Complementary Classical Fidelities as an Efficient Criterion for the Evaluation of Experimentally Realized Quantum Operations

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It is shown that a good estimate of the fidelity of an experimentally realized quantum process can be obtained by measuring the outputs for only two complementary sets of input states. The number of measurements required to test a quantum network operation is therefore only twice as high as the number of measurements required to test a corresponding classical system.

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One of the greatest challenges in quantum information science is the experimental realization of well-controlled operations on increasingly complex quantum systems. In particular, quantum computation is based on the implementation of networks of universal quantum gates operating at low noise [1]. Recently, there have been several successful experimental demonstrations of quantum controlled NOT gates that could serve as essential elements in future quantum computation networks [2–8]. Since all of these devices operate at non-negligible noise levels, there has also been an increasing interest in the quantification of noise and the development of efficient criteria for the comparison of different experiments [9–14]. However, the criteria presently discussed in the literature are mostly based on theoretical considerations, and experimentalists have usually evaluated the performance of their devices on an "*ad hoc*" basis instead of applying the more complicated and often nonintuitive procedures necessary to obtain an evaluation fulfilling the theoretical requirements for a good measure (see [14] for an interesting discussion of this problem and an overview of error measures for quantum processes). In order to bridge this gap between the experimentalists intuition and the theorists requirements for a good error measure, it may thus be useful to investigate the possibility of estimating the performance of quantum devices based on a minimal number of well-defined experimental tests.

In the following, it is shown that any unitary transform $\hat{U}_0$ is uniquely defined by its observable effects on only two complementary sets of orthogonal input states [15]. The performance of any device implementing the unitary transform $\hat{U}_0$ can therefore be tested by measuring the classical fidelities of these two complementary operations. The relationship between the complementary fidelities and the overall process fidelity is discussed and upper and lower bounds for an estimate of the process fidelity are given [20]. An estimate of the process fidelity for an N-level system can thus be obtained from only $2N$ measurement probabilities, corresponding to the successful performance of two well-defined classical operations on the respective sets of orthogonal input states.

If the desired operation of a quantum device is described by the unitary operator $\hat{U}_0$, the expected outcomes for a specific set of orthogonal input states $\{| n\rangle\}$ are given by

$$\hat{U}_0 \mid n\rangle = \mid f_n\rangle. \tag{1}$$

Since $\hat{U}_0$ is unitary, the output states also form an orthogonal set $\{| f_n\rangle\}$. It is therefore possible to verify the operation described by Eq. (1) by a conventional von Neumann measurement of the output [21]. For an experimental realization of the intended unitary operation $\hat{U}_0$, the fidelity of this classically defined operation is equal to the average probability of obtaining the correct output for each of the N possible input states. If the actual experimental process is described by the linear map $\hat{\rho}_{\text{out}} = E(\hat{\rho}_{\text{in}})$, this classical fidelity is given by

$$\begin{aligned} F_{n\to f_n} &= \frac{1}{N}\sum_{n=1}^{N}\langle n \mid \hat{U}_0^\dagger E(| n\rangle\langle n |)\hat{U}_0 \mid n\rangle \\ &= \frac{1}{N}\sum_{n=1}^{N}\langle f_n \mid E(| n\rangle\langle n |) \mid f_n\rangle \\ &= \frac{1}{N}\sum_{n=1}^{N} p(f_n|n). \end{aligned} \tag{2}$$

Since the classical concept of fidelity represents a very intuitive test of device performance, it has been commonly used to characterize the operation of experimental quantum gates in the computational basis [3,5,6]. However, it is generally recognized that the classical fidelity is not sufficient as an experimental criterion for the successful implementation of $\hat{U}_0$ since it is not sensitive to quantum coherence between different input and output states. In particular, a fidelity of one can be obtained for a large number of processes $E(\hat{\rho}_{\text{in}})$, some of which can actually have a process fidelity of zero with respect to the intended unitary operation $\hat{U}_0$.

