Nonlinearity of energy of Rankine flows on a torus ⋆

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Abstract

We study an ideal fluid flow on a torus described by the Weierstrass ζ-function. In spite of the analogy of this function to the Joukowski transformation on the plane the convex (planar) domain bounded by two streamlines passing through the stagnation points is not a disk. The energy of the flow outside the convex domain is generally nonlinear function of the strength of the dipole; in fact the energy is in only two cases a linear function of the strength, and otherwise it is a quadratic function.

Key words: Ideal fluid flows on a torus; Weierstrass ζ-function; Energy of a flow

1. Introduction

The generalized uniformization theorem — or the fundamental theorem in the theory of conformal mapping — proved by Koebe and Courant states that any plane domain can be mapped onto a so-called minimal horizontal slit domain. To be more precise, for any plane domain G and any point z_0 on G there exists a meromorphic function f such that

(i) f is univalent on G,
(ii) f has a single simple pole at z_0, and
(iii) $\hat{\mathbb{C}} \setminus f(G)$ consists of horizontal segments whose total area vanishes.

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Roughly speaking, the last condition states that $\operatorname{Im} f$ assumes a constant value on each ideal boundary component of G. In terms of physics — in particular those of hydrodynamics — the function f describes a dipole flow of an ideal fluid on G with impenetrable boundary.

The above theorem is indeed a generalization of the celebrated mapping theorem of Riemann; we can use the Joukowski transformation

$$J(z) := z + \frac{1}{z}$$

to see the relation of the Riemann mapping theorem and the theorem of Koebe-Courant.

In hydrodynamics we have a notion of "Rankine ovoids". A Rankine ovoid comes from a superposition of a uniform flow and a flow with a pair of sink and source of the same strength. Rankine ovoids are thus phenomena linearly produced, but they are connected with a nonlinear problem in the theory of conformal mapping. In fact, the quantities with which we are involved below — such as the coefficients of the resulting conformal mapping function and the energy of the flow — do not depend on the summands linearly. The dependence of the first Taylor coefficient of the regular part upon the places and the strength of the sink and source is studied in [4].

In the present article we discuss similar problems for the case of genus one. To this end we recall that the Joukowski transformation for the plane is a special case of Rankine ovoids where the sink and the source coincide, that is, it is the sum of the uniform flow z and a dipole flow $1/z$. Now, let T be a torus. We move to the universal covering surface $\mathbb{C}$ of T to construct a flow on T. The functions to be considered are thus of the form

$$F_\mu(z) := z + \mu\zeta(z), \qquad \mu > 0,$$

where ζ denotes the Weierstrass ζ function. The function F_μ with an appropriate μ defines a potential flow on T, which will be called a "dipole Rankine flow" on T. Each dipole Rankine flow on T splits into two flows as in the plane case, one flow is limited in a neighborhood B_μ of the dipole and the other is a flow outside B_μ. The domain B_μ is bounded by the streamlines passing through the stagnation points. We call B_μ a "dipole Rankine ovoid" on T.

From the function-theoretic viewpoint, F_μ is an analytic mapping of a noncompact torus $T \setminus \bar{B}_\mu$ onto another. This is based on the fact that the function F_μ has simultaneously automorphic and polymorphic properties. Among various mapping properties of F_μ we are particularly interested in the following problems: What kind of Riemann surface is the conformal image of the function

F_μ? Can B_μ be — as in the plane case — a disk?

The topics with which we are concerned here bring some new problems whose prototypes cannot be found in the planar case (see [4]), for in nonplanar cases we cannot do without the notion of moduli. We remark that a quite different algebraic method from the one employed in this paper can be found in [3].

2. Preliminaries

To save space we skip over the general theory (see e.g., [7], [11]) of conformal embedding of a noncompact Riemann surface of finite genus into compact Riemann surfaces of the same genus. Instead, we start with a concrete realization of a noncompact torus. (For general results for the case of genus one, see [8], [9].)

Let ω_1 and ω_3 be a pair of complex numbers with $\operatorname{Im}(\omega_3/\omega_1) > 0$ and consider the lattice

$$L[2\omega_1, 2\omega_3] := \{\omega \in \mathbb{C} \mid \omega = 2m\omega_1 + 2n\omega_3, \quad m, n \in \mathbb{Z}\}$$

in the z-plane and the torus

$$T_0 := \mathbb{C}/L[2\omega_1, 2\omega_3].$$

Let

$$\wp(z) = \wp(z; \omega_1, \omega_3) := \frac{1}{z^2} + \sum{}' \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

be the Weierstrass $\wp$-function with the fundamental periods $2\omega_1$ and $2\omega_3$, where

$$\sum{}' = \sum_{\omega \in L \setminus \{0\}}$$

stands for the sum for all nonzero $\omega \in L[2\omega_1, 2\omega_3]$.

