Bound entangled states violate a nonsymmetric local uncertainty relation

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As a consequence of having a positive partial transpose, bound entangled states lack many of the properties otherwise associated with entanglement. It is therefore interesting to identify properties that distinguish bound entangled states from separable states. In this paper, it is shown that some bound entangled states violate a nonsymmetric class of local uncertainty relations [H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003)]. This result indicates that the asymmetry of nonclassical correlations may be a characteristic feature of bound entanglement.

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Entanglement is an essential element of quantum information theory and is known to be responsible for a wide variety of nonclassical effects [1–5]. However, it is not generally clear what properties of entanglement are required for any particular application. In fact, the properties of mixed entangled states are so difficult to characterize that even the question whether a given density matrix is entangled or separable may be difficult to answer [6–10]. For 2×2 and 2×3 systems, this problem can be solved by testing whether the partial transpose of the density matrix is positive or not [6,7]. However, for higher dimensional Hilbert spaces, there are examples of entangled states with a positive partial transpose [8,11,12]. Since the positivity of the partial transpose is a fundamental nonlocal property of the state that cannot be changed using only local operations and classical communication, the entanglement of states with positive partial transpose cannot be distilled to singlett form and is therefore not available for standard applications in quantum information protocols [11]. For this reason, entangled states with positive partial transpose are generally referred to as bound entangled states. Nevertheless, the inseparability of bound entanglement suggests that there are other fundamental properties of entanglement that distinguish bound entanglement from separable states. In particular, it would be interesting to know whether there are some potentially useful properties of bound entanglement that can be applied without requiring distillation to pure states. In this paper, it is shown that some bound entangled states do indeed have such a property. Specifically, the correlations between the two systems in the 3×3 bound entanglement analyzed in the following overcome the local uncertainty limit [13]. This means that bound entanglement could be applied directly in quantum communication, e.g., for quantum teleportation or for dense coding, since the amount of noise in data transmission would clearly be less than that of the corresponding classical limit.

The type of bound entanglement considered in this paper is the 3×3 entanglement first presented in Ref. [8]. Since the angular momentum components of the spin-1 algebra will be used in the analysis, it is most convenient to express this state in the $\hat{l}_z$ basis as

$$\hat{\rho}_a = \frac{a}{1+8a}(|-1;0\rangle\langle -1;0| + |-1;+1\rangle\langle -1;+1| + |0;-1\rangle\langle 0;-1| + |0;+1\rangle\langle 0;+1| + |+1;0\rangle\langle +1;0|) + \frac{3a}{1+8a}|E_{\mathrm{max.}}\rangle\langle E_{\mathrm{max.}}| + \frac{1}{1+8a}|\Pi\rangle\langle\Pi|,$$

$$\text{where}\quad |E_{\mathrm{max.}}\rangle = \frac{1}{\sqrt{3}}(|-1;-1\rangle + |0;0\rangle + |+1;+1\rangle)$$

$$\text{and}\quad |\Pi\rangle = \sqrt{\frac{1+a}{2}}|+1;-1\rangle + \sqrt{\frac{1-a}{2}}|+1;+1\rangle. \tag{1}$$

The parameter a can take any value between zero and one. In order to analyze the correlations between the physical properties of the two three-level systems represented by this density matrix, it is necessary to express the statistics of the density matrix in terms of expectation values of observables. A particularily convenient description of this type can be obtained by using a set of eight Hermitian generating operators $\hat{\lambda}_i$ characterized by the relations [14]

$$\mathrm{Tr}\{\hat{\lambda}_i\} = 0,$$

$$\mathrm{Tr}\{\hat{\lambda}_i\hat{\lambda}_j\} = 2\,\delta_{i,j}, \tag{2}$$

$$\sum_i \hat{\lambda}_i^2 = \frac{16}{3}\hat{1}. \tag{3}$$

The expectation values of these generating operators can then be interpreted as a generalization of the Bloch vector. In particular, the purity of the density matrix of a three-level system can be expressed as

$$\mathrm{Tr}\{\hat{\rho}_{\mathrm{local}}^2\} = \frac{1}{3} + \frac{1}{2}\sum_i \langle\hat{\lambda}_i\rangle^2. \tag{4}$$

