

Approximate equation relevant to axial oscillations on slowly rotating relativistic stars

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Axial oscillations relevant to the r -mode instability are studied with the slow rotation formalism in general relativity. The approximate equation governing the oscillations is derived with second-order rotational corrections. The equation contains an effective “viscositylike” term, which originates from coupling to the polar g -mode displacements. The term plays a crucial role on the resonance point, where the disturbance on the rotating stars satisfies a certain condition at the lowest order equation. The effect is significant for newly born hot neutron stars, which are expected to be subject to the gravitational radiation driven instability of the r mode.

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I. INTRODUCTION

The surprising discovery of the r -mode instability in rotating stars has inspired the study of axial oscillations [1–19]. The physical mechanism of the instability is the same as that in polar modes, the so-called radiation reaction instability found by Chandrasekhar [20], and Friedman and Schutz [21]. The r -mode oscillations seem to be more important since they are unstable on an inviscid rotating fluid even for a small angular velocity. This instability can explain the spin-down process of newly born neutron stars which rotate with nearly the Kepler frequency. The gravitational waves associated with the unstable r -mode oscillations may be promising detectable sources on the ground based laser interferometric detectors. It was proposed that the unstable mode might also play a key role on the spin of the accreting white dwarfs [13,14]. However, Lindblom [15] showed that the possibility is unlikely realized. It was also proposed that the r -mode instability could provide the loss of angular momentum from the accretion disk by the gravitational waves to halt the spin-up in the low mass x-ray binaries [13,16]. The proposed conclusion crucially depends on the poorly understood dissipation mechanism [17]. In this way, the r -mode oscillations enrich the astrophysical implications. Recent review of this subject is given by Friedman and Lockitch [19].

Most of the studies are, however, limited to idealized situation. Some effects are added to the simplified models to examine the validity. For example, Rezzolla, Lamb, and Shapiro [18] suggested that the magnetic field of a neutron star is wound up during the nonlinear growth of the unstable mode, and that the energy is not transferred to the gravitational radiation so much. However, their calculation is not self-consistent nonlinear one, so that the magnetic effect on the r mode is not conclusive at moment. The relativistic effects are also important for the oscillations in neutron stars. The relativistic factor is of order 0.2, so that the frequency could slightly shift from the Newtonian calculation. More important effects of the general relativity are gravitational wave and frame dragging. Each of them leads to qualitatively a different result. Some authors [3,6] already calculated the r modes in general relativity. The frame dragging effect is regarded as a kind of differential rotation [3]. The rotational effects are, however, limited to the lowest order. Mathemati-

cally, the treatment is insufficient, since the modes are degenerate at the order. It is necessary to include the higher order rotational corrections. Such a task will be significantly complicated when the rotation and relativistic effects are simultaneously considered.

One simplification for the nonrotating case is realized as the decomposition of the spherical harmonic function. Each oscillation mode can be specified for each index l, m . Furthermore, the polar and axial modes are definitely determined by an appropriate combination of the harmonic functions. However, the functions should be mixed in the presence of the rotation. Considering the slow rotation, the coupling is weak, so that the entangled range can be restricted. It is not known *a priori* to specify the axial oscillations on the rotating stars by a few spherical harmonic functions. Indeed, Lockitch and Friedman [10] calculated the normal mode by the sum of infinite number of the spherical harmonics indices. We will consider a different approach in this paper. Our treatment is suitable for the initial-value problem. We do not consider a single Fourier mode $e^{-i\sigma t}$ with respect to time. Suppose that the initial disturbance at $t=0$ is produced with a certain symmetry, which can be assumed to be specified by a few number of spherical harmonic functions. What happens in the subsequent evolution? If the oscillation possesses the symmetry, the oscillation preserves the symmetry. Otherwise, new patterns with different spherical harmonics indices are induced. It is clear that the truncated approximation to the finite number of the spherical harmonics becomes worse for large t in general. Therefore, our method is constructing the pulsation equation adequate for small t . It is difficult to address the valid domain of t beforehand. For example, the r -mode oscillation can be well described by a few number of the spherical harmonics indices for the oscillation in the uniform density with the Cowling approximation [7,22].

The present authors applied the method to the axial oscillation in a rotating relativistic star with the Cowling approximation [7], where the metric perturbations were neglected. They took account of the rotational correction up to third order to examine the oscillation equation. They pointed out the importance of the frame-dragging effect, which causes different property unlike the Newtonian case. Thus, the perturbative approach proves useful in providing a physical un-

derstanding of many processes. In this paper, we extend the approximation scheme to include the metric perturbations. The remainder of the paper is organized as follows. In Sec. II, we formulate the perturbation scheme to solve the linearized Einstein equations. We employ the slow rotation approximation, i.e., angular velocity is assumed to be small expansion parameter. We look for the solution in which axial-led functions are dominant, where the axial-led functions mean the functions describing axial modes in the absence of the rotation. We use the terminology ‘‘polar led’’ in the same way. In our scheme, the lowest order equations are determined only by the axial-led functions. They were already derived elsewhere [3,6,10,23] but are reviewed in Sec. III. In Sec. IV, first-order corrections to the polar-led functions are shown. In Sec. V, second-order correction terms are

added to the same equations in the form as the lowest ones. In Sec. VI, concluding remarks are given. Throughout this paper, we work in the geometrical units of $c = G = 1$.

II. PERTURBATION SCHEME

We consider a rotating star with uniform angular velocity $\Omega \sim O(\varepsilon)$, where ε is a small rotational parameter. The metric and fluid quantities describing the equilibrium state can be calculated by the slow rotation formalism [24,25]. They are summarized in Appendix A. We next investigate the perturbations from the state. The metric perturbations can be described by six functions. Working in the Regge-Wheeler gauge [26], the perturbations are expressed as

$$h_{\mu\nu} = \sum_{l,m} \begin{pmatrix} e^\nu H_{0lm}(t,r) Y_{lm} & H_{1lm}(t,r) Y_{lm} & -h_{0lm}(t,r) \frac{\partial_\phi Y_{lm}}{\sin \theta} & h_{0lm}(t,r) \sin \theta \partial_\theta Y_{lm} \\ \text{sym} & e^\lambda H_{2lm}(t,r) Y_{lm} & -h_{1lm}(t,r) \frac{\partial_\phi Y_{lm}}{\sin \theta} & h_{1lm}(t,r) \sin \theta \partial_\theta Y_{lm} \\ \text{sym} & \text{sym} & r^2 K_{lm}(t,r) Y_{lm} & 0 \\ \text{sym} & \text{sym} & \text{sym} & r^2 \sin^2 \theta K_{lm}(t,r) Y_{lm} \end{pmatrix}, \quad (1)$$

where $Y_{lm} = Y_{lm}(\theta, \phi)$ represents spherical harmonics. ‘‘sym’’ indicates that the missing components of $h_{\mu\nu}$ are to be found from the symmetry $h_{\mu\nu} = h_{\nu\mu}$. The angular part is expanded with an appropriate combination of the harmonics. In the same way, the fluid perturbations are described by five functions. They are the pressure perturbation δp , density perturbation $\delta\rho$, and three components of the four-velocity ($\delta u_r, \delta u_\theta, \delta u_\phi$). The component δu_t can be determined by the condition, $\delta u_{\mu\mu} = 0$. These perturbed quantities are also expanded as

$$\delta p = \sum_{l,m} \delta p_{lm}(t,r) Y_{lm}, \quad (2)$$

$$\delta\rho = \sum_{l,m} \delta\rho_{lm}(t,r) Y_{lm}, \quad (3)$$

$$(\rho + p) \delta u_r = e^{\nu/2} \sum_{l,m} R_{lm}(t,r) Y_{lm}, \quad (4)$$

$$(\rho + p) \delta u_\theta = e^{\nu/2} \sum_{l,m} \left[V_{lm}(t,r) \partial_\theta Y_{lm} - \frac{U_{lm}(t,r)}{\sin \theta} \partial_\phi Y_{lm} \right], \quad (5)$$

$$(\rho + p) \delta u_\phi = e^{\nu/2} \sum_{l,m} [V_{lm}(t,r) \partial_\phi Y_{lm} + U_{lm}(t,r) \sin \theta \partial_\theta Y_{lm}]. \quad (6)$$

These eleven functions are determined by ten components of the linearized Einstein field equations

$$\delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu} \quad (7)$$

and the adiabatic condition for the perturbations

$$\delta p + \xi \cdot \nabla p = \frac{\Gamma p}{p + \rho} (\delta\rho + \xi \cdot \nabla \rho), \quad (8)$$

where Γ is the adiabatic index and ξ is the Lagrange displacement.