To analyze what kind of information about the experimental process $E(\hat{\rho}_{\text{in}})$ is actually obtained from a measurement of the classical fidelity defined by Eq. (2), it is useful to consider a set of N orthogonal quantum processes U_q with a fidelity of $F_{n\to f_n} = 1$ [22]. A convenient expression for such a set of orthogonal processes can be defined by

$$\hat{U}_q \mid n\rangle = \exp\left[-i\frac{2\pi}{N}qn\right] \mid f_n\rangle. \tag{3}$$

Note that this set of orthogonal unitary transformations is not unique, since the definition of phase for the output states $|f_n\rangle$ is quite arbitrary. In this sense, Eq. (3) gives only an example of how to construct an orthogonal set of N unitary transformations with a classical fidelity of one for the operation $n \rightarrow f_n$. The experimental process $E(\hat{\rho}_{\mathrm{in}})$ can then be expanded in terms of a complete set of N^2 orthogonal basis operators $\{\hat{U}_q\}$, where the first N basis operators are defined according to Eq. (3), and the remaining $N(N-1)$ operators can be any set of orthogonal unitary operators spanning the remaining process space,

$$E(\hat{\rho}_{\mathrm{in}}) = \sum_{q,r=0}^{N^2-1} \chi_{q,r}\hat{U}_q\hat{\rho}_{\mathrm{in}}\hat{U}_r^\dagger. \tag{4}$$

The fundamental properties of this expansion are most easily understood by considering the application of $E(\hat{\rho}_{\mathrm{in}})$ to a maximally entangled state $|\phi\rangle_{\mathrm{A,R}}$ of the system A and a reference R. If E is applied only to A (that is, $E \otimes I$ is applied to the joint system of A and R), the process matrix is then equal to the density matrix of the output state for the orthogonal basis states $\{\hat{U}_q \otimes \hat{1} \mid \phi\rangle_{\mathrm{A,R}}\}$ generated by applying the basis operators $\{\hat{U}_q\} \otimes \hat{1}$ to the pure state input $|\phi\rangle_{\mathrm{A,R}}$. From this observation, it follows that the process matrix is a positive Hermitian matrix with a trace of one (or less for conditional operations with a limited probability of success). Moreover, the process fidelity can be defined as the probability of obtaining the output state $\hat{U}_0 \otimes \hat{1} \mid \phi\rangle_{\mathrm{A,R}}$ corresponding to the application of the ideal process $\hat{U}_0$ to system A of the pure state input $|\phi\rangle_{\mathrm{A,R}}$. Since this probability is equal to the corresponding diagonal element of the process matrix, the overall process fidelity is then given by $F_{\mathrm{process}} = \chi_{0,0}$.

Using the expansion given by Eq. (4), the classical fidelity $F_{n\rightarrow f_n}$ can now be related directly to the elements $\chi_{q,r}$ of the process matrix,

$$F_{n\rightarrow f_n} = \chi_{0,0} + \sum_{q=1}^{N-1} \chi_{q,q}. \tag{5}$$

In terms of the linear algebra of process expansions, the classical fidelity $F_{n\rightarrow f_n}$ corresponds to a projective measure of the process components that lie within the N-dimensional subspace of the $\hat{N}^2$-dimensional process space spanned by the orthogonal basis $\{\hat{U}_q\}$. Since this subspace is larger than the one-dimensional subspace representing the ideal operation, the classical fidelity $F_{n\rightarrow f_n}$ is always equal to or greater than the process fidelity given by $\chi_{0,0}$. Each classical fidelity thus provides an upper bound for the overall process fidelity [16].

In order to experimentally distinguish the N operations $\hat{U}_q$ with classical fidelities of $F_{n\rightarrow f_n} = 1$ from each other, it is necessary to change the input basis. Optimal distinguishability is achieved when the output states of different $\hat{U}_q$ for the same input state are orthogonal to each other. This condition can be fulfilled by complementary sets of input states $|k'\rangle$ with $|\langle n \mid k'\rangle|^2 = 1/N$ for all n and k, as, e.g., given by

$$|k'\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\exp\left[-i\frac{2\pi}{N}kn\right]|n\rangle. \tag{6}$$

