Equations governing the nonradial oscillations of a slowly rotating relativistic star

Yasufumi Kojima

Department of Physics, Tokyo Metropolitan University, Hachioji, Tokyo 192-03, Japan

(Received 18 June 1992)

The nonradial perturbation equations of a slowly rotating star are derived in the framework of general relativity. We assume the stellar rotation is slow and include the rotational effect up to first order. The oscillation modes of a nonrotating star are completely separated by the spherical harmonic indices l, m and the parity. The eigenfrequencies are degenerate with respect to m at fixed l. In the presence of rotation, however, the degeneracy with respect to m is removed and different modes are mixed with each other. The odd-parity mode with l, m is coupled with the even-parity modes with $l \pm 1, m$ and vice versa. The basic equations derived here will give a new estimate for the instability of the relativistic rotating star due to gravitational radiation reaction.

PACS number(s): 04.30.+x, 97.60.Gb, 97.60.Jd

I. INTRODUCTION

Despite the long history of relativistic rotating stars, there still remains the exact evaluation of the normal modes of the nonradial oscillations and the determination of the stability limit. Chandrasekhar [1] has shown that rapidly rotating Newtonian stars with uniform density are secularly unstable against the nonaxisymmetric perturbations due to viscosity or gravitational radiation reaction. See also Ref. [2] for a combination of both effects. This instability is subsequently demonstrated to be generic for all rotating stars [3]. For the stability of the equilibrium state, it is necessary to obtain the eigenfrequencies of the oscillations or to use the variational principle. The stability limits for the rotating Newtonian polytropes are calculated in terms of the variational principle. (See, e.g., Ref. [4].) The extension to relativistic rotating stars is left open.

The study has recently developed in two aspects toward determining the eigenfrequencies for relativistic rotating stars. In Newtonian gravity, Ipser and Lindblom [5] have reformulated the perturbation equations in terms of Eulerian perturbations and obtained a simple pair of second-order eigenequations. They have succeeded in directly obtaining the eigenfrequencies of the nonradial oscillations for the inhomogeneous rotating stars and determined the stability limits. The first post-Newtonian corrections to be included in the results were recently formulated [6].

In general relativity, Chandrasekhar and Ferrari [7] and Ipser and Price [8] have reexamined the nonradial pulsations of nonrotating stars and presented simple sets of basic equations. The former use the diagonal gauge and the latter the Regge-Wheeler gauge. The relationship between them is discussed in Ref. [9]. Chandrasekhar and Ferrari [10] have extended their work to the pulsations of a slowly rotating star. The angular velocity Ω of the star is supposed to be slow and the equations are expanded in powers of $\epsilon = \Omega/\sqrt{GM/R^3}$, where M and R are the mass and radius of the star. They have included the first-order effect. They have found that the odd-parity and even-parity modes are mixed in the rotating star and that the coupling between them is subject to Laporte's rule, while both modes are completely separated in the nonrotating star. As an example, they have calculated the sextupole oscillation of the odd parity induced by the quadrupole mode of the even parity.

However, their analysis is limited in two points. (i) They have only shown how the odd-parity modes in a spherical star are affected by the coupling with the even-parity modes. The opposite problem remains, that is, how the even-parity modes are affected by the odd-parity modes. (ii) They have limited themselves to azimuthally symmetric perturbations, because their coordinate system is valid only for such perturbations. In this paper, we consider the same problem as Chandrasekhar and Ferrari [10], but in a different gauge, i.e., the Regge-Wheeler gauge, in which we can deal with the nonaxisymmetric perturbations as well as the axisymmetric ones. We also examine point (i).

In Sec. II, we summarize the basic equations governing stationary equilibrium. The linear perturbation equations are derived in Sec. III. If we neglect the rotational effect in the results of Sec. III, we have the basic equations for the nonradial pulsations of the nonrotating spherical star. The derivation and results are shown in Sec. IV for later convenience. We include the effect of rotation in Sec. V. The rotational correction can be regarded as the source terms for the equations of the nonrotating star. In Secs. VI and VII, we apply our results of Sec. III to the vacuum case outside the star. The perturbation equations correspond to those of slowly rotating black hole space-time. The equations relevant to our analysis are summarized to be self-contained. The basic equations for the Schwarzschild black hole are given in Sec. VI. The rotational correction to be included is shown in Sec. VII. Finally, the implications of our results are discussed in Sec. VIII.

A similar expansion technique in ϵ is applied to the radial pulsations [11]. The calculation is performed up to

ϵ^2. The radial oscillations couple the modes with a spherical harmonic index $l = 0, 1, 2$. In this paper, we will restrict our consideration to the nonradial oscillation with $l \geq 2$, which is associated with the gravitational wave. Otherwise, some equations in Sec. III are meaningless because there are fewer degrees of freedom. We use the geometrical units of $c = G = 1$.

II. STEADY CONFIGURATIONS

In this section, we shall summarize the basic equations which describe the steady configurations. (See, e.g., Ref. [12].) We assume the star is slowly rotating with a uniform angular velocity $\Omega = O(\epsilon)$ and keep only the effects linear in the angular velocity. In this approximation, the star is spherical because the centrifugal force deforming the shape is of the order ϵ^2. The metric can be written as

$$\begin{aligned} ds^2 &= g^{(0)}_{\mu\nu} dx^\mu dx^\nu \\ &= -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &\quad -2\omega r^2 \sin^2\theta\, dt\, d\phi, \end{aligned} \tag{1}$$

where ν and λ are functions of the radial coordinate r only and $\omega = O(\epsilon)$ is also a function of r.

The four-velocity up to the order ϵ is given by

$$[u^t, u^r, u^\theta, u^\phi] = [e^{-\nu/2}, 0, 0, \Omega e^{-\nu/2}]. \tag{2}$$

We assume a perfect fluid so that the energy-momentum tensor is given by

$$T_{\mu\nu} = (\varrho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{3}$$

where ϱ and p denote the energy density and the pressure, respectively. We shall further restrict our consideration to the barotropic case for simplicity, $p = p(\varrho)$.

From the Einstein equations, we have the equations

$$e^\lambda = \left(1 - \frac{2M}{r}\right)^{-1}, \quad M = \kappa \int_0^r \varrho r^2 dr,$$

$$\nu' = \frac{2e^\lambda (M + \kappa p r^3)}{r^2}, \quad p' = -\frac{e^\lambda(\varrho + p)(M + \kappa p r^3)}{r^2} \tag{4}$$

and

$$\varpi'' - \left[\kappa(\varrho + p)e^\lambda r - \frac{4}{r}\right]\varpi' - 4\kappa(\varrho + p)e^\lambda \varpi = 0, \tag{5}$$

where the prime means a derivative with respect to r and $\kappa = 4\pi$. The function ϖ is of the order ϵ, defined as

$$\varpi = \Omega - \omega. \tag{6}$$

The equations in (4) are the same as those of the nonrotating spherical star. That is, the star is the same configuration as the nonrotating state. Up to the first order of the rotation, the new effect is the dragging of the inertial frame due to the rotation of the star, which is determined by Eq. (5).

In the vacuum outside the star, we have

$$e^\nu = e^{-\lambda} = 1 - \frac{2M}{r}, \quad \varpi = \Omega - \frac{2J}{r^3}, \tag{7}$$

where two constants M and J are the total gravitational mass and the angular momentum.

III. PERTURBATION EQUATIONS

In this section, we shall write down ten components of the linearized Einstein equation. We first consider the metric perturbation as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{8}$$

where $g^{(0)}_{\mu\nu}$ is the background metric given in Sec. II. Assuming the metric perturbation $h_{\mu\nu}$ is small, we can linearize the Einstein tensor as

$$\begin{aligned} -2\delta G_{\mu\nu} = {} & h_{\mu\nu;\alpha}{}^{;\alpha} - (f_{\mu;\nu} + f_{\nu;\mu}) + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} h_{\alpha\beta} + h^{\alpha}{}_{\alpha;\mu;\nu} \\ & -(R^{\alpha}{}_{\nu} h_{\mu\alpha} + R^{\alpha}{}_{\mu} h_{\nu\alpha}) + g_{\mu\nu}(f_\alpha{}^{;\alpha} - h^{\alpha}{}_{\alpha;\beta}{}^{;\beta}) + R h_{\mu\nu} - g_{\mu\nu} R^{\alpha\beta} h_{\alpha\beta}, \end{aligned} \tag{9}$$

where

$$f_\mu = h_{\mu\alpha}{}^{;\alpha}. \tag{10}$$

$R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}$, $R^{\alpha\beta}$, and R are the Riemann, Ricci, and scalar curvatures calculated by the background metric $g^{(0)}_{\mu\nu}$. We may limit the calculations up to order ϵ in our present context.

Now we shall specify the gauge freedom. The following four conditions can be imposed (Regge-Wheeler gauge) [13,14]:

$$\begin{aligned} & g_{\theta\phi} = 0, \quad g_{\phi\phi} = g_{\theta\theta}\sin^2\theta, \\ & \partial_\theta(g_{t\theta}\sin\theta) + \partial_\phi(g_{t\phi}/\sin\theta) = 0, \quad \partial_\theta(g_{r\theta}\sin\theta) + \partial_\phi(g_{r\phi}/\sin\theta) = 0. \end{aligned} \tag{11}$$

Under these conditions, we can take the metric perturbation as the forms

$$
\begin{aligned}
&h_{tt} = e^{\nu}\sum_{l,m} H_{0,lm}Y_{lm}, \quad h_{tr} = h_{rt} = \sum_{l,m} H_{1,lm}Y_{lm}, \quad h_{rr} = e^{\lambda}\sum_{l,m} H_{2,lm}Y_{lm}, \\
&h_{t\theta} = h_{\theta t} = \sum_{l,m} h_{0,lm}(-\partial_\phi Y_{lm}/\sin\theta), \quad h_{t\phi} = h_{\phi t} = \sum_{l,m} h_{0,lm}(\sin\theta\partial_\theta Y_{lm}), \\
&h_{r\theta} = h_{\theta r} = \sum_{l,m} h_{1,lm}(-\partial_\phi Y_{lm}/\sin\theta), \quad h_{r\phi} = h_{\phi r} = \sum_{l,m} h_{1,lm}(\sin\theta\partial_\theta Y_{lm}), \\
&h_{\theta\theta} = h_{\phi\phi}/\sin^2\theta = r^2\sum_{l,m} K_{lm}Y_{lm}, \quad h_{\theta\phi} = h_{\phi\theta} = 0,
\end{aligned}
\tag{12}
$$

where $Y_{lm}(\theta,\phi)$ [$\propto P_{lm}(\theta)e^{im\phi}$] is the spherical harmonic and $P_{lm}(\theta)$ is the associated Legendre function. The functions $H_{0,lm}, H_{1,lm}, H_{2,lm}, K_{lm}, h_{0,lm}$, and $h_{1,lm}$ depend on r and t only.