Now, let

$$\zeta(z) := \frac{1}{z} + \sum{}' \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}$$

be the Weierstrass ζ-function. We know that

$$\zeta(z) = \frac{1}{z} - \int_0^z \left\{\wp(z) - \frac{1}{z^2}\right\} dz \quad \text{and} \quad \wp(z) = -\zeta'(z).$$

As is well known, the complex numbers

$$\eta_k := \zeta(\omega_k)$$

satisfy the functional identities

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k$$

$(k = 1, 2, 3)$, where $\omega_2 := -(\omega_1 + \omega_3)$ as routine.

The results so far are all classical and thus the readers can be refered to any textbooks on the theory of functions. For the details, see [1] or [5] for example.

3. Dipole Rankine flows

In the following we can and often do normalize the periods so that

$$2\omega_1 = 1 \text{ and } 2\omega_3 = \tau_0.$$

However, it is also convenient to continue using the general notation $2\omega_1$ and $2\omega_3$ to see more clearly what we do.

Fix a positive real number μ with

$$0 < \mu < \frac{1}{\wp(\omega_1)}$$

and consider the function

$$F_\mu(z) := z + \mu\zeta(z), \qquad z \in \mathbb{C}.$$

The function F_μ defines a potential flow on $T_0 := \mathbb{C}/L[1, \tau_0]$, since

$$F'_\mu(z) = 1 + \mu\zeta'(z) = 1 - \mu\wp(z)$$

is a singlevalued meromorphic function, that is, a complex velocity function, on T_0. The flow will be called in this paper a *dipole Rankine flow* on T_0. For further information on doubly periodic flows, see, for example, [2] and [13] (esp. p. 263, Prob. 1427).

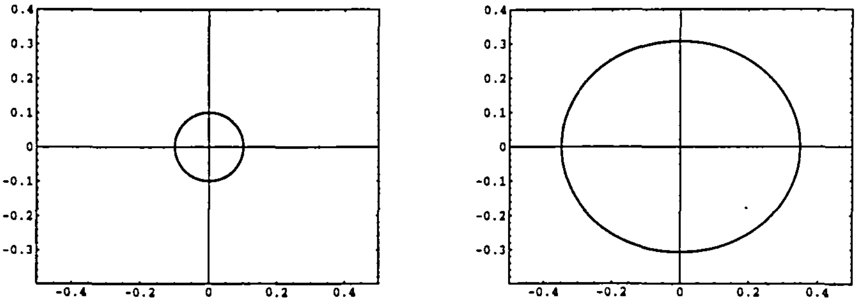

Fig. 1. Dipole Rankine ovoids for $\omega_1 = 0.5$ and $\omega_3 = 0.4i$. The values of μ are 0.01 (left) and 0.1 (right).

The flow has two stagnation points on T_0, which we denote by $\pm z_0$. They satisfy the equation

$$0 = F'_\mu(z_0) = 1 - \mu\wp(z_0).$$

We may assume that $z_0 > 0$. Because of the assumption on μ we see that the stream lines

$$L_\mu : \operatorname{Im}(z + \mu\zeta(z)) = 0$$

passing through the stagnation points bound a domain B_μ on T_0 containing the dipole. We call the domain B_μ a *dipole Rankine ovoid* on the torus T_0. As for illustrations, see Figure 1.

To observe F_μ more function-theoretically, we first confine ourselves to the upper half fundamental rectangle

$$Q^+ := \{z \in \mathbb{C} \mid |\operatorname{Re} z| < 1/2(=\omega_1),\ 0 < \operatorname{Im} z < \operatorname{Im} \tau_0/2(= \operatorname{Im} \omega_3)\}$$

(cf. [13], *loc. cit.*).

We define the lower half fundamental rectangle Q^- similarly. The flow on T_0 can then be obtained by symmetrization of Q^+ in the real axis.

The dipole Rankine ovoid B_μ on T_0 is now realized as the union of two domains

$$B_\mu^\pm := \{z \in Q^\pm \mid \operatorname{Im} F_\mu(z) \lesseqgtr 0\}$$

and the open interval $(-z_0, z_0)$ on the real axis.