For this set of states, the unitary operation $\hat{U}_0$ defines a second classical function, given by

$$\hat{U}_0 \mid k'\rangle = \mid g_k'\rangle, \tag{7}$$

where the output states $|g_k'\rangle$ are also complementary to the output states $|f_n\rangle$ according to

$$|g_k'\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\exp\left[-i\frac{2\pi}{N}kn\right]|f_n\rangle. \tag{8}$$

Since these output states are maximally sensitive to the quantum phases between the components $|f_n\rangle$, the effects of different unitary operations $\hat{U}_{q<N}$ on the quantum phases of $|f_n\rangle$ become directly observable in the output basis $|g_k'\rangle$. Specifically,

$$\hat{U}_{q<N} \mid k'\rangle = \mid g_{k+q}'\rangle. \tag{9}$$

Thus the output states for different operations $\hat{U}_q$ are indeed orthogonal, making the operation on the complementary input states $|k'\rangle$ given by Eq. (7) ideal for the task of distinguishing the operations $\hat{U}_{0<q<N}$ with $F_{n\rightarrow f_n} = 1$ from the intended operation $\hat{U}_0$.

The classical fidelity of the complementary operation $k \rightarrow g_k$ can be obtained experimentally by

$$\begin{aligned} F_{k\rightarrow g_k} &= \frac{1}{N}\sum_{k=1}^{N}\langle k' \mid \hat{U}_0^\dagger E(|k'\rangle\langle k'|)\hat{U}_0 \mid k'\rangle \\ &= \frac{1}{N}\sum_{k=1}^{N}\langle g_k' \mid E(|k'\rangle\langle k'|) \mid g_k'\rangle \\ &= \frac{1}{N}\sum_{k=1}^{N} p(g_k|k). \end{aligned} \tag{10}$$

Again, it is possible to find N orthogonal operations that all have $F_{k\rightarrow g_k} = 1$. However, the only operation that has both $F_{n\rightarrow f_n} = 1$ and $F_{k\rightarrow g_k} = 1$ is $\hat{U}_0$, since

$$\langle g_k' \mid \hat{U}_{0<q<N} \mid k'\rangle = \langle g_k' \mid g_{k+q}'\rangle = 0. \tag{11}$$

This relation also implies that any unitary operation $\hat{U}_{k\rightarrow g_k}$ with a fidelity of $F_{k\rightarrow g_k} = 1$ is orthogonal to the operations $\hat{U}_{0<q<N}$, since

$$\begin{aligned} \mathrm{Tr}\{\hat{U}_{k\rightarrow g_k}^\dagger \hat{U}_{0<q<N}\} &= \sum_k |\langle k' \mid \hat{U}_{k\rightarrow g_k}^\dagger \hat{U}_{0<q<N} \mid k'\rangle|^2 \\ &= \sum_k |\langle g_k' \mid g_{k+q}'\rangle|^2 = 0. \end{aligned} \tag{12}$$

It is therefore possible to identify the remaining $N-1$ orthogonal operations having classical fidelities of $F_{k\to g_k}=1$ with the basis operators $\hat{U}_N$ to $\hat{U}_{2(N-1)}$. In fact, it is possible to explicitly construct an orthogonal set of unitary operators in close analogy with Eq. (3),

$$\hat{U}_{N\leq q\leq 2(N-1)} \mid k'\rangle = \exp\left[-i\frac{2\pi}{N}(q+1)k\right] \mid g'_k\rangle. \quad (13)$$

The complementary classical fidelity $F_{k\to g_k}$ can then be expressed in terms of the process matrix elements $\chi_{q,r}$ of Eq. (14) as

$$F_{k\to g_k} = \chi_{0,0} + \sum_{q=N}^{2(N-1)} \chi_{q,q}. \quad (14)$$

In terms of the linear algebra of process expansions, the complementary fidelity $F_{k\to g_k}$ thus evaluates the component of the process in an N-dimensional subspace of the N^2-dimensional process space that only overlaps with the subspace defined by $F_{n\to f_n}$ in the ideal process $\hat{U}_0$. Therefore, the maximal total fidelity $F_{n\to f_n}+F_{k\to g_k}$ cannot exceed one unless there is a nonvanishing contribution from the ideal process $\hat{U}_0$.

