Applications of Darboux transformations to the self-dual Yang-Mills equations

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1 Introduction

Since the late 1970's there have been a great many articles written on methods of finding solutions to the self-dual Yang-Mills (SDYM) equations (see for example [1–6]). The main aim of this article is not to find solutions that are necessarily new but rather to find closed form expressions for a large class of solutions written in terms of determinants. Also, we aim to achieve this by employing a standard technique used in soliton theory and elsewhere; the application of Darboux transformations. In doing this we give a simple way of understanding the form of the solutions of these equations to readers familiar with Darboux transformations or Hirota's method but not with the more sophisticated methods that have been used in the literature. The original motivation of this work was to find a generalisation to the general case of the work of Sasa et al [7] on the Hirota form of SU(2) SDYM. In independent work, Darboux transformations have been used recently to study a reduced version of the SDYM equations [8].

This article is set out as follows. In §2 we describe the standard formulation of the SDYM equations as the compatibility of a Lax pair and show the equivalence of this system to Yang's equation, a single equation written in terms of an $N \times N$ matrix J. We then observe that the Lax pair lies outside the class of operators described in the standard theorem on Darboux transformations [9].

In §3 we recall this theorem and describe the dimensional reduction that is appropriate to tackling the SDYM Lax pair. In this reduction the Darboux transformation changes from a differential operator to become a multiplicative operator, linear in the spectral parameter. This discussion confirms the inapplicability of the theorem in its original form. By examining part of the proof of the reduced version of the theorem we then obtain a generalisation which can be used in the present case. In §4 we describe the binary Darboux transformation and its form in the reduced case.

Specialisation of the result of §3 and §4 to the SDYM are given in §5 and we obtain formulæ for solutions J in either of two forms. First, using the Darboux transformations expressions for the entries of J as ratios of wronskian-like determinants are obtained. We term these determinants 'wronskian-like' because in the unreduced case of the Darboux transformation they are precisely wronskians. In the reduction the derivatives in these determinants are replaces by multiplication by a spectral parameter but their structure is otherwise unchanged. In a similar way, the binary Darboux transformation gives expressions for these entries as ratios of grammian-like determinants.

Finally, we discuss the case $J \in \mathrm{SU}(N)$ in which the Lax pair is self-adjoint. Here we show that the binary Darboux transformation may be chosen to preserve this property of J in a natural way.

2 The self-dual Yang-Mills equations

We begin by describing Yang's form of these equations and its formulation as the compatibility condition of a Lax pair. Consider four $N \times N$ matrices $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ which are each functions of four complex variables x^0, x^1, x^2 and x^3. Let $\partial_a := \partial/\partial x^a$ for $a = 0,1,2,3$ and define the covariant derivative operators

$$D_a = \partial_a + \mathcal{A}_a. \tag{2.1}$$

The self-dual Yang-Mills (SDYM) equations are the compatibility conditions for the Lax pair

$$L := D_0 + \lambda D_1, \;\; M := D_2 + \lambda D_3, \tag{2.2}$$

namely $[L, M] = 0$ or, more explicitly,

$$[D_0, D_2] = 0, \quad [D_1, D_3] = 0, \tag{2.3}$$

$$[D_0, D_3] + [D_1, D_2] = 0. \tag{2.4}$$

The equations (2.3) imply that the pairs of operators D_0, D_2 and D_1, D_3 are compatible so that there exist common solutions, invertible $N \times N$ matrices h and k such that

$$D_0(h) = D_2(h) = 0$$
$$D_1(k) = D_3(k) = 0.$$

By solving these equation for $\mathcal{A}_a$, each $\mathcal{A}_a$ may be expressed in terms of h or k. Then

$$D_0 = h\partial_0 h^{-1}, \quad D_2 = h\partial_2 h^{-1}, \tag{2.5}$$

$$D_1 = k\partial_1 k^{-1}, \quad D_3 = k\partial_3 k^{-1}. \tag{2.6}$$

Note that here and elsewhere we use parentheses to indicate that an operator has acted on an argument and juxtaposition of operators to indicate an operator product. For example, in the above, $D_0(h)$ is the matrix

$$\frac{\partial h}{\partial x^0} + \mathcal{A}_0 h$$

whereas $h\partial_0 h^{-1}$ is the operator equal to

$$\frac{\partial}{\partial x^0} - \frac{\partial h}{\partial x^0}h^{-1}.$$

Furthermore, the remaining part of the SDYM equations, equation (2.4), may be expressed in terms of the matrix $J = k^{-1}h$ alone;

$$\partial_3(\partial_0(J)J^{-1}) + \partial_2(\partial_1(J)J^{-1}) = 0. \tag{2.7}$$

This is known as Yang's equation and is equivalent to the SDYM equations.