The perturbations of the energy-momentum tensor are described by an appropriate sum of three functions for the fluid perturbations: $R_{lm}(r,t)$, $V_{lm}(r,t)$, $U_{lm}(r,t)$, the density perturbation $\delta\varrho_{lm}(r,t)$, and the pressure perturbation $\delta p_{lm}(r,t)$. The explicit forms are given in Appendix A. Since we consider the barotropic case, i.e., $p = p(\varrho)$, we have one relation between the density and pressure perturbations:

$$
\delta p_{lm} = \frac{dp}{d\varrho}\delta\varrho_{lm} \equiv C^2\delta\varrho_{lm}. \tag{13}
$$

Ten components of the Einstein equation $\delta G_{\mu\nu} = 2\kappa\delta T_{\mu\nu}$ can be written as the following types: Eqs. (14)–(16), (18), and (19). From tt, tr, rr components and the sum of $\theta\theta$ and $\phi\phi$ components, we have

$$
\sum_{l,m}\{(A^{(I)}_{lm} + \tilde{A}^{(I)}_{lm}\cos\theta)Y_{lm} + B^{(I)}_{lm}\sin\theta\partial_\theta Y_{lm} + C^{(I)}_{lm}\partial_\phi Y_{lm}\} = 0 \quad (I = 0 \text{ to } 3), \tag{14}
$$

where the functions $A^{(I)}_{lm}$ and $C^{(I)}_{lm}$ are some linear combinations of $H_{0,lm}$, $H_{1,lm}$, $H_{2,lm}$, K_{lm}, R_{lm}, V_{lm}, $\delta\varrho_{lm}$, and δp_{lm} and therefore depend on r and t. On the other hand, the functions $\tilde{A}^{(I)}_{lm}$ and $B^{(I)}_{lm}$ are some linear combinations of $h_{0,lm}$, $h_{1,lm}$, and U_{lm} and, hence, depend on r and t. The former set belongs to the so-called even-parity mode and the latter the odd-parity mode. The explicit forms for these functions are given in Appendix B.

From $t\theta, t\phi, r\theta, r\phi$ components, we have

$$
\begin{aligned}
\sum_{l,m}\{&(\alpha^{(J)}_{lm} + \tilde{\alpha}^{(J)}_{lm}\cos\theta)\partial_\theta Y_{lm} - (\beta^{(J)}_{lm} + \tilde{\beta}^{(J)}_{lm}\cos\theta)(\partial_\phi Y_{lm}/\sin\theta) \\
&+\eta^{(J)}_{lm}(\sin\theta Y_{lm}) + \xi^{(J)}_{lm}X_{lm} + \chi^{(J)}_{lm}(\sin\theta W_{lm})\} = 0 \quad (J = 0,1)
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
\sum_{l,m}\{&(\beta^{(J)}_{lm} + \tilde{\beta}^{(J)}_{lm}\cos\theta)\partial_\theta Y_{lm} + (\alpha^{(J)}_{lm} + \tilde{\alpha}^{(J)}_{lm}\cos\theta)(\partial_\phi Y_{lm}/\sin\theta) \\
&+\zeta^{(J)}_{lm}(\sin\theta Y_{lm}) + \chi^{(J)}_{lm}X_{lm} - \xi^{(J)}_{lm}(\sin\theta W_{lm})\} = 0 \quad (J = 0,1)
\end{aligned}
\tag{16}
$$

where X_{lm} and W_{lm} are functions of θ and ϕ, defined as

$$
X_{lm} = 2\partial_\phi(\partial_\theta - \cot\theta)Y_{lm}, \quad W_{lm} = \left(\partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial_\phi^2\right)Y_{lm}. \tag{17}
$$

The functions $\alpha^{(J)}_{lm}$, $\tilde{\beta}^{(J)}_{lm}$, $\zeta^{(J)}_{lm}$, and $\xi^{(J)}_{lm}$ are some linear combinations of the functions belonging to the even-parity mode, while the functions $\beta^{(J)}_{lm}$, $\tilde{\alpha}^{(J)}_{lm}$, $\eta^{(J)}_{lm}$, and $\chi^{(J)}_{lm}$ belong to the odd-parity mode. The explicit forms are given in Appendix B.

From $\theta\phi$ component and the subtraction of $\theta\theta$ and $\phi\phi$ components, we have

$$
\sum_{l,m}\{f_{lm}\partial_\theta Y_{lm} + g_{lm}(\partial_\phi Y_{lm}/\sin\theta) + s_{lm}(X_{lm}/\sin^2\theta) + t_{lm}(W_{lm}/\sin\theta)\} = 0 \tag{18}
$$

and

$$
\sum_{l,m}\{g_{lm}\partial_\theta Y_{lm} - f_{lm}(\partial_\phi Y_{lm}/\sin\theta) - t_{lm}(X_{lm}/\sin^2\theta) + s_{lm}(W_{lm}/\sin\theta)\} = 0, \tag{19}
$$

where s_{lm} and f_{lm} are some linear combinations of the functions belonging to the even-parity mode and t_{lm} and g_{lm} to the odd-parity mode. The explicit forms are given in Appendix B.

Next we shall decompose the above equations to a specific mode with l, m. Multiplying $\bar{Y}_{lm}$ to Eq. (14) and integrating over the solid angle, we have

$$A_{lm}^{(I)} + imC_{lm}^{(I)} + Q_{l-1m}[\tilde{A}_{l-1m}^{(I)} + (l-1)B_{l-1m}^{(I)}] + Q_{l+1m}[\tilde{A}_{l+1m}^{(I)} - (l+2)B_{l+1m}^{(I)}] = 0, \tag{20}$$

where we have used the orthogonality condition and the following formulas for Y_{lm}:

$$\cos\theta Y_{lm} = Q_{l+1m}Y_{l+1m} + Q_{l-1m}Y_{l-1m}, \tag{21}$$

$$\sin\theta\partial_\theta Y_{lm} = Q_{l+1m}lY_{l+1m} - Q_{l-1m}(l+1)Y_{l-1m} \tag{22}$$

with

$$Q_{l-1m} = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}, \quad Q_{l+1m} = \sqrt{\frac{(l+1-m)(l+1+m)}{(2l+1)(2l+3)}}. \tag{23}$$

In a similar way, Eqs. (15) and (16) yield

$$l(l+1)\alpha_{lm}^{(J)} + im[(l-1)(l+2)\xi_{lm}^{(J)} - \tilde{\beta}_{lm}^{(J)} - \zeta_{lm}^{(J)}] + Q_{l-1m}(l+1)[(l-2)(l-1)\chi_{l-1m}^{(J)} + (l-1)\tilde{\alpha}_{l-1m}^{(J)} - \eta_{l-1m}^{(J)}]$$
$$-Q_{l+1m}l[(l+2)(l+3)\chi_{l+1m}^{(J)} - (l+2)\tilde{\alpha}_{l+1m}^{(J)} - \eta_{l+1m}^{(J)}] = 0 \tag{24}$$

and

$$l(l+1)\beta_{lm}^{(J)} + im[(l-1)(l+2)\chi_{lm}^{(J)} + \tilde{\alpha}_{lm}^{(J)} + \eta_{lm}^{(J)}] - Q_{l-1m}(l+1)[(l-2)(l-1)\xi_{l-1m}^{(J)} - (l-1)\tilde{\beta}_{l-1m}^{(J)} + \zeta_{l-1m}^{(J)}]$$
$$+Q_{l+1m}l[(l+2)(l+3)\xi_{l+1m}^{(J)} + (l+2)\tilde{\beta}_{l+1m}^{(J)} + \zeta_{l+1m}^{(J)}] = 0. \tag{25}$$

From Eqs. (18) and (19), we have

$$l(l+1)s_{lm} - imf_{lm} - Q_{l-1m}(l+1)g_{l-1m} + Q_{l+1m}lg_{l+1m} = 0 \tag{26}$$

and

$$l(l+1)t_{lm} + img_{lm} - Q_{l-1m}(l+1)f_{l-1m} + Q_{l+1m}lf_{l+1m} = 0. \tag{27}$$

We also specialize to a single Fourier mode with $e^{-i\sigma t}$. In this way, the basic equations (20), (24)–(27) depend on r only.