Now we will solve the pulsation equations by the expansion of the spherical harmonics. In the spherically symmetric star, the equations are decoupled for each harmonic index (l, m) . The perturbations can also be decoupled into the axial and polar perturbations. They are, respectively, described by the axial functions $\mathcal{A}_{lm} \equiv (U_{lm}, h_{0,lm}, h_{1,lm})$, and the polar functions $\mathcal{P}_{lm} \equiv (\delta p_{lm}, \delta\rho_{lm}, R_{lm}, V_{lm}, H_{0,lm}, H_{1,lm}, H_{2,lm}, K_{lm})$. In the presence of rotation, the perturbations are described by the mixed state of them. From now on, we call these functions as the axial-led ones for \mathcal{A}_{lm} and the polar-led ones for \mathcal{P}_{lm} . Since the slow rotation is associated with the perturbation with $l = 1$, the formal relation between \mathcal{A}_{lm} and \mathcal{P}_{lm} in Eqs. (7),(8) can schematically be expressed as

$$0 = [\mathcal{A}_{lm}] + \mathcal{E} \times [\mathcal{P}_{l\pm 1m}] + \mathcal{E}^2 \times [\mathcal{A}_{lm}, \mathcal{A}_{l\pm 2m}] + \dots, \quad (9)$$

$$0 = [\mathcal{P}_{lm}] + \mathcal{E} \times [\mathcal{A}_{l\pm 1m}] + \mathcal{E}^2 \times [\mathcal{P}_{lm}, \mathcal{P}_{l\pm 2m}] + \dots, \quad (10)$$

where the symbol \mathcal{E} denotes some functions of order ε , and the square bracket formally represents the relation among the perturbation functions therein. These selection rules follow from the addition of angular momenta. We moreover assume that the axial-led functions are dominant in the slowly rotating star, i.e., $\mathcal{A}_{lm} \gg \mathcal{P}_{lm}$. This assumption is not valid for the some cases. If the star and its perturbations obey the same one-parameter equation of state, r modes and g modes are coupled at the lowest order in general. They are, respectively, described by \mathcal{P}_{lm} and \mathcal{A}_{lm} , which have zero frequency in the spherical isentropic star. They form hybrid modes in the rotating star [10]. However, the r modes are discriminated from the g modes for nonisentropic stars, since the g -mode frequencies are nonzero in the spherical limit. In that case, we may use the assumption $\mathcal{A}_{lm} \gg \mathcal{P}_{lm}$ to solve the r -mode oscillations, and expand as

$$\mathcal{A}_{lm} = \mathcal{A}_{lm}^{(1)} + \varepsilon^2 \mathcal{A}_{lm}^{(2)} + \dots, \quad \mathcal{P}_{lm} = \varepsilon (\mathcal{P}_{lm}^{(1)} + \varepsilon^2 \mathcal{P}_{lm}^{(2)} + \dots). \quad (11)$$

Substituting these functions into Eqs. (9),(10), and comparing each order of ε , we have the following equations of ε^n ($n=0,1,2$):

$$0 = [\mathcal{A}_{lm}^{(1)}], \quad (12)$$

$$0 = [\varepsilon \mathcal{P}_{l\pm 1m}^{(1)} + \mathcal{E} \times \mathcal{A}_{lm}^{(1)}], \quad (13)$$

$$0 = [\varepsilon^2 \mathcal{A}_{lm}^{(2)} + \mathcal{E} \times [\varepsilon \mathcal{P}_{l\pm 1m}^{(1)}] + \mathcal{E}^2 \times [\mathcal{A}_{lm}^{(1)}, \mathcal{A}_{l\pm 2m}^{(1)}] \quad (14)$$

$$= [\varepsilon^2 \mathcal{A}_{lm}^{(2)} + \mathcal{E}^2 \times \mathcal{A}_{lm}^{(1)}]. \quad (15)$$

We have here assumed that the perturbation in the lowest order is described by a single component of spherical harmonic, that is, $\mathcal{A}_{l'm}^{(1)} = 0$, for $l' \neq l$. The polar-led functions in Eq. (14) are eliminated by Eq. (13). Equation (12) represents the axial oscillation in the lowest order. Equation (15) is the second-order form of it, and the term $\mathcal{E}^2 \times \mathcal{A}_{lm}^{(1)}$ can be regarded as the rotational corrections. The method to solve the equations is straightforward. The first-order equations are solved by the axial-led functions. The polar-led functions are expressed using them. We have the second-order equations with the corrections expressed by the axial-led functions in the lowest order. These equations are successively solved in the following sections. In the actual calculations, we also assume that the time variation of the oscillation is slow and proportional to Ω , i.e., $\partial_t \sim \Omega \sim O(\varepsilon)$. This is true in the r -mode frequency $\partial_t \sim [1 - 2/l(l+1)]m\Omega$.

III. LOWEST-ORDER CALCULATION

In this section, we review the equations governing the axial oscillations at the lowest order. The radial functions are decoupled for each spherical harmonic index. We denote it

as $L \equiv (l, m)$ for the abbreviation. The relevant functions are $h_{0,L}$, $h_{1,L}$, and U_L . They are calculated from three components, essentially $(t\phi)$, $(r\phi)$, and $(\theta\phi)$ components, of the Einstein equations. We define a function $\Phi_L^{(1)}$ as

$$\Phi_L^{(1)} = \frac{h_{0,L}^{(1)}}{r^2}, \quad (16)$$

where the superscript (1) denotes the lowest order term in Eq. (11). The relation between the metric functions is given by

$$h_{1,L}^{(1)} = \frac{r^4 e^{-\nu}}{[l(l+1)-2]} \left[(\partial_T - im\varpi) \Phi_L^{(1)'} + \frac{2im\varpi'}{l(l+1)} \Phi_L^{(1)} \right], \quad (17)$$

where $\partial_T = \partial_t + im\Omega$ denotes a time derivative in a corotating frame and a prime denotes a derivation with respect to r . The axial velocity function is expressed in two ways:

$$(\partial_T - im\chi) U_L^{(1)} = -4\pi(p_0 + \rho_0) r^2 e^{-\nu} \partial_T \Phi_L^{(1)}, \quad (18)$$

$$U_L^{(1)} = \frac{j^2 r^2}{4} \left[\frac{1}{jr^4} (jr^4 \Phi_L^{(1)'})' - (v + 16\pi(p_0 + \rho_0) e^\lambda) \Phi_L^{(1)} \right], \quad (19)$$

where

$$\chi = \frac{2}{l(l+1)} \varpi = \frac{2}{l(l+1)} (\Omega - \omega), \quad (20)$$

$$v = \frac{e^\lambda}{r^2} [l(l+1) - 2], \quad (21)$$

$$j = e^{-(\lambda+\nu)/2}. \quad (22)$$

Eliminating $U_L^{(1)}$ in Eqs. (18),(19), we have a second-order differential equation for $\Phi_L^{(1)}$,

$$\begin{aligned} 0 &= \mathcal{L}[\Phi_L^{(1)}] \\ &= (\partial_T - im\chi) \left[\frac{1}{jr^4} (jr^4 \Phi_L^{(1)'})' - v \Phi_L^{(1)} \right] \\ &\quad + 16\pi im\chi (p_0 + \rho_0) e^\lambda \Phi_L^{(1)}. \end{aligned} \quad (23)$$

The linear operator \mathcal{L} is defined above. This equation is reduced to a singular eigenvalue problem [3], by assuming the time dependence for the mode as the form $e^{-i\sigma t}$. The equation can be written as

$$(\varpi - \mu) \left[\frac{1}{jr^4} (jr^4 \Phi_L^{(1)'})' - v \Phi_L^{(1)} \right] = q \Phi_L^{(1)}, \quad (24)$$

where

$$\mu = -\frac{l(l+1)}{2m} (\sigma - m\Omega), \quad (25)$$

$$q = \frac{1}{jr^4} (jr^4 \varpi')' = 16\pi(p_0 + \rho_0)e^\lambda \varpi \geq 0. \quad (26)$$

There is a singular point r_0 in Eq. (24) unless $q(r_0) = 0$, corresponding to the real value of $\mu = \varpi(r_0)$. It is evident that the singularity originates from the mismatch in Eq. (23), i.e., the first term vanishes whereas the second never does. When the first term, which is formally of first order, is small enough, then higher order corrections become important. This situation is very similar to the inviscid shear flows. When the viscosity is small enough, the stability is almost determined by the Rayleigh equation, the perturbation equation for the inviscid theory. The Rayleigh equation has critical points for some mean fluid. The viscous corrections should be included to determine the behavior near the neighborhood of the critical points. Therefore, the equation should be replaced by the Orr-Sommerfeld equation derived from the Navier-Stokes equation.

