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Nonclassical correlations of phase noise and photon number in quantum nondemolition measurements

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The continuous transition from a low resolution quantum nondemolition measurement of light field intensity to a precise measurement of photon number is described using a generalized measurement postulate. In the intermediate regime, quantization appears as a weak modulation of measurement probability. In this regime, the measurement result is strongly correlated with the amount of phase decoherence introduced by the measurement interaction. In particular, the accidental observation of half integer photon numbers preserves phase coherence in the light field, while the accidental observation of quantized values increases decoherence. The quantum mechanical nature of this correlation is discussed and the implications for the general interpretation of quantization are considered.

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I. INTRODUCTION

In classical measurements, infinite precision is always desirable. Therefore there is no need for a fundamental measurement theory describing limited resolution. Instead, the lack of precision in any actual measurement is either neglected or considered to be an error that degrades the value of the measurement data obtained. In quantum mechanics, however, measurement precision always comes at a price. In particular, infinite precision requires a measurement interaction that completely randomizes some of the unobserved system properties. Consequently, limited precision may actually be desirable in quantum measurements.

For instance, a single mode of the electromagnetic field with a well-defined photon number must have a completely random phase. Therefore a precise photon number measurement destroys phase coherence and all associated interference properties of the field mode with other coherent modes. If phase coherence and interference properties are preserved the intensity of the field mode can only be determined with a precision too low to resolve single photons. On the other hand, quantization emerges only when phase coherence is lost. Nevertheless a complete characterization of the light field dynamics requires both information about the intensity and the phase distributions. In general, it is therefore realistic to consider a compromise between phase uncertainty and intensity uncertainty.

In quantum nondemoltion measurements of photon number, information about the photon number $\hat{n}$ is obtained through the interaction of the measured field with a probe field [1,2] or with probe atoms [3,4]. This interaction introduces phase noise into the measured system, as required by the uncertainty relations [5,6]. Since the purpose of the procedure is a measurement of photon number, it is very tempting to assume that a perfect resolution of photon number is the ideal case and therefore more desireable than a limited resolution. However, Kitagawa and coworkers [6] have pointed out that even if photon number states are not resolved, a quantum nondemolition measurement of photon number may produce a minimum uncertainty state of phase and photon number. There is a trade off, then, between the noise introduced and the resolution achieved, which requires the definition of a much larger class of ideal quantum measurements. By generalizing the conventional projective measurement postulate, it is possible to investigate this class of ideal quantum measurement, focussing especially on the transitional regime between classical low noise measurements at low resolution and the extreme quantum regime of fully resolved quantization and complete dephasing. It is shown in the following, that the statistical properties of such intermediate resolution measurements include nonclassical correlations between the measured photon number and the phase noise introduced in the measurement, which can only be observed in this transitional regime.

In Sec. II, a theoretical description of photon number measurements with variable resolution is given and the effective measurement postulate is derived. In particular, the measurement operator provides a description of the dephasing caused by the measurement interaction.

In Sec. III, the statistics of the measurement results are obtained. The transition from the classical limit to the quantum limit is discussed by pointing out the appearance of nonclassical correlations between the measurement result and the coherence after the measurement.

In Sec. IV, the correlations are compared to fundamental properties of the operator formalism. It is shown that the statistics of the measurement results correspond to a specific operator ordering in the evaluation of correlations.

In Sec. V, the results are summarized and possible implications are discussed. It is argued that the measurement statistics reveal that there is more to quantum reality than the integer photon number. By providing coherence, half-integer photon numbers or ''fuzzy'' photon numbers also contribute to observable fact.

II. VARIABLE RESOLUTION IN IDEAL PHOTON NUMBER MEASUREMENTS

A. Light field quantization and measurement precision

Based on the application of lasers, modern quantum optics has provided a characterization of the quantum mechani-

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cal light field, which is much closer to a classical theory of noisy fields than the operator formalism would suggest [7]. In particular, the classical property of light field coherence is much easier to control than the nonclassical property of quantized photon number. It is indeed difficult to measure the exact photon number of a single, well-defined light field mode. In multimode open systems such as lasers, Langevin equations offer a better description of the light field dynamics than photon number rate equations, even in the presence of amplitude squeezing [8]. This dominance of the classical wave properties in lasers has motivated a new kind of criticism of the photon picture, expressed especially in the notion of ''lasing without photons'' by Siegmann [9,10]. Even in the light of conventional quantum mechanics, it is questionable whether the concept of photon number has any meaning before it is definitely measured. In particular, Heisenberg emphasized that no value can be assigned to a physical property if the system is not in an eigenstate of that property [11]. After all, what photon number should be assigned to a coherent superposition of photon number states? It should be obvious that one cannot just pick out one eigenvalue while neglecting the others. Nevertheless, this point is so contrary to our natural intuition that it still raises controversies among physicists [12].

In a quantum nondemolition measurement of photon number, a nonlinear coupling mechanism is utilized to shift a noisy and continuous pointer variable by an amount proportional to the photon number. As a consequence, the measurement readout of the photon number measurement is generally both noisy and continuous. The discreteness of the photon number eigenvalues only emerges if the noise in the pointer variable is sufficiently low. Thus, the actual measurement result obtained is usually a continuous variable and not a discrete one. In order to study the emergence of photon number quantization, one should therefore examine the properties of quantum measurements with variable resolution and continuous values for the photon number measurement results. If the reality of integer photon numbers is somehow ''created'' in the measurement, there should be a transition from classical fields to quantized fields depending only on the measurement resolution. While the basic tools for such an analysis are indeed provided by the standard quantum theory of measurement [13,6], the axiomatic nature of the mathematical approach often obscures the intuitive classical limit. Therefore, it is useful to formulate a generalized measurement postulate taking into account the limited measurement resolution. This measurement postulate summarizes the conventional results while illustrating the fundamental aspects of coherence and decoherence more clearly, providing a shortcut to the derivation of quantum noise features.