Based on these results, it is possible to derive an estimate of the process fidelity $F_{\text{process}}=\chi_{0,0}$ from the measured results for the classical fidelities $F_{n\to f_n}$ and $F_{k\to g_k}$. Since the process fidelity is by definition equal to the process matrix element $\chi_{0,0}$, the relationship between the classical fidelities and the process fidelity is given by Eqs. (5) and (14). These equations show that the classical fidelities can each be interpreted as sums of process fidelities for N orthogonal (and therefore distinguishable) processes. If the two complementary classical fidelities are added, only the intended process $\hat{U}_0$ contributes twice. The lower bound of the process fidelity is therefore equal to the amount by which the total fidelity $F_{n\to f_n}+F_{k\to g_k}$ exceeds one,

$$F_{n\to f_n} + F_{k\to g_k} - 1 \leq F_{\text{process}}. \quad (15)$$

An upper bound for the process fidelity can be derived from the minimum of the two classical fidelities, since the sum of N process fidelities is necessarily equal to or greater than each individual fidelity [17]. The upper bound thus reads

$$F_{\text{process}} \leq \text{Min}\{F_{n\to f_n}, F_{k\to g_k}\}. \quad (16)$$

Note that the difference between the lower and the upper bound depends on the closeness of the maximal classical fidelity to one. Specifically, if $F_{n\to f_n}=1-\epsilon$ is greater than $F_{k\to g_k}$ and close to 1, the process fidelity will be found in an interval of width ϵ below the lower classical fidelity $F_{k\to g_k}$ given by

$$F_{k\to g_k} - \epsilon \leq F_{\text{process}} \leq F_{k\to g_k}. \quad (17)$$

The complementary classical fidelities are therefore particularly well suited for an estimate of process fidelity if the performance in one basis (e.g., the computational basis) is highly reliable and the main error source is dephasing between these basis states [18].

To place the results into a wider context, it may also be useful to convert the process fidelity into the average quantum state fidelity $\bar{F}$, as given by $\bar{F}=(NF_{\text{process}}+1)/(N+1)$ [7,13,14,19]. The inequalities (15) and (16) then establish a relation between the classical fidelities $F_{n\to f_n}$ and $F_{k\to g_k}$ obtained by averaging over a very specific limited selection of input states, and the fidelity $\bar{F}$ obtained by averaging over all possible pure state inputs. It might be interesting to consider the implications of this result for the relations between noncomplementary classical fidelities.

To illustrate the practical application of complementary classical fidelities, it may be helpful to consider the specific example of a quantum controlled-NOT gate. The effects of this gate on the computational basis (indicated by the index Z in the following) and an appropriate complementary basis (indicated by the index X in the following) can be given by

$$\begin{aligned}
&\hat{U}_{\text{CNOT}}|0_Z;0_Z\rangle = |0_Z;0_Z\rangle, \quad \hat{U}_{\text{CNOT}}|0_X;0_X\rangle = |0_X;0_X\rangle,\\
&\hat{U}_{\text{CNOT}}|0_Z;1_Z\rangle = |0_Z;1_Z\rangle, \quad \hat{U}_{\text{CNOT}}|1_X;1_X\rangle = |1_X;1_X\rangle,\\
&\hat{U}_{\text{CNOT}}|1_Z;1_Z\rangle = |1_Z;1_Z\rangle, \quad \hat{U}_{\text{CNOT}}|1_X;0_X\rangle = |1_X;0_X\rangle,\\
&\hat{U}_{\text{CNOT}}|1_Z;1_Z\rangle = |1_Z;0_Z\rangle, \quad \hat{U}_{\text{CNOT}}|1_X;1_X\rangle = |0_X;1_X\rangle,
\end{aligned} \quad (18)$$

where the basis transformation corresponds to the application of a Hadamard transformation to each qubit,

$$\mid 0_X\rangle = \frac{1}{\sqrt{2}}(\mid 0_Z\rangle + \mid 1_Z\rangle), \qquad \mid 1_X\rangle = \frac{1}{\sqrt{2}}(\mid 0_Z\rangle - \mid 1_Z\rangle). \quad (19)$$

