Klein bottle surgery and genera of knots, II

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Abstract

We study Dehn surgery on knots creating Klein bottles, and give an upper bound for such slopes in terms of the genera of knots.

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1 Introduction

In this paper, which is a sequel to [5], we continue to study the creation of Klein bottles by surgery on knots in the 3-sphere S^3. Let K be a knot in S^3, and let $E(K)$ be its exterior. A *slope* on $\partial E(K)$ is the isotopy class of an essential simple closed curve in $\partial E(K)$. As usual, the slopes on $\partial E(K)$ are parameterized by $\mathbb{Q} \cup \{1/0\}$, where $1/0$ corresponds to a meridian slope (see [9]). For a slope r on $\partial E(K)$, $K(r)$ denotes the closed 3-manifold obtained by *r-Dehn surgery* on K. That is, $K(r) = E(K) \cup V$, where V is a solid torus glued to $E(K)$ along their boundaries in such a way that a curve with slope r bounds a meridian disk in V.

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In the previous paper, we gave an upper bound for the slopes creating Klein bottles for non-cable knots. The purpose of the present paper is to fill up the remaining case.

Theorem 1 *Let K be a non-trivial knot in S^3. If $K(r)$ contains a Klein bottle, then $|r| \leq 4g(K)+4$, where $g(K)$ denotes the genus of K. Furthermore, the equality holds only when K is either the $(2,m)$-torus knot and $r = 2m \pm 2$, where the sign corresponds to that of m, or the connected sum of the $(2,m)$-torus knot and the $(2,n)$-torus knot, and $r = 2m+2n$, where m,n $(\neq \pm 1)$ have the same sign.*

For the trivial knot, the slopes $r = 4n/(2n \pm 1)$ $(n \in \mathbb{Z})$ yield lens spaces $L(4n, 2n \pm 1)$ $(n \neq 0)$ or $S^2 \times S^1$ $(n = 0)$, and these are all slopes creating Klein bottles ([1]).

2 Proof of Theorem

Let K be a non-trivial knot. Let $r = m/n$ and suppose that $K(r)$ contains a Klein bottle. In the previous paper [5], we showed that if K is a non-cable knot, then $|r| \leq 4g(K)+4$ and the equality holds only when K is the connected sum of two torus knots as stated in Theorem 1. Thus we assume that K is cabled. The proof is divided into two cases.

Case 1. K is a torus knot.

Let K be the (p,q)-torus knot. We may assume that $0 < p < |q|$.

First, suppose $q > 0$. By [7], $K(r)$ is either:

(1) $L(p,q)\sharp L(q,p)$ if $r = pq$;
(2) $L(npq \pm 1, nq^2)$ if $m = npq \pm 1$;
(3) a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices p, q and $|npq - m|$, otherwise.

Since $g(K) = (p-1)(q-1)/2$,

$$4g(K) + 4 - (pq + 1) = (p-2)(q-2) + 1 > 0.$$

Therefore, if $r = pq$ or $m = npq \pm 1$, then we have $r = |r| \leq pq+1 < 4g(K)+4$. Thus we assume $r \neq pq, pq \pm 1/n$. Then $K(r)$ is a Seifert fibered manifold over the 2-sphere with three exceptional fibers of indices $p, q, |npq - m|$. It suffices to assume $r \neq 0$ for the inequality $|r| \leq 4g(K) + 4$.

Claim 2 $p = |npq - m| = 2$.

Proof of Claim 2 Let F be a Klein bottle in $K(r)$. Then a regular neighborhood $N(F)$ of F is the twisted I-bundle over F. Since $H_1(K(r))$ is finite, $K(r)$ does not contain a two-sided incompressible surface (see [6, VI.13]). Thus the torus $\partial N(F)$ is compressible in $K(r)$. Since $K(r)$ is irreducible, $\partial N(F)$ bounds a solid torus J in $K(r)$, and so $K(r) = N(F) \cup J$. Thus $K(r)$ is a prism manifold, and in particular, $K(r)$ has finite fundamental group. Hence $\{p, q, |npq - m|\}$ is one of the platonic triples; $\{2, 2, \alpha\}$ $(\alpha > 1)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. Assume for contradiction that $\{p, q, |npq - m|\}$ is not $\{2, 2, \alpha\}$. Then it is well known that $K(r)$ has the unique Seifert fibration. By [10, Proposition 3], $K(r)$ has a Seifert fibration over the 2-sphere with three exceptional fibers of indices $2, 2, \alpha$, which is a contradiction. Thus $\{p, q, |npq - m|\} = \{2, 2, \alpha\}$. Since $q > p$, we have $p = |npq - m| = 2$.

Then $r = m/n = 2q \pm 2/n$. Since $4g(K)+4 = 2q+2$, we have $|r| \leq 4g(K)+4$. In particular, if $|r| = 4g(K)+4$, then K is the $(2, q)$-torus knot and $r = 2q+2$ as desired.