It is easy to observe the quantities of the order ϵ are involved in $\tilde{A}_{lm}^{(I)}$, $B_{lm}^{(I)}$, $C_{lm}^{(I)}$, $\tilde{\alpha}_{lm}^{(J)}$, $\tilde{\beta}_{lm}^{(J)}$, $\eta_{lm}^{(J)}$, $\zeta_{lm}^{(J)}$, $\xi_{lm}^{(J)}$, $\chi_{lm}^{(J)}$, f_{lm}, and g_{lm}, while not in $A_{lm}^{(I)}$, $\alpha_{lm}^{(J)}$, $\beta_{lm}^{(J)}$, s_{lm}, and t_{lm}. For the nonrotating case, the perturbation equations reduce to

$$\beta_{lm}^{(J)} = t_{lm} = 0 \quad (J = 0, 1) \tag{28}$$

and

$$A_{lm}^{(I)} = \alpha_{lm}^{(J)} = s_{lm} = 0 \quad (I = 0 \text{ to } 3, J = 0, 1). \tag{29}$$

Equation (28) constitutes a set of $h_{0,lm}$, $h_{1,lm}$, and U_{lm} with parity of $(-1)^{l+1}$, while Eq. (29) constitutes a set of $H_{0,lm}$, $H_{1,lm}$, $H_{2,lm}$, K_{lm}, R_{lm}, V_{lm}, $\delta\varrho_{lm}$, and δp_{lm} with parity of $(-1)^l$. The fundamental equations for the nonrotating star are decoupled by the spherical harmonic indices l, m and the parity. These equations are in fact independent of m; that is, the eigenfrequencies are degenerate with respect to m. In the presence of rotation, there are some terms proportional to m. Thus the degeneracy is removed. In addition, Eqs. (20), (24)–(27) show the couplings with opposite parity mode. The odd-parity mode with l, m couples with the even-parity modes with $l \pm 1, m$ and vice versa.

We refer to the equations for the nonrotating star as the zeroth-order equations and the equations including the terms of the order ϵ as the first-order equations. The first-order correction can be regarded as the source terms of the zeroth-order equations. In the source terms, we may use the zeroth-order solutions and simplify the source terms. In the following sections, we will show this explicitly.

IV. ZEROTH-ORDER EQUATIONS INSIDE THE STAR

In this section, we shall summarize the equations of the order ϵ^0, which describe the nonradial oscillations of nonrotating spherical stars. This problem has been studied since the work of Thorne and his collaborators [14].

We shall, however, review the derivation and the basic equations, which are necessary to include the rotational correction.

A. Odd-parity modes

We first consider the odd-parity mode and solve a set of Eq. (28). From $t_{lm} = 0$, we can solve $h_{0,lm}$ as

$$h_{0,lm} = \frac{e^{(\nu-\lambda)/2}}{-i\sigma}(X_{lm}r)', \tag{30}$$

where we have defined a function X_{lm} as

$$h_{1,lm} = e^{(\lambda-\nu)/2}X_{lm}r. \tag{31}$$

From $\beta^{(1)}_{lm} = 0$, we have a basic equation:

$$\begin{aligned} L_0[X_{lm}] \equiv & e^{(\nu-\lambda)/2}(e^{(\nu-\lambda)/2}X'_{lm})' \\ & + \left[\sigma^2 - e^{\nu}\left(\frac{2(n+1)}{r^2} - \frac{6M}{r^3} + \kappa(\varrho - p)\right)\right]X_{lm} \\ = & 0, \end{aligned} \tag{32}$$

where

$$n = (l-1)(l+2)/2. \tag{33}$$

From $\beta^{(0)}_{lm} = 0$, the fluid perturbation can be expressed as

$$U_{lm} = -\kappa(\varrho + p)e^{-\nu}h_{0,lm}. \tag{34}$$

Thus the basic equation of the odd-parity mode is a second-order differential equation, which represents the propagation of gravitational wave inside the star [14]. The oscillation mode of this parity is classified as the torsional mode and does not pulsate unless there is the shear stress. We denote $X^{(0)}_{lm}$ as the solution of Eq. (32), which will be used in the next section.

B. Even-parity modes

We shall solve a set of Eq. (29). The basic equations of even-parity mode become a fourth-order system of differential equations. There are several methods to choose a set of equations [7,8,15,16]. We here derive the basic equations obtained by Ipser and Price [8]. They choose $H_{0,lm}$, K_{lm} and their derivatives as the basic variables and all other variables are eliminated. From $s_{lm} = 0$, we have

$$H_{2,lm} = H_{0,lm}. \tag{35}$$

From $\alpha^{(1)}_{lm} = 0$, we can solve $H_{1,lm}$ as

$$\begin{aligned} i\sigma H_{1,lm} = -e^{\nu}\bigg(& H'_{0,lm} - K'_{0,lm} \\ & + \frac{2e^{\lambda}}{r^2}(M + \kappa p r^3)H_{0,lm}\bigg). \end{aligned} \tag{36}$$

From $A^{(2)}_{lm} = 0$, the pressure perturbation δp_{lm} can be written as

$$\begin{aligned} \kappa\delta p_{lm} = & \frac{e^{-\lambda}}{2r}H'_{0,lm} + \frac{1}{2r^3}(nr + 4M + 2\kappa p r^3)H_{0,lm} \\ & - \frac{1}{2r^2}(r - 3M - \kappa p r^3)K'_{lm} \\ & + \frac{1}{2r^2}(\sigma^2 e^{-\nu}r^2 - n)K_{lm}. \end{aligned} \tag{37}$$

The density perturbation $\delta\varrho_{lm}$ is given by Eq. (13) in terms by δp_{lm}. Using these equations for $A^{(0)}_{lm} = 0$ and $A^{(3)}_{lm} = 0$, we have the basic equations

$$\begin{aligned} L^{(1)}_e[H_{0,lm}, K_{lm}] \equiv & (K_{lm} - H_{0,lm})'' - \frac{e^{\lambda}}{r^2}[2r - 10M + \kappa(\varrho - 5p)r^3](K_{lm} - H_{0,lm})' \\ & + \frac{e^{\lambda}}{r^2}(\sigma^2 e^{-\nu}r^2 - 2n)(K_{lm} - H_{0,lm}) + \frac{4e^{\lambda}}{r^4}[3Mr - \kappa\varrho r^4 - e^{\lambda}(M + \kappa p r^3)^2]H_{0,lm} \\ = & 0 \end{aligned} \tag{38}$$

and

$$\begin{aligned} L^{(2)}_e[H_{0,lm}, K_{lm}] \equiv & K''_{lm} - \frac{e^{\lambda}}{r^2}[(r - 3M - \kappa p r^3)C^{-2} - 3r + 5M + \kappa\varrho r^3]K'_{lm} \\ & + \frac{e^{\lambda}}{r^2}[\sigma^2 e^{-\nu}r^2 C^{-2} - n(C^{-2} + 1)]K_{lm} + \frac{1}{r}(C^{-2} - 1)H'_{0,lm} \\ & + \frac{e^{\lambda}}{r^3}[(nr + 4M + 2\kappa p r^3)C^{-2} - (n+2)r + 2\kappa\varrho r^3]H_{0,lm} \\ = & 0. \end{aligned} \tag{39}$$

From $A^{(1)}_{lm} = 0$ and $\alpha^{(0)}_{lm} = 0$, the quantities describing perturbations of the fluid R_{lm} and V_{lm} can be expressed as

$$\begin{aligned} i\sigma R_{lm} = & \frac{n+1}{2r^2}H'_{0,lm} - \frac{1}{2r^4}[\sigma^2 e^{-\nu}r^3 - 2(n+1)e^{\lambda}(M + \kappa p r^3)]H_{0,lm} \\ & + \frac{1}{2r^2}(\sigma^2 e^{-\nu}r^2 - n - 1)K'_{lm} + \frac{\sigma^2 e^{\lambda-\nu}}{2r^2}(r - 3M - \kappa p r^3)K_{lm} \end{aligned} \tag{40}$$

and

$$i\sigma V_{lm} = \frac{e^{-\lambda}}{2r}H'_{0,lm} + \frac{1}{2r^3}[nr + 4M - \kappa(\varrho - p)r^3]H_{0,lm} - \frac{1}{2r^2}(r - 3M - \kappa pr^3)K'_{lm} + \frac{1}{2r^2}(\sigma^2 e^{-\nu}r^2 - n)K_{lm}. \tag{41}$$

In this way, we have a fourth-order system of differential equations for $H_{0,lm}$ and K_{lm}, which govern the oscillation of the even-parity mode. We will denote $H^{(0)}_{0,lm}$ and $K^{(0)}_{lm}$ as the solutions of Eqs. (38) and (39), which will be used in the next section.

V. FIRST-ORDER EQUATIONS INSIDE THE STAR

In this section, we shall include the rotational effect up to the first order. We have to solve Eqs. (20), (24)–(27) instead of Eqs. (28) and (29). We can regard the additional terms $\tilde{A}^{(I)}_{lm}$, $B^{(I)}_{lm}$, $C^{(I)}_{lm}$, $\tilde{\alpha}^{(J)}_{lm}$, $\tilde{\beta}^{(J)}_{lm}$, $\eta^{(J)}_{lm}$, $\zeta^{(J)}_{lm}$, $\xi^{(J)}_{lm}$, $\chi^{(J)}_{lm}$, f_{lm}, and g_{lm} as the source terms for Eqs. (28) and (29). In these source terms, we may use the zeroth-order solutions and eliminate the higher-order derivative of some functions, e.g., $H''_{0,lm}$. In this way, the source terms can be expressed by $X^{(0)}_{lm}$, $H^{(0)}_{0,lm}$, and $K^{(0)}_{lm}$ and their first-order derivatives. The procedure to derive basic equations is the same as shown in Sec. IV.