B. Generalized measurement postulate for pointer measurements

In a quantum nondemolition measurement, a pointer variable n_m of the probe system is shifted by an amount corresponding to the photon number n of the light field. However, since the pointer variable n_m is itself noisy, there is some error in this procedure. Assuming Gaussian noise, the probability distribution of n_m subject to an uncertainty of δn reads

$$P(n_m)=(2\pi\delta n^2)^{-1/2}\exp\left(-\frac{(n-n_m)^2}{2\,\delta n^2}\right). \tag{1}$$

This distribution applies to a photon number eigenstate. In order to describe the effects of a measurement on superpositions of photon number states, it is necessary to define an operater $\hat{P}_{\delta n}(n_m)$, such that the general effect of a measurement result n_m with a quantum mechanical uncertainty δn on an initial state $|\psi_i\rangle$ is given by $\hat{P}_{\delta n}(n_m)|\psi_i\rangle$. The probability of obtaining the result n_m and the state $|\psi_f(n_m)\rangle$ after the measurement are then given by

$$P(n_m)=\langle\psi_i|\hat{P}^{\dagger}_{\delta n}(n_m)\hat{P}_{\delta n}(n_m)|\psi_i\rangle$$

$$|\psi_f(n_m)\rangle=[1/\sqrt{P(n_m)}]\hat{P}_{\delta n}(n_m)|\psi_i\rangle. \tag{2}$$

Note that the measurement thus described is ideal, since a pure state remains pure and no additional decoherence is introduced. It is assumed that the measurement system is prepared in a well-defined quantum state and that the readout is accurate. The source of the uncertainty in the measurement is the quantum noise in the pointer variable n_m before the measurement interaction takes place. By increasing this noise, the phase noise introduced in the measurement interaction is reduced and vice versa. In a realistic situation, there may be additional measurement uncertainties due to an inaccurate readout of the pointer or due to additional phase noise introduced in the measurement interaction. Such additional noise sources cause decoherence and change the pure state $|\psi_f(n_m)\rangle$ into a mixture that would have to be represented by a density matrix. In the following, however, it is assumed that such additional noise sources can be avoided. It is then possible to deduce the correct measurement operator by comparing equations (1) and (2). It reads

$$\hat{P}_{\delta n}(n_m)=(2\pi\delta n^2)^{-1/4}\exp\left(-\frac{(\hat{n}-n_m)^2}{4\,\delta n^2}\right). \tag{3}$$

This operator describes the relation of the photon number operator $\hat{n}$ with the value n_m obtained in the measurement. Thus the connection between the quantum system and the classical measurement readout is established. Although the standard measurement postulate as formulated by von Neumann [13] can be recovered by either letting δn approach zero or by applying $\hat{P}_{\delta n}(n_m)$ many times, the generalized concept of measurement represented by $\hat{P}_{\delta n}(n_m)$ describes a much wider range of physical situations and is definitely closer to the kind of perception we know from everyday experience. In particular, it describes the classical limit of the uncertainty relations in the case of low resolution, $\delta n \gg 1$.

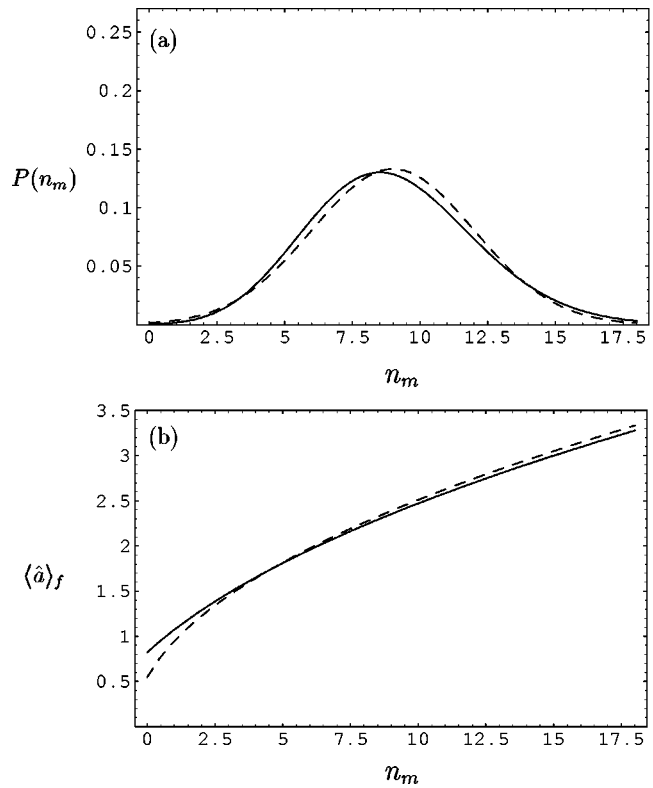

FIG. 1. Photon number measurement statistics of a coherent state with an average amplitude of $\alpha=3$ at a photon number resolution of $\delta n=0.7$. (a) shows the probability distribution over measurement results n_m. Quantization is not resolved yet. The dashed curve corresponds to the approximate result using a Gaussian photon number distribution as explained in the text. (b) shows the expectation value $\langle \hat{a} \rangle_f$ after the measurement as a function of the measurement result n_m. The dashed curve is the result obtained by multiplying a coherent amplitude of $(n_m+1/2)^{1/2}$ with the dephasing factor.