After the gauge transformation $L \to k^{-1}Lk$, $M \to k^{-1}Mk$, the Lax pair (2.2) of the SDYM equations is also expressed in terms of J, and we obtain a simpler Lax pair for Yang's equation

$$L = J\partial_0 J^{-1} + \lambda\partial_1, \quad M = J\partial_2 J^{-1} + \lambda\partial_3. \tag{2.8}$$

From the above we see that in each of the formulations of the SDYM equations we have described, the Lax pairs comprise operators of the form

$$L := \sum_{i=1}^{M} a_i \lambda^i \tag{2.9}$$

where a_i are first order linear operators and, in this case, $M = 1$. In the next section it will be seen that such an operator lies outside the class to which Darboux transformations are usually applied. As a result, an extension of the normal theory needs to be obtained in order to study the SDYM equations.

3 The standard Darboux transformation

In a paper on a Darboux transformation for the time-dependent Schrödinger operator [9], it was indicated that the standard Darboux transformation which applies to, for example, the Schrödinger equation, may be used for a rather general class of operators. It does not however apply to the operators in the Lax pair of the SDYM equations. In this section we will examine the proof of this general result and investigate how it may be extended to deal with the case we wish to consider.

Theorem 1. *Let*

$$L = \sum_{i=0}^{M} a_i \partial^i \tag{3.1}$$

where $\partial = \partial/\partial x$ *and* a_i *are operators independent of* ∂*, let* θ *be an invertible matrix such that* $L(\theta) = \theta A$ *where* $\partial(A) = 0$ *and* $G_\theta := \theta\partial\theta^{-1}$*. Then* L *is form invariant under the Darboux transformation*

$$L \to \widetilde{L} = G_\theta L G_\theta^{-1}$$

if and only if

$$a_0 = \alpha\partial_y + m_0$$
$$a_i = m_i, \quad i > 0$$

where α *is scalar and constant* ($\partial(\alpha) = 0$)*,* m_i *are matrices and* ∂_y *denotes a differential operator independent of* ∂*.*

As usual with Darboux transformations, the theorem implies that given any solution ϕ of $L(\phi) = 0$, $\widetilde{\phi} := G(\phi)$ is a solution of $\widetilde{L}(\widetilde{\phi}) = 0$ and the coefficients $\widetilde{a}_i$ in $\widetilde{L}$ are given in terms of the coefficients a_j in L and θ.

There is a natural dimensional reduction of this theorem obtained by making the x-dependence explicit in the solutions. Let $\phi = \phi^r e^{\lambda x}$, λ a constant scalar and $\theta = \theta^r e^{\Lambda x}$, Λ a constant matrix, where ϕ^r and θ^r are independent of x. In the following we will omit the superscript ^{r}. If Λ is a diagonal matrix then the entries are eigenvalues and the corresponding columns eigenfunctions, but there is in general no requirement that Λ is restricted to these circumstances.

Corollary 1. *Let* $L = \sum_{i=0}^{M} a_i\lambda^i$ *where* a_i *are operators independent of* λ*, let* θ *be an invertible matrix such that* $\sum_{i=0}^{M} a_i(\theta)\Lambda^i = 0$ *and* $G_\theta := \theta(\lambda I - \Lambda)\theta^{-1} = \lambda - \Theta$*, where* $\Theta = \theta\Lambda\theta^{-1}$*. Then* L *is form invariant under the Darboux transformation*

$$L \to \widetilde{L} = G_\theta L G_\theta^{-1} \tag{3.2}$$

if

$$a_0 = \alpha\partial_y + m_0 \tag{3.3}$$
$$a_i = m_i, \quad i > 0 \tag{3.4}$$

where α *is scalar and constant* ($\partial(\alpha) = 0$) *and* m_i *are matrices.*

While this does not on the face of it include operators of the form (2.9), a generalisation of this Corollary may be found which does. To see how this comes about it is instructive to consider part of the proof of the Theorem.

Sketch proof of Theorem 1.