A. Odd-parity modes

Using the same definition for X_{lm} as in Eq. (31), we can solve $h_{0,lm}$, as Eq. (30) but with some additional terms. Corresponding to Eq. (32), the basic equation of the odd-parity mode becomes

$$L_0[X_{lm}] = \frac{m}{\sigma}N_{lm} + e^{(\lambda+3\nu)/2}\left[\frac{iQ_{l-1m}}{(n-n_-)\sigma}F_{l-1m} + \frac{iQ_{l+1m}}{(n-n_+)\sigma}F_{l+1m}\right], \tag{42}$$

where L_0 is the second-order differential operator defined in Eq. (32). The source terms are given by

$$N_{lm} = -2\omega' e^{\nu-\lambda}X^{(0)\prime}_{lm} + \left[2\sigma^2\omega + \frac{4n}{n+1}\kappa(\varrho+p)e^{\nu}\varpi + \frac{2\omega' e^{\nu}}{(n+1)r^2}\{n(2r - 5M - \kappa pr^3) - re^{-\lambda}\}\right]X^{(0)}_{lm} \tag{43}$$

and

$$\begin{aligned} F_{l\pm1m} = {} & \left[\frac{2e^{-\lambda}\omega'}{r}(r - 3M - \kappa pr^3) + \frac{2\varpi}{r^3}\{2\kappa(\varrho+p)r^4e^{-\lambda} + (r - 3M - \kappa pr^3)(M + \kappa pr^3)(C^{-2} - 1)\}\right]K^{(0)\prime}_{l\pm1m} \\ & + \left[\frac{2n_\pm e^{-\lambda}\omega'}{r} - \frac{2\varpi}{r^3}(\sigma^2 e^{-\nu}r^2 - n_\pm)(C^{-2} - 1)(M + \kappa pr^3)\right]K^{(0)}_{l\pm1m} \\ & - \left[2e^{-2\lambda}\omega' + \frac{2e^{-\lambda}\varpi}{r^2}\{(M + \kappa pr^3)(C^{-2} - 1) + 2\kappa(\varrho+p)r^3\}\right]H^{(0)\prime}_{0,l\pm1m} \\ & - \left[\frac{2\omega' e^{-\lambda}}{r^2}\{n_\pm r + 4M - \kappa(\varrho - p)r^3\}\right. \\ & \left. + \frac{2\varpi(M + \kappa pr^3)}{r^4}[4\kappa(\varrho+p)r^3 + \{n_\pm r + 4M - \kappa(\varrho - p)r^3\}(C^{-2} - 1)]\right]H^{(0)}_{0,l\pm1m}, \end{aligned} \tag{44}$$

where

$$n_- = (l-2)(l+1)/2, \quad n_+ = l(l+3)/2. \tag{45}$$

The fluid motion, corresponding to Eq. (34), is also modified, but the expression is omitted here.

B. Even-parity modes

Next we shall derive the basic equations for the even-parity mode. The relation (13) is unchanged if the first-order rotational effect is included. Corresponding to Eqs. (35)–(37), we can solve $H_{2,lm}, H_{1,lm}$, and δp_{lm} with some additional terms. Eliminating $H_{2,lm}, H_{1,lm}, \delta p_{lm}$, and $\delta\varrho_{lm}$ in terms of these equations, we eventually have a pair of second-order differential equations. The basic equations can be written as

$$L^{(J)}_e[H_{0,lm}, K_{lm}] = -\frac{m}{(n+1)\sigma}E^{(J)}_{lm} + e^{-(\lambda+\nu)/2}\left[\frac{iQ_{l-1m}D^{(J)}_{l-1m}}{(n-n_-)\sigma} + \frac{iQ_{l+1m}D^{(J)}_{l+1m}}{(n-n_+)\sigma}\right] \quad (J = 1, 2), \tag{46}$$

where $L^{(1)}_e$ and $L^{(2)}_e$ are the second-order differential operators defined in Eqs. (38) and (39), respectively. The source terms for $J = 1$ are given by

$$E^{(1)}_{lm} = \Bigg[\frac{2\omega'}{r}(r-M+\kappa p r^3) + (2\varpi+\omega)\sigma^2 e^{-\nu} r (C^{-2}-1)$$
$$+\frac{2\varpi e^{\lambda}}{r^3}[(M+\kappa p r^3)(r-M+\kappa p r^3)(C^{-2}-1) - 4Mre^{-\lambda}$$
$$+2(r-M+\kappa p r^3)\kappa\varrho r^3 - 2(r-3M-\kappa p r^3)\kappa p r^3]\Bigg] H^{(0)\prime}_{0,lm}$$
$$+\Bigg[-2\sigma^2 r\omega' e^{-\nu} + \frac{2\omega' e^{\lambda}}{r^3}(r-M+\kappa p r^3)\{nr+4M-\kappa(\varrho-p)r^3\}$$
$$+\frac{(2\varpi+\omega)\sigma^2 e^{\lambda-\nu}}{r}\{(nr+4M)(C^{-2}-1)+2\kappa(pC^{-2}+\varrho)r^3\} + 2\omega\sigma^2 e^{-\nu}\{(n+1)e^{\lambda}-1\}$$
$$+\frac{2\varpi e^{2\lambda}(M+\kappa p r^3)}{r^5}[\{nr+4M-\kappa(\varrho-p)r^3\}(r-M+\kappa p r^3)(C^{-2}-1) - 4(nr+4M)re^{-\lambda}$$
$$+4\{M+\kappa(\varrho+p)r^3\}\kappa p r^3 + 4(2r-3M)\kappa\varrho r^3]\Bigg] H^{(0)}_{0,lm}$$
$$+\Bigg[2\sigma^2 r^2\omega' e^{-\nu} - \frac{2\omega' e^{\lambda}}{r^2}(r-M+\kappa p r^3)(r-3M-\kappa p r^3) - \sigma^2 e^{\lambda-\nu}(2\varpi+\omega)(r-3M-\kappa p r^3)(C^{-2}-1)$$
$$-\frac{2\varpi e^{\lambda}(M+\kappa p r^3)}{r^3}\Bigg(\frac{e^{\lambda}}{r}(r-3M-\kappa p r^3)(r-M+\kappa p r^3)(C^{-2}-1)$$
$$-3r+10M+2\kappa(\varrho+3p)r^3\Bigg) + \frac{2\varpi}{r^2}\{M-\kappa(2\varrho+p)r^3\}\Bigg] K^{(0)\prime}_{lm}$$
$$+\Bigg[-\frac{2n\omega' e^{\lambda}}{r^2}(r-M+\kappa p r^3) + (2\varpi+\omega)\sigma^4 e^{\lambda-2\nu} r^2 (C^{-2}-1)$$
$$-\frac{(2\varpi+\omega)\sigma^2 e^{\lambda-\nu}}{r}\{nr(C^{-2}+1)+4(M+\kappa p r^3)\} + \frac{4n\varpi e^{\lambda}}{r^3}(\sigma^2 e^{-\nu} r^3 + 2M + 2\kappa p r^3)$$
$$+\frac{2\varpi e^{2\lambda}}{r^4}(\sigma^2 r^2 e^{-\nu} - n)(M+\kappa p r^3)(r-M+\kappa p r^3)(C^{-2}-1)\Bigg] K^{(0)}_{lm} \tag{47}$$

and

$$D^{(1)}_{l\pm 1m} = \Bigg[\frac{4\omega'}{r}\{(-3n+5n_{\pm}+4)r + (n+n_{\pm})e^{\lambda}(M+\kappa p r^3)\}$$
$$-16\kappa(n-n_{\pm}-1)\varpi e^{\lambda}(\varrho+p)r - \frac{8(n_{\pm}+1)\omega e^{\lambda}}{r^2}\{(n-n_{\pm})r + re^{-\lambda} - 2\kappa p r^3\}\Bigg] X^{(0)\prime}_{l\pm 1m}$$
$$+\Bigg[-4(n-n_{\pm})\sigma^2 e^{\lambda-\nu} r\omega'$$
$$+\frac{4e^{2\lambda}\omega'}{r^3}[(nn_{\pm}+3n_{\pm}^2-3n+n_{\pm}+4)r^2 e^{-\lambda} + (7n+2n_{\pm}-8)Mre^{-\lambda}$$
$$+n_{\pm}Mr + \kappa p r^3\{nre^{-\lambda} + n_{\pm}(r+2M+2\kappa p r^3)\}]$$
$$-\frac{16\kappa(\varrho+p)e^{\lambda}\varpi}{r}\{(n-1)r + n_{\pm}e^{\lambda}(M+\kappa p r^3)\}$$
$$-\frac{8(n_{\pm}+1)e^{\lambda}\omega}{r^3}\{(n+n_{\pm}+1)r - 2(n_{\pm}e^{\lambda}+1)(M+\kappa p r^3)\}\Bigg] X^{(0)}_{l\pm 1m}. \tag{48}$$

The source terms for $J=2$ are given by

$$E^{(2)}_{lm} = \Big[-\frac{\omega'}{2}\{(n+1)(C^{-2}-1) - 4e^{-\lambda}\} + \frac{2\varpi}{r^2}[\{(n+2)r - M + \kappa p r^3\}(C^{-2}-1) - 2re^{-\lambda} + 4\kappa(\varrho+p)r^3]\Big] H^{(0)\prime}_{0,lm}$$
$$+\Big[-\frac{\omega'}{r^2}\{(n+1)e^{\lambda}(M+\kappa p r^3)(C^{-2}-1) - 2nr - 8M + 2\kappa(\varrho-p)r^3\}$$
$$-2\varpi\sigma^2 e^{-\nu}(C^{-2}-1) - \frac{2\varpi e^{\lambda}}{C^2 r^4}\{(r-M+\kappa p r^3)[r-4M+\kappa(\varrho-p)r^3] - (n+1)(r+M+3\kappa p r^3)r\}$$
$$-\frac{2\varpi e^{\lambda}}{r^4}[(n+1)r\{3r-3M-\kappa(2\varrho-p)r^3\} + 2(\kappa\varrho r^3)^2 - 7\kappa M\varrho r^3 - 5\kappa^2\varrho p r^6 - \kappa\varrho r^4$$
$$-20M^2 - 13\kappa M p r^3 + 17Mr - 5(\kappa p r^3)^2 + 4\kappa p r^4 - 3r^2]\Big] H^{(0)}_{0,lm}$$
$$+\Big[\frac{\omega'}{2r}\{(n+1)r(C^{-2}-1) - 4(r-3M-\kappa p r^3)\} + \sigma^2 e^{-\nu} r(2\varpi+\omega)(C^{-2}-1)$$
$$-\frac{2\varpi e^{\lambda}}{r^3}[\{(n+1)r^2e^{-\lambda} + (r-M+\kappa p r^3)(r-3M-\kappa p r^3)\}(C^{-2}-1) + 2\kappa\varrho r^3(2r-5M-\kappa p r^3)$$
$$-2(re^{-\lambda}-\kappa p r^3)(r-3M-\kappa p r^3) + 2\kappa p r^4 e^{-\lambda}]\Big] K^{(0)\prime}_{lm}$$
$$+\Big[-\frac{\omega'}{r}\{\sigma^2 r^2 e^{-\nu}(C^{-2}-1) + 2n\} + 4\varpi\sigma^2 e^{-\nu}\{C^{-2} - 2 + \kappa(\varrho+p)r^2 e^{\lambda}\}$$
$$-\frac{2n\varpi e^{\lambda}}{r^3}\{(r-M+\kappa p r^3)(C^{-2}-1) - 2re^{-\lambda} + 2\kappa(\varrho+p)r^3\}$$
$$-\omega\sigma^2 e^{\lambda-\nu}\{(n+1-e^{-\lambda})(C^{-2}+1) + 4e^{-\lambda} - 2\kappa(\varrho+p)r^2\}\Big] K^{(0)}_{lm} \qquad (49)$$