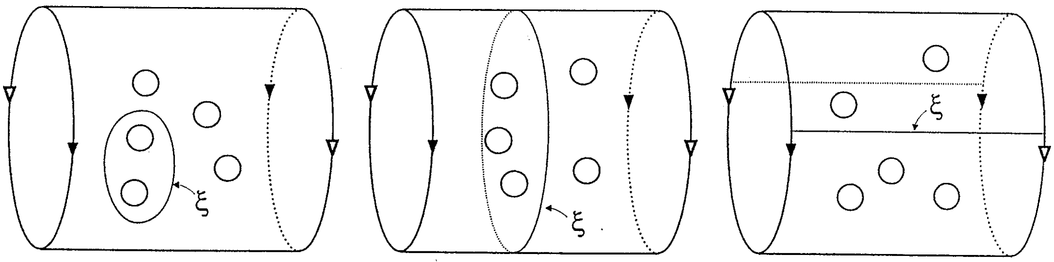
Next, when $q < 0$, take a mirror image $K^!$ of K, which is the $(2, -q)$-torus knot. Then $K(r)$ is homeomorphic to $K^!(-r)$ by an orientation-reversing homeomorphism. By the above argument, $|r| = |-r| \leq 4g(K^!) + 4 = 4g(K) + 4$. Also, if the equality holds, then $-r = 2(-q) + 2$, and so $r = 2q - 2$. This completes the proof of Case 1.

Remark that the $(2, q)$-torus knot bounds a Möbius band B whose boundary slope is $2q$. If a small half-twisted band (right-handed or left-handed according to the sign of q) is attached to B locally, then we have a once-punctured Klein bottle bounded by the knot with boundary slope $2q \pm 2$.

Case 2. K is a non-torus cable knot.

Let U be a standard solid torus in S^3, and let k be the (p, q)-torus knot in $\operatorname{Int} U$, which is isotopic to a (p, q)-curve on ∂U. We can assume that k runs p times along a core of U and $p \geq 2$. Let W be a knotted solid torus in S^3, and $f : U \to W$ be a faithful orientation-preserving homeomorphism, that is, it sends a preferred framing of U to that of W. Then put $K = f(k)$.

Let $\widehat{S}$ be a Klein bottle in $K(r)$. We can assume that $\widehat{S}$ meets the attached solid torus V in mutually disjoint meridian disks, and that the number s of these disks is minimal among all Klein bottles in $K(r)$. Note that $s \geq 1$. Let $S = \widehat{S} \cap E(K)$. Then S is a punctured Klein bottle properly embedded in $E(K)$ with s boundary components, each of which has slope r on $\partial E(K)$. By the minimality of s, S is incompressible in $E(K)$ (see [12]).



(i) (ii) (iii)

Let $T = \partial W$. We may assume that $S \cap T$ consists of loops, and that no loop of $S \cap T$ bounds a disk in S or T by the incompressibility of S and T in $E(K)$.

Subcase 1. $S \cap T = \emptyset$.

Then S is contained in W, and so $f^{-1}(S)$ lies in $U \subset S^3$. Thus r-surgery on the (p, q)-torus knot k yields a Klein bottle. (Since f is faithful, the slope is preserved.) By Case 1, $|r| \leq 4g(k) + 4$. Since $g(k) < g(K)$ [11], we have $|r| < 4g(K) + 4$.

Subcase 2. $S \cap T \neq \emptyset$.

Since the loops of $S \cap T$ are essential on T, they are mutually parallel on T. Let ξ be a loop of $S \cap T$. Note that ξ is orientation-preserving (i.e., bicollared) on S. There are three possibilities for ξ on S ([8]); ξ bounds a disk on $\widehat{S}$, ξ is non-separating, or bounds a Möbius band on $\widehat{S}$. See Figure 1. (Here, the two end circles of a cylinder are identified to form a Klein bottle as indicated by the arrows. The figures show ξ after a homeomorphism of $\widehat{S}$.)

Assume that ξ is of type (i) in Figure 1. We may assume that ξ is innermost. That is, ξ bounds a disk D on $\widehat{S}$ such that $\operatorname{Int} D$ contains no loop of $S \cap T$. Let $W(K; r)$ denote the manifold obtained from W by r-surgery on K. Recall that $T = \partial W(K; r)$ and ξ is essential on T. Hence D gives a compressing disk for T in $W(K; r)$. Then $r = pq$ or $m = npq \pm 1$ by [3]. By the same calculation as in Case 1, we have $|r| < 4g(K) + 4$. Therefore we hereafter assume that there is no loop of type (i).

Clearly, all the loops of $S \cap T$ are either of type (ii) or (iii). In the former case, S is divided into annuli and punctured annuli by the loops. Since T is separating, it divides $E(K)$ into black and white sides. We assume $\partial E(K)$ is contained in the black side. Thus the above (punctured) annuli are colored by either black or white. Note that the boundary components of ∂S lie in the black regions. Hence any white region is an annulus.