The complementary classical fidelities of the quantum controlled-NOT gate thus correspond to the fidelities of two classical controlled-NOT operations, where the Hadamard transform of the input and output basis causes an exchange of the roles of control and target qubit [23]. The complementary classical fidelities of the quantum controlled-NOT gate can then be obtained from eight measurement probabilities,

$$\begin{aligned}
F_Z &= \frac{1}{4}[P_{ZZ|ZZ}(00|00) + P_{ZZ|ZZ}(01|01)\\
&\quad + P_{ZZ|ZZ}(11|10) + P_{ZZ|ZZ}(10|11)],\\
F_X &= \frac{1}{4}[P_{XX|XX}(00|00) + P_{XX|XX}(11|01)\\
&\quad + P_{XX|XX}(10|10) + P_{XX|XX}(01|11)].
\end{aligned} \quad (20)$$

As discussed above, these eight measurement results are already sufficient to obtain reliable estimates of the process fidelity F_{process}. In particular, the lower bound of the process fidelity given by $F_{\text{process}} \geq F_Z + F_X - 1$ can be used to obtain estimates of the gate performance for other sets of orthogonal input states, since the classical fidelities of such operations are always greater than or equal to the process fidelity. For example, an estimate of the entanglement capability can be obtained by considering the classical fidelity F_{entangle} for the generation of maximally entangled outputs if the control qubit input is an eigenstate of X and the target qubit is an eigenstate of Z. The classical fidelity F_{entangle} of this entanglement generation process represents the average overlap of the output states with the corresponding maximally entangled states. This average overlap therefore defines a minimal amount of entanglement that can be generated by the operation. In terms of the concurrence C, this lower bound of the entanglement capability is given by

$$C \geq 2F_{\text{entangle}} - 1. \quad (21)$$

Since $F_{\text{entangle}} \leq F_{\text{process}}$, the lower bound of the process fidelity given by $F_Z + F_X - 1$ also applies to F_{entangle} and the entanglement capability can be estimated directly by

$$C \geq 2(F_Z + F_X) - 3. \quad (22)$$

If $F_Z = 1 - \epsilon$ is close to 1, the gate is thus capable of generating entanglement if F_X is greater than $0.5 - \epsilon$. Note that this estimate of the entanglement capability can be obtained without actually generating any entanglement when the device is tested. The possibility of entanglement generation is simply a necessary consequence of the high fidelity observed in the complementary local operations of the quantum gate.

In summary, it has been shown that an efficient test of experimentally realized quantum operations can be performed by measuring the classical fidelities for only two complementary sets of orthogonal input states. This simplified test criterion can provide good estimates of the process fidelity and other characteristic properties of the noisy experimental process from only $2N$ measurement probabilities. In the case of a quantum controlled-NOT operation, the complementary classical fidelities can be determined from the measurement probabilities of eight pairs of local inputs and outputs. For comparison, the precise determination of process fidelity from local inputs and outputs reported in [13] was based on 71 measurement probabilities out of the 256 probabilities required for complete quantum process tomography. The complementary classical fidelities therefore provide a compact and intuitive measure of how well a given experimental device performs a desired quantum process.

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[15] The simplest example is a quantum channel, $\hat{U}_0 = \hat{1}$. The results in this paper thus also apply to fidelities in quantum communication (see [16–19]).

[16] Note that a related result on quantum channel capacities was previously derived in H. Barnum, E. Knill, and M. A. Nielsen, IEEE Trans. Inf. Theory **46**, 1317 (2000).

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[20] The process fidelity is also often referred to as entanglement fidelity (see B. Schumacher, quant-ph/9604023). The terminology used here is that of recent experimental papers such as Refs. [7,13].

[21] In fact, Eq. (1) can be interpreted in terms of the Heisenberg picture as a transformation of the physical property $\hat{n}$ with the eigenstates $| n\rangle$ into a different physical property $\hat{f}$ with eigenstates $| f_n\rangle$. It therefore provides a clear physical picture of the quantum process $\hat{U}_0$.

[22] Here, orthogonality is defined by the product trace of two operators, that is, $\hat{A}$ and $\hat{B}$ are orthogonal if $\mathrm{Tr}\{\hat{A}^\dagger \hat{B}\} = 0$.

[23] See also H. F. Hofmann, quant-ph/0407165.