and

$$D^{(2)}_{l\pm1m} = \Big[-4\omega'\{(n-n_\pm-1)C^{-2} - 2n_\pm - 1\}$$
$$-\frac{4(n_\pm+1)e^{\lambda}\omega}{r}\{2\kappa(\varrho+p)r^2 + (n+1+e^{-\lambda})(C^{-2}-1) - 2e^{-\lambda}\}\Big] X^{(0)\prime}_{l\pm1m}$$
$$+\Big[\frac{2\omega' e^{\lambda}}{r^2}[\{(n_\pm^2+n_\pm)r - 2(n-1)re^{-\lambda}\}(C^{-2}+1) + 2nr(n_\pm+e^{-\lambda}) + 4n_\pm(2M-\kappa\varrho r^3)]$$
$$+\frac{4(n_\pm+1)\omega e^{\lambda}}{r^3}[\{\sigma^2 e^{-\nu} r^3 - (n+2n_\pm+2)r + 2M\}(C^{-2}-1) + 2re^{-\lambda} - 2\kappa(\varrho+p)r^3]$$
$$-16\kappa n_\pm\varpi(\varrho+p)e^{\lambda}\Big] X^{(0)}_{l\pm1m}. \qquad (50)$$

In a similar way, some terms due to the rotational effect are included in the equations describing the fluid motion, Eqs. (40) and (41), but those expressions are omitted here.

VI. ZEROTH-ORDER EQUATIONS IN VACUUM

In order to determine the eigenfrequency of the nonradial oscillation, it is necessary to connect the interior perturbation equations with the exterior ones. In this section, we review the basic equations of the order ϵ^0, which correspond to the perturbation equations of a Schwarzschild black hole [17,18]. The perturbations of Einstein equations are given by the condition (7) and $\varrho = p = \delta\varrho_{lm} = \delta p_{lm} = R_{lm} = V_{lm} = U_{lm} = 0$. The basic equations are obtained by solving Eqs. (28) and (29) under these conditions.

A. Odd-parity modes

It is easy to derive the basic equation from the analysis of Sec. IV. Equations (30) and (31) reduce to

$$h_{0,lm} = \frac{e^{-\lambda}}{-i\sigma}(X_{lm}r)', \qquad (51)$$

and

$$h_{1,lm} = e^{\lambda} X_{lm} r. \qquad (52)$$

The basic equation (32) becomes

$$\hat{L}_0[X_{lm}] \equiv e^{-\lambda}(e^{-\lambda}X'_{lm})' + \Big[\sigma^2 - e^{-\lambda}\Big(\frac{2(n+1)}{r^2} - \frac{6M}{r^3}\Big)\Big] X_{lm} = 0. \qquad (53)$$

It is easily checked that the remaining equation $\beta_{lm}^{(0)} = 0$ is automatically satisfied. We denote $X_{lm}^{(0)}$ as the solution of Eq. (53), which will be used in the next section.

B. Even-parity modes

Next we consider the even-parity mode. We can reduce the order of differential equation, because there is no dynamical degree of the freedom for the fluid motion. From $s_{lm} = 0$, we have

$$H_{2,lm} - H_{0,lm} = 0. \tag{54}$$

From $A_{lm}^{(1)} = 0$, $\alpha_{lm}^{(0)} = 0$, and $\alpha_{lm}^{(1)} = 0$, we have the following differential equations for $H_{0,lm}$, $H_{1,lm}$, and K_{lm}:

$$\hat{L}_e^{(1)}[H_{0,lm}, H_{1,lm}, K_{lm}] \equiv K'_{lm} + \frac{e^\lambda (r-3M)}{r^2} K_{lm} - \frac{1}{r} H_{0,lm} + \frac{n+1}{i\sigma r^2} H_{1,lm} = 0, \tag{55}$$

$$\hat{L}_e^{(2)}[H_{0,lm}, H_{1,lm}, K_{lm}] \equiv \frac{1}{i\sigma} H'_{1,lm} + e^\lambda (K_{lm} + H_{0,lm}) + \frac{2Me^\lambda}{i\sigma r^2} H_{1,lm} = 0, \tag{56}$$

$$\hat{L}_e^{(3)}[H_{0,lm}, H_{1,lm}, K_{lm}] \equiv H'_{0,lm} + \frac{e^\lambda (r-3M)}{r^2} K_{lm} - \frac{e^\lambda (r-4M)}{r^2} H_{0,lm} - \left(\sigma^2 e^\lambda - \frac{n+1}{r^2}\right) \frac{1}{i\sigma} H_{1,lm} = 0. \tag{57}$$

Using these equations in $A_{lm}^{(2)} = 0$, we have an identity among $H_{0,lm}$, $H_{1,lm}$, and K_{lm}:

$$\hat{I}[H_{0,lm}, H_{1,lm}, K_{lm}] \equiv (\sigma^2 r^4 e^\lambda - nr^2 - Mr + M^2 e^\lambda) K_{lm} + (nr+3M) r H_{0,lm} + \frac{1}{i\sigma}[\sigma^2 r^3 - (n+1)M] H_{1,lm} = 0. \tag{58}$$

We can also check the remaining equations $A_{lm}^{(0)} = A_{lm}^{(3)} = 0$ are automatically satisfied. Thus we have three first-order differential equations and one identity. We can therefore reduce to a second-order system of the differential equations. For example, we may solve Eqs. (55) and (57) after eliminating $H_{1,lm}$ by Eq. (58). If we prefer a second-order differential equation, we can transform to the Regge-Wheeler function X_{lm} or the Zerilli function Z_{lm} from $H_{0,lm}$ and K_{lm} as

$$X_{lm} = \frac{1}{\sigma^2 r^3 - (n+1)M} \left[\{3\sigma^2 M r^3 - n(n+1) r (r-3M)\} K_{lm} + (n+1) r (nr+3M) e^{-\lambda} H_{0,lm} \right], \tag{59}$$

$$X'_{lm} = \frac{1}{\sigma^2 r^3 - (n+1)M} \Big[\{-\sigma^2 r e^\lambda (nr+3M) + n(n+1)\}\{(n+1)r - 3M\} K_{lm} + \{3\sigma^2 M r^2 - n(n+1)^2 r - 3(n+1) M e^{-\lambda}\} H_{0,lm} \Big], \tag{60}$$

$$Z_{lm} = \frac{r}{\sigma^2 r^3 - (n+1)M} \left[\frac{nr^2 - 3nMr - 3M^2}{nr+3M} K_{lm} - r e^{-\lambda} H_{0,lm} \right], \tag{61}$$

$$Z'_{lm} = \frac{\sigma^2 r^3 e^\lambda (nr+3M)^2 - n^2(n+1) r^3 - 3n^2 M r^2 - 9nM^2 r - 9M^3}{[\sigma^2 r^3 - (n+1)M](nr+3M)^2} K_{lm} + \frac{n^2(n+1) r^3 + 3n(2n+1) M r^2 + 15 n M^2 r + 18 M^3}{[\sigma^2 r^3 - (n+1)M](nr+3M)^2} H_{0,lm}. \tag{62}$$

The function X_{lm} satisfies the Regge-Wheeler equation (53) and Z_{lm} satisfies the Zerilli equation

$$e^{-\lambda}(e^{-\lambda} Z'_{lm})' + \left[\sigma^2 - \frac{2e^{-\lambda}}{(nr+3M)^2 r^3} \{n^2(n+1) r^3 + 3n^2 M r^2 + 9nM^2 r + 9M^3\} \right] Z_{lm} = 0. \tag{63}$$

In the following argument, however, we will use $K_{lm}^{(0)}$, $H_{0,lm}^{(0)}$, and $H_{1,lm}^{(0)}$, the solutions of Eqs. (55)–(58), to avoid the confusion concerning the parity.

VII. FIRST-ORDER EQUATIONS IN VACUUM

In this section, we shall include the effect of the order ϵ in the perturbation equations for vacuum. The procedure to derive the basic equations is the same as shown in Sec. VI. Some source terms will be included in the basic equations of the nonrotating case. We can represent the source terms by $X^{(0)}_{lm}, X^{(0)\prime}_{lm}, K^{(0)}_{lm}, H^{(0)}_{0,lm}$, and $H^{(0)}_{1,lm}$.

