac{1}{\alpha},1)\cong L'_4(0,1)\cong L_4(\infty)$.

(6) For most previously known classifications, the Lie algebra $L'_7(\alpha,\beta)$ defined by

$$[Y_1,Y_2]=Y_2,\quad [Y_1,Y_3]=\alpha Y_3,\quad [Y_1,Y_4]=\beta Y_4.$$

is adopted as one normal form. Clearly, the bracket operation of $L'_7(\alpha,\beta)$ is simpler than that of $L_7(\alpha,\beta)$, and this Lie algebra is isomorphic to $L_7(\alpha,\beta)$ if $\alpha\neq\beta$, $\alpha\neq 1$ and $\beta\neq 1$. But, for the remaining singular cases, $L'_7(\alpha,\beta)$ is isomorphic to other Lie algebras:

$$L'_7(\alpha,\beta)\cong\begin{cases} L_4(\infty) & \alpha=\beta=0,\\ L_3 & \alpha=\beta=1,\\ L_4(\frac{1}{\alpha}) & \alpha=\beta\neq 0,1,\\ L_4(\beta) & \alpha=1,\ \beta\neq 1,\\ L_4(\alpha) & \beta=1,\ \alpha\neq 1.\end{cases}$$

For most classifications, the Lie algebras $L_7(\alpha,\alpha)$ ($\cong L_7(1,\frac{1}{\alpha})$ if $\alpha\neq 0$) and $L_7(1,0)$, not appearing in this family $\{L'_7(\alpha,\beta)\}$ are treated as other separate normal forms. But actually, by calculating the dimension of $GL(V)$-orbits of these Lie algebras, we know that the above Lie algebras L_3 and $L_4(\alpha)$ are singular. (For example, the Lie algebra L_3 is a degeneration of $L_7(1,1)$. See Table 2 and Figure 1 in §5.) On the contrary, the dimension of the $GL(V)$-orbit of $L_7(\alpha,\beta)$ is constant for any α, β. And hence, $L_7(\alpha,\alpha)$ and $L_7(1,0)$ should be included in the continuous family of Lie algebras, instead of L_3 and $L_4(\alpha)$. Therefore, the family of Lie algebras $L_7(\alpha,\beta)$ is better than $L'_7(\alpha,\beta)$ in describing deformations and degenerations. (In terms of the language of matrices, we may symbolically say that $L'_7(\alpha,\beta)$ corresponds to a "diagonal" matrix and $L_7(\alpha,\beta)$ corresponds to a matrix with a non-trivial "Jordan block". Clearly, the former is simple and the latter is generic in the set of matrices with multiple eigenvalues.)

As for the Lie algebra $L_8(\alpha)$, it is isomorphic to the Lie algebra $L'_8(\alpha)$ defined by

$$[Y_1, Y_2] = Y_2, \ [Y_1, Y_3] = \alpha Y_3, \ [Y_1, Y_4] = (\alpha+1)Y_4, \ [Y_2, Y_3] = Y_4,$$

if $\alpha \neq 1$. And the bracket operation of this Lie algebra is simpler than that of $L_8(\alpha)$. But $L'_8(\alpha)$ degenerates to L_5 in the case $\alpha = 1$. A similar phenomenon occurs for the Lie algebra $L_4(\alpha)$. The Lie algebra defined by

$$[Y_1, Y_2] = Y_2, \ [Y_1, Y_3] = Y_3, \ [Y_1, Y_4] = \alpha Y_4$$

is isomorphic to $L_4(\alpha)$ if $\alpha \neq 1$, and to L_3 if $\alpha = 1$. By the same reason as above, it is better to adopt the Lie algebras $L_8(\alpha)$ and $L_4(\alpha)$ as our normal forms.

Next, as one peculiar feature of 4-dimensional Lie algebras, we consider the ratio of the eigenvalues of $\mathrm{ad}\, X$ ($X \in \mathfrak{g}$). The following proposition is quite important, especially in defining the invariants of 4-dimensional Lie algebras in §4.

Proposition 2. *Assume $\mathfrak{g}$ is not isomorphic to L_9. Then, the ratio of the eigenvalues of $\mathrm{ad}\, X$ does not depend on the choice of X if X is sufficiently generic.*

Proof. We can easily show this fact by using Table 1. For example, for the Lie algebra $L_7(\alpha, \beta)$, we have

$$\mathrm{ad}\,(aX_1 + bX_2 + cX_3 + dX_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -b-c & a & a & 0 \\ -c\alpha - d & 0 & a\alpha & a \\ -d\beta & 0 & 0 & a\beta \end{pmatrix},$$

and the eigenvalues of this matrix are $\{0, a, a\alpha, a\beta\}$. Hence, if $a \neq 0$, the ratio of the eigenvalues of $\mathrm{ad}\,(aX_1 + bX_2 + cX_3 + dX_4)$ is always equal to $0 : 1 : \alpha : \beta$. We can easily calculate the ratio for the remaining Lie algebras. Results are summarized in Table 2. □

Remark. (1) For the Lie algebra L_9, the eigenvalues of $\mathrm{ad}\,(aX_1 + bX_2 + cX_3 + dX_4)$ are given by $\{0, 0, a, c\}$, and this ratio essentially depends on the choice of X. Among 4-dimensional comlex Lie algebras, L_9 is uniquely characterized by this property. We also remark that the ratios for real solvable 4-dimensional Lie algebras are listed up in [36; p.180 ~ 181].

(2) If possible, it is desirable to prove Proposition 2 without the help of the classification. But unfortunately, we do not know such a proof at present.

In Table 2, we summarize fundamental quantities of L_i, including the ratios of the eigenvalues of $\mathrm{ad}X$. We remark that our normal forms are selected such that the dimension of the $GL(V)$-orbit of L_i does not depend on the parameters (α and β), as we stated

Table 2

	$\dim \mathfrak{g}^{(1)}$	$\dim \mathfrak{g}^{(2)}$	$\dim \mathcal{O}(\mathfrak{g})$	ratio
L_0	0	0	0	$0:0:0:0$
L_1	1	0	6	$0:0:0:0$
L_2	2	0	9	$0:0:0:0$
L_3	3	0	4	$0:1:1:1$
$L_4(\alpha)$	$\begin{cases} 1 & \alpha=\infty \\ 2 & \alpha=0 \\ 3 & \alpha\neq 0,\infty \end{cases}$	0	8	$0:1:1:\alpha$ $(*)_1$
L_5	3	1	9	$0:1:1:2$
L_6	3	3	12	$0:0:1:-1$
$L_7(\alpha,\beta)$	$\begin{cases} 2 & \alpha=0 \text{ or } \beta=0 \\ 3 & \alpha,\beta\neq 0 \end{cases}$	0	10	$0:1:\alpha:\beta$
$L_8(\alpha)$	$\begin{cases} 2 & \alpha=0 \\ 3 & \alpha\neq 0 \end{cases}$	$\begin{cases} 0 & \alpha=0 \\ 1 & \alpha\neq 0 \end{cases}$	11	$0:1:\alpha:\alpha+1$
L_9	2	0	12	$(*)_2$

$\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}^{(2)}=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]$.
$(*)_1$: We consider $0:1:1:\infty = 0:0:0:1$.
$(*)_2$: Two eigenvalues are 0. But the remaining two eigenvalues essentially depend on the choice of $X \in L_9$.

above. Perhaps, this is the most important feature of our normal forms. But instead, the dimensions of $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ may vary for singular α and β, where $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}^{(2)} = [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]$. In the following, we denote by $\mathcal{O}(\mathfrak{g})$ the $GL(V)$-orbit of $\mathfrak{g}$ in $\wedge^2V^*\otimes V$. Note that the dimension of $\mathcal{O}(\mathfrak{g})$ can be calculated by the formula:

$$\dim \mathcal{O}(\mathfrak{g}) = \dim \mathfrak{g}^*\otimes\mathfrak{g} - \dim Der(\mathfrak{g}),$$

where $Der(\mathfrak{g})$ is the space of derivations of $\mathfrak{g}$, i.e.,

$$Der(\mathfrak{g}) = \{\, A\in\mathfrak{g}^*\otimes\mathfrak{g} \mid A[X,Y]=[AX,Y]+[X,AY],\ {}^{\forall}X,Y\in\mathfrak{g}\,\}.$$

Theoretically, the value $\dim\mathcal{O}(\mathfrak{g})$ may serve as one measure to determine the isomorphism class of $\mathfrak{g}$. But for general (un-normalized) $\mathfrak{g}$, the determination of $\dim\mathcal{O}(\mathfrak{g})$ requires many calculations, and we do not use this value as our device.

Finally, it should be remarked that the dimensions of $GL(V)$-orbits are not preserved by summations of Lie algebras. For example, we can show that the dimension of the $GL(V)$-orbit of the 3-dimensional complex Lie algebra $M_3(\alpha)$ is 5 for any $\alpha\in \boldsymbol{C}$. But curiously, by adding a 1-dimensional abelian center, a singular parameter α appears. In fact, we have

$$\dim \mathcal{O}(M_3(\alpha)\oplus \boldsymbol{C}) = \begin{cases} \dim\mathcal{O}(L_4(\infty))=8 & \alpha=0, \\ \dim\mathcal{O}(L_7(\alpha,0))=10 & \alpha\neq 0. \end{cases}$$

2. Polynomial ring of $\wedge^2 V^* \otimes V$

In order to give an algorithm to determine the isomorphism classes of Lie algebras, we need several intrinsic concepts of $\mathfrak{g}$, by which we can distinguish non-isomorphic Lie algebras. These intrinsic concepts are all characterized by the vanishing of $GL(V)$-invariant sets of polynomials of structure constants up to degree three. For example, by fixing a basis $\{X_1, \cdots, X_4\}$ of $\mathfrak{g}$ and by putting $[X_i, X_j] = \Sigma\, c_{ij}^k X_k$, the unimodularity of $\mathfrak{g}$ is characterized by the vanishing of four linear polynomials $\Sigma_k\, c_{ik}^k = 0$ $(i = 1 \sim 4)$, as stated in Introduction. These four polynomials $\{\Sigma_k\, c_{ik}^k\}$ constitute a $GL(V)$-invariant irreducible subspace of $(\wedge^2 V^* \otimes V)^*$. Other intrinsic concepts of $\mathfrak{g}$ which we use in this paper are also characterized by the vanishing of some $GL(V)$-irreducible components of the polynomial ring $\Sigma_p\, S^p(\wedge^2 V^* \otimes V)^*$.

In this section, we give the explicit $GL(V)$-irreducible decomposition of the space $S^p(\wedge^2 V^* \otimes V)^*$ for $p = 1 \sim 3$, and calculate their generators. In addition, we evaluate these generators for each normal form in Table 1. From these results, we obtain several nice devices to distinguish the isomorphism classes of $\mathfrak{g}$. Intrinsic meaning defined by the vanishing of these $GL(V)$-invariant sets of polynomials is explained in detail in the next section.

We will calculate the generator of each $GL(V)$-irreducible component of $S^p(\wedge^2 V^* \otimes V)^*$ by the method stated in [1; p.115 $\sim$ 116]. For this purpose, we modify the space $\wedge^2 V^* \otimes V$ in the following way. We fix a volume form Ω of V once for all. Since V is 4-dimensional, we can naturally identify two spaces $\wedge^2 V^*$ and $\wedge^2 V$ by using this volume form Ω. Hence $S^p(\wedge^2 V^* \otimes V)^*$ is isomorphic to $S^p(\wedge^2 V \otimes V)^*$ as $SL(V)$-modules. In particular, they have the same $SL(V)$-irreducible decompositions. As $GL(V)$-modules, irreducible components of $S^p(\wedge^2 V^* \otimes V)^*$ and the corresponding components of $S^p(\wedge^2 V \otimes V)^*$ are isomorphic to each other by multiplying some powers of $\det g$ $(g \in GL(V))$. In this paper, we only use the concepts determined by the vanishing of polynomials, or the concepts determined by the ratio of two polynomials with the same degree. And hence, our arguments do not depend on the choice of the volume form of V, and in the following, we use the space $\wedge^2 V \otimes V$ instead of $\wedge^2 V^* \otimes V$.

Now, under this situation, we give the explicit $GL(V)$-irreducible decomposition of the space of polynomials on $\wedge^2 V \otimes V$ up to degree three. We express the $GL(V)$-irreducible representation space corresponding to the partition $\lambda = (\lambda_1, \cdots, \lambda_4)$ by the symbol $S_\lambda = S_\lambda(V^*)$. (For the representation theory of the group $GL(V)$, see [23], [17], [1], etc.) For example, the symbol S_2 expresses the space of symmetric 2-forms $S^2 V^*$. Strictly speaking, this space $S^2 V^*$ should be expressed as $S_{0,0,0,-2}$. But, for simplicity, we use the above dual notation throughout this paper.