The function F_μ is auto- and polymorphic; it satisfies

$$F_\mu(z + 2\omega_k) = (z + 2\omega_k) + \mu\zeta(z + 2\omega_k) = F_\mu(z) + (2\omega_k + \mu \cdot 2\eta_k)$$

for each $k = 1, 2, 3$. This means that F_μ gives a holomorphic bijection of $T_0 \setminus \bar{B}_\mu$ onto a *horizontal slit torus*

$$R_0(\mu) := T_0(\mu) \setminus \Sigma_0(\mu),$$

where $T_0(\mu)$ is a torus and $\Sigma_0(\mu)$ is a straight horizontal line segment on $T_0(\mu)$. Here, the "straightness" is understood in terms of the metric of *some* (equivalently, *any*) holomorphic differential on $T_0(\mu)$, and the "horizontal" means that the slit can be realized on the plane as an interval on the real axis.

By the choice of μ and by the well known properties of the functions ζ and $\wp$, we see that

$$F_\mu(\omega_1) > 0 \quad \text{or} \quad \omega_1 + \mu\eta_1 > 0.$$

Taking this fact into account, we know that the modulus $\tau_0(\mu)$ of $T_0(\mu)$ is — under our normalization $2\omega_1 = 1$ and $2\omega_3 = \tau_0$ — given by

$$\frac{\omega_3 + \mu\eta_3}{\omega_1 + \mu\eta_1} = \frac{\tau_0 + \mu \cdot 2\zeta(\tau_0/2)}{1 + \mu \cdot 2\zeta(1/2)}.$$

The length $\ell_0(\mu)$ of the slit $\Sigma_0(\mu)$ is

$$\frac{2|F_\mu(z_0)|}{1 + \mu \cdot 2\zeta(1/2)} = \frac{2|z_0 + \mu\zeta(z_0)|}{1 + \mu \cdot 2\zeta(1/2)},$$

where the length is measured in comparison with the geodesic length $2\omega_1 = 1$. The energy of the dipole Rankine flow on $T_0 \setminus \bar{B}_\mu$ is given by

$$\mathcal{E}(\mu) := 4 \cdot \frac{1}{i} \left\{ \frac{\tau_0}{2} + \mu\zeta\left(\frac{\tau_0}{2}\right) \right\} \cdot \left\{ \frac{1}{2} + \mu\zeta\left(\frac{1}{2}\right) \right\},$$

while the energy of the flow inside B_μ is always infinite.

We shall later ask if the dipole Rankine ovoid can be a disk. We also want to know how the energy of the dipole Rankin flows depend on the strength μ. Before discussing these problems we see in the next section what kind of roles the problems play in the theory of conformal mapping.

4. The span of a slit torus

The slit torus $R_0(\mu)$ obtained in the previous section can be mapped onto another slit torus, a *vertical slit torus*,

$$R_1(\mu) := T_1(\mu) \setminus \Sigma_1(\mu),$$

where $T_1(\mu)$ is a torus and $\Sigma_1(\mu)$ is a straight vertical line segment on $T_1(\mu)$.

The modulus $\tau_1(\mu)$ of the torus $T_1(\mu)$ and the length of the slit $\Sigma_1(\mu)$ can be computed explicitly in terms of elliptic functions. The length is again measured compared with the geodesic length. See [12].

The quantity $\sigma := \mathrm{Im}\,[\tau_1(\mu) - \tau_0(\mu)]$ is known as the span of the noncompact torus $T_0 \setminus \bar{B}_\mu$. For the basic properties of the span, see [9]. Cf. also [12] and [6].

To recall the importance of the span in the theory of conformal mapping we quote here only the following three theorems([8], [9]):

Theorem 1. *The horizontal slit torus $R_0(\mu)$ (resp. the vertical slit torus $R_1(\mu)$) is minimal (resp. maximal) in the sense that its modulus has the minimum (resp. maximum) imaginary part among all the (compact) tori into which the noncompact torus $T_0 \setminus \bar{B}_\mu$ is conformally embedded.*

Theorem 2. *The disk M whose diameter σ is the closed interval* $[\mathrm{Im}\,\tau_0(\mu), \mathrm{Im}\,\tau_1(\mu)]$ *on the imaginary axis describes the totality of all the possible conformal embeddings of the noncompact torus $T_0 \setminus \bar{B}_\mu$.*

Theorem 3. *The maximum of the omitted area among all the possible conformal embeddings of $T_0 \setminus \bar{B}_\mu$ into tori — measured by the normalized holomorphic differential — is attained at the euclidean center of the disk M and the maximum is equal to $\sigma/4$.*

In the above three theorems, any conformal mapping is supposed to preserve the cycles corresponding to the periods $2\omega_1$ and $2\omega_3$ respectively. As a matter of fact we should have considered an abstract noncompact Riemann surface R which is conformally equivalent to $T_0 \setminus \bar{B}_\mu$. But we could actually choose from the outset a noncompact torus which is obtained as the complement of a dipole Rankine ovoid on a torus as one of such surfaces. Also we should have carefully observed the correspondence of the homology basis of tori. In our present case, however, the correspondence is rather obvious, so that we have not been concerned with the complicated definitions. See, for the details, [7], [8] and [9].