As we will show in the subsequent sections, the function $\Phi_L^{(1)}$ can also affect the polar-led functions. When one considers the equations of the next order $\Phi_L^{(2)}$ additional terms appear in the form (23). The terms depend on different aspect of the background flow, and play an important role as the viscosity terms.

IV. FIRST-ORDER CORRECTIONS IN POLAR-LED FUNCTIONS

In this section, we will derive the equations governing the polar-led functions $H_0^{(1)}$, $H_1^{(1)}$, $H_2^{(1)}$, $K^{(1)}$, $\delta\rho^{(1)}$, $\delta\rho_\pm^{(1)}$, $R^{(1)}$, and $V^{(1)}$. As shown in Sec. II, the functions with $(l \pm 1, m)$ are coupled with the axial-led functions with (l, m) . We will shorten the subscript of the spherical harmonic index as, e.g., $\delta\rho_\pm^{(1)} \equiv \delta\rho_{l \pm 1, m}^{(1)}$. These eight functions are determined by seven components of the linearized Einstein field equations and one thermodynamical relation. The calculations are straightforward, but results are sometimes messy. The pressure $\delta p_\pm^{(1)}$ and density perturbations $\delta\rho_\pm^{(1)}$ are expressed by

$$4\pi\delta\rho_\pm^{(1)} = 2\pi(p_0 + \rho_0)(H_{0,\pm}^{(1)} + 2T_\pm\Omega r^2 e^{-\nu}\Phi_L^{(1)}) + 2S_\pm\varpi U_L^{(1)}, \quad (27)$$

$$4\pi\delta\rho_\pm^{(1)} = -4\pi\frac{\rho_0'}{\nu'}(H_{0,\pm}^{(1)} + 2T_\pm\Omega r^2 e^{-\nu}\Phi_L^{(1)}) - 4S_\pm\frac{e^{-\nu/2}}{\nu'}(e^{\nu/2}\varpi U_L^{(1)})' + 2T_\pm\frac{e^\nu}{\nu' r^2}(\varpi r^2 e^{-\nu})' U_L^{(1)}, \quad (28)$$

where

$$S_+ = \frac{l}{l+1}Q_+, \quad S_- = \frac{l+1}{l}Q_-, \quad T_+ = lQ_+,$$

$$T_- = -(l+1)Q_-, \quad (29)$$

$$Q_+ = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}, \quad Q_- = \sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}. \quad (30)$$

These quantities are expressed by the axial-led functions $(\Phi_L^{(1)}, U_L^{(1)})$ and $H_{0,\pm}^{(1)}$. There is another relation among $H_{0,\pm}^{(1)}$, $K_\pm^{(1)}$, and $\delta\rho_\pm^{(1)}$ in the field equations. Eliminating $\delta\rho_\pm^{(1)}$ by Eq. (28), we have a second-order differential equation for the metric perturbations. The equation can be regarded as relativistic version of the Poisson equation $\nabla^2\delta\phi = 4\pi\delta\rho$, for the gravitational potential $\delta\phi$ and the density perturbation $\delta\rho$. In the relativistic case, the Newtonian potential is replaced by $H_0^{(1)}$ or $K^{(1)}$. The second-order differential equation is explicitly given by

$$K_\pm^{(1)'} - \frac{e^\lambda}{\nu' r^2} \{n_\pm(K_\pm^{(1)} - H_{0,\pm}^{(1)}) + 2[4\pi(p_0 + \rho_0)r^2 + e^{-\lambda} - 1]H_{0,\pm}^{(1)}\} = s_1, \quad (31)$$

$$(K_\pm^{(1)} - H_{0,\pm}^{(1)})' - \nu' H_{0,\pm}^{(1)} = \left(1 + \frac{\nu' r}{2}\right) s_1 + s_2, \quad (32)$$

where

$$n_\pm = -2 + l'(l' + 1)|_{l'=l \pm 1}, \quad (33)$$

$$s_1 = -Q_\pm[8\varpi r U_L^{(1)} + 32\pi(p_0 + \rho_0)\varpi r^3 e^{-\nu}\Phi_L^{(1)} + 2\varpi' j^2 r^3 \Phi_L^{(1)'}] + S_\pm \left[8 \left\{ r + \frac{e^\lambda}{\nu'} \right\} \varpi U_L^{(1)} - \frac{4e^{-\nu}}{\nu'} \{l(l+1)\omega e^\lambda + 16\pi p_0' \varpi r^3 - \varpi' r\} \Phi_L^{(1)} + 2\varpi' j^2 r^3 \Phi_L^{(1)'} \right] + T_\pm \left[\frac{2e^{-\nu}}{\nu'} \{[2 - 2e^\lambda - l(l+1)e^\lambda]\omega + 8\pi\Omega(p_0 + \rho_0)r^2 e^\lambda - \varpi' r\} \Phi_L^{(1)} - \frac{\varpi' r^2 e^{-\nu}}{\nu'} \Phi_L^{(1)'} \right], \quad (34)$$

$$\begin{aligned}
s_2 = & \left(r + \frac{2}{\nu'} \right) \left[-S_{\pm} [4\varpi e^{\lambda} U_L^{(1)} - 2\{l(l+1)\omega e^{\lambda} \right. \\
& - 2\varpi' r\} e^{-\nu} \Phi_L^{(1)}] + T_{\pm} \left\{ \frac{1}{2} \varpi' r^2 e^{-\nu} \Phi_L^{(1)'} \right. \\
& + [\{(l(l+1)+2)\omega - 8\pi(p_0 + \rho_0)\Omega r^2\} e^{\lambda - \nu} \\
& + 2\omega e^{-\nu}] \Phi_L^{(1)} \left. \right\} + \left[4S_{\pm} \frac{\varpi' r e^{-\nu}}{\nu'} \Phi_L^{(1)} \right. \\
& \left. + T_{\pm} \left(2\omega r^2 e^{-\nu} \Phi_L^{(1)'} + \frac{(2\varpi' r - 8\omega) e^{-\nu}}{\nu'} \Phi_L^{(1)} \right) \right]. \tag{35}
\end{aligned}$$

When the axial-led function at the lowest order $\Phi_L^{(1)}$ is given, the functions $H_{0,\pm}^{(1)}$ and $K_{\pm}^{(1)}$ are solved with appropriate boundary conditions. In a similar way, we can solve the other polar-led functions $H_{1,\pm}^{(1)}$, $H_{2,\pm}^{(1)}$, $R_{\pm}^{(1)}$, and $V_{\pm}^{(1)}$ by $(\Phi_L^{(1)}, U_L^{(1)})$ and $(H_{0,\pm}^{(1)}, K_{\pm}^{(1)})$. The expressions for these four polar-led functions are omitted here, since they are eliminated in the following calculations, and never appear in the final results. However, here is a comment on using the adiabatic condition Eq. (8). The time derivative of it can be written as

$$\begin{aligned}
4\pi\partial_T \left(\delta p_{\pm}^{(1)} - \frac{\Gamma p_0}{p_0 + \rho_0} \delta \rho_{\pm}^{(1)} \right) \\
= \frac{A\Gamma p_0 e^{\nu}}{p_0 + \rho_0} \left(e^{-\lambda} R_{\pm}^{(1)} - \frac{3im\xi_2}{r^2} Q_{\pm} U_L^{(1)} \right), \tag{36}
\end{aligned}$$

where

$$\xi_2 = -\frac{2}{\nu'} \left(h_2 + \frac{1}{3} \varpi^2 r^2 e^{-\nu} \right), \tag{37}$$

$$A = \frac{\rho_0'}{p_0 + \rho_0} - \frac{p_0'}{\Gamma p_0}. \tag{38}$$

This thermodynamical relation determines the function $R_{\pm}^{(1)}$ unless the Schwarzschild discriminant A vanishes. Otherwise, we have one constraint for the function $U_L^{(1)}$ through Eqs. (27),(28), and the function $R_{\pm}^{(1)}$ should be specified in another way. (For example, see the method in Ref. [7] in the Cowling approximation.) The mathematical drawback for the isentropic case $A=0$ is related with the coupling of the g modes. Both r modes and g modes are degenerate to zero frequency in the non-rotating star, and hence a particular treatment is necessary [10]. From now on, we will consider the case $A \neq 0$ only.