C. Photon number squeezing and phase noise

Although, strictly speaking, the phase of a light field mode is not an observable since no phase operator can be constructed, approximate operators and phase space distributions show that there is an uncertainty relation between photon number and phase given by $\delta n\, \delta\phi \geq 1/2$ [14,15]. The role of this uncertainty in quantum nondemolition measurements of photon number has been investigated in the context of measurements using the optical Kerr effect [5,6]. It will be shown in the following that the generalized measurement operator $\hat{P}_{\delta n}(n_m)$ faithfully reproduces these experimentally confirmed results.

Since the phase itself cannot be represented by an operator, it is more realistic to illustrate the decoherence induced by the phase noise by analyzing the reduction in the expectation value of the complex field amplitude $\langle \hat{a} \rangle$. Adding Gaussian phase noise with a variance of $\delta\phi^2$ to an arbitrary field state reduces the initial expectation value of the amplitude $\langle \hat{a} \rangle_i$ to a final value of

$$\langle \hat{a} \rangle_f = \exp\left(-\frac{\delta\phi^2}{2} \right) \langle \hat{a} \rangle_i . \tag{4}$$

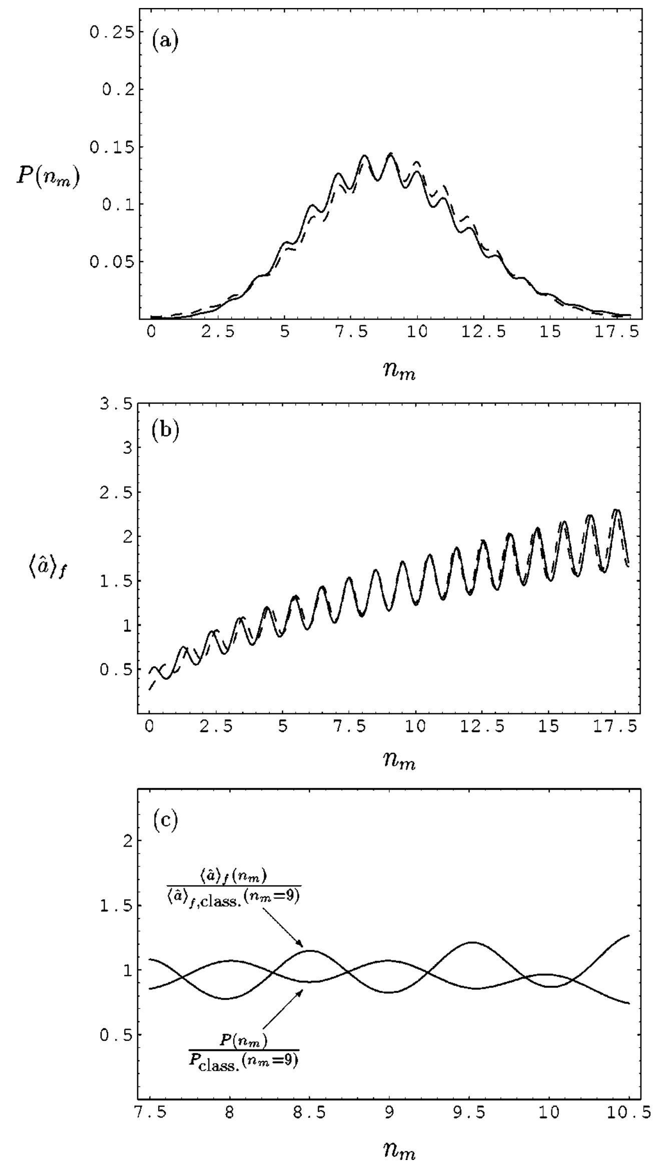

FIG. 2. Photon number measurement statistics of a coherent state with an average amplitude of $\alpha=3$ at a photon number resolution of $\delta n=0.4$. (a) shows the probability distribution over measurement results and (b) shows the expectation value $\langle \hat{a} \rangle_f(n_m)$ after the measurement. The dashed curves correspond to the approximate formulas given in the text. (c) shows details of the quantum mechanical modulations of measurement probability and coherence after the measurement near $n_m=9$, normalized by the respective classical results $P_{\text{class.}}(n_m=9)$ and $\langle \hat{a} \rangle_{f,\text{class.}}(n_m=9)$.

The overall average $\langle \hat{a} \rangle_f(\text{av.})$ of the field expectation value after the measurement is given by

$$\begin{aligned}\langle \hat{a} \rangle_f(\text{av.}) &= \int \langle \psi_f(n_m)|\hat{a}|\psi_f(n_m)\rangle P(n_m)\, dn_m \\ &= \int \langle \psi_i|\hat{P}_{\delta n}(n_m)\hat{a}\hat{P}_{\delta n}(n_m)|\psi_i\rangle\, dn_m \\ &= \exp\left(-\frac{1}{8\,\delta n^2} \right) \langle \psi_i|\hat{a}|\psi_i\rangle . \end{aligned} \tag{5}$$