Then, in the case of degree = 1, by using Littlewood-Richardson's rule, we obtain the irreducible decomposition immediately: $(\wedge^2 V \otimes V)^* = S_{11} \otimes S_1 = S_{21} + S_{111}$. For higher degree cases, we use the formula $S^p(\wedge^2 V \otimes V)^* \cong \Sigma_\lambda\, S_\lambda(\wedge^2 V^*) \otimes S_\lambda(V^*)$, where λ runs all over the partitions with $|\lambda| = p$ and depth ≤ 4. The decomposition of the plethysm $S_\lambda(\wedge^2 V^*)$ $(= \{1^2\} \otimes \{\lambda\}$, in the classical notation) for small $|\lambda|$ is given for example in [3], [8]. (Or one can calculate it by using the software "SYMMETRICA":

http://www.mathe2.uni-bayreuth.de.) As a result, we have:

Proposition 3. *$GL(V)$-irreducible decompositions of $S^p(\wedge^2 V \otimes V)^*$ for $p = 1 \sim 3$ are given by*

$p = 1 : S_{21} + S_{111}$,
$p = 2 : S_{42} + 2S_{321} + 2S_{3111} + 2S_{222} + S_{2211}$,
$p = 3 : S_{63} + 2S_{531} + S_{522} + 2S_{5211} + S_{441} + 3S_{432} + 3S_{4311}$
$\qquad + 5S_{4221} + 3S_{333} + 4S_{3321} + 3S_{3222}$.

Remind that the partitons λ with depth > 4 do not appear in the above decomposition because $\dim V = 4$. The coefficient of S_λ implies its multiplicity. In the following, we often call the irreducible component S_λ in $S^p(\wedge^2 V \otimes V)^*$ *covariant.*

Remark. Theoretically, we can continue to decompose the space $S^p(\wedge^2 V \otimes V)^*$ for large p. But unfortunately, closed decomposition formulas of $S^p(\wedge^2 V \otimes V)^*$ for general p are not known yet even in the case $\dim V = 4$. We often encounter this type of difficulty in considering multi-tensor spaces (cf. [1], [2], [13], [34]).

Next, of all irreducible components of $S^p(\wedge^2 V \otimes V)^*$ $(p \leq 3)$, we give here generators p_λ of S_λ for nine components, by which we can determine the isomorphism classes of 4-dimensional Lie algebras. (We omit the generators of the remaining irreducible components because some of them are quite lenghty and we do not use them in this paper.) Incidentally, among the components with multiplicity > 1 in the above decomposition, we use at most one component in this paper, and we may express it simply as S_λ in the following. Then, by fixing a basis $\{X_1, \cdots, X_4\}$ of V, we have the following list:

- $p = 1$:

$p_{21} = c_{34}^1 \in S_{21}$,

$p_{111} = c_{14}^1 + c_{24}^2 + c_{34}^3 = -\mathrm{Tr}\,\mathrm{ad}\, X_4 \in S_{111}$,

- $p = 2$:

$p_{321} = c_{24}^1 c_{34}^2 - c_{34}^1 c_{24}^2 \in S_{321}$,

$p_{3111} = c_{12}^1 c_{34}^1 - c_{13}^1 c_{24}^1 + c_{14}^1 c_{23}^1 \in S_{3111}$,

$$\begin{aligned} p_{222} &= c_{14}^1 c_{24}^2 + c_{14}^1 c_{34}^3 - c_{24}^1 c_{14}^2 - c_{34}^1 c_{14}^3 + c_{24}^2 c_{34}^3 - c_{34}^2 c_{24}^3 \\ &= 1/2 \cdot \{(\mathrm{Tr}\,\mathrm{ad}\, X_4)^2 - \mathrm{Tr}\,(\mathrm{ad}\, X_4)^2\} \in S_{222}, \end{aligned}$$

- $p = 3$:

$p_{441} = c_{14}^1 c_{34}^1 c_{34}^2 + c_{24}^1 (c_{34}^2)^2 - (c_{34}^1)^2 c_{14}^2 - c_{34}^1 c_{24}^2 c_{34}^2 \in S_{441}$,

$$p_{4221} = \begin{vmatrix} c_{23}^1 & c_{23}^2 & c_{23}^3 \\ c_{24}^1 & c_{24}^2 & c_{24}^3 \\ c_{34}^1 & c_{34}^2 & c_{34}^3 \end{vmatrix} \in S_{4221},$$

$$p_{333} = \begin{vmatrix} c_{14}^1 & c_{14}^2 & c_{14}^3 \\ c_{24}^1 & c_{24}^2 & c_{24}^3 \\ c_{34}^1 & c_{34}^2 & c_{34}^3 \end{vmatrix} = -1/6 \cdot \{(\mathrm{Tr\,ad}\, X_4)^3 - 3\,\mathrm{Tr\,ad} X_4 \cdot \mathrm{Tr}\,(\mathrm{ad}\, X_4)^2 + 2\,\mathrm{Tr}\,(\mathrm{ad}\, X_4)^3\} \in S_{333},$$

$$p_{3222} = \begin{vmatrix} c_{12}^1 & c_{12}^2 & c_{12}^3 \\ c_{23}^1 & c_{23}^2 & c_{23}^3 \\ c_{34}^1 & c_{34}^2 & c_{34}^3 \end{vmatrix} + \begin{vmatrix} c_{12}^1 & c_{12}^2 & c_{12}^4 \\ c_{24}^1 & c_{24}^2 & c_{24}^4 \\ c_{34}^1 & c_{34}^2 & c_{34}^4 \end{vmatrix} + \begin{vmatrix} c_{13}^1 & c_{13}^3 & c_{13}^4 \\ c_{24}^1 & c_{24}^3 & c_{24}^4 \\ c_{34}^1 & c_{34}^3 & c_{34}^4 \end{vmatrix} - \begin{vmatrix} c_{13}^1 & c_{13}^2 & c_{13}^3 \\ c_{23}^1 & c_{23}^2 & c_{23}^3 \\ c_{24}^1 & c_{24}^2 & c_{24}^3 \end{vmatrix}$$
$$- \begin{vmatrix} c_{14}^1 & c_{14}^2 & c_{14}^4 \\ c_{23}^1 & c_{23}^2 & c_{23}^4 \\ c_{24}^1 & c_{24}^2 & c_{24}^4 \end{vmatrix} - \begin{vmatrix} c_{14}^1 & c_{14}^3 & c_{14}^4 \\ c_{23}^1 & c_{23}^3 & c_{23}^4 \\ c_{34}^1 & c_{34}^3 & c_{34}^4 \end{vmatrix} + 2 \begin{vmatrix} c_{23}^2 & c_{23}^3 & c_{23}^4 \\ c_{24}^2 & c_{24}^3 & c_{24}^4 \\ c_{34}^2 & c_{34}^3 & c_{34}^4 \end{vmatrix} \in S_{3222}.$$

These generators can be obtained by applying the method satated in [1; p.115 $\sim$ 116]. We give here one example. For details, see [1]. We denote by $\{\omega_1, \cdots, \omega_4\}$ the dual basis of $\{X_1, \cdots, X_4\}$. Then, the bracket $[\ ,\ ] \in \wedge^2 V^* \otimes V$ is expressed as $\Sigma_{i<j}\, c_{ij}^k \omega_i \wedge \omega_j \otimes X_k$. In terms of the volume form $\Omega = \omega_1 \wedge \cdots \wedge \omega_4$, this bracket is transformed into the element $\Sigma_{i<j}\, a_{ijk} X_i \wedge X_j \otimes X_k \in \wedge^2 V \otimes V$, where $a_{12k} = c_{34}^k$, $a_{13k} = -c_{24}^k$, $a_{14k} = c_{23}^k$, $a_{23k} = c_{14}^k$, $a_{24k} = -c_{13}^k$, $a_{34k} = c_{12}^k$. Then, the generator of the space $S_{321} \subset S^2(\wedge^2 V \otimes V)^*$ is given by

$$\begin{aligned} \textstyle\sum_{\sigma \in \mathfrak{S}_3, \tau \in \mathfrak{S}_2} (-1)^\sigma (-1)^\tau a_{\sigma(1)\tau(1)\sigma(2)} a_{1\tau(2)\sigma(3)} &= a_{121}a_{132} - a_{122}a_{131} \\ &= c_{24}^1 c_{34}^2 - c_{34}^1 c_{24}^2 \\ &= p_{321}. \end{aligned}$$

Here, $\mathfrak{S}_n$ denotes the symmetric group with degree n and $(-1)^\sigma$ denotes the sign of $\sigma \in \mathfrak{S}_n$. Other generators can be calculated in the same way. But the above repeated sum requires many computations on polynomials, and for most cases, we used computers to obtain p_λ. Of course, the polynomial p_λ itself essentially depends on the choice of a basis of V.

Note that the set of sixteen polynomials appearing in the Jacobi identity splits into two irreducible components of $S^2(\wedge^2 V \otimes V)^*$: S_{2211} and one component of $2S_{3111}$ in Proposition 3. (We have dim $S_{2211} = 6$ and dim $S_{3111} = 10$.) Note that the above S_{3111} generated by p_{3111} does not involve the Jacobi identity.

Among the above generators, three polynomials p_{111}, p_{222}, p_{333} play a special role in §4, where we define the invariants of 4-dimensional Lie algebras. In addition, the cubic polynomial

$$\varphi = 8p_{333} - 4p_{111}p_{222} + p_{111}^3$$

also plays an important role. (Namely, it gives a part of the defining equations of a variety of Lie algebras. See Proposition 9 in §5.) The polynomial φ generates an irreducible subspace of $S^3(\wedge^2 V \otimes V)^*$, which is equivalent to S_{333} as a $GL(V)$-module. In the following, we denote this space by $\langle \varphi \rangle$.

We express the eigenvalues of ad X ($X \in \mathfrak{g}$) as $\{0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Then, by putting $X_4 = X$ in the above list of generators, we have immediately

$$p_{111} = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),$$
$$p_{222} = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1,$$
$$p_{333} = -\varepsilon_1\varepsilon_2\varepsilon_3.$$

By substituting these values to the above φ, we have

$$\varphi = (\varepsilon_1 + \varepsilon_2 - \varepsilon_3)(\varepsilon_1 + \varepsilon_3 - \varepsilon_2)(\varepsilon_2 + \varepsilon_3 - \varepsilon_1).$$

Hence, the vanishing of this polynomial also gives some intrinsic property of the Lie algebra $\mathfrak{g}$ (cf. Proposition 4).

As stated above, the generator $p_\lambda \in S_\lambda$ depends on the coice of a basis $\{X_1, \cdots, X_4\}$. Hence, the vanishing or non-vanishing of the generator p_λ itself has no intrinsic meaning. But the vanishing of *all* polynomials in S_λ generated by p_λ possesses an intrinsic meaning of $\mathfrak{g}$. In the following, we express this situation symbolically as "$S_\lambda = 0$", and often say that the covariant S_λ vanishes for $\mathfrak{g}$. Our next task is to clarify the meaning of the intrinsic property defined by "$S_\lambda = 0$". But, before stating this meaning, we give a table summarizing the vanishing or non-vanishing of S_λ for each Lie algebra L_i (Table 3). The symbol "0" in Table 3 implies "$S_\lambda = 0$", and the symbol "$*$" implies that there exists a non-vanishing polynomial in S_λ. To check these results, we evaluate the generator p_λ of S_λ in terms of a generic basis of L_i. Namely, if $p_\lambda \neq 0$ for some basis, we write "$*$" in the table, and if $p_\lambda = 0$ for a generic (and hence any) basis, we have "$S_\lambda = 0$". In the actual calculations, we used computers. We use Table 3 frequently in the subsequent sections. It is easy to see that at the present stage, we can distinguish ten classes of Lie algebras $L_0 \sim L_9$ by using this table. But the value of the parameters α and β in L_4, L_7 and L_8 cannot be determined by only using these concepts. Explicit determination of the parameters will be carried out in §4.

3. Intrinsic concepts determined by the vanishing of covariants

In this section, we state several intrinsic concepts of $\mathfrak{g}$ determined by the vanishing of covariants appeared in §2. The results are summarized in the following proposition. Most concepts appearing in this proposition are actually used in the algorithm to determine the isomorphism classes of Lie algebras, which we will state in detail in §6. We denote by $\{0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the eigenvalues of $\mathrm{ad}\, X$ for generic $X \in \mathfrak{g}$, as before.

Proposition 4. (1) $S_{21} = 0$ *if and only if there exists an element* $f \in \mathfrak{g}^*$ *such that* $[X, Y] = f(X)Y - f(Y)X$.
(2) $S_{111} = 0$ *if and only if* $\mathfrak{g}$ *is unimodular.*
(3) $S_{3111} = 0$ *if and only if* $d\alpha \wedge d\alpha = 0$ *for any* $\alpha \in \mathfrak{g}^*$. (*The condition* $d\alpha \wedge d\alpha = 0$ *is equivalent to the decomposability of* $d\alpha$ *for the* 4*-dimensional case*).
(4) $S_{222} = 0$ *if and only if* $\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 = 0$.
(5) $S_{441} = 0$ *if and only if* $\dim \langle X, Y, [X, Y], [X, [X, Y]] \rangle \leq 3$ *for any* $X, Y \in \mathfrak{g}$.
(6) $S_{4221} = 0$ *if and only if* $\dim \langle [X, Y], [Y, Z], [Z, X] \rangle \leq 2$ *for any* $X, Y, Z \in \mathfrak{g}$. *This condition is also equivalent to* $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$.