V. INCLUDING SECOND-ORDER CORRECTIONS

So far we have considered ten components of the Einstein equations and one thermodynamical relation. They were limited to the lowest-order form with respect to the rotational

parameter. In this section, we consider how the next order terms modify Eq. (23) derived in Sec. III. The relevant equations for this purpose are three components of the Einstein equations, i.e., $(t\phi)$, $(r\phi)$, and $(\theta\phi)$ components, which are used in the leading order equation in Sec. III. These equations contain the relations among the axial-led functions of second-order $h_{0,L}^{(2)}$, $h_{1,L}^{(2)}$, $U_L^{(2)}$ and the polar-led functions calculated in the previous section. We follow the same procedure as done in Sec. III. Defining the function $\Phi_L^{(2)} = h_{0,L}^{(2)}/r^2$ and eliminating $h_{1,L}^{(2)}$ and $U_L^{(2)}$, we eventually have the equation governing the axial oscillations. It can be written in the following form:

$$\mathcal{L}[\Phi_L^{(2)}] = \mathcal{D}[\Phi_L^{(1)}, h_{1,L}^{(1)}, U_L^{(1)}] + \mathcal{G}[H_{0,\pm}^{(1)}, K_{\pm}^{(1)}], \tag{39}$$

where the operator \mathcal{L} is defined in Eq. (23) and the right hand side means the second-order rotational corrections. They consist of several terms as

$$\begin{aligned}
\mathcal{D}[\Phi_L^{(1)}, h_{1,L}^{(1)}, U_L^{(1)}] = & \mathcal{D}_0 + \alpha_1 h_{1,L}^{(1)} + \alpha_2 \partial_T \Phi_L^{(1)} + im(\beta_1 U_L^{(1)} \\
& + \beta_2 \Phi_L^{(1)} + \beta_3 \Phi_L^{(1)}), \tag{40}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}[H_{0,\pm}^{(1)}, K_{\pm}^{(1)}] = & (A_1 \partial_T + imA_2) K_{\pm}^{(1)} \\
& + (B_1 \partial_T + imB_2) H_{0,\pm}^{(1)}. \tag{41}
\end{aligned}$$

The explicit forms of the coefficients α_i , β_i , A_i , and B_i are given in Appendix B. They are expressed by the quantities determined by the stellar model in the equilibrium. Since the term \mathcal{D}_0 contains higher order derivatives, we explicitly show

$$\begin{aligned}
\mathcal{D}_0 = & 32c_3 \frac{\varpi e^{-\nu}}{jr^2} \left\{ \frac{(\rho_0 + p_0)r^2}{jA\nu'} \left(\frac{\varpi \partial_T U_L^{(1)'}}{\rho_0 + p_0} \right)' \right. \\
& + \left. \left\{ \frac{16c_2 \varpi^2 e^{-\nu}}{(\rho_0 + p_0)jr^2} \left[\frac{(\rho_0 + p_0)e^{\nu}}{A\varpi j\nu'} (\varpi r^2 e^{-\nu})' \right] \right. \right. \\
& - \left. \left. \left(\frac{8c_1 e^{\nu}}{Aj^2 \nu' r^4} \right) [(\varpi r^2 e^{-\nu})']^2 \right\} \partial_T U_L^{(1)} + (\partial_T - im\chi) \right. \\
& \times \left\{ \left(\frac{\nu'}{r^2} - \frac{2}{r^3} \right) (\partial_T - im\varpi) + 2im \frac{\varpi'}{r^2} \right\} h_{1,L}^{(1)} \\
& - e^{\lambda - \nu} (\partial_T - im\chi) (\partial_T - im\varpi)^2 \Phi_L^{(1)}, \tag{42}
\end{aligned}$$

where

$$c_n = \frac{l+1}{l^n} Q_-^2 + (-1)^{n-1} \frac{l}{(l+1)^n} Q_+^2. \tag{43}$$

From Eqs. (23) and (39), a function $\Phi_L = \Phi_L^{(1)} + \varepsilon^2 \Phi_L^{(2)}$ satisfies the following equation, which is correct up to $O(\varepsilon^2)$,

$$\mathcal{L}[\Phi_L] = \mathcal{D}[\Phi_L, h_{1,L}, U_L] + \mathcal{G}[H_{0,\pm}, K_{\pm}]. \tag{44}$$

The quantities without the superscript satisfy the same relations as in the leading order, i.e., Eqs. (17), (19), (31), and

(32), which are adequate approximation to this order. This equation is of course reduced to the leading order equation (23), when the second-order rotational effects and the coupling to the polar modes are neglected.

The first term in \mathcal{D}_0 contains fourth derivative of Φ_L with respect to r , since U_L can be expressed by the second derivative of Φ_L as shown in Eq. (19). The highest derivative term of Φ_L with respect to r is therefore given by

$$\mathcal{D}_*[\Phi_L] = 8c_3 \frac{\varpi e^{-\nu}}{jr^2} \times \left\{ \frac{(\rho_0 + p_0)r^2}{jA\nu'} \left(\frac{j\varpi}{(\rho_0 + p_0)r^2} (jr^4 \partial_r \Phi_L') \right)' \right\}'. \quad (45)$$

Neglecting \mathcal{G} and \mathcal{D} except \mathcal{D}_* in Eq. (44) leads to $\mathcal{L}[\Phi_L] = \mathcal{D}_*[\Phi_L]$, which is analogous to the Orr-Sommerfeld equation in the incompressible shear flow (see Ref. [27]). The term (45) effectively gives the ‘‘viscosity’’ in the viscous fluid. The viscosity is important for the stability of the flows. For the small Reynolds number, the laminar flow is realized, whereas the flow becomes turbulence above a critical Reynolds number. The effective Reynolds number R_e in Eq. (45) is estimated from dimensional argument as

$$R_e \sim \frac{A\nu'}{\varpi^2}. \quad (46)$$

This is roughly the square of the ratio of the g mode frequency to rotational one. The viscosity term will play a key role on the singular point of the first-order equation, but the consequence is not clear at moment. It is necessary to explore further how the effective Reynolds number should operate in the stability and so on.

VI. CONCLUDING REMARKS

In this paper, we have explored an effective theory to describe the axial oscillation on a slowly rotating stars. The approximate equation governing the oscillation is constructed from the Einstein equations. The equation is derived by assuming that the angular dependence of the oscillation is dominated by a single component of spherical harmonics. The assumption in general breaks down as the evolution of oscillation. There are coupling terms of order ε in the rotating fluids. Other oscillation patterns with different spherical harmonics will gradually be produced through the rotational coupling. For this reason, the equation is valid for small t , and can be used to examine the time evolution as the initial-value problem. The rotational effects up to third order are involved in this paper, so that the regime of application is enlarged. The equation derived here is also irrelevant to the singular point found in the first order one.

The equation also shows a remarkable property. It is evident that the axial oscillation strongly couples to g -mode oscillations. A viscositylike term arises from the polar pieces related to the g -mode oscillations. The ‘‘viscosity’’ term

originated from considering the subsystem only, i.e., a single component of the spherical harmonics. There is no production or extinction in the whole system, but, e.g., the ‘‘energy’’ of a component is partially transferred to the others. This transportation is regarded as dissipative effect so far as a particular subsystem is concerned. The term also has a significant implication. The condition $A=0$ is a good approximation for cold neutron stars, so that the coupling may be neglected. On the other hand, it is not clear that the condition holds, in particular for newly born hot neutron stars, in which the r -mode instability sets in.