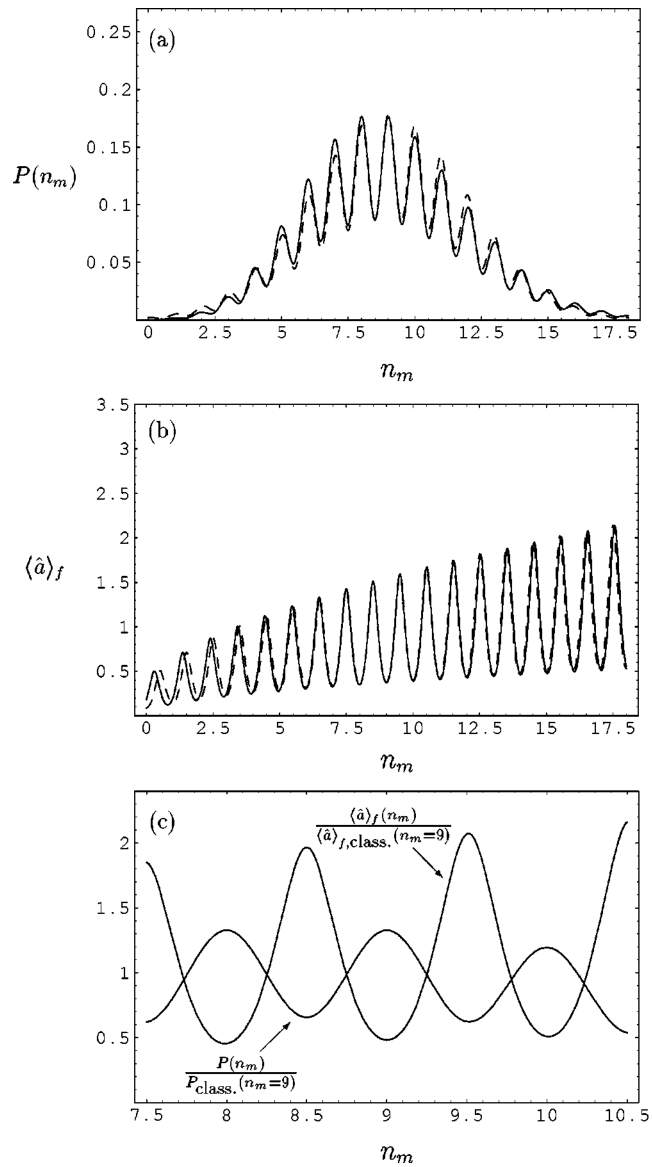

FIG. 3. Photon number measurement statistics of a coherent state with an average amplitude of $\alpha=3$ at a photon number resolution of $\delta n=0.3$. (a) to (c) are as in the previous figure.

According to equation (4), this reduction in amplitude corresponds to a Gaussian phase noise with a variance of

$$\delta\phi^2=\frac{1}{4\,\delta n^2}. \tag{6}$$

Thus the amount of phase noise introduced in the measurement corresponds to the minimum noise required by the uncertainty relation of phase and photon number for a measurement resolution of δn. This is a direct consequence of assuming an ideal quantum mechanical measurement, which does not introduce additional phase noise. In a realistic situation, it is likely that the phase noise introduced is somewhat higher than this ideal quantum limit. Relation (6) may then be used to determine how much excess phase noise is introduced in a given experimental setup. Note that this excess noise may originate not only from an additional source of decoherence, but also from an inaccurate readout of the pointer variable.

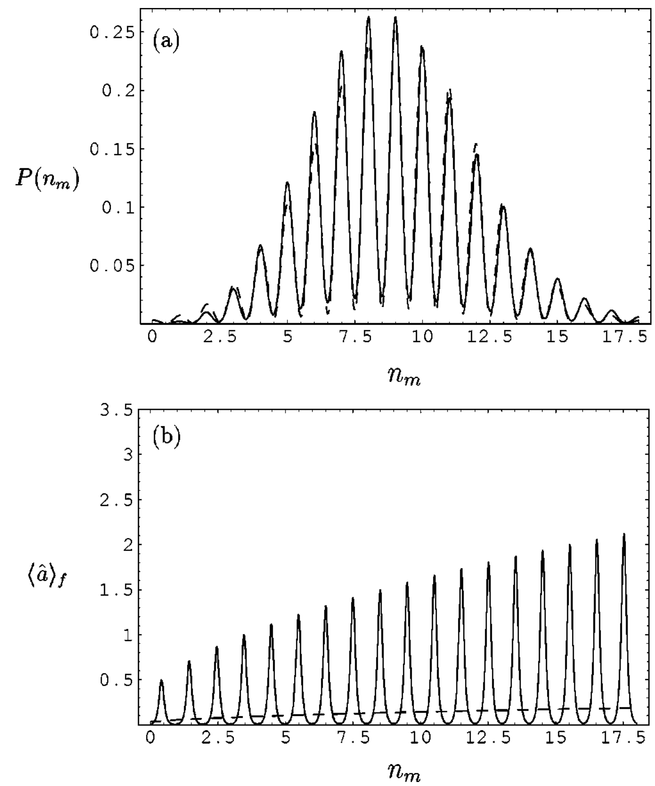

FIG. 4. Photon number measurement statistics of a coherent state with an average amplitude of $\alpha=3$ at a photon number resolution of $\delta n=0.2$. (a) shows the probability distribution over measurement results. The dashed curve corresponds to the approximate formulas given in the text. (b) shows the expectation value $\langle\hat{a}\rangle_f(n_m)$ after the measurement. The dashed curve shows the classical result, $\langle\hat{a}\rangle_{f,\mathrm{class.}}(n_m)$.