Table 3

	S_{21}	S_{111}	S_{321}	S_{3111}	S_{222}
L_0	0	0	0	0	0
L_1	$*$	0	0	0	0
L_2	$*$	0	$*$	0	0
L_3	0	$*$	0	0	$*$
$L_4(\alpha)$	$*$	$\begin{cases} 0 & \alpha = -2 \\ * & \alpha \neq -2 \end{cases}$	$\begin{cases} 0 & \alpha = \infty \\ * & \alpha \neq \infty \end{cases}$	0	$\begin{cases} 0 & \alpha = -1/2, \infty \\ * & \alpha \neq -1/2, \infty \end{cases}$
L_5	$*$	$*$	$*$	$*$	$*$
L_6	$*$	0	$*$	0	$*$
$L_7(\alpha, \beta)$	$*$	$\begin{cases} 0 & \alpha + \beta = -1 \\ * & \alpha + \beta \neq -1 \end{cases}$	$*$	0	$\begin{cases} 0 & \alpha\beta + \alpha + \beta = 0 \\ * & \alpha\beta + \alpha + \beta \neq 0 \end{cases}$
$L_8(\alpha)$	$*$	$\begin{cases} 0 & \alpha = -1 \\ * & \alpha \neq -1 \end{cases}$	$*$	$\begin{cases} 0 & \alpha = -1 \\ * & \alpha \neq -1 \end{cases}$	$\begin{cases} 0 & \alpha^2 + 3\alpha + 1 = 0 \\ * & \alpha^2 + 3\alpha + 1 \neq 0 \end{cases}$
L_9	$*$	$*$	$*$	$*$	$*$

	S_{441}	S_{4221}	S_{333}	S_{3222}	φ
L_0	0	0	0	0	0
L_1	0	0	0	0	0
L_2	$*$	0	0	0	0
L_3	0	0	$*$	0	$*$
$L_4(\alpha)$	0	0	$\begin{cases} 0 & \alpha = 0, \infty \\ * & \alpha \neq 0, \infty \end{cases}$	0	$\begin{cases} 0 & \alpha = 0, 2 \\ * & \alpha \neq 0, 2 \end{cases}$
L_5	0	$*$	$*$	0	0
L_6	$*$	$*$	0	$*$	0
$L_7(\alpha, \beta)$	$*$	0	$\begin{cases} 0 & \alpha \text{ or } \beta = 0 \\ * & \alpha, \beta \neq 0 \end{cases}$	0	$\begin{cases} 0 & \lvert\alpha - \beta\rvert = 1 \text{ or } \alpha + \beta = 1 \\ * & \lvert\alpha - \beta\rvert \neq 1, \alpha + \beta \neq 1 \end{cases}$
$L_8(\alpha)$	$*$	$\begin{cases} 0 & \alpha = 0 \\ * & \alpha \neq 0 \end{cases}$	$\begin{cases} 0 & \alpha = 0, -1 \\ * & \alpha \neq 0, -1 \end{cases}$	0	0
L_9	$*$	0	0	0	$*$

(7) $S_{333}=0$ *if and only if some* $\varepsilon_i=0$. *This condition is also equivalent to* $\operatorname{rank}\operatorname{ad}X \leq 2$ *for any* $X\in\mathfrak{g}$.
(8) $S_{3222}=0$ *if and only if* $\dim[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]\leq 1$ (*or equivalently* ≤ 2). *This condition is also equivalent to the solvability of* $\mathfrak{g}$.
(9) $S_{21}=S_{111}=0$ *if and only if* $\mathfrak{g}$ *is abelian.*
(10) $S_{111}=S_{222}=S_{333}=0$ *if and only if* $\mathfrak{g}$ *is nilpotent.*
(11) $S_{321}=S_{222}=0$ *if and only if* $\dim[\mathfrak{g},\mathfrak{g}]\leq 1$. *This condition is also equivalent to* $\operatorname{rank}\operatorname{ad}X\leq 1$ *for any* $X\in\mathfrak{g}$.
(12) $S_{4221}=S_{333}=0$ *if and only if* $\dim[\mathfrak{g},\mathfrak{g}]\leq 2$.
(13) $\langle\varphi\rangle=0$ *if and only if* $\varepsilon_i=\varepsilon_j+\varepsilon_k$ *for some distinct* i, j, k.

Proof. (1), (2), (9). We put $f(X)=\frac{1}{3}\operatorname{Tr}\operatorname{ad}X$, and define a new bracket operation $[\ ,\ ]'$ by $[X,Y]'=[X,Y]-f(X)Y+f(Y)X$. Then, we have $[X,Y]=[X,Y]'+\{f(X)Y-f(Y)X\}$, and this gives the $GL(V)$-irreducible decomposition of the space $\wedge^2V^*\otimes V$. In fact, we already know that the space $\wedge^2V^*\otimes V$ splits into two irreducible components (Proposition 3). And it is easy to see that the trace of the adjoint map $Y\mapsto[X,Y]'$ is zero for any X. Hence, $[\ ,\ ]'$ gives the traceless part, and $f(X)Y-f(Y)X$ gives the contracted part of $[\ ,\ ]\in\wedge^2V^*\otimes V$, respectively. By definition, the condition $S_{111}=0$ is equivalent to the unimodularity of $\mathfrak{g}$, i.e., the vanishing of f. And hence, the remaining condition $S_{21}=0$ is equivalent to $[\ ,\ ]'=0$, i.e., $[X,Y]$ is expressed as $f(X)Y-f(Y)X$. Clearly, combined conditions $S_{21}=S_{111}=0$ are equivalent to $c_{ij}^k=0$, which implies that $\mathfrak{g}$ is abelian.

(3) The condition $S_{3111}=0$ is equivalent to the vanishing of ten polynomials $c_{12}^ic_{34}^j-c_{13}^ic_{24}^j+c_{14}^ic_{23}^j+c_{34}^ic_{12}^j-c_{24}^ic_{13}^j+c_{23}^ic_{14}^j$. Since $d\alpha(X_i,X_j)=-\alpha([X_i,X_j])=-\Sigma\, c_{ij}^k\alpha(X_k)$ for $\alpha\in\mathfrak{g}^*$ and $X_i\in\mathfrak{g}$, this condition is equivalent to $(d\alpha\wedge d\alpha)(X_1,\cdots,X_4)=0$.

(4), (7), (13). These are clear from the definition of p_{222}, p_{333}, φ. For the second statement in (7), we can directly check that the Lie algebras satisfying the condition $\operatorname{rank}\operatorname{ad}X\leq 2$ for any $X\in\mathfrak{g}$ are exhausted by L_0, L_1, L_2, $L_4(0)$, $L_4(\infty)$, L_6, $L_7(\alpha,0)$, $L_8(0)$, $L_8(-1)$, L_9, and by using Table 3, we can show that these Lie algebras are just characterized by the condition $S_{333}=0$.

(5) We can rewrite the polynomial p_{441} in the form

$$p_{441}=\begin{vmatrix} c_{34}^1 & \Sigma_k\, c_{4k}^1c_{34}^k \\ c_{34}^2 & \Sigma_k\, c_{4k}^2c_{34}^k \end{vmatrix}.$$

Then, it is easy to check that this polynomial is equal to the determinant of the matrix $(X_3,X_4,[X_4,X_3],[X_4,[X_4,X_3]])$ in case $X_1\sim X_4$ are linearly independent. Hence, the condition $S_{441}=0$ is equivalent to $\dim\langle X,Y,[X,Y],[X,[X,Y]]\rangle\leq 3$ for any X, $Y\in\mathfrak{g}$.

(6) The polynomial p_{4221} is equal to the principal minor of the $(4,3)$-matrix $([X_2,X_3],[X_2,X_4],[X_3,X_4])$. Hence, the condition $S_{4221}=0$ is equivalent to $\dim\langle[X,Y],[Y,Z],[Z,X]\rangle\leq 2$ for any X, Y, $Z\in\mathfrak{g}$. In view of Table 2 and Table 3, we can check that this condition is just equivalent to $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=0$.

(8), (10), (11), (12). These statements can be directly verified by using Table 1 $\sim$ Table 3. □

Remark. (1) In the above proof, we often used the classification table. But, it is desirable to prove this proposition by an intrinsic way without the help of the classification.

(2) Among the above conditions, the condition $S_{3111} = 0$ in (3) plays a crucial role in considering the existence or non-existence of left invariant symplectic strucutres on 4-dimensional complex Lie groups. For details, see [6].

(3) Unfortunately, we do not know the intrinsic meaning defined by the single condition "$S_{321} = 0$".

(4) As we stated before, 4-dimensional Lie algebras are unimodular or solvable. The same fact also holds for 3-dimensional Lie algebras. In the 3-dimensional case, all Lie algebras satisfy the so-called "fundamental identity": $\mathfrak{S}_{X,Y,Z}$ $(\operatorname{Tr}\operatorname{ad}X)\cdot[Y,Z] = 0$, where $\mathfrak{S}$ implies the cyclic sum. And by using this identity, we can show the above fact directly without the help of the classification (cf. [5; p.6 $\sim$ 8]). In the 4-dimensional case, the Jacobi identity implies $S_{111} = 0$ or $S_{3222} = 0$, as a result of the classification. But, we do not know whether there exists a similar "fundamental identity" for 4-dimensional Lie algebras, by which we can directly prove the above fact.

4. Invariants of 4-dimensional Lie algebras

To distinguish the isomorphism classes of Lie algebras, we need more delicate additional devices. For example, as stated in Proposition 1, two Lie algebras $L_4(\alpha)$ and $L_4(\alpha')$ are isomorphic if and only if $\alpha = \alpha'$. Hence, we must extract the value α from the Lie algebra structure of $L_4(\alpha)$ in order to determine the isomorphism class. For this purpose, we introduce three fundamental invariants of 4-dimensional Lie algebras taking values in $\boldsymbol{C} \cup \{\infty\}$ as follows:

$$\chi_1(\mathfrak{g}) = \frac{p_{222}}{p_{111}^2}, \quad \chi_2(\mathfrak{g}) = \frac{p_{333}}{p_{111}^3}, \quad \chi_3(\mathfrak{g}) = \frac{p_{333}^2}{p_{222}^3}.$$

Clearly, these invariants satisfy the relation $\chi_1(\mathfrak{g})^3\chi_3(\mathfrak{g}) = \chi_2(\mathfrak{g})^2$. We express the eigenvalues of $\operatorname{ad}X$ ($X \in \mathfrak{g}$) as $\{0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ as before. Then, by substituting $p_{111} = -(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $p_{222} = \varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1$, $p_{333} = -\varepsilon_1\varepsilon_2\varepsilon_3$ into the above, we know that $\chi_i(\mathfrak{g})$ are essentially determined by the ratio of ε_i, which indicates that $\chi_i(\mathfrak{g})$ are the intrinsic invariants of $\mathfrak{g}$, except for the following exceptional cases: For the nilpotent Lie algebras L_0, L_1 and L_2, the invariants $\chi_i(\mathfrak{g})$ are undetermined because $p_{111} = p_{222} = p_{333} = 0$. Similarly, the following cases are also undetermined because both denominators and numerators simultaneously vanish.

$$\begin{cases} \chi_1(\mathfrak{g}) & : \; L_7(\alpha,\beta) \;\; (\alpha+\beta=-1,\; \alpha\beta=1), \\ \chi_2(\mathfrak{g}) & : \; L_6, \;\; L_7(-1,0), \;\; L_8(-1), \\ \chi_3(\mathfrak{g}) & : \; L_4(\infty), \;\; L_7(0,0). \end{cases}$$

As for the Lie algebra L_9, the ratio of ε_i has not an intrinsic meaning, and hence $\chi_1(\mathfrak{g})$ is also undetermined. But, since one ε_i is always zero, we have $p_{333} = 0$. Hence, we may put $\chi_2(\mathfrak{g}) = \chi_3(\mathfrak{g}) = 0$ for this Lie algebra.

We can easily calculate the explicit values of $\chi_i(L_j)$ in view of Table 2. For a general (un-normalized) Lie algebra $\mathfrak{g}$, we first calculate the values $\mathrm{Tr}\,(\mathrm{ad}\,X)^k$ $(k = 1 \sim 3)$ for generic $X \in \mathfrak{g}$. Then, by the definition of p_{kkk}, we have

$$\begin{aligned} p_{111} &= -\mathrm{Tr}\,\mathrm{ad}\,X, \\ p_{222} &= \tfrac{1}{2}\{(\mathrm{Tr}\,\mathrm{ad}\,X)^2 - \mathrm{Tr}\,(\mathrm{ad}\,X)^2\}, \\ p_{333} &= -\tfrac{1}{6}\{(\mathrm{Tr}\,\mathrm{ad}\,X)^3 - 3\,\mathrm{Tr}\,\mathrm{ad}\,X \cdot \mathrm{Tr}\,(\mathrm{ad}\,X)^2 + 2\,\mathrm{Tr}\,(\mathrm{ad}\,X)^3\}, \end{aligned}$$

and $\chi_i(\mathfrak{g})$ are obtained from these values, though it requires not a little computations in general. (See the examples at the end of §6.) Of course, we can know the value of p_{kkk} by calculating the characteristic polynomial

$$|\lambda I - \mathrm{ad}\,X| = \lambda(\lambda^3 + p_{111}\lambda^2 + p_{222}\lambda + p_{333}).$$

In general, the value $\chi_3(\mathfrak{g})$ is automatically determined by $\chi_1(\mathfrak{g})$ and $\chi_2(\mathfrak{g})$. This third invariant $\chi_3(\mathfrak{g})$ is mainly used in case the denominator p_{111} of $\chi_1(\mathfrak{g})$ and $\chi_2(\mathfrak{g})$ vanishes, i.e., for unimodular Lie algebras. (See Proposition 5 (3).)

We summarize the explicit values of $\chi_i(L_j)$ in Table 4. The symbol " $-$ " in this table implies that it is undetermined.