In this paper, we concentrated the equations only inside a star, to be more precise, the equations for the region $A \neq 0$. The pulsation equation derived here should be solved with appropriate boundary conditions. The boundary conditions are determined from matching with the equations outside, or regularity conditions. For example, the regularity condition of a function Φ is given by $\Phi \sim r^{l-1}$ near the center. Depending on details of the stellar structure, the solution for the $A \neq 0$ region may be matched to the solution for isentropic region, $A=0$. Furthermore, the interior solution should be matched to the exterior one at the stellar surface. The exterior perturbation equation in vacuum is not derived here, but the form should be reduced to the wave equation describing gravitational wave. The perturbation equation should be solved by out-going boundary condition at infinity. One question may arise. Is it possible to calculate the radiation reaction at this order? Newtonian estimate indicates that the backreaction is of order ε^{2m+2} for $m \geq 2$. Our expansion of the rotational parameter is limited to the third order, and higher order corrections are necessary to examine the effect in a consistent way. The radiation-reaction effect is also a kind of dissipative one, so that the accurate evolution for a long period is necessary.

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APPENDIX A: EQUILIBRIUM CONFIGURATION OF A SLOWLY ROTATING PERFECT FLUID

We here summarize the equilibrium of a slowly rotating star to explain our notation. The equilibrium state with uniform angular velocity $\Omega \sim O(\varepsilon)$ can be described by stationary and axisymmetric metric $g_{\mu\nu}$, four-velocity $u^\mu = (u^t, 0, 0, u^\phi)$, pressure p , and energy density ρ of the fluid. The rotational corrections up to $O(\varepsilon^3)$ is needed to assure the consistency in our analysis. The metric is given by

$$ds^2 = -e^\nu [1 + 2(h_0 + h_2 P_2)] dt^2 + e^\lambda \left[1 + \frac{2e^\lambda}{r} \times (m_0 + m_2 P_2) \right] dr^2 + r^2 (1 + 2k_2 P_2) \left\{ d\theta^2 + \sin^2 \theta \times \left[d\phi - \left(\omega + W_1 - W_3 \frac{1}{\sin \theta} \frac{dP_3}{d\theta} \right) dt \right]^2 \right\}, \quad (A1)$$

where $P_l = P_l(\cos\theta)$ ($l=2,3$) is the Legendre polynomial of order l . The metric functions introduced above obey the following ordering in ε : $\omega \sim O(\varepsilon)$, $h_0, h_2, m_0, m_2, k_2 \sim O(\varepsilon^2)$, $W_1, W_3 \sim O(\varepsilon^3)$. These are functions of radial coordinate r only. The components of the four-velocity are

$$u^t = (-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2)^{-1/2}, \quad u^\phi = \Omega u^t. \quad (\text{A2})$$

The pressure and energy density are, respectively, given by

$$p = p_0 + \{p_{20} + p_{22}P_2\}, \quad \rho = \rho_0 + \{\rho_{20} + \rho_{22}P_2\}, \quad (\text{A3})$$

where p_0 and ρ_0 are the pressure and energy density of non-rotating fluid. The centrifugal force of order ε^2 alters the configuration shape, which corresponds to the quantities in the braces. The functions $p_{20}, p_{22}, \rho_{20}$, and ρ_{22} are related with the metric functions of order ε^2 . We rather use the metric functions to eliminate the pressure and density of order ε^2 in the oscillation equations.

APPENDIX B: THE SECOND-ORDER TERMS

1. Coefficient of $h_{1L}^{(1)}$, α_1

$$\begin{aligned} \alpha_1 = & c_2 \left[32\pi(\rho_0 + p_0) \left(\frac{2}{\nu'^2 j^2 r^2} \varpi(\omega e^{-\nu})' + \frac{2}{r} \varpi \omega - (\omega + 3\varpi)\varpi' \right) - \frac{8}{r^2 e^\lambda} (\omega r)' \varpi' \right] \\ & + c_1 \left[-5\pi(\rho_0' + 3p_0')\varpi^2 + \left(\frac{28}{3r^2 e^\lambda} - \frac{5}{12r^2} + \frac{5}{2r^3 \nu'} \right) \omega \varpi' - \frac{97}{12r e^\lambda} \varpi'^2 \right. \\ & \left. + 4\pi(p_0 + \rho_0) \left(-\frac{5\varpi^2}{r} + 23\varpi\varpi' + \frac{28}{3}\omega\varpi' - \frac{56}{3r}\omega\varpi - \frac{12}{j^2 r^2 \nu'^2} \varpi(\omega e^{-\nu})' \right) \right] \\ & + \frac{m^2}{l(l+1)} \left[2\pi \left(l(l+1) - \frac{11}{2} \right) (\rho_0' + 3p_0')\varpi^2 + \frac{4\pi}{r^2} \left(l(l+1) - \frac{15}{2} \right) (\rho_0 + p_0) (\varpi^2 r^2)' - \frac{e^\nu}{2\nu'} \left(l(l+1) + \frac{3}{2} \right) (e^{-\nu} r^{-2})' \omega \varpi' \right. \\ & \left. - \frac{1}{2r e^\lambda} \left(l(l+1) - \frac{1}{6} \right) \varpi'^2 + \frac{1}{3r^2} (8e^{-\lambda} - 7)\omega\varpi' + 16\pi(\rho_0 + p_0) \right. \\ & \left. \times \left(\frac{\varpi^2}{r} - \frac{e^\lambda}{r^2 \nu'^2} \varpi\varpi' - \frac{e^\lambda}{\nu' r^2} \omega\varpi + \frac{2}{3} \omega r^2 (\varpi r^{-2})' + 3\varpi\varpi' \right) \right] + \frac{4}{r^3} e^{(6\nu-\lambda)/4} (e^{(-2\nu-3\lambda)/4} h_0') - \frac{e^\nu}{r^4} \{ l(l+1) - 4 - [(\nu' r)^2 \\ & - 6] e^{-\lambda} \} h_0' - \frac{1}{j^2 r^9} \left[\nu' r^7 e^{-2\lambda} \left(\frac{e^\lambda}{r} m_0 \right)' \right]' - \frac{e^\nu}{r^4} [l(l+1) - 2 + 4e^{-\lambda}] \left(\frac{e^\lambda}{r} m_0 \right)' - \frac{2\pi}{3} (\rho_0' + 3p_0') \\ & \times \left\{ \left[l(l+1) - 4 \frac{(e^\nu r^2)'}{r e^\nu} \right] \varpi^2 - \frac{12\nu'}{j^2 r^2} m_0 \right\} + \{ 4\pi(\rho_0 + p_0) r^2 [2(\nu' r)^2 + \nu' \lambda' r^2 + 12\nu' r + 2\lambda' r + 4] + 8 - 2\lambda' r \} \frac{m_0}{j^2 r^6} \\ & + \frac{32\pi^2}{3} (\rho_0 + p_0)^2 \varpi^2 e^{\lambda-\nu} (r^2 e^\nu)' - \frac{16\pi}{3r^5} e^{\nu/2} (r^6 \varpi^2 e^{-\nu/2})' p_0' - \frac{8\pi}{3r} (\rho_0 + p_0) [l(l+1)\varpi(\varpi r)' - 8\omega\varpi + 4r\Omega\varpi' - 2(3 \\ & - e^\lambda)\varpi^2] - \left\{ \frac{l(l+1)}{2r^2} \left(1 + \frac{2}{\nu' r} \right) + \frac{4}{3r^2} (2e^\lambda - 1) \right\} \omega \varpi' + \frac{1}{6r e^\lambda} \{ l(l+1) - 4 - 3\nu' r \} \varpi'^2. \end{aligned} \quad (\text{B1})$$