III. THE EMERGENCE OF QUANTIZATION

A. Measurement of a coherent state

If the initial state $|\psi_i\rangle$ is a coherent state $|\alpha\rangle$ with the photon number state expansion

$$|\alpha\rangle=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{7}$$

then the measurement statistics defined by equation (2) reads

$$\begin{aligned}P(n_m)&=\langle\alpha|\hat{P}^2_{\delta n}(n_m)|\alpha\rangle\\&=[\exp(-|\alpha|^2)/\sqrt{2\pi\delta n^2}]\\&\quad\times\sum_n \frac{|\alpha|^{2n}}{n!}\exp\left(-\frac{(n-n_m)^2}{2\,\delta n^2}\right),\end{aligned} \tag{8}$$

and the coherent amplitude $\langle\hat{a}\rangle_f$ after the measurement reads

$$\langle \hat{a} \rangle_f(n_m) = \frac{\langle \alpha | \hat{P}_{\delta n}(n_m) \hat{a} \hat{P}_{\delta n}(n_m) | \alpha \rangle}{\langle \alpha | \hat{P}^2_{\delta n}(n_m) | \alpha \rangle}$$

$$= \alpha \exp\left(-\frac{1}{8\,\delta n^2} \right) \times \frac{\sum_n \frac{|\alpha|^{2n}}{n!} \exp\left(-\frac{\left(n + \frac{1}{2} - n_m \right)^2}{2\,\delta n^2} \right)}{\sum_n \frac{|\alpha|^{2n}}{n!} \exp\left(-\frac{(n - n_m)^2}{2\,\delta n^2} \right)}. \quad (9)$$

The results shown in Figs. 1–4 have been calculated using these exact results. However, it is helpful to apply some approximations in order to identify the quantization effects.

For $|\alpha|^2 \gg 1$, the photon number distribution may be approximated by a Gaussian distribution with a mean photon number $|\alpha|^2$ and a photon number fluctuation of $|\alpha|$. The application of the measurement operator $\hat{P}_{\delta n}(n_m)$ then results in a convolution of two Gaussians. If the resolved photon number δn is much smaller than the photon number fluctuation $|\alpha|$, then the amplitude of the photon number state components of $|\alpha\rangle$ does not change much within the measurement interval of $n_m \pm \delta n$ and the convolution may be approximately factorized into a product reading

$$\hat{P}_{\delta n}(n_m)|\alpha\rangle \approx \overbrace{(2\pi|\alpha|^2)^{-1/4} \exp\left(-\frac{(n_m - |\alpha|^2)^2}{4|\alpha|^2} \right)}^{\text{Gaussian intensity distribution of } |\alpha\rangle} \times \underbrace{\sum_n (2\pi\,\delta n^2)^{-1/4} \exp\left(-\frac{(n - n_m)^2}{4\,\delta n^2} \right) \exp(-i\phi n)|n\rangle}_{\text{decoherence and quantization effects}}, \quad (10)$$

where the phase ϕ is defined by $\alpha = |\alpha| \exp(-i\phi)$.

It is thus possible to separate the state dependent photon number distribution from the fundamental effects of decoherence and quantization. By applying the approximations of Eq. (10) to the measurement statistics described by Eqs. (8) and (9), an even clearer separation of classical noise properties and quantization effects is obtained. The approximate results read

$$P(n_m) \approx \underbrace{(2\pi|\alpha|^2)^{-1/2} \exp\left(-\frac{(n_m - |\alpha|^2)^2}{2|\alpha|^2} \right)}_{\text{classical intensity distribution}} \underbrace{\sum_n (2\pi\,\delta n^2)^{-1/2} \exp\left(-\frac{(n - n_m)^2}{2\,\delta n^2} \right)}_{\text{quantization effects}} \quad (11)$$

for the probability, and

$$\langle \hat{a} \rangle_f(n_m) \approx \underbrace{\exp(-i\phi) \sqrt{n_m + \frac{1}{2}} \exp\left(-\frac{1}{8\,\delta n^2} \right)}_{\text{classical amplitude average}} \underbrace{\frac{\sum_n \exp\left(-\frac{\left(n - \frac{1}{2} - n_m \right)^2}{2\,\delta n^2} \right)}{\sum_n \exp\left(-\frac{(n - n_m)^2}{2\,\delta n^2} \right)}}_{\text{quantization effects}} \quad (12)$$

for the coherent amplitude. Note that only the phase of the coherent amplitude expectation value $\langle \hat{a} \rangle_f$ after the measurement depends on the initial value of α. The absolute value is determined by the measurement result and is proportional to $(n_m + 1/2)^{1/2}$. This result corresponds to the classical notion that the absolute value of the coherent amplitude should be the square root of the intensity.

The sums that express the quantization effects in Eqs. (11) and (12) are periodic functions of n_m. In other words, quantization effects only depend on how close the measurement result n_m is to an integer value. Because of this periodicity, the sums can be expressed as Fourier series. Specifically,

$$(2\pi\delta n^2)^{-1/2}\sum_n \exp\left(-\frac{(n-n_m)^2}{2\delta n^2}\right) = 1+2\sum_{k=1}^{\infty} \exp(-2\pi^2\delta n^2 k^2)\cos(2\pi k n_m) \tag{13}$$

and

$$(2\pi\delta n^2)^{-1/2}\sum_n \exp\left(-\frac{\left(n-\frac{1}{2}-n_m\right)^2}{2\delta n^2}\right) = 1-2\sum_{k=1}^{\infty} \exp(-2\pi^2\delta n^2 k^2)\cos(2\pi k n_m). \tag{14}$$

Note that the Fourier coefficients are Gaussians in the modulation frequency variable k. The high frequency components of the periodic modulations are therefore strongly suppressed. Depending on the measurement resolution δn, it is reasonable to limit the expansion to only the first few contributions. This resolution dependent truncation of the Fourier series defines the transition from the classical regime to the quantum regime.