Table 4

	$\chi_1(\mathfrak{g})$	$\chi_2(\mathfrak{g})$	$\chi_3(\mathfrak{g})$
L_3	$\frac{1}{3}$	$\frac{1}{27}$	$\frac{1}{27}$
$L_4(\alpha)$	$\frac{2\alpha+1}{(\alpha+2)^2}$	$\frac{\alpha}{(\alpha+2)^3}$	$\begin{cases} - & \alpha = \infty \\ \frac{\alpha^2}{(2\alpha+1)^3} & \alpha \neq \infty \end{cases}$
L_5	$\frac{5}{16}$	$\frac{1}{32}$	$\frac{4}{125}$
L_6	∞	$-$	0
$L_7(\alpha,\beta)$	$\frac{\alpha\beta+\alpha+\beta}{(\alpha+\beta+1)^2}$ $(*)_1$	$\frac{\alpha\beta}{(\alpha+\beta+1)^3}$ $(*)_2$	$\frac{\alpha^2\beta^2}{(\alpha\beta+\alpha+\beta)^3}$ $(*)_3$
$L_8(\alpha)$	$\frac{\alpha^2+3\alpha+1}{4(\alpha+1)^2}$	$\begin{cases} - & \alpha = -1 \\ \frac{\alpha}{8(\alpha+1)^2} & \alpha \neq -1 \end{cases}$	$\frac{\alpha^2(\alpha+1)^2}{(\alpha^2+3\alpha+1)^3}$
L_9	$-$	0	0

$(*)_1$: undetermined in case $\alpha+\beta=-1$, $\alpha\beta=1$.
$(*)_2$: undetermined in case $(\alpha,\beta)=(-1,0)$, $(0,-1)$.
$(*)_3$: undetermined in case $\alpha=\beta=0$. If $\beta=-\alpha-1$, we have $\chi_3(\mathfrak{g}) = -\frac{\alpha^2(\alpha+1)^2}{(\alpha^2+\alpha+1)^3}$.

In terms of these invariants, the parameters α and β in L_4, L_7, L_8 are essentially uniquely determined as follows.

Proposition 5. (1) *$L_4(\alpha)$ is isomorphic to $L_4(\alpha')$ if and only if $\chi_i(L_4(\alpha)) = \chi_i(L_4(\alpha'))$ for $i=1,2$. In this case, the parameter α is given by*

$$\alpha = \begin{cases} \infty & \chi_1 = \chi_2 = 0, \\ -2 & \chi_1 = \infty, \\ 1 & \chi_1 = \frac{1}{3}, \\ \frac{2\chi_2(1-3\chi_1)}{\chi_1{}^2+3\chi_1\chi_2-4\chi_2} & otherwise. \end{cases}$$

For this Lie algebra, χ_1 and χ_2 satisfy the relation $3(3\chi_2 - \chi_1)^2 + 4(\chi_1{}^3 - \chi_1{}^2 + \chi_2) = 0$ in case they have finite values.

(2) $L_7(\alpha, \beta)$ $(\alpha + \beta \neq -1,\ \alpha,\ \beta \neq 0)$ is isomorphic to $L_7(\alpha', \beta')$ $(\alpha' + \beta' \neq -1,\ \alpha',\ \beta' \neq 0)$ if and only if $\chi_i(L_7(\alpha, \beta)) = \chi_i(L_7(\alpha', \beta'))$ for $i = 1, 2$. The parameters α and β are determined from χ_1, χ_2 by two conditions $\alpha + \beta = a - 1$ and $\alpha\beta = \chi_2 a^3$, where a is a complex number satisfying the condition $\chi_2 a^3 - \chi_1 a^2 + a - 1 = 0$.

(3) $L_7(\alpha, -(\alpha + 1))$ $(\alpha \neq 0, -1)$ is isomorphic to $L_7(\alpha', -(\alpha' + 1))$ $(\alpha' \neq 0, -1)$ if and only if $\chi_3(L_7(\alpha, -(\alpha + 1))) = \chi_3(L_7(\alpha', -(\alpha' + 1)))$. The parameter α is determined from the equation $\chi_3 = -\alpha^2(\alpha + 1)^2/(\alpha^2 + \alpha + 1)^3$ if $\chi_3 \neq \infty$. In case $\chi_3 = \infty$, we have $\alpha = (-1 \pm \sqrt{3}i)/2$, both of which define the isomorphic Lie algebras.

(4) $L_7(\alpha, 0)$ is isomorphic to $L_7(\alpha', 0)$ if and only if $\chi_1(L_7(\alpha, 0)) = \chi_1(L_7(\alpha', 0))$. The parameter α is determined from the equation $\chi_1 = \alpha/(\alpha+1)^2$ if $\chi_1 \neq \infty$. In case $\chi_1 = \infty$, we have $\alpha = -1$.

(5) $L_8(\alpha)$ is isomorphic to $L_8(\alpha')$ if and only if $\chi_1(L_8(\alpha)) = \chi_1(L_8(\alpha'))$. The parameter α is determined from the equation $\chi_1 = (\alpha^2 + 3\alpha + 1)/(4(\alpha + 1)^2)$ if $\chi_1 \neq \infty$. In case $\chi_1 = \infty$, we have $\alpha = -1$.

Proof. We prove the "if" part of this proposition. The "only if" part is clear from Proposition 2 and the definition of $\chi_i(\mathfrak{g})$.

(1) Assume $\alpha \neq 1, -2, \infty$. Then, from the definition of $\chi_1(\mathfrak{g})$ and $\chi_2(\mathfrak{g})$, we have

$$2\chi_2(1 - 3\chi_1) = \frac{2\alpha(\alpha-1)^2}{(\alpha+2)^5},$$
$$\chi_1{}^2 + 3\chi_1\chi_2 - 4\chi_2 = \frac{2(\alpha-1)^2}{(\alpha+2)^5}$$

for $\mathfrak{g} = L_4(\alpha)$. Hence, the value α is uniquely determined from $\chi_1(\mathfrak{g})$ and $\chi_2(\mathfrak{g})$. For the remaining cases, we can easily check that α is also uniquely determined from the values $\chi_1(\mathfrak{g})$, $\chi_2(\mathfrak{g})$.

(2) Assume $\chi_1(L_7(\alpha, \beta)) = \chi_1(L_7(\alpha', \beta'))$ and $\chi_2(L_7(\alpha, \beta)) = \chi_2(L_7(\alpha', \beta'))$. From the condition α, $\beta \neq 0$, we have $\chi_2 \neq 0$, and it is easy to see that the solutions of the cubic equation $\chi_2 t^3 - \chi_1 t^2 + t - 1 = 0$ are $\alpha + \beta + 1$, $(\alpha + \beta + 1)/\alpha$ and $(\alpha + \beta + 1)/\beta$ if $\chi_1 = (\alpha\beta + \alpha + \beta)/(\alpha + \beta + 1)^2$ and $\chi_2 = \alpha\beta/(\alpha + \beta + 1)^3$. Replacing α, β by α', β' and considering the same cubic equation, we know that two sets of solutions

$$\left\{\alpha + \beta + 1,\ \frac{\alpha+\beta+1}{\alpha},\ \frac{\alpha+\beta+1}{\beta}\right\} \text{ and } \left\{\alpha' + \beta' + 1,\ \frac{\alpha'+\beta'+1}{\alpha'},\ \frac{\alpha'+\beta'+1}{\beta'}\right\}$$

must coincide because two invariants have the same values. There are six combinations of correspondence between these two sets, and by checking them, it follows that (α', β') is equal to one of the following:

$$(\alpha, \beta),\ (\beta, \alpha),\ \left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right),\ \left(\frac{\beta}{\alpha}, \frac{1}{\alpha}\right),\ \left(\frac{1}{\beta}, \frac{\alpha}{\beta}\right),\ \left(\frac{\alpha}{\beta}, \frac{1}{\beta}\right).$$

For any case, two unordered ratios $1 : \alpha : \beta$ and $1 : \alpha' : \beta'$ coincide, and hence we have $L_7(\alpha, \beta) \cong L_7(\alpha', \beta')$.

Next, we determine the values of parameters α, β from χ_1 and χ_2. We assume that the invariants are expressed as $\chi_1 = (\alpha_0\beta_0 + \alpha_0 + \beta_0)/(\alpha_0 + \beta_0 + 1)^2$, $\chi_2 = \alpha_0\beta_0/(\alpha_0 + \beta_0 + 1)^3$ for some α_0, β_0 ($\alpha_0 + \beta_0 \neq -1$, α_0, $\beta_0 \neq 0$). And let a be a solution of the cubic equation $\chi_2 t^3 - \chi_1 t^2 + t - 1 = 0$. Then, as we see above, a is equal to one of $\alpha_0 + \beta_0 + 1$, $(\alpha_0 + \beta_0 + 1)/\alpha_0$ and $(\alpha_0 + \beta_0 + 1)/\beta_0$. First, we take a solution $a = \alpha_0 + \beta_0 + 1$. Then, the equations $\alpha + \beta = a - 1$ and $\alpha\beta = \chi_2 a^3$ are equivalent to $\alpha + \beta = \alpha_0 + \beta_0$ and $\alpha\beta = \alpha_0\beta_0$, which implies that $(\alpha, \beta) = (\alpha_0, \beta_0)$ or (β_0, α_0). If we take a different solution $a = (\alpha_0 + \beta_0 + 1)/\alpha_0$, then we have $\alpha + \beta = a - 1 = (\beta_0 + 1)/\alpha_0$ and $\alpha\beta = \chi_2 a^3 = \beta_0/{\alpha_0}^2$. In this case, we have $(\alpha, \beta) = (\frac{\beta_0}{\alpha_0}, \frac{1}{\alpha_0})$ or $(\frac{1}{\alpha_0}, \frac{\beta_0}{\alpha_0})$, and hence, $L_7(\alpha, \beta) \cong L_7(\frac{\beta_0}{\alpha_0}, \frac{1}{\alpha_0}) \cong L_7(\alpha_0, \beta_0)$. By using the third solution $a = (\alpha_0 + \beta_0 + 1)/\beta_0$, we obtain the same conclusion. Hence, we may say that the parameters α and β are essentially determined from χ_1 and χ_2, by the procedure stated in (2).

(3) Assume $\chi_3(L_7(\alpha, -(\alpha+1))) = \chi_3(L_7(\alpha', -(\alpha'+1))) \neq \infty$. Then, we have $-\frac{\alpha^2(\alpha+1)^2}{(\alpha^2+\alpha+1)^3} = -\frac{\alpha'^2(\alpha'+1)^2}{(\alpha'^2+\alpha'+1)^3}$, and by solving this sextic equation, we have

$$\alpha' = \alpha,\ \frac{1}{\alpha},\ -(\alpha+1),\ \frac{-1}{\alpha+1},\ -\frac{\alpha+1}{\alpha},\ \frac{-\alpha}{\alpha+1}.$$

Hence, for any case, the unordered ratios $1 : \alpha : -(\alpha + 1)$ and $1 : \alpha' : -(\alpha' + 1)$ coincide, which implies $L_7(\alpha, -(\alpha+1)) \cong L_7(\alpha', -(\alpha'+1))$. If $\chi_3(L_7(\alpha, -(\alpha+1))) = \infty$, then we have $\alpha = (-1 \pm \sqrt{3}i)/2$, and this Lie algebras is isomorphic to $L_7((-1+\sqrt{3}i)/2, (-1-\sqrt{3}i)/2)$.

(4) Assume $\chi_1(L_7(\alpha, 0)) = \chi_1(L_7(\alpha', 0)) \neq 0, \infty$. Then, from this condition, we have easily $\alpha = \alpha'$ or $\alpha\alpha' = 1$, and hence $L_7(\alpha, 0)$ is isomorphic to $L_7(\alpha', 0)$. If $\chi_1(L_7(\alpha, 0)) = 0$ (resp. ∞), then we have $\alpha = 0$ (resp. -1), and $L_7(\alpha, 0)$ is also uniquely determined.

(5) If $\chi_1(L_8(\alpha)) = \chi_1(L_8(\alpha')) \neq \infty$, then we have $\alpha = \alpha'$ or $\alpha\alpha' = 1$ from this condition, which implies $L_8(\alpha) \cong L_8(\alpha')$. In case $\chi_1(L_8(\alpha)) = \infty$, we have $\alpha = -1$, and $L_8(\alpha)$ is also uniquely determined. □

Remark. The invariant $\chi_3(\mathfrak{g})$ for the unimodular Lie algebra $L_7(\alpha, -(\alpha+1))$ resembles the j-invariant of the elliptic curve $y^2 = x(x+1)(x-\alpha)$, where $j = 2^8\frac{(\alpha^2+\alpha+1)^3}{\alpha^2(\alpha+1)^2}$ (cf. [16], [28; p.140]). The invariant $\chi_3(L_7(\alpha, -(\alpha+1)))$ is a rational function of α, and it is invariant under the action of the symmetric group $\mathfrak{S}_3$ consisting of transformations $\alpha \mapsto \alpha'$ given in the proof of (3). This is essentially the unique rational function of α possessing this property.

We can also describe several intrinsic properties of $\mathfrak{g}$ in terms of these invariants. For example, we can verify that the rank of the exterior differential map $d : \wedge^2\mathfrak{g}^* \longrightarrow \wedge^3\mathfrak{g}^*$ is 3 for "generic" 4-dimensional Lie algebras, and the set of "singular" Lie algebras satisfying rank $d \leq 2$ constitutes two irreducible subvarieties of $\wedge^2 V^* \otimes V$. We can characterize these varieties in terms of the covariants S_λ and invariants χ_i appeared in §2 and §4. (For details, see [6].) This result plays an essential role in considering left invariant symplectic structures on 4-dimensional complex Lie groups.