The transformed operator

$$\widetilde{L} = \theta\partial\theta^{-1}\sum_{i=0}^{M} a_i\partial^i\theta\partial^{-1}\theta^{-1}.$$

has a pseudo-differential tail unless the ∂-independent part of

$$R := \theta\partial\theta^{-1}\sum_{i=0}^{M} a_i\partial^i\theta$$

is zero.

This can come about in one of two ways;

1. $a_i = m_i$ The multiplicative part of R is then

$$\theta\partial\left(\theta^{-1}\sum_{i=0}^{M} m_i\partial^i(\theta)\right) = \theta\partial(\theta^{-1}L(\theta))$$

which vanishes since $L(\theta) = \theta A$.

2. $a_i = \alpha_i\partial_i$ where $\partial_i := \partial/\partial x^i$ are differential operators independent of ∂ and α_i are scalar constants. In this case

$$\begin{aligned} R &= \theta\partial\theta^{-1}\sum_{i=0}^{M}\alpha_i\big(\partial_i(\partial^i(\theta) + \partial^i(\theta)\partial_i\big) + O(\partial) \\ &= \theta\partial(\theta^{-1}L(\theta)) + \sum_{i=0}^{M}\alpha_i\theta\partial\theta^{-1}\partial^i(\theta)\partial_i + O(\partial). \end{aligned}$$

This first term on the right hand side vanishes as in case 1. and so R is $O(\partial)$ if and only if, for each i, $\alpha_i = 0$ or ∂ commutes with $\theta^{-1}\partial^i(\theta)$. Thus it is seen that, without imposing further constraints on θ, only α_0 may be non-zero.

When the same technique of proof is applied to the Corollary, the dimensional reduction replaces $\partial^i(\theta)$ with $\theta\Lambda^i$ and ∂ with λ. Clearly in the second case, $\partial\theta^{-1}\partial^i(\theta) \to \lambda\theta^{-1}\theta\Lambda^i = \Lambda^i\lambda$ and so we make take $\alpha_i \neq 0$ for all i, not just if $i = 0$. Thus the Corollary has the generalisation

Theorem 2. *Let $L = \sum_{i=0}^{M} a_i\lambda^i$ where a_i are operators independent of λ, let θ be an invertible matrix such that $\sum_{i=0}^{M} a_i(\theta)\Lambda^i = 0$ and $G_\theta := \theta(\lambda I - \Lambda)\theta^{-1} = \lambda - \Theta$, where $\Theta = \theta\Lambda\theta^{-1}$. Then L is form invariant under the Darboux transformation*

$$L \to \widetilde{L} = G_\theta L G_\theta^{-1}$$

if and only if

$$a_i = \alpha_i\partial_i + m_i, \quad i \geq 0$$

where α_i are scalar and constant and m_i are matrices.

4 The standard binary Darboux transformation

As well as the Darboux transformation there is a standard binary Darboux transformation constructed, in one form, by composing a Darboux transformation with inverse of another. We use S to denote the set of eigenfunctions for L and similarly $\widetilde{S}$ for $\widetilde{L}$ etc. With L and G_θ as given in Theorem 1, if

$$\widetilde{L} := G_\theta L G_\theta^{-1}$$

then $G_\theta \colon S \to \widetilde{S}$. Taking formal operator adjoints gives

$$\widetilde{L}^\dagger = G_{\theta^{-\dagger}}^{-1} L^\dagger G_{\theta^{-\dagger}} \tag{4.1}$$

where the notation $(\cdot)^{-\dagger}$ is shorthand for $\left((\cdot)^\dagger\right)^{-1}$. Thus $G_{\theta^{-\dagger}} \colon \widetilde{S}^\dagger \to S^\dagger$. Furthermore, since for any ρ, $G_\rho(\rho) = 0$, it follows from (4.1) that

$$\widetilde{L}^\dagger(\theta^{-\dagger}) = 0. \tag{4.2}$$

Let $\widehat{L}$ be a third copy of L, with new coefficients. Suppose that $\widehat{\theta} \in \widehat{S}$ and $G_{\widehat{\theta}} \widehat{L} G_{\widehat{\theta}}^{-1} = \widetilde{L}$ (i.e. the same as $G_\theta L G_\theta^{-1}$.) From (4.2) one has $\theta^{-\dagger}, \widehat{\theta}^{-\dagger} \in \widetilde{S}^\dagger$. Given $\rho \in S^\dagger$, $\widehat{\theta}$ may be defined up to a constant through the expression $G_{\theta^{-\dagger}}(\widehat{\theta}^{-\dagger}) = \rho$. Specifically,

$$\widehat{\theta} = \theta \Omega(\theta, \rho)^{-1} \tag{4.3}$$

where Ω is the potential defined by

$$\partial(\Omega(\phi, \psi)) = \psi^\dagger \phi. \tag{4.4}$$

One may then construct a *binary Darboux transformation* $B_{\theta,\rho} = G_{\widehat{\theta}}^{-1} \circ G_\theta$ which can be written explicitly as

$$B_{\theta,\rho} = I - \theta \Omega(\theta, \rho)^{-1} \Omega(\cdot, \rho). \tag{4.5}$$