2. Coefficient of $\partial_t \Phi_L^{(1)}$, α_2

$$\begin{aligned} \alpha_2 = & c_2 \left[\frac{32\pi e^{\lambda-\nu}}{\nu'} (\varpi + \Omega) \varpi r^2 (\rho_0' + 3p_0') - 32\pi(p_0 + \rho_0) \frac{e^{\lambda-\nu}}{(r\nu')^2} ((3\varpi + \chi)\{4r^2 e^\lambda \varpi - 2r\varpi e^\nu (r^2 e^{-\nu})' - r^2 (r^2 e^\nu)' (\varpi e^{-\nu})' \right. \\ & \left. + 4r^2 e^\lambda \omega \varpi - \nu' [r^4 \omega \varpi' + 2(r^4 \varpi^2)'] \} - \frac{4}{r\nu'} (r^2 e^{-\nu})' \omega \varpi' \right] + c_1 \left[16\pi(\rho_0' + 3p_0') \Omega (\varpi + \Omega) \frac{r^2 e^{\lambda-\nu}}{\nu'} \right. \\ & \left. - 16\pi(p_0 + \rho_0) \frac{e^{\lambda-\nu}}{\nu'^2} \{ 4e^\lambda \omega \varpi - 2r\nu' (\varpi + \Omega)^2 - r^2 \nu' \varpi' (\varpi + 3\Omega) \} + \frac{8e^{-\nu}}{r\nu'} \omega^2 - 2 \left(r + \frac{2}{\nu'} \right) e^{-\nu} \omega \varpi' - \frac{49}{24} r^2 e^{-\nu} \varpi'^2 \right] \end{aligned}$$

$$\begin{aligned}
& -2f_1+f_2+f_4 \Big] + \frac{m^2}{l(l+1)} \left[l(l+1) \left(\frac{2}{rv'} - 1 \right) e^{\lambda-\nu} \omega^2 - 16\pi(p_0+\rho_0)r^2 e^{\lambda-\nu} \omega^2 - \frac{71}{24} r^2 e^{-\nu} \omega'^2 - f_1+f_2-3f_3-f_4 \right] \\
& + l(l+1) \left[-\frac{2e^{2\lambda}}{r^3} m_0 + \left(1 - \frac{2}{rv'} \right) e^{\lambda-\nu} \omega^2 \right] - \nu' \left[\left(\frac{e^\lambda m_0}{r} \right)' + h_0' \right] + \frac{4e^{2\lambda} m_0}{r^3} - 16\pi p_0' e^\lambda \left(e^\lambda m_0 + \frac{r^3}{3} e^{-\nu} \omega^2 \right) \\
& + 2r^2 e^{-\nu} \omega'^2 + f_1 - \frac{2}{3} f_2 + f_3, \tag{B2}
\end{aligned}$$

where

$$\begin{aligned}
f_1 = & \frac{8\pi e^\lambda}{3} (p_0 + \rho_0) \left[\left\{ 8e^{-\nu} r^2 - 12r^2 j^2 + \frac{64e^{-\nu}}{(\nu')^2} (4\pi r^2 e^\lambda p_0 - r\nu') \right\} \omega^2 + \frac{8r^2}{\nu'} e^{-\nu} \left(1 + \frac{2}{\nu' r} \right) \omega \omega' \right. \\
& \left. + r^{12} j^2 \left(\frac{\omega}{r^8} \right)' \omega' \right] + \frac{4r}{3} e^{-\nu} \omega \omega', \tag{B3}
\end{aligned}$$

$$\begin{aligned}
f_2 = & 4\pi(\rho_0' + 3p_0')(3e^\lambda \xi_2 + 2r^3 e^{-\nu} \omega^2) - 16\pi(p_0 + \rho_0)^2 r^4 e^{\lambda-\nu} \omega^2 - 12 \left(1 - \frac{2}{rv'} \right) \frac{e^\lambda}{r^2} k_2 + \frac{6}{r^3} [1 - 3e^\lambda + 4\pi(\rho_0 + p_0)r^2 e^\lambda] \xi_2 \\
& + \frac{4}{rv'} (1 - 3e^\lambda) e^{-\nu} \omega^2 + \left(\frac{r}{\nu'} - r^2 e^{-\lambda} \right) e^{-\nu} \omega'^2, \tag{B4}
\end{aligned}$$

$$f_3 = l(l+1) \left[\frac{e^\lambda}{2r^2} \left(k_2 - \frac{\nu'}{2} \xi_2 \right) - \frac{4\pi}{3} (p_0 + \rho_0) r^2 e^{\lambda-\nu} \omega^2 - \frac{e^{\lambda-\nu}}{6} \omega^2 - \frac{r^2}{12} e^{-\nu} \omega'^2 \right] + \frac{2\nu' e^\lambda}{r^2} \xi_2 + \frac{4}{3} e^{\lambda-\nu} \omega^2, \tag{B5}$$

$$\begin{aligned}
f_4 = & \frac{3e^\lambda}{4r^2} \left(k_2 - \frac{\nu'}{2} \xi_2 \right) - 4\pi(p_0 + \rho_0) \left[\left(\frac{e^\lambda}{6} + 4 \right) \omega^2 r^2 e^{-\nu} - \frac{8}{\nu'^2} e^{2\lambda-\nu} \omega \omega' - \frac{r^{12}}{3} e^{-\nu} \left(\frac{\omega}{r^8} \right)' \omega' \right] \\
& - \frac{e^{\lambda-\nu}}{4} \omega^2 + \left(\frac{1}{rv'} - \frac{1}{2} \right) e^{\lambda-\nu} \omega^2. \tag{B6}
\end{aligned}$$

3. Coefficient of $U_L^{(1)}$, β_1

$$\begin{aligned}
\beta_1 = & c_3 \left[32\chi \omega^2 \left\{ \frac{e^{\lambda/2}}{r^2} \left(\frac{e^{\lambda/2} r^2}{\nu'} \right)' + \left(\frac{8\pi e^{2\lambda}}{(\nu')^2} \right) (\rho_0 + p_0) \right\} \right] - c_2 \left[\frac{16}{\nu' j^2 r^4} (\chi \omega^2 r^4 e^{-\nu})' + 4\chi \omega' \left\{ (\omega r^2)' + \frac{2e^\lambda}{\nu'} \omega \right\} \right] \\
& + c_1 \left[\frac{12\chi}{jr} \left(\frac{\xi_2}{jr} \right)' - 18 \frac{\xi_2 \chi'}{j^2 r^2} - 12 \frac{\chi k_2}{j^2 r^2} + \frac{120W_3}{l(l+1)j^2 r^2} + 4\chi \omega^2 e^\lambda \{ 1 + 4\pi(\rho_0 + p_0)r^2 \} + \frac{2}{3r^6} \chi \omega' (\omega r^8)' \right] \\
& + \frac{m^2}{l(l+1)} \left[6 \frac{(\chi r^2)'}{j^2 r^4} \xi_2 + 12 \frac{\chi k_2}{j^2 r^2} - \frac{120W_3}{l(l+1)j^2 r^2} + \frac{4}{3} \chi \omega' (\omega r^2)' \right] + 2\chi (e^\nu)' \left\{ h_0' + \frac{1}{j^2} \left(\frac{m_0}{re^\nu} \right)' \right\} + \frac{8W_1 + 48W_3}{l(l+1)j^2 r^2} \\
& + \frac{32\pi}{3} \chi \omega^2 p_0' r^3 e^\lambda - \frac{4}{3} \chi \omega' (\omega r^2)'. \tag{B7}
\end{aligned}$$