Our invariants $\chi_1(\mathfrak{g})$ and $\chi_2(\mathfrak{g})$ are essentially related to the (i, j)-invariants introduced in [7; p.734]. In fact, it is easy to see that any (i, j)-invariant of [7] can be expressed as a rational function of χ_1 and χ_2 because it is equal to

$$\frac{({\varepsilon_1}^i + {\varepsilon_2}^i + {\varepsilon_3}^i)({\varepsilon_1}^j + {\varepsilon_2}^j + {\varepsilon_3}^j)}{{\varepsilon_1}^{i+j} + {\varepsilon_2}^{i+j} + {\varepsilon_3}^{i+j}}$$

in our notation. For example, we have

$$(1,1)\text{-}invariant = \frac{1}{1-2\chi_1}, \qquad (2,1)\text{-}invariant = \frac{1-2\chi_1}{1-3\chi_1+3\chi_2},$$
$$(3,1)\text{-}invariant = \frac{1-3\chi_1+3\chi_2}{1-4\chi_1+2{\chi_1}^2+4\chi_2}, \quad (2,2)\text{-}invariant = \frac{(1-2\chi_1)^2}{1-4\chi_1+2{\chi_1}^2+4\chi_2},$$

etc.

5. Varieties of 4-dimensional Lie algebras and their degenerations

As a by-product of the results in the previous sections, we can describe the varieties and the degenerations of 4-dimensional Lie algebras. It is convenient to summarize these results before exhibiting an algorithm to determine the isomorphism classes because they give one basis of the understanding of the algorithm. The results in Proposition 6 and Proposition 8 are essentially already known. But, these results can be summarized in a comparatively simple form, on account of our normal forms that are fitted to describe degenerations.

We first recall the definition of degeneration. We say that a Lie algebra $\mathfrak{g}_1$ *degenerates* to $\mathfrak{g}_2$ if $\mathfrak{g}_1 \not\cong \mathfrak{g}_2$ and $\mathfrak{g}_2 \in \overline{\mathcal{O}(\mathfrak{g}_1)}$, where $\overline{\mathcal{O}(\mathfrak{g}_1)}$ denotes the Zariski closure of the $GL(V)$-orbit $\mathcal{O}(\mathfrak{g}_1)$. In this case, we have $\dim \mathcal{O}(\mathfrak{g}_1) > \dim \mathcal{O}(\mathfrak{g}_2)$. Degenerations of 4-dimensional complex Lie algebras are already completely determined in [7; p.736]. Here, we re-summarize the results in terms of our normal forms. In the following, the symbol $\mathfrak{g}_1 \xrightarrow{deg} \mathfrak{g}_2$ implies that $\mathfrak{g}_1$ degenerates to $\mathfrak{g}_2$. (We sometimes drop the symbol "*deg*" on the arrow if there is no danger of confusion.) Note that the notion of degeneration is transitive, i.e., if $\mathfrak{g}_1 \xrightarrow{deg} \mathfrak{g}_2$ and $\mathfrak{g}_2 \xrightarrow{deg} \mathfrak{g}_3$, then we have $\mathfrak{g}_1 \xrightarrow{deg} \mathfrak{g}_3$.

Proposition 6. (cf. [7; p.736].) *Essential degenerations of* 4*-dimensional complex Lie algebras are exhausted by the following, i.e., all degenerations are obtained by composing the following degenerations:*

$$\begin{array}{lllllll}
L_4(1) & \longrightarrow & L_3 & \longrightarrow & L_0, & & \\
L_6 & \longrightarrow & L_8(-1), & & & & \\
L_7(\alpha,\beta) & \longrightarrow & L_2 & \longrightarrow & L_1 & \longrightarrow & L_0, \\
L_7(\alpha,1) & \longrightarrow & L_4(\alpha) & \longrightarrow & L_1 & (\alpha \in \boldsymbol{C} \cup \{\infty\}), & \\
L_8(\alpha) & \longrightarrow & L_7(\alpha,\alpha+1), & & & & \\
L_8(1) & \longrightarrow & L_5 & \longrightarrow & L_4(2), & & \\
L_9 & \longrightarrow & L_7(\alpha,0), & & & & \\
L_9 & \longrightarrow & L_8(0). & & & &
\end{array}$$

(*Note that we use the notational convention* $L_7(\infty,1) = L_7(0,0)$ *as stated in* §1.)

Outline of the proof. We can explicitly construct a curve in $\wedge^2 V^* \otimes V$ which expresses a degeneration for each case. For example, the Lie algebra

$$\begin{aligned}
&[X_1,X_2] = X_3, \quad [X_1,X_3] = X_4, \\
&[X_1,X_4] = t^3\alpha\beta X_2 - t^2(\alpha\beta+\alpha+\beta)X_3 + t(\alpha+\beta+1)X_4
\end{aligned}$$

is isomorphic to $L_7(\alpha,\beta)$ in case $t \neq 0$, and to L_2 in case $t = 0$. The Lie algebra

$$[X_1,X_2] = X_2 + X_4, \quad [X_1,X_3] = X_2 + \alpha X_3, \quad [X_1,X_4] = (\alpha+1)X_4, \quad [X_2,X_3] = tX_4$$

is isomorphic to $L_8(\alpha)$ in case $t \neq 0$, and to $L_7(\alpha,\alpha+1)$ in case $t = 0$. These facts show that there exist degenerations $L_7(\alpha,\beta) \xrightarrow{deg} L_2$ and $L_8(\alpha) \xrightarrow{deg} L_7(\alpha,\alpha+1)$. Remaining degenerations can be checked in a similar way.

On the other hand, we need several devices to show the non-existence of degenerations. First, as we stated before, if $\dim \mathcal{O}(\mathfrak{g}_1) \leq \dim \mathcal{O}(\mathfrak{g}_2)$, then $\mathfrak{g}_1$ cannot degenerate to $\mathfrak{g}_2$. In case $\dim \mathcal{O}(\mathfrak{g}_1) > \dim \mathcal{O}(\mathfrak{g}_2)$, we show the non-existence of degenerations by using the covariants and invariants which we introduced in §2 and §4. For example, for the Lie algebra L_2, we have $S_{111} = S_{222} = 0$ from Table 3. But, for the Lie algebra $L_4(\alpha)$, two covariants S_{111} and S_{222} cannot simultaneously vanish, which implies that a degeneration $L_2 \xrightarrow{deg} L_4(\alpha)$ does not exist for any α. (Note that $\dim \mathcal{O}(L_2) = 9 > \dim \mathcal{O}(L_4(\alpha)) = 8$.) As another example, we consider the case $L_7(\alpha,\beta) \xrightarrow{deg} L_3$. In case $\alpha+\beta+1 = 0$, we can show the non-existence of degenerations in the same way as above by using the covariant S_{111}. In case $\alpha+\beta+1 \neq 0$, we use two invariants χ_1 and χ_2 to check the non-existence of degenerations. Note that in this case, χ_1 and χ_2 are well-defined for both Lie algebras. If there exists a degeneration $L_7(\alpha,\beta) \xrightarrow{deg} L_3$, then the values of invariants of $L_7(\alpha,\beta)$ must coincide with that of L_3 because L_3 is contained in the Zariski closure of the $GL(V)$-orbit of $L_7(\alpha,\beta)$. Hence, from Table 4, we obtain two conditions

$$\begin{aligned}
&(\alpha+\beta)^2 - 3\alpha\beta - (\alpha+\beta) + 1 = 0, \\
&(\alpha+\beta)^3 + 3(\alpha+\beta)^2 - 27\alpha\beta + 3(\alpha+\beta) + 1 = 0.
\end{aligned}$$

From these conditions, we have immediately $\alpha = \beta = 1$, which implies that $L_7(\alpha,\beta)$ cannot degenerate to L_3 in case $(\alpha,\beta) \neq (1,1)$. (We already know the existence of a degeneration

$L_7(1,1) \xrightarrow{deg} L_3$.) For the remaining cases not listed in Proposition 6, we can similarly prove the non-existence of degenerations. Note that for this purpose, we have only to use the covariants S_{111}, S_{222}, S_{333} (and three invariants χ_1, χ_2, χ_3), except for the cases $L_7(\alpha,\beta) \xrightarrow{deg} L_5$ and $L_9 \xrightarrow{deg} L_8(\alpha)$ $(\alpha \neq 0)$. For these two exceptional cases, we use the covariant S_{4221} to show the non-existence of degenerations. □

Remark. (1) Two Lie algebras L_6 and L_9 are sums of lower dimensional Lie algebras. But their degenerate Lie algebras are not necessarily expressed as sums of Lie algebras such as $L_8(-1)$, $L_8(0)$, L_2 (cf. Remark (1) after Proposition 1).

(2) Nilpotent Lie algebras are all contained in $\overline{\mathcal{O}(L_2)}$. Namely, any nilpotent Lie algebra is obtained as a degeneration of L_2 (or L_2 itself) in the 4-dimensional case.

As a corollary of Proposition 6, we can show several facts on the variety of 4-dimensional Lie algebras. But, before stating them, we first calculate the cohomology space of $\mathfrak{g}$ for later use.

Proposition 7. *The dimensions of the second cohomology space $H^2(\mathfrak{g},\mathfrak{g})$ with coefficients in the adjoint representation are given as follows:*

	$\dim H^2(\mathfrak{g},\mathfrak{g})$	
L_0	24	
L_1	13	
L_2	6	
L_3	8	
$L_4(\alpha)$	7	$\alpha = 0$
	6	$\alpha = \infty$
	5	$\alpha = 2$
	4	$\alpha \neq 0, 2, \infty$
L_5	3	

	$\dim H^2(\mathfrak{g},\mathfrak{g})$	
L_6	0	
$L_7(\alpha,\beta)$	5	$(\alpha,\beta) = (-1,0),(0,-1)$
	3	$\alpha \neq -1, \beta = 0$ *or* $\alpha \neq -1, \beta = \alpha + 1$
	2	$\alpha + \beta \neq 1, \lvert\alpha - \beta\rvert \neq 1,\ \alpha,\beta \neq 0$
$L_8(\alpha)$	2	$\alpha = 0, -1$
	1	$\alpha \neq 0, -1$
L_9	0	

We can check this result by direct calculations. By the deformation theory of Lie algebras [26], the dimension of the space $H^2(\mathfrak{g},\mathfrak{g})$ indicates the degree of freedom of non-trivial infinitesimal deformations of $\mathfrak{g}$. In particular, if $H^2(\mathfrak{g},\mathfrak{g}) = 0$ (such as L_6, L_9), then $\mathfrak{g}$ is rigid, i.e., the orbit space $\mathcal{O}(\mathfrak{g})$ is Zariski open in the set of Lie algebra structures in $\wedge^2 V^* \otimes V$, and its closure $\overline{\mathcal{O}(\mathfrak{g})}$ is irreducible.

From Proposition 6, the Lie algebras that cannot be expressed as degenerations of other Lie algebras are exhausted by L_6, $L_7(\alpha,\beta)$ $(|\alpha - \beta| \neq 1,\ \alpha \neq 0,\ \beta \neq 0)$, $L_8(\alpha)$ $(\alpha \neq 0, -1)$ and L_9. (Note that the Lie algebra $L_7(\alpha,\beta)$ with $\alpha + \beta = 1$ also should be excluded. But this Lie algebra is isomorphic to $L_7(\gamma, \gamma+1)$ for some γ. See Remark (3) after Proposition 1.) For two families of non-rigid Lie algebras $L_7(\alpha,\beta)$ and $L_8(\alpha)$, the number of parameters just coincide with the dimension of $H^2(\mathfrak{g},\mathfrak{g})$ for generic parameters. Hence, we obtain the following well known fact on the irreducible decomposition of the variety of Lie algebras.

Proposition 8. (cf. [9], [12], [19], [27].) *The algebraic set of $\wedge^2 V^* \otimes V$ consisting of all 4-dimensional Lie algebras is a union of four irreducible varieties $\Sigma_1 \sim \Sigma_4$. These varieties are the Zariski closures of the following $GL(V)$-orbits.*

$$\Sigma_1 = \overline{\mathcal{O}(L_6)}, \qquad \Sigma_2 = \overline{\cup_{\alpha,\beta}\, \mathcal{O}(L_7(\alpha,\beta))},$$
$$\Sigma_3 = \overline{\cup_{\alpha}\, \mathcal{O}(L_8(\alpha))}, \qquad \Sigma_4 = \overline{\mathcal{O}(L_9)}.$$

In view of Table 3, we can characterize these varieties in terms of $GL(V)$-invariant sets of polynomials with degree at most three.

Proposition 9. *The defining equations of the varieties $\Sigma_1 \sim \Sigma_4$ are given by:*

$$\begin{array}{lll} \Sigma_1 : S_{111} = S_{333} = 0 & & (\textit{linear and cubic polynomials}), \\ \Sigma_2 : S_{3111} = S_{4221} = 0 & & (\textit{quadratic and cubic polynomials}), \\ \Sigma_3 : \langle \varphi \rangle = S_{3222} = 0 & & (\textit{cubic polynomials}), \\ \Sigma_4 : S_{4221} = S_{333} = 0 & & (\textit{cubic polynomials}), \end{array}$$

in addition to the Jacobi identity.