A. Odd-parity modes

The odd-parity equation becomes

$$\hat{L}_0[X_{lm}] = \frac{\omega}{\sigma}\left(m\hat{N}_{lm} + \frac{iQ_{l-1m}\hat{F}_{l-1m}}{n-n_-} + \frac{iQ_{l+1m}\hat{F}_{l+1m}}{n-n_+}\right), \tag{64}$$

where $\hat{L}_0$ is the second-order differential operator defined in Eq. (53). The source terms are given by

$$\hat{N}_{lm} = \frac{6e^{-2\lambda}}{r}X^{(0)\prime}_{lm} + \left[2\sigma^2 - \frac{6e^{-\lambda}}{(n+1)r^3}\{n(2r-5M) - re^{-\lambda}\}\right]X^{(0)}_{lm} \tag{65}$$

and

$$\hat{F}_{l\pm1m} = -6\sigma^2 e^{-\lambda}K^{(0)}_{l\pm1m}. \tag{66}$$

We can check the consistency of the remaining equation, in which the first-order effect is included corresponding to $\beta^{(0)}_{lm} = 0$.

B. Even-parity modes

We turn our consideration to the even-parity mode. The relation (54) between $H_{2,lm}$ and $H_{0,lm}$ becomes

$$\begin{aligned} H_{2,lm} - H_{0,lm} = {} & \frac{m\sigma\omega r^2 e^{\lambda}}{n+1}K^{(0)}_{lm} + \frac{4iQ_{l-1m}\omega}{\sigma(n-n_-)}[(n_-+1)rX^{(0)\prime}_{l-1m} - (2n_- - 1)X^{(0)}_{l-1m}] \\ & + \frac{4iQ_{l+1m}\omega}{\sigma(n-n_+)}[(n_+ +1)rX^{(0)\prime}_{l+1m} - (2n_+ - 1)X^{(0)}_{l+1m}]. \end{aligned} \tag{67}$$

The basic equations, (55)–(57), become

$$\hat{L}^{(J)}_e[H_{0,lm}, H_{1,lm}, K_{lm}] = \frac{\omega}{\sigma}\left(m\hat{E}^{(J)}_{lm} + \frac{iQ_{l-1m}\hat{D}^{(J)}_{l-1m}}{n-n_-} + \frac{iQ_{l+1m}\hat{D}^{(J)}_{l+1m}}{n-n_+}\right) \quad (J = 1 \text{ to } 3), \tag{68}$$

where $\hat{L}^{(1)}_e$, $\hat{L}^{(2)}_e$, and $\hat{L}^{(3)}_e$ are the first-order differential operators, defined in Eqs. (55)–(57). The source terms for $J = 1$ are given by

$$\hat{E}^{(1)}_{lm} = \frac{e^{\lambda}}{2r^2}\left(\frac{2\sigma^2 r^3}{n+1} - r + 3M\right)K^{(0)}_{lm} + \frac{1}{r}H^{(0)}_{0,lm} - \frac{n+1}{2i\sigma r^2}H^{(0)}_{1,lm}, \tag{69}$$

$$\hat{D}^{(1)}_{l\pm1m} = 4(n_\pm + 1)X^{(0)\prime}_{l\pm1m} + \frac{2}{r}[(n+1)(n_\pm+1)e^{\lambda} - 2(2n_\pm - 1)]X^{(0)}_{l\pm1m}. \tag{70}$$

The source terms for $J = 2$ are given by

$$\begin{aligned} \hat{E}^{(2)}_{lm} = {} & \frac{e^{2\lambda}}{2(n+1)r^2}[-2\sigma^2 r^4 + (n+1)r^2 e^{-\lambda} + r^2 - 9Mr + 15M^2]K^{(0)}_{lm} \\ & + \frac{e^{\lambda}}{2(n+1)r}[(n-2)r + 7M]H^{(0)}_{0,lm} - \frac{e^{\lambda}}{2i\sigma r^2}\left(\frac{\sigma^2 r^3}{n+1} - 2r + 5M\right)H^{(0)}_{1,lm}, \end{aligned} \tag{71}$$

$$\hat{D}^{(2)}_{l\pm1m} = -2(n_\pm + 1)re^{\lambda}X^{(0)\prime}_{l\pm1m} - (6n - 16n_\pm - 4)e^{\lambda}X^{(0)}_{l\pm1m}. \tag{72}$$

The source terms for $J = 3$ are given by

$$\hat{E}^{(3)}_{lm} = -\frac{e^{\lambda}}{2r^2}\left(\frac{\sigma^2 r^2 e^{\lambda}}{n+1}(2r - M) + r - 3M\right)K^{(0)}_{lm} - \frac{1}{r}\left(\frac{\sigma^2 r^2 e^{\lambda}}{2(n+1)} - 1\right)H^{(0)}_{0,lm} - \frac{1}{2i\sigma r^2}(\sigma^2 r^2 e^{\lambda} + n + 1)H^{(0)}_{1,lm}, \tag{73}$$

$$\hat{D}^{(3)}_{l\pm1m} = (6n - 4n_{\pm} - 4)X^{(0)\prime}_{l\pm1m} + \frac{2e^{\lambda}}{r}[(n_{\pm}+1)(\sigma^2 r^2 e^{\lambda} + n) - 4n_{\pm}^2 + 1 + (3n - 5n_{\pm} - 2)e^{-\lambda}]X^{(0)}_{l\pm1m}. \quad (74)$$

The identity (58) becomes

$$\hat{I}[H_{0,lm}, H_{1,lm}, K_{lm}] = \frac{\omega}{\sigma}\left(m\hat{E}^{(4)}_{lm} + \frac{iQ_{l-1m}\hat{D}^{(4)}_{l-1m}}{n - n_-} + \frac{iQ_{l+1m}\hat{D}^{(4)}_{l+1m}}{n - n_+}\right), \quad (75)$$

where $\hat{I}$ is the algebraic equation for $H_{0,lm}, H_{1,lm}$, and K_{lm}, defined in (58). The source terms are given by

$$\hat{E}^{(4)}_{lm} = \frac{e^{\lambda}}{2}\left(\frac{\sigma^2 r^3}{n+1}(2nr + 3M) + (r - 3M)M\right)K^{(0)}_{lm} - \frac{r}{2}\left(\frac{\sigma^2 r^3}{n+1} + 2M\right)H^{(0)}_{0,lm} + \frac{(n+1)M}{2i\sigma}H^{(0)}_{1,lm}, \quad (76)$$

$$\hat{D}^{(4)}_{l\pm1m} = -2r^3\{(2n+3)(n_{\pm}+1) - 3(n - n_{\pm} - 1)e^{-\lambda}\}X^{(0)\prime}_{l\pm1m}$$
$$+2re^{\lambda}[(n_{\pm}+1)\sigma^2 r^3 + 3(n - n_{\pm} - 1)re^{-2\lambda} - (n_{\pm}+1)\{(2n + 3n_{\pm} + 3)re^{-\lambda} + (n+1)M\}]X^{(0)}_{l\pm1m}. \quad (77)$$

We can also check the consistency of the two remaining equations, in which the first-order effect is included corresponding to $A^{(0)}_{lm} = A^{(3)}_{lm} = 0$. In this way, the basic equation for the even-parity mode is the second-order system of the differential equation. It is possible to eliminate one function, e.g., $H_{1,lm}$ by Eq. (75), and solve two differential equations for the remaining functions.

VIII. CONCLUDING REMARKS

The basic equations governing nonradial oscillations of a slowly rotating relativistic star are Eqs. (42), (46), (64), (68), and (75). These equations explicitly represent how the modes in a spherical symmetric star are mixed due to stellar rotation. We expect that the degeneracy with respect to the azimuthal spherical harmonic index m is removed like the Zeeman splitting, because some terms depend on m . The basic equations for the odd-parity modes are a second-order system both inside and outside the star. There is a regularity condition at the center and a boundary condition at infinity. The outgoing wave condition is normally used for the latter condition. On the other hand, those for the even-parity modes are a fourth-order system inside the star, and a second-order system outside the star. There are two regularity conditions at the center. We impose the boundary condition that the Lagrangian change in pressure must vanish at the stellar surface. The interior perturbations are smoothly connected with the exterior perturbations. The outgoing wave condition is imposed for the perturbation at infinity. In this way, we have an eigenvalue problem. In a subsequent paper, we will solve these equations numerically and obtain the eigenfrequency as a function of the angular velocity. The prospects and the implication are given here.

The basic equations describe two important phenomena. One is the rotationally induced oscillation of the opposite parity mode, as partially examined by Chandrasekhar and Ferrari [10]. We can generalize their calculations, that is, the oscillation of the even-parity mode excited by the odd-parity mode as well as that of the odd-parity mode by the even-parity mode.

The other is the rotational shift of the eigenfrequency. We shall assume there exists only the even-parity mode for the zeroth solution for simplicity. The first-order correction changes the eigenfrequency of the even-parity mode with $m \neq 0$. The quasinormal modes of nonradial oscillations in a nonrotating star are so far studied. The eigenfrequency is a complex number. The real part means the oscillatory frequency and the imaginary part represents the damping rate due to the gravitational radiation reaction. Thus, the nonradial pulsations are stable for the nonrotating star. As the stellar rotation increases, the eigenfrequency and hence damping rate changes. We expect that the sign of the imaginary part changes for the large angular velocity for particular modes; that is, the modes become unstable. This corresponds to the secular instability of the nonaxisymmetric perturbations due to the gravitational radiation reaction. In this way, we will have a new estimate for the critical angular velocity for a stably rotating star by the approximation of the slow rotation, but fully relativistic treatment. The instability sets in for a more slowly rotating star as m increases. This means our approximation is good for such large m modes.

ACKNOWLEDGMENTS

I would like to thank B. F. Schutz for useful discussions at the beginning of this work. This work was supported in part by the Grant-in-Aid for Scientific Research on Priority Area of Japan (03250212, 04234209).