4. Coefficient of $\Phi_L^{(1)}$, β_2

$$\begin{aligned}
\beta_2 = & 4c_2\chi\left[4\pi(\rho_0+p_0)\{4\varpi(\omega r)'-\varpi'\omega r\}r^3e^{-\nu}-\varpi'(\omega r)'j^2r^2\right]+c_1\chi r^2e^{-\nu}\left[-\frac{5\pi}{2}(\rho'_0+3p'_0)\varpi^2r^2+\varpi'\left\{\left(\frac{5}{4\nu'r}-\frac{5}{24}\right)\omega\right.\right. \\
& \left.\left.+\left(\frac{14}{3}\omega-\frac{97}{24}\varpi'r\right)e^{-\lambda}\right\}-4\pi(\rho_0+p_0)\left\{\frac{5}{2}\varpi^2r+\frac{44}{3}\varpi\omega r-\frac{85}{6}\varpi'\varpi r^2-\frac{14}{3}\varpi'\omega r^2+\frac{6}{\nu'^2j^2}\varpi(\omega e^{-\nu})'\right\}\right] \\
& +m^2\left[4\pi(\rho'_0+3p'_0)\chi\varpi r^4e^{-\nu}\left(\frac{1}{4}\varpi-\frac{11}{16}\chi\right)+\chi'\omega e^{-\nu}\left\{\frac{(r^2e^\nu)'}{4\nu'}\varpi e^{-\nu}+\frac{3}{8\nu'}\chi r-\frac{19}{48}\chi r^2\right\}\right. \\
& \left.-\chi j^2r^2\left\{\frac{1}{4}\varpi'^2r-\frac{2}{3}\chi'\omega-\frac{1}{48}\chi'\varpi'r\right\}+4\pi(\rho_0+p_0)\chi r^2e^{-\nu}\left\{(\varpi r)'\varpi r-\frac{11}{4}\chi\varpi r-\frac{8}{3}\chi\omega r+\frac{1}{j^2\nu'^2}\chi(\omega e^{-\nu})'\right.\right. \\
& \left.\left.-\frac{1}{12}\chi\varpi'r^2+\frac{2}{3}\chi'\omega r^2\right\}\right]+\left(\chi-\varpi+\frac{\nu'}{2}\chi'r^2e^{-\lambda}\right)\left[h'_0+\left(\frac{e^\lambda m_0}{r}\right)'\right]+\frac{2\pi}{3}(\rho'_0+3p'_0)(4\chi-\varpi)\varpi^2r^4e^{-\nu} \\
& -(\lambda'+\nu')j^2\left[\frac{1}{9}[(\varpi r)^3]'+\frac{4}{3}\left(\frac{\varpi}{r}\right)'\chi\varpi r^4+\frac{2}{3}\left(\frac{\varpi}{r^4}\right)'\chi\omega r^7+\frac{\nu'}{6}\varpi^2\chi'r^4+\frac{\nu'r}{2j^2}\chi'm_0\right]+\left[\frac{2}{3}\chi-\frac{1}{2}\varpi-\frac{1}{\nu'r}\varpi\right]\varpi'\omega r^2e^{-\nu} \\
& +\frac{1}{6}(\varpi'^2r-8\chi'\omega+4\chi'\varpi'r)\varpi j^2r^2. \tag{B8}
\end{aligned}$$

5. Coefficient of $\Phi_L^{(1)}$, β_3

$$\begin{aligned}
\beta_3 = & c_2\left[16\pi r^2e^{-\nu}\chi\varpi(r^2\omega)'(\rho'_0+3p'_0)-16\pi(\rho_0+p_0)e^{-\nu}\chi\left[16\pi(\rho_0+p_0)r^4e^\lambda\varpi(\varpi+\Omega)+2r^2e^\lambda\Omega\varpi\right.\right. \\
& \left.\left.\times\left(1+\frac{8}{r\nu'}-\frac{32\pi}{\nu'^2}e^\lambda p_0\right)-6r\omega(r^2\varpi)'\right]-\frac{2re^\lambda}{\nu'^2}\varpi'(r\nu'\varpi+2\Omega-r\varpi')+r^3\varpi'(4\varpi+5r\varpi')\right]+\frac{4}{r\nu'}(r^2e^{-\nu})'\chi\omega\varpi' \\
& -4e^{-\nu}r^2\chi'\varpi'[4\pi(\rho_0+p_0)r^2\omega+e^{-\lambda}(r\omega)']\left]+c_1\left(\frac{16\pi e^{\lambda-\nu}}{\nu'^2}(\rho_0+p_0)\Omega\{112\pi r^2e^\lambda\varpi\chi p_0-4e^\lambda\chi\omega-2r\nu'(9\varpi+\omega)\chi\right.\right. \\
& \left.\left.+6r\chi\varpi'+7r^2\nu'\chi\varpi'+r^2\nu'\omega\chi'\right\}+2re^{-\nu}\omega\chi'\left[2\varpi+\left(1-\frac{2}{r\nu'}\right)\omega\right]-r^2e^{-\nu}\varpi'\chi'\left[\varpi-\left(1+\frac{2}{r\nu'}\right)\omega\right]+g_1\right) \\
& +\frac{m^2}{l(l+1)}\left[-4\pi(\rho_0+p_0)e^{\lambda-\nu}\varpi\left(\frac{23}{4}r^2\varpi^2+\frac{15}{4}r^2\varpi\omega+\frac{4e^\lambda}{\nu'^2}\Omega\omega\right)+\frac{e^{\lambda-\nu}}{2}(5\varpi+\omega)\left(1-\frac{2}{r\nu'}\right)\omega^2\right. \\
& \left.+\frac{16\pi}{\nu'}(\rho'_0+3p'_0)r^2e^{\lambda-\nu}\Omega^2(\varpi+\Omega)-32\pi(\rho_0+p_0)\varpi\chi e^{\lambda-\nu}\left((\Omega+\omega)r^2+\frac{\Omega}{\nu'^2}(1-e^\lambda)\right)+r^2\Omega\varpi'^2e^{-\nu}+4\chi e^{-\nu}\right. \\
& \left.\times\left(e^\lambda\varpi^2-\frac{2}{r\nu'}\omega^2\right)-g_1+3g_2-g_3\right]-m^2\left(1-\frac{2}{r\nu'}\right)e^{\lambda-\nu}\Omega\omega^2+\left(l(l+1)(\Omega-\chi)\left\{\left(1-\frac{2}{\nu'r}\right)e^{\lambda-\nu}\omega^2-\frac{2e^{2\lambda}}{r^3}m_0\right\}\right) \\
& +\frac{8\pi}{3}(\rho_0+p_0)r^2e^{\lambda-\nu}\varpi\{\varpi^2-4(\varpi-\omega)\chi\}+\Omega\left[-4\pi e^\lambda(\rho_0+p_0)\varpi\left(\frac{r^2}{12}e^{-\nu}\varpi+\frac{4}{\nu'^2}e^{\lambda-\nu}\omega\right)+\frac{4}{r^3}e^{2\lambda}m_0\right. \\
& \left.-\nu'\left[\left(\frac{e^\lambda}{r}m_0\right)'+h'_0\right]\right]+\frac{4}{r^3}e^{2\lambda}\{4\pi[8\pi(\rho_0+p_0)r^2\chi+\chi-\Omega]p'_0r^3-\chi\}\left(m_0+\frac{r^3}{3}j^2\varpi^2\right) \\
& +\frac{1}{r}\{(4+\nu'r-16\pi p'_0r^3)\chi+\chi'r\}\left[\left(\frac{e^\lambda}{r}m_0\right)'+h'_0\right]-g_2+g_3\}, \tag{B9}
\end{aligned}$$

where

$$\begin{aligned}
l(l+1)g_1 = & 4\pi(\rho'_0 + 3p'_0) \left\{ \frac{3}{4}(8\varpi + 3\Omega)e^\lambda \xi_2 + \frac{r^2\Omega}{\nu'} e^{\lambda-\nu}(\varpi + \Omega)(4\varpi + 3\Omega) + \frac{5}{6} e^{-\nu} r^2 \varpi^2 (4(r^2\varpi)' - 11\Omega r) \right\} \\
& + \frac{20\pi}{3} (\rho_0 + p_0)^2 e^{\lambda-\nu} r^4 \varpi^2 (8\varpi + 11\Omega) + 4\pi(\rho_0 + p_0) e^\lambda \left\{ -\frac{3}{2r} (5\varpi - 3\omega) \xi_2 + \frac{5}{12} r^2 j^2 (8\varpi - 11\Omega) \right. \\
& \times [12\varpi^2 - (r\varpi')^2] + \frac{44}{3} r^2 j^2 \varpi \varpi' (r^2\varpi)' - 26r^3 j^2 \Omega \varpi \varpi' + \frac{17}{3} e^{-\nu} r^2 \varpi^3 - \frac{1123}{24} e^{-\nu} r^2 \Omega \varpi^2 + \frac{\Omega e^{-\nu}}{\nu'^2} \\
& \left. \times (-22e^\lambda \varpi \omega + 2r\nu'(20\varpi^2 + 12\varpi\omega + 3\omega^2) + 8r\varpi\varpi' + (32\varpi + 13\omega)r^2\nu'\varpi') \right\} + (5\varpi - 3\omega) \\
& \times \left\{ \left(\frac{45}{16r^2} - \frac{6}{\nu'r^3} \right) e^\lambda k_2 - \frac{3}{2r^3} \left(1 - 3e^\lambda - \frac{\nu're^\lambda}{16} \right) \xi_2 + \frac{e^{\lambda-\nu}}{16} \varpi^2 + \left(1 - \frac{2}{r\nu'} \right) \frac{e^{\lambda-\nu}}{8} \omega^2 - \frac{2e^{-\nu}}{r\nu'} \omega^2 \right. \\
& \left. + (3e^\lambda - 1) \frac{\varpi^2 e^{-\nu}}{r\nu'} - \frac{r}{4\nu'} e^{-\nu} (\varpi')^2 \right\} - \frac{7}{\nu'} e^{-\nu} \Omega \omega \varpi' + \frac{re^{-\nu}}{6} (23\varpi - 9\omega) \omega \varpi' + \frac{r^2 e^{-\nu}}{96} (437\varpi - 403\omega) \varpi'^2 \\
& + \frac{5}{12} r^2 j^2 \varpi'^2 [11\Omega - 8(r\varpi)'], \tag{B10}
\end{aligned}$$