Remark. By definition, Lie algebras are defined by the vanishing of polynomials of $\{c_{ij}^k\}$ corresponding to the Jacobi identity. But by this proposition, $\{c_{ij}^k\}$ must satisfy the additional different types of polynomial identities.

The orbit decompositions of Σ_i and their degenerations are summarized in Figure 1. For each variety Σ_i, it is clear that there exists one principal line of degenerations. But, there also appear several singular Lie algebras such as L_3, L_5, etc., and these degenerate Lie algebras make Figure 1 a little complicated.

Note that the variety Σ_2 mainly consists of the Lie algebras $L_7(\alpha,\beta)$. Among them, Lie algebras which satisfy $\dim H^2(\mathfrak{g},\mathfrak{g}) > 2$ constitute a family $\{L_7(\alpha,0), L_7(\alpha,\alpha+1)\}$ (cf. Proposition 7). And they just coincide with the ones that are situated in the intersection of other varieties Σ_1, Σ_3, Σ_4. Similar phenomenon occurs for the variety $\Sigma_3 = \overline{\cup_\alpha \mathcal{O}(L_8(\alpha))}$.

In addition, from Table 3 and Figure 1, we can easily see that the set of all 4-dimensional unimodular Lie algebras splits into two varieties Σ_1 and $\overline{\cup_\alpha \mathcal{O}(L_7(\alpha,-\alpha-1))}$ ($\subset \Sigma_2$) with dimensions 12 and 11, respectively.

6. An algorithm to determine the isomorphism classes of 4-dimensional Lie algebras

Now, in this final section, we give an algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras, by using the devices prepared in the previous sections. This algorithm is the main result of the present paper.

First, among 4-dimensional Lie algebras, L_0, L_6, L_9 and non-abelian nilpotent Lie algebras L_1, L_2 can be characterized by simple properties. In fact, the Lie algebra L_6 has a characteristic property $\dim [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 3$, and the Lie algebra L_9 is uniquely

Figure 1

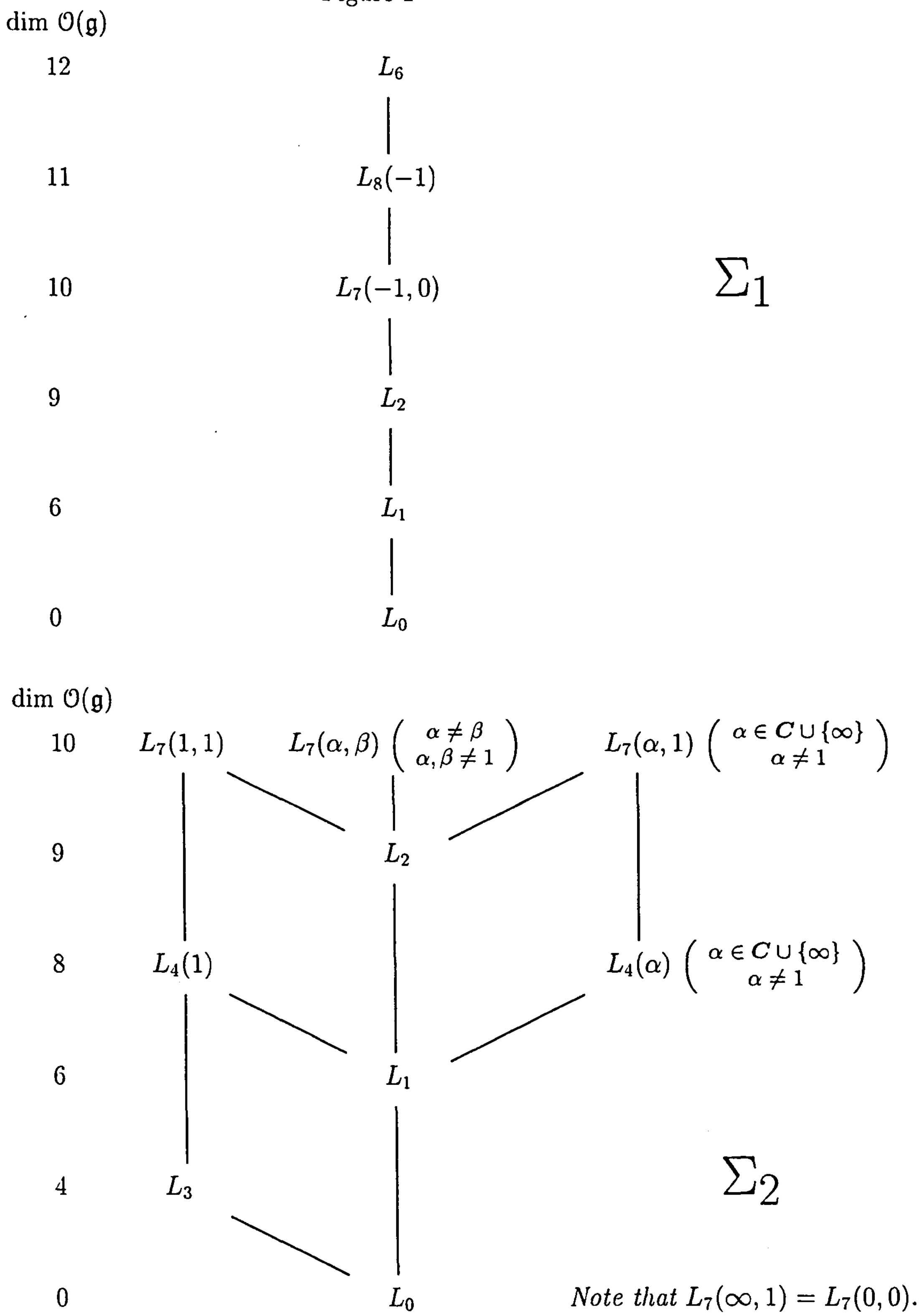
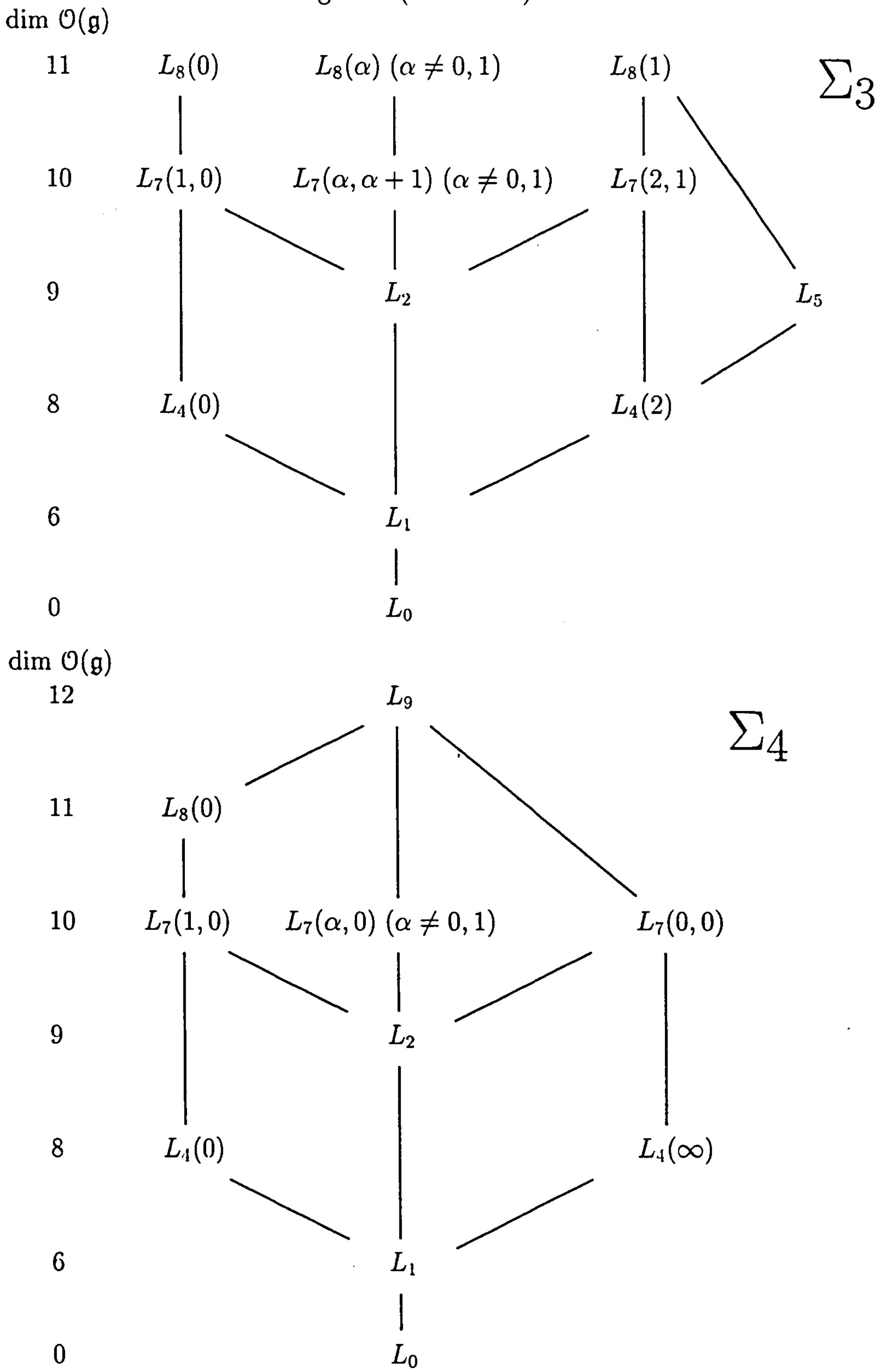

dim $\mathcal{O}(\mathfrak{g})$
12
11
10
9
6
0
L_6
$L_8(-1)$
$L_7(-1,0)$
L_2
L_1
L_0
Σ_1
dim $\mathcal{O}(\mathfrak{g})$
10
9
8
6
4
0
$L_7(1,1)$
$L_7(\alpha,\beta)$ $\begin{pmatrix} \alpha \neq \beta \\ \alpha,\beta \neq 1 \end{pmatrix}$
$L_7(\alpha,1)$ $\begin{pmatrix} \alpha \in C \cup \{\infty\} \\ \alpha \neq 1 \end{pmatrix}$
L_2
$L_4(1)$
$L_4(\alpha)$ $\begin{pmatrix} \alpha \in C \cup \{\infty\} \\ \alpha \neq 1 \end{pmatrix}$
L_1
L_3
L_0
Σ_2
Note that $L_7(\infty,1) = L_7(0,0)$.

Figure 1 (continued)



characterized by the property that the ratio of the eigenvalues of ad X essentially depends on the choice of X. Nilpotent Lie algebras are characterized by the properties $S_{111} = S_{222} = S_{333} = 0$, which are also equivalent to $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, where $\{0, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ are the eigenvalues of ad X for generic X (cf. Proposition 4 (10)). Three nilpotent Lie algebras L_0, L_1, L_2 can be distinguished by the value dim $[\mathfrak{g}, \mathfrak{g}]$.

Next, we state a method to determine the isomorphism classes for the remaining Lie algebras L_3, $L_4(\alpha)$, L_5, $L_7(\alpha,\beta)$ and $L_8(\alpha)$. These Lie algebras are roughly classified into four classes by the values $\dim [\mathfrak{g}, \mathfrak{g}] = \dim \mathfrak{g}^{(1)}$ and $\dim [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = \dim \mathfrak{g}^{(2)}$:

$$\begin{array}{lcl} \dim \mathfrak{g}^{(1)} = 1 & : & L_4(\infty), \\ \dim \mathfrak{g}^{(1)} = 2 & : & L_4(0),\ L_7(\alpha,0),\ L_8(0), \\ \dim \mathfrak{g}^{(1)} = 3,\ \dim \mathfrak{g}^{(2)} = 0 & : & L_3,\ L_4(\alpha)\ (\alpha \neq 0, \infty),\ L_7(\alpha,\beta)\ (\alpha, \beta \neq 0), \\ \dim \mathfrak{g}^{(1)} = 3,\ \dim \mathfrak{g}^{(2)} = 1 & : & L_5,\ L_8(\alpha)\ (\alpha \neq 0). \end{array}$$

In particular, $L_4(\infty)$ is uniquely characterized by the property $\dim [\mathfrak{g}, \mathfrak{g}] = 1$. We give a method to determine the isomorphism classes for the remaining Lie algebras in terms of several covariants and invariants.

(i) We first consider three Lie algebras L_3, $L_4(1)$ and $L_7(1,1)$. These Lie algebras constitute special degenerations: $L_7(1,1) \xrightarrow{deg} L_4(1) \xrightarrow{deg} L_3$, and they are characterized by the properties $\chi_1 = \frac{1}{3}$ and $\chi_2 = \frac{1}{27}$ among the above remaining Lie algebras (in case the invariants have definite values). From Table 3, we have

	S_{21}	S_{441}
L_3	0	0
$L_4(1)$	*	0
$L_7(1,1)$	*	*

and hence, these three Lie algebras can be distinguished to each other by two covariants S_{21} and S_{441}.