On the other hand, in the following theorem ([9]) we do not need to prescribe any correspondence between homology groups, and hence the theorem is more intrinsic.

Theorem 4. *The ratio of the omitted area to the total area is maximized at the hyperbolic center of the disk M and the maximum is equal to $\sigma_H/2$, where*

$$\sigma_H := \log \frac{\operatorname{Im} \tau_1}{\operatorname{Im} \tau_0} = \log \frac{\tau_1}{\tau_0}$$

stands for the hyperbolic diamter of the (hyperbolic) disk M.

To explain the results hydrodynamically, we need a new definition. Let R be a noncompact torus. An irrotational and solenoidal (two-dimensional) flow on R will be called *uniformly extendable*, if there is another torus T containing R as a subsurface such that the flow is the restriction of a uniform flow on T.

The following theorem follows at once.

Theorem 5. *Any dipole Rankine flow is uniformly extendable, if it is restricted to the outside of the dipole Rankine ovoid.*

Now we ask: How large can the energy of a uniformly extendable flow be? To eliminate the influence of a multiplicative factor, we consider the ratio

(Energy of the flow on R)/(Energy of the flow on T),

or, equivalently,

(Energy of the flow on $T \setminus R$)/(Energy of the flow on T).

For this question we have, for example,

Theorem 6. *Let E_0 be the energy of a uniformly extendable flow on a noncompact torus R. Let E_1 be the energy of the flow outside R (that is, on $T \setminus R$, where T is as above). Then, E_1/E_0 is maximized by a torus whose modulus is located at the hyperbolic center of M.*

5. Energy of a dipole Rankine flow

The Weierstrass ζ-function is a kind of generalization of a dipole flow on the plane. However, the function F_μ is not so simple as the Joukowski transformation on the plane, as the following example shows.

Example. *A dipole Rankine ovoid on a torus which is not a disk. Here, a set on a torus is said to be a disk, if its lift to the universal covering surface $\mathbb{C}$ is a euclidean disk.*

This can be computationally verified. Figure 1 (with the aid of Mathematica)

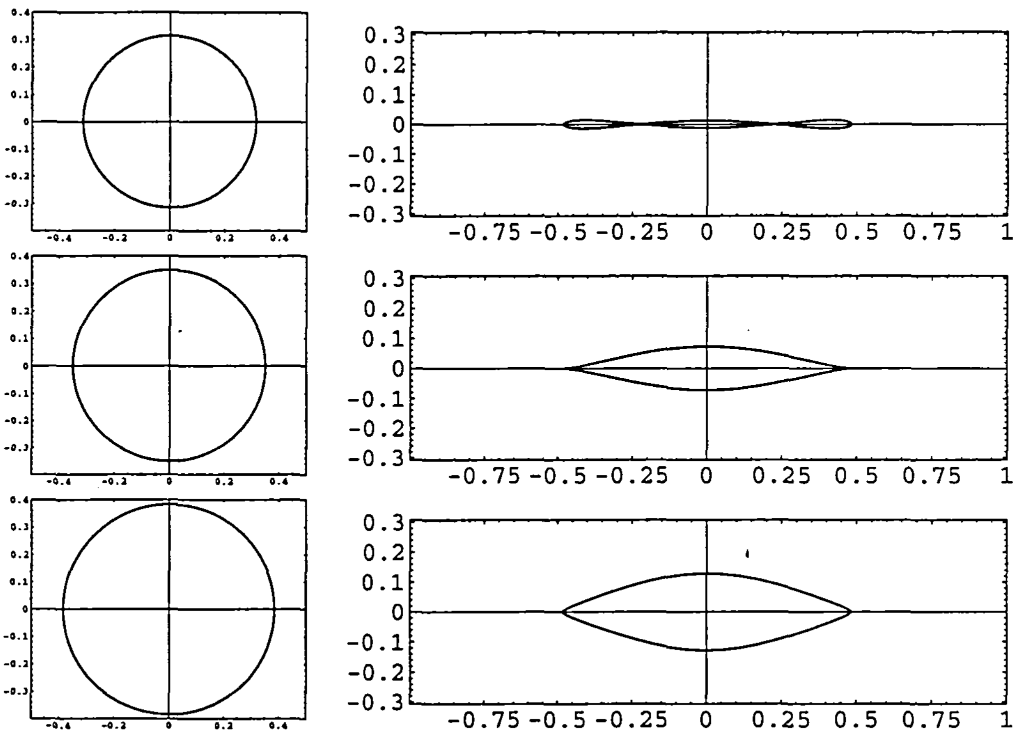

Fig. 2. The images of the concentric circles about the dipole. The second circle passes through the stagnation points.