This relation implies that the length of the eight-dimensional Bloch vector is limited to $\sqrt{4/3}$. Using Eqs. (3) and (4), it is then possible to formulate the sum uncertainty relation [13] for the generating operators $\hat{\lambda}_i$,

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$$\sum_i \delta\lambda_i^2 \geqslant 4. \tag{5}$$

As explained in Ref. [13], this purity uncertainty can be used to define a sufficient condition for entanglement. Specifically, no separable state of the 3×3 system can violate any local uncertainty relation of the form

$$\sum_i \delta(\lambda_i(1)-\lambda_i(2))^2 \geqslant 8, \tag{6}$$

where $\lambda_i(1/2)$ represent the measurement outcomes for the observables corresponding to the operators $\hat{\lambda}_i(1/2)$, respectively. It should be noted, however, that $\hat{\lambda}_i(1)$ and $\hat{\lambda}_i(2)$ do not have to be the same operators. Indeed, it is an important part of the result presented in this paper that they can have completely different properties.

It is now possible to define an optimal selection of operators $\hat{\lambda}_i(1)$ and $\hat{\lambda}_i(2)$ for the bound entangled state (1). In order to obtain both a compact formulation and a direct connection with the physical properties of a spin-1 system, one fundamental set of generating operators can be defined using the operators of the spin components, $\hat{l}_x$, $\hat{l}_y$, and $\hat{l}_z$, and their quadratic functions,

$$\hat{Q}_{ij} = \hat{l}_i\hat{l}_j + \hat{l}_j\hat{l}_i,$$

$$\hat{S}_{xy} = \hat{l}_x^2 - \hat{l}_y^2,$$

$$\hat{G}_z = \sqrt{3}\left(\hat{l}_z^2 - \frac{2}{3}\right). \tag{7}$$

With these basic definitions, the correlations of the bound entanglement described by Eq. (1) can be expressed in terms of optimally aligned operator pairs. This optimal alignment is determined by maximizing the total correlation given by

$$K_{\text{total}} = \sum_i \langle \hat{\lambda}_i(1) \otimes \hat{\lambda}_i(2) \rangle. \tag{8}$$

The choices of $\hat{\lambda}_1(1)$ to $\hat{\lambda}_5(1)$ are determined by the fact that these operator properties are only correlated with the respective operator properties $\hat{\lambda}_1(2)$ to $\hat{\lambda}_5(2)$ in system-2. However, there are some cross correlations in the remaining three operator properties, making it necessary to determine a nontrivial selection of operators. In general, the optimized operator alignment can be given by the following set of operator pairs,

$$\hat{\lambda}_1(1) = \hat{l}_x(1) \quad \hat{\lambda}_1(2) = \hat{l}_x(2),$$

$$\hat{\lambda}_2(1) = -\hat{l}_y(1) \quad \hat{\lambda}_2(2) = \hat{l}_y(2),$$

$$\hat{\lambda}_3(1) = -\hat{Q}_{xy}(1) \quad \hat{\lambda}_3(2) = \hat{Q}_{xy}(2),$$

$$\hat{\lambda}_4(1) = -\hat{Q}_{yz}(1) \quad \hat{\lambda}_4(2) = \hat{Q}_{yz}(2),$$

$$\hat{\lambda}_5(1) = \hat{Q}_{zx}(1) \quad \hat{\lambda}_5(2) = \hat{Q}_{zx}(2),$$

$$\hat{\lambda}_6(1) = \hat{Z}(1) \quad \hat{\lambda}_6(2) = \hat{l}_z(2),$$

$$\hat{\lambda}_7(1) = \hat{F}_{xy}(1) \quad \hat{\lambda}_7(2) = \hat{S}_{xy}(2),$$