B. From the classical limit to full quantization

In the classical limit, all Fourier components with $k>1$ are negligible. The measurement probability and the expectation value of the coherent field after the measurement read

$$P_{\text{class.}}(n_m) = (2\pi|\alpha|^2)^{-1/2}\exp\left(-\frac{(n_m-|\alpha|^2)^2}{2|\alpha|^2}\right)$$

$$\langle\hat{a}\rangle_{f,\text{class.}}(n_m) = \sqrt{n_m+1/2}\exp(-i\phi)\exp\left(-\frac{1}{8\delta n^2}\right). \tag{15}$$

These results correspond to the classical assumption of continuous light field intensity and equally continuous Gaussian noise in the light field phase and amplitude. A typical example is shown in Fig. 1 for a coherent state with an amplitude of $\alpha=3$. The measurement resolution is at $\delta n=0.7$, quite close to the quantum limit. Nevertheless, the approximate results of Eqs. (15) correspond quite well to the more precise results of Eqs. (8) and (9). Indeed, the main discrepancy between the probability distribution $P(n_m)$ given by Eq. (15) and the exact result is due to the asymmetry of the Poissonian photon number distribution which has been neglected by assuming a Gaussian photon number distribution in Eqs. (11) and (12). This deviation gets much smaller as the average photon number of the coherent state is increased. However, it is already a good approximation at the average photon number of nine shown in the examples.

As the quantum limit is approached, the classical results are modulated by quantum effects. In the probability distribution of measurement results, this modulation appears as a fringe pattern similar to that caused by an interference effect. At the same time, a complementary fringe pattern emerges in the coherence after the measurement as given by $\langle\hat{a}\rangle_f(n_m)$. The lowest order contributions to these quantization effects read

$$P(n_m) = P_{\text{class.}}(n_m)(1+2\exp(-2\pi^2\delta n^2)\cos(2\pi n_m))$$

$$\langle\hat{a}\rangle_f(n_m) = \langle\hat{a}\rangle_{f,\text{class.}}(n_m)\frac{1-2\exp(-2\pi^2\delta n^2)\cos(2\pi n_m)}{1+2\exp(-2\pi^2\delta n^2)\cos(2\pi n_m)}. \tag{16}$$

The accuracy of this approximation is worst for $\langle\hat{a}\rangle_f(n_m)$ at integer or half-integer values of n_m. At these points, it is accurate to within 1% for $\delta n\geqslant 0.27$ and accurate to within 10% for $\delta n\geqslant 0.23$. Thus, the reliability of the lowest order approximation is generally very high above $\delta n\approx 0.25$. Figure 2 shows the probability distribution and the coherent amplitude after the measurement at a resolution of $\delta n=0.4$. This resolution corresponds to a modulation factor of $2\exp(-2\pi^2\delta n^2)=0.085$. The modulation is still very weak and the likelihood of obtaining an integer result is only about 1.2 times higher than the likelihood of obtaining a half integer result. Nevertheless, the quantization fringes in $P(n_m)$ and the decoherence fringes in $\langle\hat{a}\rangle_f(n_m)$ are clearly visible. The anticorrelation of the probability peaks and the coherence maxima is illustated in Fig. 2(c), which shows the respective modulations near $n_m=9$, normalized using the classical results at $n_m=9$. Figure 3 shows the probability distribution and the coherent amplitude after the measurement at a resolution of $\delta n=0.3$. This resolution corresponds to a modulation factor of $2\exp(-2\pi^2\delta n^2)=0.338$. The likelihood of obtaining an integer result is about twice as high as that of obtaining a half-integer result and the reduction in the coherent amplitude is about four times greater for integer n_m than for half-integer n_m. At an average decoherence factor of $\exp(-1/(8\delta n^2))=0.25$, the average coherent amplitude after the measurement is still quite significant. A measurement resolution of $\delta n=0.3$ thus combines aspects of photon number quantization and aspects of phase coherence, defining the center of the transitional regime between continuous field intensities and quantized photon numbers.

Between a resolution of $\delta n=0.3$ and a resolution of $\delta n=0.2$, the approximation given by Eq. (16) breaks down. For $\delta n<0.2$, the probability distribution is given by isolated Gaussians centered around integer measurement results n_m. Half-integer results become extremely unlikely. However, if such an unlikely result is obtained, there still is coherence even in extremely precise measurments. This fact is usually obscured by the assumption of infinite precision inherent in the conventional projective measurement postulate. Figure 4 shows the probability distribution and the coherence after the measurement for a resolution of $\delta n=0.2$. Note that the approximation given by Eq. (16) is still very good for the probability distribution. However, the relative error in the peak values of the coherent amplitude $\langle \hat{a} \rangle_f$ after the measurement is nearly 100%. Therefore, the dashed curve in Fig. 4(b) does not show the approximate result, but instead shows the classical approximation $\langle \hat{a} \rangle_{f,\text{class.}}$ given by Eq. (15). This comparison illustrates the relatively high coherence at half-integer measurement results n_m. At half-integer measurement results n_m, the expectation value $\langle \hat{a} \rangle_{f,\text{class.}}$ of the coherent amplitude is equal to $(n_m+1/2)^{1/2}/2$, or one half of the amplitude corresponding to a classical light field intensity of $n_m+1/2$. This result is valid for all $\delta n<0.2$, regardless of the average dephasing induced by the measurement interaction. Therefore, the peak values of the coherence after the measurement are much higher than the classical results, while the minima at integer photon number are actually closer to zero than the classical interpretation of dephasing would suggest. In the case of $\delta n=0.2$ shown in Fig. 4, the classical approximation predicts an average decoherence factor of $\exp(-1/(8\,\delta n^2))=0.044$. However, the peak values of coherence at half-integer photon number are more than ten times higher and the minima at integer photon number are more than ten times lower than the classically expected coherence after dephasing. Since the likelihood of integer results is about ten times higher than the likelihood of half-integer results, the main contribution to the average coherence after the measurement still originates from half-integer photon number results. Even at fully resolved quantization, the half-integer photon number results thus provide a contribution to the dephasing statistics.