By writing the formula for $B_{\theta,\rho}$ in this way one sees that the transformation makes sense for any $N \times n$ matrices θ, ρ such that $L(\theta) = L^\dagger(\rho) = 0$ and not just $N \times N$ ones. Indeed one may prove that, for any operator L of the form given in Theorem 1,

$$\widehat{L} = B_{\theta,\rho} L B_{\theta,\rho}^{-1} \tag{4.6}$$

is an operator of the same form as L.

As for the dimensionally reduced Darboux transformation described in Theorem 2 again we take $\phi = \phi^r e^{\lambda x}$, λ a constant scalar and $\theta = \theta^r e^{\Lambda x}$, Λ a constant $n \times n$ matrix and the adjoint eigenfunction is $\rho = \rho^r e^{\Xi x}$ where Ξ is another constant $n \times n$ matrix. From this it follows that the x dependence of the potential Ω can also be made explicit

$$\Omega(\theta,\rho) = e^{\Xi^{\dagger}x}\Omega^{r}(\theta^{r},\rho^{r})e^{\Lambda x} \quad \text{where} \quad \Xi^{\dagger}\Omega^{r}(\theta^{r},\rho^{r}) + \Omega^{r}(\theta^{r},\rho^{r})\Lambda = \rho^{r\dagger}\theta^{r} \tag{4.7}$$

and

$$\Omega(\phi,\rho) = e^{(\Xi^{\dagger}+\lambda I)x}\Omega^{r}(\phi^{r},\rho^{r}) \quad \text{where} \quad (\Xi^{\dagger}+\lambda I)\Omega^{r}(\phi^{r},\rho^{r}) = \rho^{r\dagger}\phi^{r}. \tag{4.8}$$

Then the dimensionally reduced binary Darboux transformation is given by (4.5) with all terms replaces with the reduced versions described above. From now on, for notational simplicity, we will omit the superscript ^{r} denoting reduced objects and only discuss the Darboux transformation in this case.

So now, in the reduced case Ω is an *algebraic* potential satisfying the condition

$$\Xi^{\dagger}\Omega(\theta,\rho) + \Omega(\theta,\rho)\Lambda = \rho^{\dagger}\theta. \tag{4.9}$$

For example, if we were to choose the constant matrices to be diagonal, $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ and $\Xi = \mathrm{diag}(\xi_1,\ldots,\xi_n)$, then we could obtain the explicit expressions

$$\Omega(\theta,\rho) = \left(\frac{(\rho^{\dagger}\theta)_{ij}}{\lambda_j + \bar{\xi}_i}\right) \quad \text{and} \quad \Omega(\phi,\rho) = \left(\frac{(\rho^{\dagger}\phi)_{ij}}{\lambda + \bar{\xi}_i}\right). \tag{4.10}$$

5 Explicit version for the case of Yang's equation

For Yang's formulation of the SDYM equations the Lax pair is given by (2.8), and the action of the Darboux transformation defined in Theorem 2 is encapsulated in a transformation of the matrix J.

Theorem 3. *Let $G = \lambda - \Theta$ where $\Theta := \theta\Lambda\theta^{-1}$, Λ is a constant matrix and θ an invertible matrix satisfying*

$$\begin{aligned} J\partial_0(J^{-1}\theta) + \partial_1(\theta)\Lambda = 0, \\ J\partial_2(J^{-1}\theta) + \partial_3(\theta)\Lambda = 0. \end{aligned} \tag{5.1}$$

The operators L, M given by (2.8)*, the Lax pair of Yang's equation, are form invariant under the Darboux transformation $L \to GLG^{-1}$, $M \to GMG^{-1}$ and this transformation is expressed entirely as the transformation $J \to \Theta J$.*

Note that the matrix Θ arises as the dimensional reduction of $\partial(\theta)\theta^{-1}$.

It is also possible to write down compact explicit formulae for solutions J_n obtained after n iterations of this Darboux transformation.