$$\begin{aligned}
g_2 = & l(l+1)e^\lambda \Omega \left\{ \frac{4\pi}{3} (\rho_0 + p_0) r^2 e^{-\nu} \varpi^2 - \frac{1}{2r^2} \left(k_2 - \frac{\nu'}{2} \xi_2 \right) + \frac{e^{-\nu}}{6} \varpi^2 + \frac{r^2 j^2}{12} \varpi'^2 \right\} + 4\pi(\rho'_0 + 3p'_0) e^\lambda \\
& \times \left\{ \left(2\chi + \frac{\Omega}{2} \right) \xi_2 + \frac{j^2}{3} \varpi [r^3 \varpi (\Omega - 4\chi) - 2\chi (r^4 \omega)'] - \frac{2}{\nu'} r^2 e^{-\nu} \Omega^2 (\varpi + \Omega) \right\} - \frac{32\pi^2}{3} (\rho_0 + p_0)^2 r^4 e^{\lambda-\nu} \varpi \\
& \times [\Omega \varpi - 4\chi(\varpi + 2\Omega)] + 4\pi(\rho_0 + p_0) \left\{ e^\lambda \Omega \left(\frac{\xi_2}{r} + \frac{r^2 j^2}{6} [12\varpi^2 + 8r\varpi'\varpi - (r\varpi')^2] \right) - \frac{4}{r} e^\lambda \chi \xi_2 - \frac{2}{3} r^2 e^{-\nu} \right. \\
& \times [12(\varpi + 2\omega) \varpi \chi + 4r(\varpi + 3\omega) \chi \varpi' - (5\varpi + 2\omega) r^2 \varpi' \chi'] \left. \right\} - \frac{1}{r^3} \left[r\nu' e^\lambda \left(\frac{\varpi}{2} + \frac{29}{16} \Omega - 2\chi \right) - (1 - 3e^\lambda)(\Omega - 4\chi) \right] \xi_2 \\
& + \frac{e^\lambda}{r^2} \left(\varpi - \frac{19}{8} \Omega + 8\chi + \frac{4}{r\nu'} (\Omega - 4\chi) \right) k_2 - \frac{r^2}{6} \varpi'^2 e^{-\nu} \left(e^{-\lambda} \Omega + \frac{47}{8} \Omega + \varpi - \frac{\Omega}{r\nu'} \right) - e^{\lambda-\nu} \varpi^2 \\
& \times \left(\frac{\varpi}{3} + \frac{29}{24} \Omega + \frac{2}{3\nu'r} (3 - e^{-\lambda})(\Omega - 4\chi) \right) + \frac{2r}{3\nu'} e^{-\nu} \chi' \{ e^{-\lambda} \nu' \varpi' (r\varpi + (r^2 \omega)') - \varpi \varpi' + 2\nu' \varpi \omega + r\nu' \varpi' (3\varpi - \omega) \}, \tag{B11}
\end{aligned}$$

$$g_3 = \Omega e^{-\nu} \left\{ \frac{8\pi r e^\lambda}{\nu'} (\rho_0 + p_0) [2(\Omega + \varpi)^2 + r\varpi' (3\Omega + \varpi)] - \frac{e^\lambda}{4} \left(1 - \frac{2}{r\nu'} \right) \omega^2 + \frac{4}{\nu'r} \omega^2 - \left(r + \frac{2}{\nu'} \right) \omega \varpi' \right\}. \tag{B12}$$

6. $\mathcal{G}[H_{0,\pm}^{(1)}, K_{\pm}^{(1)}]$

$$\mathcal{G}[H_{0,\pm}^{(1)}, K_{\pm}^{(1)}] = A_1 \partial_T K_{\pm}^{(1)} + B_1 \partial_T H_{\pm}^{(1)} + imA_2 K_{\pm}^{(1)} + imB_2 H_{\pm}^{(1)}, \tag{B13}$$

where

$$A_1 = Q_{\pm} \left[\frac{2e^{\lambda}}{r^2} \omega + 64\pi(\rho_0 + p_0) \left(\frac{e^{-\nu}}{j^2 r \nu'} \right)^2 \varpi \right] + S_{\pm} \left[16\pi e^{\lambda} (\rho_0 + p_0) \left\{ \chi - \frac{2e^{\lambda}}{r^2 \nu'^2} (\varpi + \chi) \right\} - \frac{2e^{\lambda}}{r^2} \left(1 - \frac{2}{r \nu'} \right) \omega \right] + T_{\pm} \left[- \frac{e^{\lambda-\nu}}{\nu'} \left(\frac{e^{\nu}}{r^2} \right)' \omega \right], \quad (\text{B14})$$

$$B_1 = Q_{\pm} B_3 + S_{\pm} \left[\frac{16\pi e^{\lambda}}{r^2 \nu'^2} (\rho_0 + p_0) \left\{ 2\chi(e^{\lambda} + r \nu') + 2e^{\lambda} \varpi - r^2 \nu' \varpi' - (2\varpi + \omega) \left(1 - \frac{3}{4} r \nu' \right) r \nu' \right\} - (2\varpi + \omega) \left(\frac{8\pi e^{-\nu} \rho_0'}{j^2 \nu'} \right) - \frac{4}{r^3 \nu'} \omega \right] + T_{\pm} \left[- \frac{2e^{\lambda}}{r^3 \nu'} \left(1 - \frac{r \nu'}{2} \right) \omega \right], \quad (\text{B15})$$

$$A_2 = Q_{\pm} \left[+ 2\chi e^{\lambda} \left(8\pi \Omega (\rho_0 + p_0) - \frac{\omega}{r^2} \right) \right] + S_{\pm} \left[- 2\chi e^{\lambda} \left\{ 8\pi \Omega (\rho_0 + p_0) - \left(1 - \frac{2}{\nu' r} \right) \omega \right\} \right] + T_{\pm} \left[+ \chi e^{\lambda} \left(\frac{1}{r^2} - \frac{2}{\nu' r^3} \right) \omega \right], \quad (\text{B16})$$

$$B_2 = Q_{\pm} \left[- \chi B_3 + 8\pi \chi e^{\lambda} (\rho_0 + p_0) \varpi \right] + S_{\pm} \left[\chi B_3 - 2\chi e^{\lambda} \left\{ \frac{e^{-\nu} \omega}{\nu'} \left(\frac{e^{\nu}}{r^2} \right)' + 4\pi (\rho_0 + p_0) \varpi \right\} \right] + T_{\pm} \left[\frac{e^{\lambda-\nu}}{\nu'} \left(\frac{e^{\nu}}{r^2} \right)' \chi \omega + 4\pi (\rho_0 + p_0) \left\{ \frac{(r^4 \chi^2 e^{-\nu})'}{j^2 r^4 \nu'} - 8 \left(\frac{\chi e^{-\nu}}{r \nu'} \right)^2 \right\} \right], \quad (\text{B17})$$

$$B_3 = (2\varpi + \omega) \left\{ \frac{8\pi e^{-\nu} \rho_0'}{j^2 \nu'} + 4\pi e^{\lambda} (\rho_0 + p_0) \left(\frac{4}{r \nu'} - 3 \right) \right\} + \left(\frac{4}{r^3 \nu'} (1 - e^{\lambda}) + \frac{2e^{\lambda}}{r^2} \right) \omega + \frac{16\pi e^{\lambda}}{\nu'} (\rho_0 + p_0) \left(\varpi' - \frac{4e^{-\nu} \varpi}{j^2 r^2 \nu'} \right). \quad (\text{B18})$$

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