C. Correlation between quantization and dephasing

The discussion above reveals a clear qualitative difference between measurement results n_m of integer photon number and of half-integer photon number. To obtain a quantitative expression, it is necessary to define a measure of quantization associated with each measurement result n_m. In the following, the quantization Q of a measurement result n_m is therefore defined as

$$Q(n_m)=\cos(2\pi n_m). \tag{17}$$

Thus, the quantization Q of integer values of n_m is $+1$ and the quantization of half-integer values is -1. In the classical case, this results in an average quantization of zero. The average quantization $\bar{Q}$ of the measurement results is given by

$$\bar{Q}=\int dn_m\, Q(n_m)P(n_m) = \exp(-2\pi^2\delta n^2). \tag{18}$$

Since $\bar{Q}$ depends only on δn, it may be used as an experimental measure of the resolution obtained in quantum nondemolition measurements of photon number. It is now possible to evaluate the correlation between the quantization observed and the coherence after the measurement by averaging the product,

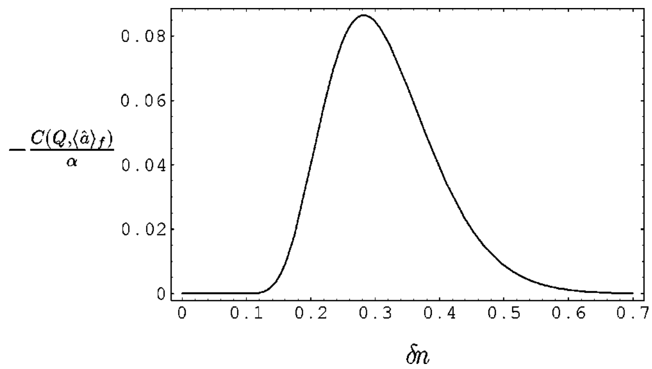

FIG. 5. Normalized anticorrelation of the quantization Q of the measurement result n_m and the coherence $\langle \hat{a} \rangle_f(n_m)$ after the measurement as a function of measurement resolution δn.

$$\begin{aligned}\overline{Q\langle \hat{a}\rangle_f} &= \int dn_m\, Q(n_m)\langle \hat{a}\rangle_f(n_m)P(n_m)\\ &= -\exp(-2\pi^2\delta n^2)\exp\left(-\frac{1}{8\,\delta n^2}\right)\alpha\\ &= -\bar{Q}\langle \hat{a}\rangle_f(\text{av.}).\end{aligned} \tag{19}$$

The average of the product of quantization and coherence is exactly equal to the negative product of the averages. Therefore, quantization and coherence are strongly anticorrelated. The correlation $C(Q,\langle \hat{a}\rangle_f)$ is given by

$$\begin{aligned}C(Q,\langle \hat{a}\rangle_f) &= \overline{Q\langle \hat{a}\rangle_f}-\bar{Q}\langle \hat{a}\rangle_f(\text{av.})\\ &= -2\bar{Q}\langle \hat{a}\rangle_f(\text{av.})\\ &= -2\exp(-2\pi^2\delta n^2)\exp\left(-\frac{1}{8\,\delta n^2}\right)\alpha.\end{aligned} \tag{20}$$

Figure 5 shows this correlation as a function of measurement resolution δn. The correlation is maximal at $\delta n=1/(2\sqrt{\pi})$, which is a resolution of about 0.282 photons. At this point, the average quantization $\bar{Q}$ is equal to $\exp(-\pi/2)=0.208$ and the average coherent amplitude $\langle \hat{a}\rangle_f(\text{av.})$ after the measurement is equal to $\exp(-\pi/2)=0.208$ times the original amplitude α.

There appears to be a well-defined transition from the classical limit to the quantum limit of measurement resolution at $\delta n=1/(2\sqrt{\pi})$, which is characterized by statistical properties not observable in either the extreme quantum limit or in the classical limit. Since it should be possible to obtain these statistical properties from experimental results, some measure of reality must be attributed to the concept of variable quantization Q. Specifically, even though it is clear that only measurement results of full quantization $Q=1$ remain as the resolution is increased, the reduced decoherence at $Q=-1$ demonstrates that such results cannot be interpreted as measurement errors due to either a higher or a lower photon number. This measurement scenario thus highlights the problem of assuming the existence of an integer photon number before the photon number is actually measured. Obviously, quantization is not a property of the system, which is simply hidden by the noise of the low precision measurement in the classical limit. Some very real physical properties are associated with noninteger values of photon number measurement results. Possibly, it is necessary to consider operator values other than the eigenvalues as part of the physical reality associated with quantum mechanical operator variables.

IV. FUNDAMENTAL PROPERTIES OF THE OPERATOR FORMALISM

A. Quantization and the parity operator

The generalized measurement operator $\hat{P}_{\delta n}(n_m)$ describes both classical and quantum mechanical features of measurements in terms of a quantum mechanical operator. Classically, it would be possible to distinguish between the measurement result n_m and the actual photon number n. In quantum mechanics, however, the photon number $\hat{n}$ is an operator which does not have a well-defined value unless the field is in a photon number eigenstate. Therefore, the relationship between the measurement result n_m and the photon number operator $\hat{n}$ is quite different from the classical relationship between a noisy measurement result and the true value of the measured quantity.