Corollary 2. *For each $k = 1, \ldots, n$ take a constant matrix Λ_k and a non-singular solution (matrix) θ_k of* (5.1) *with $\Lambda = \Lambda_k$. From these $N \times N$ matrices, the $N \times nN$ matrix $\theta = (\theta_1, \ldots, \theta_n)$ and $nN \times nN$ matrix $\Lambda =$* $\mathrm{diag}(\Lambda_1, \ldots, \Lambda_n)$ *are constructed. The following notation is also introduced; $\theta^{(k)} := \theta\Lambda^k$ (i.e. what is obtained in the dimensional reduction from the kth x-derivative of θ) and $\theta^{(k)}_{j\to i} :=$ ($\theta^{(k)}$ with the jth row replaced by the ith row of $\theta^{(k+1)}$). Then define the scalar F and the $N \times N$ matrix G*

$$F = \det \begin{pmatrix} \theta^{(0)} \\ \theta^{(1)} \\ \vdots \\ \theta^{(n-1)} \end{pmatrix} \quad \text{and} \quad G_{ij} = \det \begin{pmatrix} \theta^{(0)} \\ \theta^{(1)} \\ \vdots \\ \theta^{(n-2)} \\ \theta^{(n-1)}_{j\to i} \end{pmatrix}. \tag{5.2}$$

After n iterations of the Darboux transformation defined in Theorem 2 the solution of Yang's equation is

$$J_n = \frac{G}{F} J \tag{5.3}$$

Using Theorem 3, the binary Darboux transformation as constructed in §4 arises from the composition of two Darboux transformations, $J \to \widetilde{J}$ and $\widehat{J} \to \widetilde{J}$. Thus we have

$$\widetilde{J} = \Theta J \quad \text{and} \quad \widetilde{J} = \widehat{\Theta}\widehat{J} \tag{5.4}$$

where $\Theta = \theta\Lambda\theta^{-1}$ as in Theorem 3 and $\widehat{\Theta} = -\theta\Omega^{-1}\Xi^{\dagger}\Omega\theta^{-1}$. This expression for $\widehat{\Theta}$ is the dimensionally reduced version of $\partial(\widehat{\theta})\widehat{\theta}^{-1}$ with $\widehat{\theta} = \theta\Omega^{-1}$ as given in (4.3). Then, from (5.4), we get

$$\widehat{J} = -\theta\Omega^{-1}\Xi^{-\dagger}\Omega\Lambda\theta^{-1}J. \tag{5.5}$$

and using (4.9) this can be written as

$$\widehat{J} = (I - \theta\Omega^{-1}\Xi^{-\dagger}\rho^{\dagger})J. \tag{5.6}$$

This second formula for $\widehat{J}$ has the advantage that it is valid even when the inverse of θ is not defined and in particular is valid when θ and ρ are chosen to be $N \times n$ matrices where n is arbitrary rather than $N \times N$.

Using (5.5) we may determine the inverse of $\widehat{J}$ in an obvious way and then use (4.9) to obtain an expression valid in the more general case when θ and ρ are not invertible

$$\widehat{J}^{-1} = J^{-1}(I - \theta\Lambda^{-1}\Omega^{-1}\rho^{\dagger}). \tag{5.7}$$

Theorem 4. *Let Λ, Ξ be invertible $n \times n$ constant matrices and let θ satisfy* (5.1) *and ρ satisfy the adjoint equations*

$$\begin{aligned} J^{-\dagger}\partial_0(J^\dagger \rho) + \partial_1(\rho)\Xi = 0, \\ J^{-\dagger}\partial_2(J^\dagger \rho) + \partial_3(\rho)\Xi = 0. \end{aligned} \tag{5.8}$$

Then define B by (4.5) *where Ω is defined by* (4.9).