(ii) Next, we consider the case $\dim [\mathfrak{g}, \mathfrak{g}] = 2$. Among these Lie algebras, there exist degenerations $L_8(0) \xrightarrow{deg} L_7(1,0) \xrightarrow{deg} L_4(0)$. The value of χ_1 for these three Lie algebras is $\frac{1}{4}$. Hence, if $\chi_1 \neq \frac{1}{4}$, it is isomorphic to the remaining Lie algebra $L_7(\alpha, 0)$ $(\alpha \neq 1)$, and from Proposition 5 (4), the value of the parameter of $L_7(\alpha,0)$ is uniquely determined by χ_1. For the above three Lie algebras with $\chi_1 = \frac{1}{4}$, we have from Table 3,

	S_{3111}	S_{441}
$L_4(0)$	0	0
$L_7(1,0)$	0	*
$L_8(0)$	*	*

Hence, they are distinguished by two covariants S_{3111} and S_{441}.

(iii) The case $\dim[\mathfrak{g},\mathfrak{g}]=3$ and $\dim[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=1$. Among these Lie algebras, there is a degeneration $L_8(1)\xrightarrow{deg} L_5$. For these two Lie algebras, the value of χ_1 is equal to $\frac{5}{16}$, and hence, if $\chi_1\neq\frac{5}{16}$, it is isomorphic to $L_8(\alpha)$ $(\alpha\neq 0,1)$. In this case, from Proposition 5 (5), the value of the parameter α is uniquely determined by χ_1. To distinguish two Lie algebras $L_8(1)$ and L_5, we have only to use the covariant S_{441} (cf. Table 3).

(iv) The case $\dim[\mathfrak{g},\mathfrak{g}]=3$ and $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]=0$. Remaining Lie algebras are exhausted by $L_4(\alpha)$ $(\alpha\neq 0,1,\infty)$ and $L_7(\alpha,\beta)$ $(\alpha,\beta\neq 0,\ (\alpha,\beta)\neq(1,1))$. From Table 3, two Lie algebras $L_4(\alpha)$ and $L_7(\alpha,\beta)$ are distinguished by the covariant S_{441}. And from Proposition 5 (1), (2) and (3), we can determine the values of the parameters of $L_4(\alpha)$ and $L_7(\alpha,\beta)$ in terms of $\chi_1\sim\chi_3$.

By these procedures, in terms of covariants and invariants which we introduced in this paper, we can uniquely determine the isomorphism classes of 4-dimensional Lie algebras without constructing the explicit isomorphisms. Summarizing these results, we obtain the following main theorem of this paper.

Theorem 10. *The isomorphism classes of* 4*-dimensional complex Lie algebras are determined uniquely in terms of the following quantities*:

- $\dim[\mathfrak{g},\mathfrak{g}]$, $\dim[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]$.
- S_{21}, S_{3111}, S_{441}.
- *The ratio of the eigenvalues of* $\mathrm{ad}\,X$ *for generic* $X\in\mathfrak{g}$, *(i.e., three invariants* χ_1, χ_2 *and* χ_3.*)*

An algorithm to determine the isomorphism classes of 4*-dimensional Lie algebras is summarized in Figure* 2.

Concerning three covariants appeared in this theorem, remind the results in Proposition 4:

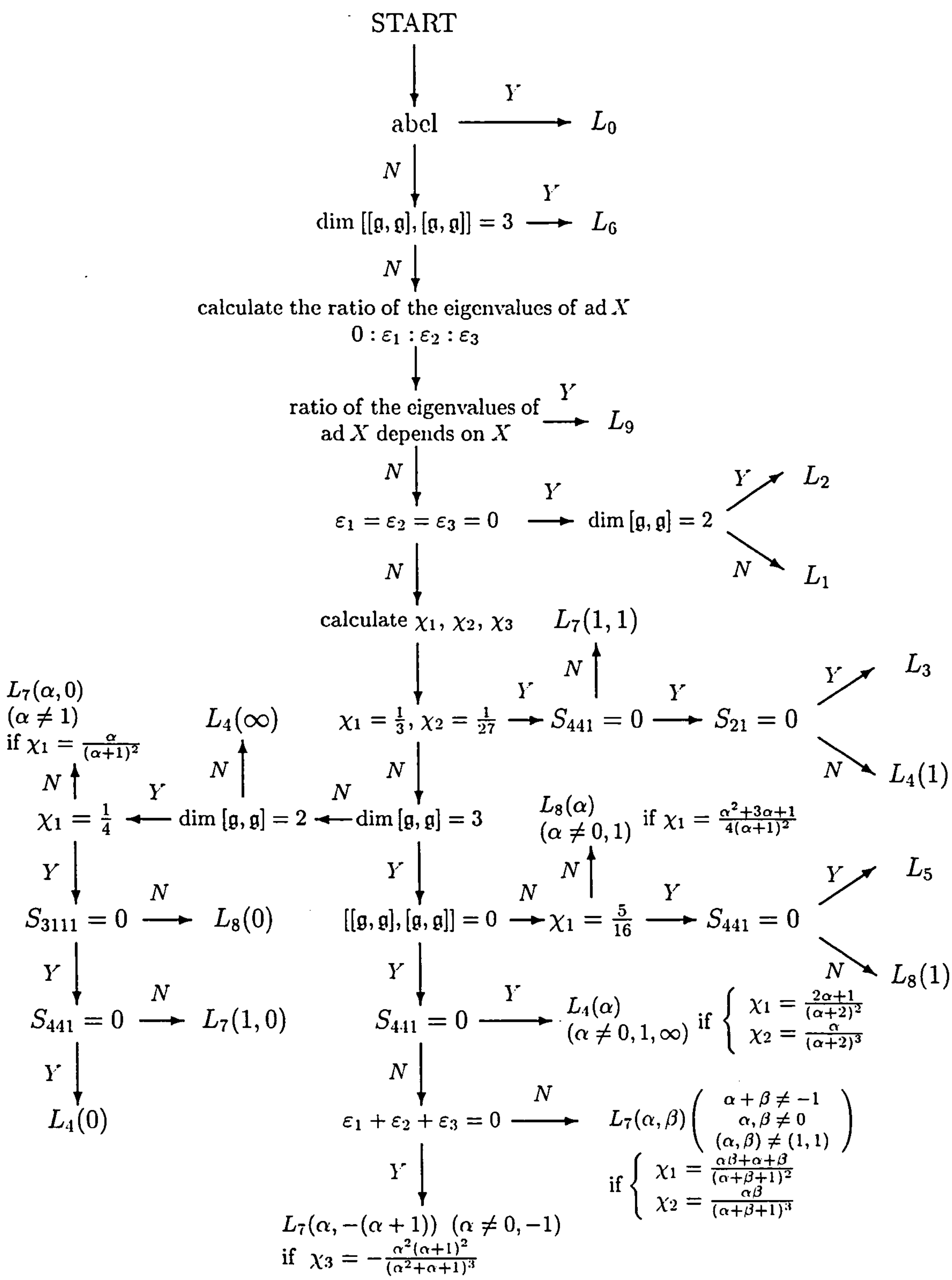
- $S_{21}=0$ if and only if there exists an element $f\in\mathfrak{g}^*$ such that $[X,Y]=f(X)Y-f(Y)X$.
- $S_{3111}=0$ if and only if $d\alpha\wedge d\alpha=0$ for any $\alpha\in\mathfrak{g}^*$.
- $S_{441}=0$ if and only if $\dim\langle X,Y,[X,Y],[X,[X,Y]]\rangle\leq 3$ for any X, $Y\in\mathfrak{g}$.

Hence it is now an easy task to verify whether a given Lie algebras satisfies the condition $S_\lambda=0$ or not. (Notice that once we proved Proposition 4, we need not to repeat a hard polynomial calculation which we carried out in § 2.) Of course, the above two values $\dim[\mathfrak{g},\mathfrak{g}]$ and $\dim[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]$ are also characterized by the vanishing or non-vanishing of several covariants, as stated in Proposition 4.

Example. We give here two examples, which shows the usefulness of our algorithm. We first consider the following Lie algebra:

$$\begin{aligned}
&[X_1,X_2]=-X_1-X_2+X_3, && [X_1,X_3]=-6X_2+4X_3,\\
&[X_1,X_4]=2X_1-X_2+X_4, && [X_2,X_3]=3X_1-9X_2+5X_3,\\
&[X_2,X_4]=4X_1-2X_2+2X_4, && [X_3,X_4]=6X_1-3X_2+3X_4.
\end{aligned}$$

Figure 2


START
abel
Y
L_0
N
$\dim [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 3$
Y
L_6
N
calculate the ratio of the eigenvalues of ad X
$0 : \varepsilon_1 : \varepsilon_2 : \varepsilon_3$
ratio of the eigenvalues of ad X depends on X
Y
L_9
N
$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$
Y
$\dim [\mathfrak{g},\mathfrak{g}] = 2$
Y
L_2
N
L_1
N
calculate χ_1, χ_2, χ_3
$L_7(1,1)$
$\chi_1 = \frac{1}{3}, \chi_2 = \frac{1}{27}$
Y
$S_{441} = 0$
N
Y
$S_{21} = 0$
Y
L_3
N
$L_4(1)$
$L_7(\alpha,0)$
$(\alpha \neq 1)$
if $\chi_1 = \frac{\alpha}{(\alpha+1)^2}$
$L_4(\infty)$
N
N
N
N
$\chi_1 = \frac{1}{4}$
Y
$\dim [\mathfrak{g},\mathfrak{g}] = 2$
$\dim [\mathfrak{g},\mathfrak{g}] = 3$
$L_8(\alpha)$
$(\alpha \neq 0,1)$ if $\chi_1 = \frac{\alpha^2+3\alpha+1}{4(\alpha+1)^2}$
Y
$S_{3111} = 0$
N
$L_8(0)$
Y
$[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$
N
$\chi_1 = \frac{5}{16}$
N
Y
$S_{441} = 0$
Y
L_5
N
$L_8(1)$
Y
$S_{441} = 0$
N
$L_7(1,0)$
Y
$S_{441} = 0$
Y
$L_4(\alpha)$
$(\alpha \neq 0,1,\infty)$ if $\chi_1 = \frac{2\alpha+1}{(\alpha+2)^2}$, $\chi_2 = \frac{\alpha}{(\alpha+2)^3}$
Y
$L_4(0)$
N
$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$
N
$L_7(\alpha,\beta)$ $\left(\alpha+\beta \neq -1;\ \alpha,\beta \neq 0;\ (\alpha,\beta) \neq (1,1)\right)$
if $\chi_1 = \frac{\alpha\beta+\alpha+\beta}{(\alpha+\beta+1)^2}$, $\chi_2 = \frac{\alpha\beta}{(\alpha+\beta+1)^3}$
Y
$L_7(\alpha,-(\alpha+1))$ $(\alpha \neq 0,-1)$
if $\chi_3 = -\frac{\alpha^2(\alpha+1)^2}{(\alpha^2+\alpha+1)^3}$

We can easily check that $[\mathfrak{g},\mathfrak{g}] = \langle 2X_1 - X_2, 3X_1 - X_3, X_4\rangle$ and $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$. And by putting $X = aX_1 + bX_2 + cX_3 + dX_4$, we have

$$\operatorname{ad} X = \begin{pmatrix} b-2d & -a-3c-4d & 3b-6d & 2a+4b+6c \\ b+6c+d & -a+9c+2d & -6a-9b+3d & -a-2b-3c \\ -b-4c & a-5c & 4a+5b & 0 \\ -d & -2d & -3d & a+2b+3c \end{pmatrix}.$$

Hence, after some calculations, we have $\mathrm{Tr\,ad}X = 4(a+2b+3c)$, $\mathrm{Tr}(\mathrm{ad}X)^2 = 6(a+2b+3c)^2$, $\mathrm{Tr}\,(\mathrm{ad}\,X)^3 = 10(a+2b+3c)^3$. From these values (or by calculating the characteristic polynomial of ad X directly), we have

$$\begin{aligned} p_{111} &= -\mathrm{Tr\,ad}\,X = -4(a+2b+3c), \\ p_{222} &= \tfrac{1}{2}\{(\mathrm{Tr\,ad}\,X)^2 - \mathrm{Tr}\,(\mathrm{ad}\,X)^2\} = 5(a+2b+3c)^2, \\ p_{333} &= -\tfrac{1}{6}\{(\mathrm{Tr\,ad}\,X)^3 - 3\,\mathrm{Tr\,ad}\,X\cdot\mathrm{Tr}\,(\mathrm{ad}\,X)^2 + 2\,\mathrm{Tr}\,(\mathrm{ad}\,X)^3\} \\ &= -2(a+2b+3c)^3, \end{aligned}$$

and hence, $\chi_1 = p_{222}/p_{111}^2 = \frac{5}{16}$, $\chi_2 = p_{333}/p_{111}^3 = \frac{1}{32}$. In addition, four elements X_1, X_4, $[X_1, X_4] = 2X_1 - X_2 + X_4$ and $[X_1,[X_1,X_4]] = 3X_1 - X_3 + X_4$ are linearly independent, and hence we have $S_{441} \neq 0$. Since the value of p_{111} is non-zero, this Lie algebra is not unimodular. Therefore, by applying the algorithm in Figure 2, we know that this Lie algebra is isomorphic to $L_7(\alpha,\beta)$ with $\chi_1 = \frac{5}{16}$, $\chi_2 = \frac{1}{32}$ and $\alpha+\beta \neq -1$, $(\alpha,\beta)\neq(1,1)$, α, $\beta \neq 0$. By solving the equations

$$\frac{\alpha\beta+\alpha+\beta}{(\alpha+\beta+1)^2} = \frac{5}{16}, \quad \frac{\alpha\beta}{(\alpha+\beta+1)^3} = \frac{1}{32},$$

we have for example, $(\alpha, \beta) = (2,1)$, and hence this Lie algebra is isomorphic to $L_7(2,1)$.