APPENDIX A

The perturbations of the energy-momentum tensor are given in this appendix. Ten independent components $\delta T_{\mu\nu}$ in the Regge-Wheeler gauge [Eq. (12)] are given by

$$\delta T_{tt} = e^{\nu} \sum_{lm} [(\delta \varrho_{lm} - \varrho H_{0,lm}) Y_{lm} + 2\Omega \kappa^{-1} (U_{lm} \sin\theta \partial_\theta Y_{lm} + V_{lm} \partial_\phi Y_{lm})], \tag{A1}$$

$$\delta T_{tr} = -\sum_{lm} [(\varrho H_{1,lm} + e^{\nu} \kappa^{-1} R_{lm}) Y_{lm} + (\varrho + p) \Omega h_{1,lm} (\sin\theta \partial_\theta Y_{lm})], \tag{A2}$$

$$\delta T_{rr} = e^{\lambda} \sum_{lm} (\delta p_{lm} + p H_{2,lm}) Y_{lm}, \tag{A3}$$

$$\delta T_{t\theta} = \sum_{lm} [(\varrho h_{0,lm} + e^{\nu} \kappa^{-1} U_{lm}) (\partial_\phi Y_{lm} / \sin\theta) - e^{\nu} \kappa^{-1} V_{lm} \partial_\theta Y_{lm}], \tag{A4}$$

$$\delta T_{t\phi} = -\sum_{lm} [\, (\varrho h_{0,lm} + e^{\nu} \kappa^{-1} U_{lm}) (\sin\theta \partial_\theta Y_{lm}) + e^{\nu} \kappa^{-1} V_{lm} \partial_\phi Y_{lm} \\ + \{\Omega \delta p_{lm} - (\varrho + p) \Omega (H_{0,lm}/2 - K_{lm}) + \varpi \delta \varrho_{lm}\} r^2 \sin^2\theta Y_{lm}], \tag{A5}$$

$$\delta T_{r\theta} = -p \sum_{lm} h_{1,lm} (\partial_\phi Y_{lm} / \sin\theta), \tag{A6}$$

$$\delta T_{r\phi} = \sum_{lm} [p h_{1,lm} (\sin\theta \partial_\theta Y_{lm}) + \varpi r^2 \sin^2\theta \{\kappa^{-1} R_{lm} + (\varrho + p) e^{-\nu} H_{1,lm}\} Y_{lm}], \tag{A7}$$

$$\delta T_{\theta\phi} = \varpi r^2 \sin^2\theta \sum_{lm} [\kappa^{-1} V_{lm} (\partial_\theta Y_{lm}) - \{\kappa^{-1} U_{lm} + (\varrho + p) e^{-\nu} h_{0,lm}\} (\partial_\phi Y_{lm} / \sin\theta)], \tag{A8}$$

$$\delta T_{\theta\theta} = r^2 \sum_{lm} (\delta p_{lm} + p K_{lm}) Y_{lm}, \tag{A9}$$

$$\delta T_{\phi\phi} = r^2 \sin^2\theta \sum_{lm} [(\delta p_{lm} + p K_{lm}) Y_{lm} + 2\varpi \{((\varrho + p) e^{-\nu} h_{0,lm} + \kappa^{-1} U_{lm}) (\sin\theta \partial_\theta Y_{lm}) + \kappa^{-1} V_{lm} \partial_\phi Y_{lm}\}], \tag{A10}$$

where we have used the definitions

$$\kappa(\varrho + p) e^{\lambda - \nu/2} \delta u^r = \sum_{lm} R_{lm} Y_{lm}, \tag{A11}$$

$$\kappa(\varrho + p) e^{-\nu/2} r^2 \delta u^\theta = \sum_{lm} [V_{lm} \partial_\theta Y_{lm} - U_{lm} (\partial_\phi Y_{lm} / \sin\theta)], \tag{A12}$$

$$\kappa(\varrho + p) e^{-\nu/2} r^2 \sin^2\theta \delta u^\phi = \sum_{lm} [V_{lm} \partial_\phi Y_{lm} + U_{lm} (\sin\theta \partial_\theta Y_{lm})]. \tag{A13}$$

We have also expanded the density and pressure perturbations as

$$\delta\varrho = \sum_{lm} \delta\varrho_{lm} Y_{lm}, \quad \delta p = \sum_{lm} \delta p_{lm} Y_{lm}. \tag{A14}$$

APPENDIX B

In this appendix, the functions appeared in Eqs. (14)–(27) are given. A prime and a dot mean a derivative with respect to r and t, respectively.

The coefficients in Eqs. (14) and (20) are

$$A^{(0)}_{lm} = 2e^{\nu-\lambda} K''_{lm} + \frac{2e^{\nu}}{r^2} (3r - 5M - \kappa\varrho r^3) K'_{lm} - \frac{2e^{\nu-\lambda}}{r} H'_{2,lm} \\ - \frac{(l-1)(l+2)e^{\nu}}{r^2} K_{lm} - \frac{e^{\nu}}{r^2} [l(l+1) + 2 - 4\kappa\varrho r^2] H_{2,lm} + 4\kappa e^{\nu} \delta\varrho_{lm}, \tag{B1}$$

$$A^{(1)}_{lm} = 2\dot{K}'_{lm} + \frac{2e^{\lambda}}{r^2} (r - 3M - \kappa p r^3) \dot{K}_{lm} - \frac{2}{r} \dot{H}_{2,lm} - \frac{l(l+1)}{r^2} H_{1,lm} - 4e^{\nu} R_{lm}, \tag{B2}$$

$$A^{(2)}_{lm} = 2e^{\lambda-\nu}\ddot{K}_{lm} - \frac{2e^{\lambda}}{r^2}(r - M + \kappa p r^3)K'_{lm} - \frac{4e^{-\nu}}{r}\dot{H}_{1,lm} + \frac{2}{r}H'_{0,lm}$$
$$+\frac{(l-1)(l+2)e^{\lambda}}{r^2}K_{lm} - \frac{l(l+1)e^{\lambda}}{r^2}H_{0,lm} + \frac{2e^{\lambda}}{r^2}(1 + 2\kappa p r^2)H_{2,lm} + 4\kappa e^{\lambda}\delta p_{lm}, \tag{B3}$$

$$A^{(3)}_{lm} = e^{-\lambda}(H''_{0,lm} - K''_{lm}) + e^{-\nu}(\ddot{K}_{lm} + \ddot{H}_{2,lm}) - 2e^{-\lambda-\nu}\dot{H}'_{1,lm}$$
$$+\frac{1}{r^2}[r + M - \kappa(\varrho - 2p)r^3]H'_{0,lm} + \frac{1}{r^2}(r - M + \kappa p r^3)H'_{2,lm}$$
$$-\frac{1}{r^2}[2r - 2M - \kappa(\varrho - p)r^3]K'_{lm} - \frac{2e^{-\nu}}{r^2}(r - M - \kappa\varrho r^3)\dot{H}_{1,lm} - \frac{l(l+1)}{2r^2}H_{0,lm}$$
$$+\frac{1}{2r^2}[l(l+1) + 8\kappa p r^2]H_{2,lm} + 4\kappa\delta p_{lm}, \tag{B4}$$

$$\tilde{A}^{(0)}_{lm} = 0, \tag{B5}$$

$$\tilde{A}^{(1)}_{lm} = -\frac{2l(l+1)\omega}{r^2}h_{1,lm}, \tag{B6}$$

$$\tilde{A}^{(2)}_{lm} = -\frac{4l(l+1)\omega e^{\lambda-\nu}}{r^2}h_{0,lm}, \tag{B7}$$

$$\tilde{A}^{(3)}_{lm} = -\frac{2l(l+1)\omega e^{-\nu}}{r^2}h_{0,lm}, \tag{B8}$$

$$B^{(0)}_{lm} = -2\omega e^{-\lambda}(h''_{0,lm} - \dot{h}'_{1,lm}) - [e^{-\lambda}\omega' - 2\kappa(\varrho + p)\omega r](h'_{0,lm} - \dot{h}_{1,lm}) + \frac{4e^{-\lambda}\omega}{r}\dot{h}_{1,lm}$$
$$+\frac{2}{r^3}[e^{-\lambda}\omega' r^2 + \kappa(\varrho + p)r^3(4\Omega - 2\omega) + \{l(l+1)r - 4M\}\omega]h_{0,lm} + 8e^{\nu}\Omega U_{lm}, \tag{B9}$$

$$B^{(1)}_{lm} = -\omega e^{-\nu}(\dot{h}'_{0,lm} - \ddot{h}_{1,lm}) + \frac{2\omega e^{-\nu}}{r}\dot{h}_{0,lm} - \frac{2\omega}{r^2}h_{1,lm}, \tag{B10}$$

$$B^{(2)}_{lm} = \frac{e^{-\nu}}{r}(\omega' r + 4\omega)(h'_{0,lm} - \dot{h}_{1,lm}) + \frac{2e^{-\nu}}{r^3}[\omega' r^2 - e^{\lambda}\omega\{l(l+1)r + 4M + 4\kappa p r^3\}]h_{0,lm}, \tag{B11}$$

$$B^{(3)}_{lm} = 2\omega e^{-\lambda-\nu}(h''_{0,lm} - \dot{h}'_{1,lm}) + \frac{2e^{-\nu}}{r^2}[e^{-\lambda}r^2\omega' + \omega\{r - 3M - \kappa(\varrho + 2p)r^3\}]h'_{0,lm}$$
$$-\frac{2e^{-\nu}\omega}{r^2}(r - M - \kappa\varrho r^3)\dot{h}_{1,lm} - \frac{e^{-\nu}}{r^2}[2(r - M + \kappa p r^3)\omega' + 4\kappa(\varrho + p)r^2\Omega$$
$$-\frac{\omega}{r^2}\{4e^{\lambda}(M + \kappa p r^3)^2 - l(l+1)r^2 + 4Mr - 4\kappa p r^4\}]h_{0,lm} + 4\varpi U_{lm}, \tag{B12}$$

$$C^{(0)}_{lm} = 2\omega(e^{-\lambda}H'_{1,lm} - \dot{H}_{2,lm} - \dot{K}_{lm}) - \frac{1}{r^2}[e^{-\lambda}r^2\omega' - \omega\{4M - 2\kappa(\varrho - p)r^3\}]H_{1,lm} + 8\Omega e^{\nu}V_{lm}, \tag{B13}$$

$$C^{(1)}_{lm} = \omega(H'_{0,lm} - e^{-\nu}\dot{H}_{1,lm}) - \frac{1}{2r^2}[r^2\omega' + 2e^{\lambda}\omega(r - 3M - \kappa p r^3)](H_{0,lm} + H_{2,lm}), \tag{B14}$$

$$C^{(2)}_{lm} = 2\omega e^{\lambda-\nu}\dot{K}_{lm} - \frac{e^{-\nu}}{r}(\omega' r + 4\omega)H_{1,lm}, \tag{B15}$$

$$C^{(3)}_{lm} = -\omega e^{-\nu}(2e^{-\lambda}H'_{1,lm} - 2\dot{H}_{2,lm} - \dot{K}_{lm}) - \frac{e^{-\nu}}{r^2}[\omega' e^{-\lambda}r^2 + 2\omega(r - M - \kappa\varrho r^3)]H_{1,lm} + 4\varpi V_{lm}. \tag{B16}$$