Let A be a white annulus. Since the annulus A is properly embedded in S^3 –

Int W, it separates $S^3 - \text{Int}\, W$ and we may assume that A is innermost among white annuli. That is, there is an annulus B on T, whose interior is disjoint from S, such that $\partial A = \partial B$. Let $S' = (S - A) \cup B$. We push S' slightly into W away from T. By repeating this cut-and-paste procedure, we can eliminate all white annuli. Then we have a punctured Klein bottle contained in Int W whose boundary components have slope r. Thus the same calculation as in Subcase 1 gives $|r| < 4g(K) + 4$.

Finally, we consider the situation where $S \cap T$ consists of loops of type (iii). In this case, S is divided into annuli, punctured annuli and two (punctured) Möbius bands. By the same procedure as above, we can eliminate all white annuli. We keep to use S to denote the resulting surface. Then $|S \cap T| = 1$ or 2.

Claim 3 $|S \cap T| = 1$.

Proof of Claim 3 If $|S \cap T| = 2$, then there are two white Möbius bands B_1 and B_2. Then S^3 contains a Klein bottle as the union of B_1, B_2 and an annulus on T bounded by ∂B_1 and ∂B_2, a contradiction. Hence $|S \cap T| = 1$.

Let ξ be the loop of $S \cap T$. It divides S into a white Möbius band B and a black s-punctured Möbius band Q.

Claim 4 *ξ has an integral slope on T with respect to a preferred framing of W.*

Proof of Claim 4 Let K_1 be a core of W. Then $S^3 - \text{Int}\, W$ can be identified with the exterior $E(K_1)$ of K_1. Thus ξ is a loop on $\partial E(K_1)$ which bounds a Möbius band B in $E(K_1)$. Let r' be the slope represented by ξ. Then r'-surgery $K_1(r')$ contains a projective plane, and so $K_1(r')$ is a reducible manifold or $\mathbb{R}P^3$. In the former, r' is integral by [4] as desired. If $K_1(r') = \mathbb{R}P^3$, then K_1 is not a torus knot [7]. Then r' is integral by the cyclic surgery theorem [2].

Thus ξ has slope $c/1$, where c is an even integer. By pulling Q back into U by f^{-1}, we see that $U - \text{Int}\, N(k)$ contains an s-punctured Möbius band Q', where $\partial Q' \cap \partial U$ is a loop with slope $c/1$. Let L be a core of $S^3 - U$. Then $1/c$-surgery on L yields S^3 again, since L is trivial. But k is changed to the $(p, q - pc)$-torus knot k'. This process is the same that U is added $(-c)$-full twists. Thus we see that $(r - p^2c)$-surgery on k' yields a projective plane.

Consider the case where k' is trivial. Then $q - pc = \pm 1$ and $r - p^2c = 2/\ell$ for some odd integer ℓ. In particular, $|r| \leq p|q| + p + 2$. Since $g(k) = (p-1)(|q|-1)/2$

and $g(K) \geq g(k) + p$ [11], $4g(K) + 4 - |r| \geq 4g(K) + 4 - (p|q| + p + 2) = (p-2)(|q|+1) + 6 > 0$.

Hence we may assume that k' is non-trivial. By [7], the resulting manifold is a reducible manifold with $\mathbb{R}P^3$ summand, and $r - p^2c = p(q - pc)$, and so $r = pq$. Since $g(K) \geq g(k) + p$, $4g(K) + 4 - |r| \geq 4(g(k) + p) + 4 - |r| = (p-2)(|q|+2) + 10 > 0$. Thus $|r| < 4g(K) + 4$ as desired.

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References

[1] G. Bredon and Wood, Non-orientable surfaces in orientable 3-manifolds, *Invent. Math.* **7** (1969) 83–110.

[2] M. Culler, C. McA. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots, *Ann. Math.* **125** (1987) 237–300.

[3] C. McA. Gordon, Dehn surgery and satellite knots, *Trans. Amer. Math. Soc.* **275** (1983) 687–708.

[4] C. McA. Gordon and J. Luecke, Only integral Dehn surgeries can yield reducible manifolds, *Math. Proc. Cambridge Philos. Soc.* **102** (1987) 97–101.

[5] K. Ichihara and M. Teragaito, Klein bottle surgery and genera of knots, to appear in *Pacific J. Math.*

[6] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Math. **43** (Amer. Math. Soc., Providence, RI, 1980).

[7] L. Moser, Elementary surgery along a torus knot, *Pacific J. Math.* **38** (1971) 737–745.

[8] T. Price, Homeomorphisms of quaternion space and projective planes in four space, *J. Austral. Math. Soc.* **23** (1977) 112–128.

[9] D. Rolfsen, *Knots and Links* (Publish or Perish, Berkeley, California, 1976).

[10] J. H. Rubinstein, On 3-manifolds that have finite fundamental group and contain Klein bottles, *Trans. Amer. Math. Soc.* **251** (1979) 129–137.

[11] H. Schubert, Knoten und Vollringe, *Acta Math.* **90** (1953) 131–286.

[12] M. Teragaito, Creating Klein bottles by surgery on knots, *J. Knot Theory Ramifications* **10** (2001) 781–794.