show that the dipole Rankine ovoid for a small μ looks like a disk, while this is not the case for a large μ. We can more efficiently see this fact by considering the circle C whose diamter is determined by the stagnation points. The image of C, together with some other concentric circles about the dipole, are shown in Figure 2, by which we can see the behavior of the mapping function F_μ in a neighborhood of C. For the area problem of the Rankine ovoid we can use the results in §4 to verify the following property computationally:

The area of a dipole Rankine ovoid is an increasing function of the strength μ of the dipole. However, the area function is not linear in μ, as the graph (Figure 3) shows.

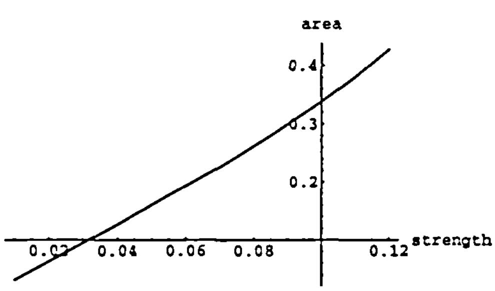

Fig. 3. The nonlinearity of the area of a dipole Rankine ovoid.

We come back again to the situation in §3. That is, we start with a fixed torus T_0 and consider dipole Rankine flows on T_0 with parameter μ (strength). We recall

$$\mathcal{E}(\mu) = \frac{4}{i} \cdot \left\{ \frac{\tau_0}{2} + \mu\zeta\left(\frac{\tau_0}{2}\right) \right\} \cdot \left\{ \frac{1}{2} + \mu\zeta\left(\frac{1}{2}\right) \right\} = \frac{4}{i} \cdot \left(\frac{\tau_0}{2} + \mu\eta_3 \right) \cdot \left(\frac{1}{2} + \mu\eta_1 \right)$$

is the energy of the dipole Rankine flow on $T_0 \setminus \bar{B}_\mu$.

In general the function $\mathcal{E}$ is quadratic in μ. In the exceptional cases where

$$\eta_1 = 0 \quad \text{or} \quad \eta_3 = 0,$$

we have by the Legendre relation

$$\eta_1\omega_3 - \eta_3\omega_1 = \frac{\pi}{2}i$$

that

$$\eta_3 = -\pi i \quad \text{or} \quad \eta_1 = \frac{\pi i}{\tau_0}$$

respectively, so that

$$\mathcal{E}(\mu) = \begin{cases} -2\pi\mu + \operatorname{Im}\hat{\tau}_0 & \text{if } \eta_1 = 0, \\ \\ 2\pi\mu + \operatorname{Im}\check{\tau}_0 & \text{if } \eta_3 = 0, \end{cases}$$

where $\hat{\tau}_0$ and $\check{\tau}_0$ are the complex numbers such that

$$\zeta\left(\frac{1}{2}; \frac{1}{2}, \frac{\hat{\tau}_0}{2}\right) = 0, \quad \zeta\left(\frac{\check{\tau}_0}{2}; \frac{1}{2}, \frac{\check{\tau}_0}{2}\right) = 0$$

respectively.

Hence we have

Theorem 7. *The energy of a dipole Rankine flow is a quadratic function of the strength μ of the dipole, unless $\eta_1\eta_3 = 0$. In the exceptional cases the energy is a linear function of μ.*

The moduli $\tau_0(\mu)$ of $T_0(\mu)$ can be similarly considered. Since

$$\tau_0(\mu) = \frac{\omega_3 + \mu\zeta(\omega_3)}{\omega_1 + \mu\zeta(\omega_1)} = \frac{\omega_3 + \mu\eta_3}{\omega_1 + \mu\eta_1},$$

we have, again by virtue of Legendre's relation,

$$\tau_0'(\mu) = \frac{\eta_3\omega_1 - \eta_1\omega_3}{(\omega_1 + \mu\eta_1)^2} = -\frac{\pi i}{2(\omega_1 + \mu\eta_1)^2},$$

so that we have

Theorem 8. $\operatorname{Im}\tau_0(\mu)$ *is a decreasing function of* μ.

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