The coefficients in Eqs. (15), (16), (24), and (25) are

$$\alpha^{(0)}_{lm} = -e^{-\lambda}H'_{1,lm} + \dot{H}_{2,lm} + \dot{K}_{lm} - \frac{1}{r^2}[2M - \kappa(\varrho - p)r^3]H_{1,lm} - 4e^{\nu}V_{lm}, \tag{B17}$$

$$\alpha_{lm}^{(1)} = -H'_{0,lm} + K'_{lm} + e^{-\nu}\dot{H}_{1,lm} + \frac{e^{\lambda}}{r^2}(r - 3M - \kappa p r^3)H_{0,lm} - \frac{e^{\lambda}}{r^2}(r - M + \kappa p r^3)H_{2,lm}, \tag{B18}$$

$$\tilde{\alpha}_{lm}^{(0)} = -2\omega e^{-\lambda}h'_{1,lm} - \frac{1}{r^2}[3e^{-\lambda}r^2\omega' + 2\omega(r - M - \kappa\varrho r^3)]h_{1,lm}, \tag{B19}$$

$$\tilde{\alpha}_{lm}^{(1)} = -\omega e^{-\nu}(h'_{0,lm} + \dot{h}_{1,lm}) - \frac{e^{-\nu}}{r^2}[3\omega' r^2 - 2\omega e^{\lambda}(M + \kappa p r^3)]h_{0,lm}, \tag{B20}$$

$$\beta_{lm}^{(0)} = e^{-\lambda}(h''_{0,lm} - \dot{h}'_{1,lm}) - \kappa(\varrho + p)r(h'_{0,lm} - \dot{h}_{1,lm}) - \frac{2e^{-\lambda}}{r}\dot{h}_{1,lm} - \frac{1}{r^3}[l(l+1)r - 4M + 2\kappa(\varrho + p)r^3]h_{0,lm} - 4e^{\nu}U_{lm}, \tag{B21}$$

$$\beta_{lm}^{(1)} = e^{-\nu}(\dot{h}'_{0,lm} - \ddot{h}_{1,lm}) - \frac{2e^{-\nu}}{r}\dot{h}_{0,lm} - \frac{(l-1)(l+2)}{r^2}h_{1,lm}, \tag{B22}$$

$$\tilde{\beta}_{lm}^{(0)} = \omega(2K_{lm} - H_{2,lm}), \tag{B23}$$

$$\tilde{\beta}_{lm}^{(1)} = -\omega e^{-\nu}H_{1,lm}, \tag{B24}$$

$$\xi_{lm}^{(0)} = \frac{\omega}{2}H_{0,lm}, \tag{B25}$$

$$\xi_{lm}^{(1)} = \frac{\omega e^{-\nu}}{2}H_{1,lm}, \tag{B26}$$

$$\chi_{lm}^{(0)} = -\omega e^{-\nu}\dot{h}_{0,lm} - \frac{1}{2r^2}[e^{-\lambda}r^2\omega' + 2\omega(r - 3M - \kappa p r^3)]h_{1,lm}, \tag{B27}$$

$$\chi_{lm}^{(1)} = -\frac{\omega e^{-\nu}}{2}(h'_{0,lm} + \dot{h}_{1,lm}) - \frac{e^{-\nu}}{2r^2}[\omega' r^2 - 2\omega e^{\lambda}(M + \kappa p r^3)]h_{0,lm}, \tag{B28}$$

$$\eta_{lm}^{(0)} = -\frac{l(l+1)}{2r^2}[e^{-\lambda}r^2\omega' + 2\omega(r - 3M - \kappa p r^3)]h_{1,lm}, \tag{B29}$$

$$\eta_{lm}^{(1)} = \frac{3l(l+1)\omega e^{-\nu}}{2}(h'_{0,lm} - \dot{h}_{1,lm}) + \frac{l(l+1)e^{-\nu}}{2r^2}[r^2\omega' - 2\omega e^{\lambda}(2r - 3M + \kappa p r^3)]h_{0,lm}, \tag{B30}$$

$$\begin{aligned}\zeta_{lm}^{(0)} = {} & -\omega r^2 e^{-\lambda}(H''_{0,lm} - 2K''_{lm} - 2e^{-\nu}\dot{H}'_{1,lm}) - \omega r^2 e^{-\nu}(\ddot{H}_{2,lm} + \ddot{K}_{lm}) \\ & - \left(\frac{\omega' r^2 e^{-\lambda}}{2} + \omega\{r + M - \kappa(\varrho - 2p)r^3\}\right)H'_{0,lm} + 2\omega e^{-\nu}(r - M - \kappa\varrho r^3)\dot{H}_{1,lm} \\ & + \left(\frac{\omega' r^2 e^{-\lambda}}{2} - \omega(r - M + \kappa p r^3)\right)H'_{2,lm} + 2\omega(3r - 5M - \kappa\varrho r^3)K'_{lm} \\ & + \left(\frac{l(l+1)\omega}{2} + 2\kappa(\varrho + p)\Omega r^2\right)H_{0,lm} - [l(l+1)\omega + 4\kappa(\varrho\varpi + p\Omega)r^2]H_{2,lm} \\ & - [l(l+1)\omega + 4\kappa(\varrho + p)\Omega r^2]K_{lm} - 4\kappa(\varpi\delta\varrho_{lm} + \Omega\delta p_{lm})r^2,\end{aligned} \tag{B31}$$

$$\zeta_{lm}^{(1)} = \omega r^2 e^{-\nu}\dot{K}'_{lm} - \frac{\omega' r^2 e^{-\nu}}{2}(\dot{H}_{0,lm} - \dot{H}_{2,lm} + 2\dot{K}_{lm}) - \frac{l(l+1)\omega e^{-\nu}}{2}H_{1,lm} + 4\varpi r^2 R_{lm}. \tag{B32}$$

The coefficients in Eqs. (18), (19), (26) and (27) are

$$s_{lm} = -\frac{1}{2}(H_{0,lm} - H_{2,lm}), \tag{B33}$$

$$t_{lm} = e^{-\nu}\dot{h}_{0,lm} - e^{-\lambda}h'_{1,lm} - \frac{1}{r^2}[2M - \kappa(\varrho - p)r^3]h_{1,lm}, \tag{B34}$$

$$f_{lm} = \omega r^2 e^{-\nu}\dot{K}_{lm} + 4\varpi r^2 V_{lm}, \tag{B35}$$

$$g_{lm} = e^{-\lambda-\nu}\omega' r^2(h'_{0,lm} - \dot{h}_{1,lm}) - e^{-\nu}[2e^{-\lambda}r\omega' - l(l+1)\omega + 4\kappa(\varrho + p)\varpi r^2]h_{0,lm} - 4\varpi r^2 U_{lm}. \tag{B36}$$

[1] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press, New Haven, 1969); Phys. Rev. Lett. **24**, 611 (1970).
[2] L. Lindblom and S. L. Detweiler, Astrophys. J. **211**, 565 (1977).
[3] J.L. Friedman and B.F. Schutz, Astrophys. J. **222**, 281 (1978); J.L. Friedman, Commun. Math. Phys. **62**, 247 (1978).
[4] R.A. Managan, Astrophys. J. **294**, 463 (1985); J.N. Imamura, J.L. Friedman, and R.H. Durisen, *ibid.* **294**, 474 (1985).
[5] J.R. Ipser and L. Lindblom, Phys. Rev. Lett. **62**, 2777 (1989); Astrophys. J. **355**, 226 (1990).
[6] C. Cutler, Astrophys. J. **374**, 248 (1991).
[7] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. London **A432**, 247 (1991).
[8] J.R. Ipser and R.H. Price, Phys. Rev. D **43**, 1768 (1991).
[9] R.H. Price and J.R. Ipser, Phys. Rev. D **44**, 307 (1991).
[10] S. Chandrasekhar and V. Ferrari, Proc. R. Soc. London **A433**, 423 (1991).
[11] J.B. Hartle, K.S. Thorne, and S.M. Chitre, Astrophys. J. **176**, 177 (1972).
[12] J.B. Hartle, Astrophys. J. **150**, 1005 (1967); S. Chandrasekhar and J.C. Miller, Mon. Not. R. Astron. Soc. **167**, 63 (1974).
[13] T. Regge and J.A.Wheeler, Phys. Rev. **108**, 1063 (1957).
[14] K.S. Thorne and A. Campolattro, Astrophys. J. **149**, 591 (1967); R. Price and K.S. Thorne, *ibid.* **155**, 163 (1969); K.S. Thorne, *ibid.* **158**, 1 (1969); J.R. Ipser and K.S. Thorne, *ibid.* **181**, 181 (1973).
[15] S. Detweiler and L. Lindblom, Astrophys. J. **292**, 12 (1985).
[16] Y. Kojima, Prog. Theor. Phys. **77**, 297 (1987); **79**, 665 (1988).
[17] F.J. Zerilli, Phys. Rev. D **2**, 2141 (1970).
[18] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon, Oxford, 1983).