The operators L, M given by (2.8) *are form invariant under the binary Darboux transformation $L \to BLB^{-1}$, $M \to BMB^{-1}$ and this transformation is expressed entirely as*

$$J \to (I - \theta\Omega^{-1}\Xi^{-\dagger}\rho^\dagger)J = \frac{G}{F}J$$

where G is an $N \times N$ matrix and F a scalar given by

$$F = |\Omega(\theta, \rho)| \tag{5.9}$$

and

$$G_{ij} = \begin{vmatrix} \Omega(\theta,\rho) & (\rho\Xi^{-1})^\dagger_{.j} \\ \theta_{i.} & \delta_{ij} \end{vmatrix}. \tag{5.10}$$

Here we use the notation $(\cdots)_{i.}$ to denote the ith row and $(\cdots)_{.j}$ the jth column. In a similar way,

$$J^{-1} \to J^{-1}(I - \theta\Lambda^{-1}\Omega^{-1}\rho^\dagger) = J^{-1}\frac{H}{F}$$

where H is an $N \times N$ matrix given by

$$H_{ij} = \begin{vmatrix} \Omega(\theta,\rho) & \rho^\dagger_{.j} \\ (\theta\Lambda)_{i.} & \delta_{ij} \end{vmatrix}. \tag{5.11}$$

6 The SU(N) reduction

It is well known that in general a symmetry in the Lax pair is not preserved by a Darboux transformation but can be preserved by a binary one. This is possible because of the extra freedom allowed in the choice eigenfunction and and adjoint eigenfunction. This situation arises here if wish to construct solutions J in SU(N). Before showing how this done, we first comment that recently Sasa et al [7] were able to construct solutions similar to the wronskian like ones given in Corollary 2 but only for the case of SU(2). This was done by building the SU(2) symmetry into J from the start but it not clear how one could do this in any other case.

Let $J \in \mathrm{U}(N)$ so that $J^{-\dagger} = J$. Then Lax pair (2.8) is self-adjoint and hence, if we choose $\Xi = \Lambda$, equations (5.1) and (5.8) are identical and we may choose $\rho = \theta$. As a consequence of this, $\Omega(\theta, \rho)$ is a Hermitian matrix.

After applying the binary Darboux transformation with these choices we get a new solution given by (from Theorem 4)

$$\widehat{J} = (I - \theta\Omega^{-1}\Lambda^{-\dagger}\theta^{\dagger})J \quad \text{and} \quad \widehat{J}^{-1} = J^{-1}(I - \theta\Lambda^{-1}\Omega^{-1}\theta^{\dagger})$$

where the relation (4.9) now reads $\Lambda^{\dagger}\Omega + \Omega\Lambda = \theta^{\dagger}\theta$. It is clear from this that $\widehat{J}^{\dagger} = \widehat{J}^{-1}$ and so $J \in \mathrm{U}(N)$.

Since $\widehat{J} \in \mathrm{U}(N)$ if follows that $\det \widehat{J} = \pm 1$. To show that $\widehat{J} \in \mathrm{SU}(N)$ we have to prove that $\det J = 1 \implies \det \widehat{J} = 1$. In the general case that θ is an $N \times n$ matrix, it is not possible to show this except, maybe, for special choices of θ which are difficult to identify. However, if θ is an invertible $N \times N$ matrix, and hence the binary Darboux transformation can be considered as the composition of Darboux transformations, then we can use the formula (5.5) for $\widehat{J}$. From this we see that

$$\det \widehat{J} = (-1)^N \frac{\det \Lambda}{\overline{(\det \Lambda)}} \det J$$

so that choosing the $N \times N$ matrix Λ such that $\det \Lambda = (-1)^N \overline{(\det \Lambda)}$ realises the required property. In other words, $\widehat{J} \in \mathrm{SU}(N)$ whenever $J \in \mathrm{SU}(N)$ provided

$$\det \Lambda \in \mathbb{R} \quad \text{for } N \text{ even}, \quad \det \Lambda \in i\mathbb{R} \quad \text{for } N \text{ odd}. \tag{6.1}$$

This restricted form of binary Darboux transformation may also be iterated to obtain solutions as described in Theorem 4 with n a multiple of N.

Corollary 3. *For $i = 1, \ldots, m$, let Λ_i be invertible $N \times N$ constant matrices satisfying* (6.1) *and let $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_m)$. Take $J \in SU(N)$ so that* (5.1) *are self-adjoint, and choose a $N \times mN$ solution matrix θ. Set $\Xi = \Lambda$ and $\rho = \theta$ so that Ω is Hermitian.*

Subject to the above modifications, the solutions constructed in Theorem 4 are in $SU(N)$.

7 Conclusion

In this article we have show that after minor modifications, Darboux transformations can be applied to construct solutions of the self-dual Yang-Mills equations. These solutions take the form of the wronskian type or grammian type determinants. As well as arising through Darboux transformations, such determinants are familiar as ansätze used in Hirota's method such as the recent work of Sasa et al [7].

It is also shown that $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ solutions can be readily constructed by suitably specialised versions of the binary Darboux transformation.

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