$$\hat{\lambda}_8(1) = \hat{F}_z(1) \quad \hat{\lambda}_8(2) = \hat{G}_z(2), \tag{9}$$

where $\hat{Z}(1)$, $\hat{F}_{xy}(1)$, and $\hat{F}_z(1)$ are linear combinations of $\hat{l}_z(1)$, $\hat{S}_{xy}(1)$, and $\hat{G}_z(1)$ that have to be optimized depending on the specific value of a chosen for $\hat{\rho}$ in Eq. (1). The result of this optimization reads

$$\hat{Z} = \frac{1}{2}\hat{l}_z + \frac{\sqrt{3}}{2}\hat{G}_z,$$

$$\hat{F}_{xy} = \frac{1+2a}{2+a}\hat{S}_{xy} + \frac{\sqrt{3(1-a^2)}}{2+a}\left(\frac{\sqrt{3}}{2}\hat{l}_z - \frac{1}{2}\hat{G}_z\right),$$

$$\hat{F}_z = \frac{1+2a}{2+a}\left(\frac{\sqrt{3}}{2}\hat{l}_z - \frac{1}{2}\hat{G}_z\right) - \frac{\sqrt{3(1-a^2)}}{2+a}\hat{S}_{xy}. \tag{10}$$

With this choice of operator properties, the maximal correlation achieved is always equal to $K_{\text{total}} = 4/3$. Since this result is exactly equal to the square of the maximal length of the local Bloch vector, the optimized correlation already corresponds to the maximal correlation that can be achieved in separable systems. The local uncertainty defined by Eqs. (6) and (9) can now be evaluated by using this correlation,

$$\sum_i \delta(\lambda_i(1)-\lambda_i(2))^2 = \underbrace{\left(\sum_i \langle[\hat{\lambda}_i(1)-\hat{\lambda}_i(2)]^2\rangle\right)}_{32/3-2K_{\text{total}}} - \left(\sum_i \langle\hat{\lambda}_i(1)-\hat{\lambda}_i(2)\rangle^2\right)$$

$$= 8 - \left(\sum_i \langle\hat{\lambda}_i(1)-\hat{\lambda}_i(2)\rangle^2\right) < 8. \tag{11}$$

The local uncertainty relation (6) is therefore violated because of the nonvanishing mismatch in the local expectation values given by

$$\langle\lambda_7(1)-\lambda_7(2)\rangle = -\left(\frac{3a\sqrt{1-a^2}}{(2+a)(1+8a)}\right),$$

$$\langle\lambda_8(1)-\lambda_8(2)\rangle = \frac{\sqrt{3}a(1-a)}{(2+a)(1+8a)}. \tag{12}$$

Any separable states with a maximal correlation total of $K_{\text{total}} = 4/3$ must have perfectly aligned local Bloch vectors. However, the mismatch given by Eq. (12) shows that this is not the case for the bound entangled state ρ_a given by Eq.

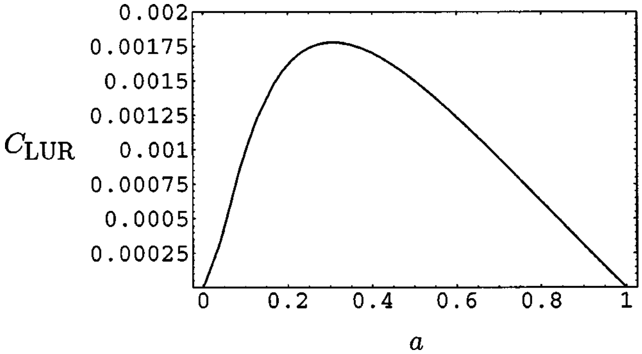

FIG. 1. Relative violation of local uncertainty C_{LUR} as a function of the parameter a defining the bound entangled state $\hat{\rho}_a$.

(1). Therefore, this class of bound entangled states violates the local uncertainty relation (6).