As another example, we consider the following Lie algebra:

$$\begin{aligned} &[X_1,X_2] = 4X_1 + 3X_2 - 6X_3 + 2X_4, && [X_1,X_3] = 15X_1 + 5X_2 - 15X_3 + 5X_4, \\ &[X_1,X_4] = 50X_1 + 15X_2 - 48X_3 + 16X_4, && [X_2,X_3] = 21X_1 + 2X_2 - 15X_3 + 5X_4, \\ &[X_2,X_4] = 93X_1 + 21X_2 - 81X_3 + 27X_4, && [X_3,X_4] = 90X_1 + 25X_2 - 84X_3 + 28X_4. \end{aligned}$$

Then, for $X = aX_1 + bX_2 + cX_3 + dX_4$, we have

$$\operatorname{ad} X = \begin{pmatrix} -4b-15c-50d & 4a-21c-93d & 15a+21b-90d & 50a+93b+90c \\ -3b-5c-15d & 3a-2c-21d & 5a+2b-25d & 15a+21b+25c \\ 6b+15c+48d & -6a+15c+81d & -15a-15b+84d & -48a-81b-84c \\ -2b-5c-16d & 2a-5c-27d & 5a+5b-28d & 16a+27b+28c \end{pmatrix}.$$

In this case, by putting $p = a+2b+3c+4d$, $q = 3a+6b+8c+9d$, we have $\mathrm{Tr\,ad}X = p+q$, $\mathrm{Tr}\,(\mathrm{ad}\,X)^2 = p^2+q^2$, $\mathrm{Tr}\,(\mathrm{ad}\,X)^3 = p^3+q^3$. Hence, the eigenvalues of ad X is given by $\{0,0,p,q\}$, and the ratio essentially depends on X. (cf. Remark (1) after Proposition 2.) Thus, by the algorithm in Figure 2, it follows that this Lie algebra is isomorphic to L_9.

Appendix. Relation to normal forms in [7]

There are already several classifications of 4-dimensional real or complex Lie algebras. (For example, [7], [20], [22], [24], [27], [29], [30], [31], [33], [36], etc. But, as for the classification table in [27; p.209], it seems that it contains some mistakes. See also the comments in [7; p.732].) In this appendix, we give explicit isomorphisms between our normal forms in Table 1 and the normal forms in [7; p.733] which was essentially taken from [31]. By checking these correspondences in detail, we see that the list of degenerations in Proposition 6 just coincides with the result in [7; p.736]. Note that to find the isomorphic Lie algebra among several normal forms is now an easy task for us on account of the algorithm in Figure 2. But, to construct the explicit isomorphism is another problem, which requires many tedious trials.

In this appendix, $\{X_1, \cdots, X_4\}$ denotes the basis of L_i in Table 1, and $\{e_1, \cdots, e_4\}$ denotes the basis of the Lie algebras in [7]. We use the same symbols as in [7]. But, for three Lie algebras $\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_8$, we replace the parameters α and β in [7] by λ and μ, respectively. We drop the parameter restrictions in [7] because singular cases often give good examples of deformations of Lie algebras (cf. Figure 1). In the following list, we give the isomorphisms only for non-trivial cases. For the explicit bracket operations $[e_i, e_j]$, see [7; p.733].

- $\boldsymbol{C}^4 \cong L_0$.
- $\mathfrak{n}_3(\boldsymbol{C}) \oplus \boldsymbol{C} \cong L_1$.
- $\mathfrak{r}_2(\boldsymbol{C}) \oplus \boldsymbol{C}^2 \cong L_4(\infty)$.
- $\mathfrak{r}_3(\boldsymbol{C}) \oplus \boldsymbol{C} \cong L_7(1,0)$,

$$e_1 = X_1,\ e_2 = X_2,\ e_3 = X_3,\ e_4 = X_2 - X_3 + X_4.$$

- $\mathfrak{r}_{3,\lambda}(\boldsymbol{C}) \oplus \boldsymbol{C} \cong \begin{cases} L_4(\infty), & \lambda = 0, \\ L_4(0), & \lambda = 1, \\ L_7(\lambda, 0), & \lambda \neq 0, 1, \end{cases}$

$$\begin{cases} \lambda = 1 & : \ e_1 = X_1,\ \ e_2 = X_2,\ \ e_3 = X_3,\ \ e_4 = X_3 - X_4, \\ \lambda \neq 0, 1 & : \ e_1 = X_1,\ \ e_2 = X_2,\ \ e_3 = X_2 + (\lambda - 1)X_3, \\ & \quad\ e_4 = X_2 - X_3 + \lambda X_4. \end{cases}$$

- $\mathfrak{r}_2(\boldsymbol{C}) \oplus \mathfrak{r}_2(\boldsymbol{C}) \cong L_9$.
- $\mathfrak{sl}_2(\boldsymbol{C}) \oplus \boldsymbol{C} \cong L_6$,

$$e_1 = X_2,\ e_2 = 2X_3,\ e_3 = 2X_1,\ e_4 = X_4.$$

- $\mathfrak{n}_4(\boldsymbol{C}) \cong L_2$.
- $\mathfrak{g}_1(\alpha) \cong \begin{cases} L_3, & \alpha = 1, \\ L_4(\alpha), & \alpha \neq 1, \end{cases}$

$$\alpha \neq 1 \quad : \quad e_1 = X_1,\ e_2 = X_2,\ e_3 = X_3,\ e_4 = X_3 + (\alpha - 1)X_4.$$

- $\mathfrak{g}_2(\lambda, \mu) \cong L_7(\alpha, \beta), \qquad (\alpha + \beta \neq -1),$

$$e_1 = \tfrac{1}{a}X_1,\ e_2 = a^2 X_4,\ e_3 = a(X_3 + \beta X_4),\ e_4 = X_2 + (\alpha + \beta)X_3 + \beta^2 X_4.$$

Here, for given λ and μ, we define two complex numbers α, β by $\alpha + \beta = a - 1$ and $\alpha\beta = a^3\lambda$, where a is a non-zero complex number satisfying $\lambda a^3 - \mu a^2 + a - 1 = 0$. Since $a \neq 0$, we have $\alpha + \beta \neq -1$. And from the definition of α and β, we can easily show the equalities

$$\lambda = \frac{\alpha\beta}{(\alpha + \beta + 1)^3} = \chi_2(\mathfrak{g}), \quad \mu = \frac{\alpha\beta + \alpha + \beta}{(\alpha + \beta + 1)^2} = \chi_1(\mathfrak{g}).$$

Remark. In case $\lambda \neq 0$, other solutions of the cubic equation $\lambda x^3 - \mu x^2 + x - 1 = 0$ are given by a/α, a/β. And if we use a/α instead of a in the above isomorphism, then the solutions of $\alpha' + \beta' = \frac{a}{\alpha} - 1$, $\alpha'\beta' = \left(\frac{a}{\alpha}\right)^3 \lambda$ are $(\alpha', \beta') = (\frac{1}{\alpha}, \frac{\beta}{\alpha})$, $(\frac{\beta}{\alpha}, \frac{1}{\alpha})$. But the unordered ratio $1 : \alpha' : \beta'$ corresponding to this new solution coincides with the original ratio $1 : \alpha : \beta$, and hence we may use any solution of $\lambda x^3 - \mu x^2 + x - 1 = 0$ in constructing the above isomorphism. (See the proof of Proposition 5 (2).) In case $\lambda = 0$, we can easily show that the same fact holds.

- $\mathfrak{g}_3(\lambda) \cong \begin{cases} L_2, & \lambda = 0, \\ L_7(\alpha, -(\alpha+1)), & \lambda \neq 0, \quad (\alpha \neq 0, -1,\ \alpha^2 + \alpha + 1 \neq 0), \end{cases}$

$$\begin{aligned} \lambda \neq 0 \quad : \quad & e_1 = kX_1, \quad e_2 = \alpha^2(\alpha+1)^2 X_4, \\ & e_3 = \alpha(\alpha+1)(\alpha^2+\alpha+1)\{-X_3 + (\alpha+1)X_4\}, \\ & e_4 = (\alpha^2+\alpha+1)^2\{X_2 - X_3 + (\alpha+1)^2 X_4\}, \\ & (k = -\tfrac{\alpha^2+\alpha+1}{\alpha(\alpha+1)} \neq 0). \end{aligned}$$

Here, in the case $\lambda \neq 0$, the number α is a solution of the equation $(x^2 + x + 1)^3 = \lambda x^2(x+1)^2$. Clearly, we have $\alpha \neq 0, -1$ and $\alpha^2 + \alpha + 1 \neq 0$. In this case, the parameter λ satisfies the equality $\lambda = \frac{(\alpha^2+\alpha+1)^3}{\alpha^2(\alpha+1)^2} = -\frac{1}{\chi_3(\mathfrak{g})}$.

Remark. Other solutions of the equation $(x^2 + x + 1)^3 = \lambda x^2(x+1)^2$ are given by $\frac{1}{\alpha}$, $-(\alpha+1)$, $\frac{-1}{\alpha+1}$, $-\frac{\alpha+1}{\alpha}$, $\frac{-\alpha}{\alpha+1}$, and it is easy to check that the unordered ratio $1 : \alpha : -(\alpha+1)$ does not depend on the choice of these solutions. Hence, as above, we may use any solution of this equation in constructing the isomorphism. (See also the proof of Proposition 5 (3).)

- $\mathfrak{g}_4 \cong L_7(\omega, \omega^2), \qquad (\omega^3 = 1,\ \omega \neq 1),$

$$e_1 = X_1,\ e_2 = X_4,\ e_3 = X_3 + \omega^2 X_4,\ e_4 = X_2 - X_3 + \omega X_4.$$

- $\mathfrak{g}_5 \cong L_4(1),$

$$e_1 = \tfrac{1}{3}X_1,\ e_2 = 3X_4,\ e_3 = X_3,\ e_4 = X_2.$$

- $\mathfrak{g}_6 \cong L_5.$

- $\mathfrak{g}_7 \cong L_8(-1)$,

$$e_1 = X_1,\ e_2 = X_3,\ e_3 = X_2 - X_3,\ e_4 = -X_4.$$

- $\mathfrak{g}_8(\lambda) \cong L_8(\alpha)$, $\lambda = \frac{\alpha}{(\alpha+1)^2}$ $(= 8\chi_2(\mathfrak{g}))$, $(\alpha \neq -1)$,

$$e_1 = \frac{1}{\alpha+1}X_1,\ e_2 = (\alpha+1)X_3,\ e_3 = X_2 + \alpha X_3,\ e_4 = -(\alpha+1)X_4.$$

Remark. In case $\lambda \neq 0$, the solutions of the equation $\frac{x}{(x+1)^2} = \lambda$ are of the form α, $\frac{1}{\alpha}$ for each fixed λ. Since $L_8(\alpha) \cong L_8(\frac{1}{\alpha})$, we may use any solution of $\frac{x}{(x+1)^2} = \lambda$ in constructing the above isomorphism.

Finally, we add some comments. For the Lie algebra $\mathfrak{g}_2(\lambda, \mu)$, the parameters λ and μ just coincide with our invariants $\chi_2(\mathfrak{g})$, $\chi_1(\mathfrak{g})$, and they appear in the coefficients of the bracket $[e_1, e_4] = \lambda e_2 - \mu e_3 + e_4$ in a natural way (cf. [7; p.733]). Normal forms of $\mathfrak{g}_3(\lambda)$ and $\mathfrak{g}_8(\lambda)$ also possess this property. These facts imply that the normal forms in [7] (or [31]) are elegantly selected from the invariant theoretic viewpoint because the bracket operations are simply and uniquely expressed by their invariants. (In the 3-dimensional case, the normal form in [35] also possesses this property.)

But on the other hand, the normal forms in [7] are not necessarily fitted to describe deformations or degenerations of Lie algebras. For example, in view of the above isomorphism list, the Lie algebras $\mathfrak{r}_3(\boldsymbol{C}) \oplus \boldsymbol{C}$, $\mathfrak{r}_{3,\lambda}(\boldsymbol{C}) \oplus \boldsymbol{C}$ $(\lambda \neq 0, 1)$, $\mathfrak{g}_2(\lambda, \mu)$, $\mathfrak{g}_3(\lambda)$ $(\lambda \neq 0)$ and $\mathfrak{g}_4$ should be gathered together to construct one family of Lie algebras because they are continuously deformable, possessing the same dimensional $GL(V)$-orbits. As another example, the family of Lie algebras $\mathfrak{g}_1(\alpha)$ contains a degenerate Lie algebra L_3 in case $\alpha = 1$. And as normal forms, it is desirable to adopt a family of Lie algebras such that $\mathfrak{g}_1(\alpha)$ corresponds to $\mathfrak{g}_5 \cong L_4(1)$ in case $\alpha = 1$. (Note that there exists a degeneration $\mathfrak{g}_5 \xrightarrow{deg} \mathfrak{g}_1(1)$.) In addition, this family should contain the Lie algebra $\mathfrak{r}_2(\boldsymbol{C}) \oplus \boldsymbol{C}^2$ because $\lim_{\alpha\to\infty} \mathfrak{g}_1(\alpha) \cong \mathfrak{r}_2(\boldsymbol{C}) \oplus \boldsymbol{C}^2$. Our family of Lie algebras $L_4(\alpha)$ is selected to satisfy these conditions.

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