As discussed in Ref. [13], a useful measure of the relative violation of local uncertainty can be obtained by normalizing the amount by which the uncertainty is violated with the uncertainty limit,

$$\begin{aligned} C_{\text{LUR}} &= 1 - \frac{1}{8}\sum_i \delta(\lambda_i(1) - \lambda_i(2))^2 \\ &= \frac{1}{8}[\langle \lambda_7(1) - \lambda_7(2)\rangle^2 + \langle \lambda_8(1) - \lambda_8(2)\rangle^2] \\ &= \frac{3a^2(1-a)}{4(2+a)(1+8a)^2}. \end{aligned} \tag{13}$$

The dependence of this relative violation of local uncertainty on the parameter a that defines the bound entangled state is shown in Fig. 1. This result shows that the nonclassical correlations of the bound entangled states considered here are about one-thousand times weaker than the nonclassical correlations of maximally entangled states. It is also interesting to note that the maximal amount of bound entanglement is obtained for $a \approx 0.3077$, with a relative local uncertainty violation of $C_{\text{LUR}} \approx 0.001\,78$.

Using the violation of local uncertainty as a criterion, it is also possible to extend the class of bound entangled states given by Eq. (1) to mixtures of $\hat{\rho}_a$ and white noise,

$$\hat{\rho}(a;p_N) = p_N \hat{1} \otimes \hat{1} + (1-p_N)\hat{\rho}_a. \tag{14}$$

Such states still violate the local uncertainty relation (6) as long as the noise level is below the limit given by

$$\frac{p_N}{3(1-p_N)^2} < C_{\text{LUR}}(p_N = 0), \tag{15}$$

where $C_{\text{LUR}}(p_N=0)$ is the relative violation of local uncertainty for $\hat{\rho}_a$ given by Eq. (13) above. This result indicates that an addition of noise to bound entanglement is not critical if the noise level is well below 0.5%, thus determining the level of precision required for an experimental investigation of bound entangled states.

Besides the possibility of quantifying the nonclassical properties of bound entanglement, the violation of local uncertainty relations also provides insights into the physical properties of bound entanglement. The local uncertainty relation (6) defines a correspondence between the properties $\hat{\lambda}_i(1)$ of system 1 and the properties $\hat{\lambda}_i(2)$ of system 2. However, Eq. (10) defines the last three operator pairs in a highly asymmetric fashion. In particular, the correlated operators do not even have the same eigenvalue spectrum. It may well be that this lack of symmetry in bound entanglement is the main practical obstacle preventing the construction of entanglement purification protocols for bound entangled states [11]. Nevertheless the fact that bound entanglement overcomes the uncertainty limit given by relation (6) suggests that it may actually be used directly to realize a kind ofquantum teleportation. Specifically, quantum teleportation using bound entanglement would transfer the properties of the input state to properties of the output state according to a trace preserving map defined by the Bell measurement and the pair correlations given by Eq. (9). Since this map changes the eigenvalue spectrum of operators, it is necessarily nonpositive. The positivity of the output state is only preserved by the noise added in the transfer process. Even though this kind of asymmetric teleportation therefore requires a certain minimum of noise, the violation of the local uncertainty relation (6) shows that the transfer of properties would still be more precise than local operations and classical communication would allow. Quantum teleportation using bound entangled states can thus be seen as a nonclassical implementation of a nonpositive map, similar to quantum cloning or the universal NOT operation [15–17].

In conclusion, it has been shown that the bound entangled states given by Eq. (1) violate the local uncertainty relation (6) defined by the choice of generating operators (9). It is then possible to quantify the amount of entanglement in terms of the relative violation of local uncertainty and to identify the nonclassical correlations between the two systems in terms of local physical properties. The result indicates a specific kind of asymmetry between the correlated operator properties, characterized by the fact that the correlated operators do not share the same eigenvalue spectrum. It may well be that this kind of asymmetry is largely responsible for the lack of distillability in bound entanglement. However, the violation of local uncertainty itself shows that bound entanglement may overcome the classical limit in applications such as quantum teleportation or dense coding.

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