A quantum mechanical property that may provide a connection between the definition of quantization Q based on the measurement result n_m and the properties of the photon number operator $\hat{n}$ is the parity $\hat{\Pi}$ defined as

$$\hat{\Pi}=(-1)^{\hat{n}}. \tag{21}$$

The square of the parity $\hat{\Pi}^2$ may then be associated with the quantization Q. Of course, the quantum mechanical value of quantization is always one. However, by ‘‘breaking apart’’ the square of the parity, a correlation between quantization and coherent field amplitude may be established. It reads

$$\langle\hat{\Pi}\hat{a}\hat{\Pi}\rangle-\langle\hat{\Pi}^2\rangle\langle\hat{a}\rangle=-2\langle\hat{\Pi}^2\rangle\langle\hat{a}\rangle. \tag{22}$$

If $\langle\hat{\Pi}^2\rangle$ is identified with $\overline{Q}$ and $\langle\hat{a}\rangle$ is identified with $\langle\hat{a}\rangle_f(\mathrm{av.})$, this correlation corresponds to the one given in Eq. (20). The relationship between coherence and quantization can thus be traced to the anticommutation between parity and field amplitude, $\hat{\Pi}\hat{n}=-\hat{n}\hat{\Pi}$. One could indeed argue that the correlation that appears in the measurement is hidden in the commutation relations of the operator formalism.

B. Ambiguous correlations in the operator formalism

The correlation given in Eq. (22) is of course a result of the specific order in which the operators have been applied. Since $\hat{\Pi}^2$ is always one, there is no correlation as soon as both parity operators are placed on the same side of the field operator $\hat{a}$. In principle, it is not possible to determine the correlation between noncommuting quantum variables directly from the operator formalism because of this ambiguity concerning the ordering of the operators.

In particular, the case of photon number quantization and parity belongs to a general class of correlations based on the inequality

$$\frac{1}{2}\langle\hat{A}\hat{B}^2+\hat{B}^2\hat{A}\rangle\neq\langle\hat{B}\hat{A}\hat{B}\rangle, \tag{23}$$

where $\hat{A}$ and $\hat{B}$ represent arbitrary noncommuting operator variables. The operator ordering $\hat{B}\hat{A}\hat{B}$ allows correlations even if the quantum state is an eigenstate of $\hat{A}$ or $\hat{B}^2$. This property definitely contradicts any assumption of classical statistics. Nevertheless, such correlations can be obtained in experiment, even though the outcome of a direct measurement of $\hat{A}$ or $\hat{B}$ performed on the initial state would be perfectly predictable. Thus the quantum nondemolition measurement discussed in this paper represents an example of a more general class of measurements revealing fundamental nonclassical properties of quantum statistics.

C. Operator ordering and physical reality

In the theory of quantum mechanics, the classical values of physical variables are replaced by operators. Consequently, it is not possible to assign a well-defined value to an operator variable if the system is not in an eigenstate of the operator. This situation calls for a review of our concepts of physical reality, as can be seen from the arguments concerning entanglement and the debate of hidden variables [12,16]. Quantum mechanical uncertainty is definitely quite different from a classical lack of knowledge [17], and this difference is revealed in the correlations between noncommuting variables. For instance, the EPR argument basically uses the entanglement of two particles to establish a correlation between position and momentum of the same particle—thus trying to circumvent the restrictions imposed by uncertainty on Einsteins arguments in the Bohr-Einstein dialogue [18]. However, as Bell has shown, the correlations between noncommuting variables thus obtained cannot be represented by a classical probability distribution [19]. Since this paradox is an inherent property of the operator formalism, it should be possible to trace its origin directly to the fundamental nonclassical properties of quantum mechanical measurements.

In principle it would be desirable to know the value of a correlation between noncommuting variables such as the parity $\hat{\Pi}$ and the coherent amplitude $\hat{a}$ without reference to a measurement. If there were hidden variables defining classical values for both operators, there should also be a well-defined correlation. However, the formalism itself introduces an ambiguity. A formal calculation of correlations based on the expectation values of operator products raises the question of operator ordering. A particularly striking ambiguity is represented by Eq. (22), since it permits a correlation of $\hat{\Pi}^2$ with the coherent amplitude even though the eigenvalues of $\hat{\Pi}^2$ are all one. Of course one could argue that it should not be allowed to separate the square of the parity operator. However, such a postulate would not be based on any physical observation but only on preconceived notions of what reality should be like. It is therefore important to note that unusual correlations such as the one given by Eq. (22) can have a real physical meaning in measurement statistics.

Since quantum mechanics does not allow the simultaneous assignment of well-defined physical values to noncommuting observables, it is not possible to discuss correlations between such observables without a definition of the measurement by which such correlations are obtained. The futility of trying a more general approach is clearly revealed by the ambiguity of the correlations caused by the commutation relations between operators.

V. CONCLUSIONS AND OUTLOOK

A. Interpretation of the nonclassical correlations

The results presented above show that a quantum nondemolition measurement reveals much more than just the photon number of a light field at an intermediate measurement resolution close to $\delta n = 0.3$. In this intermediate regime, the property that phase coherence in the field requires quantum coherence between neighboring photon number states emerges visibly as a correlation between the continuous measurement result n_m and the coherence after the measurement $\langle \hat{a} \rangle_f$. This measurement scenario thus reveals the difference between quantum mechanical uncertainty and a classical lack of precision. In particular, there is a real physical difference between the measurement results of half-integer photon number and the measurement results of integer photon number, which makes it impossible to argue that the measurement of half-integer photon number is merely an error. By introducing the variable Q to denote the quantization of the measurement result, it is possible to evaluate the correlation between quantization and decoherence in the measurement. In the operator formalism, the quantization can be interpreted as the square of the parity operator $\hat{\Pi}$. It is then possible to derive the observed correlation directly from the operator formalism.

The correlation obtained both from the statistics of the quantum nondemolition measurement and from the operator statistics suggests the reality of half-integer photon number results. Depending on the circumstances, quantum measurements may therefore reveal physical values of operator variables, which are quite different from the eigenvalues of the corresponding operators. At the same time, the ambiguity of the correlations between operator variables shows that an identification of neither eigenvalues n nor measurement results n_m with elements of reality can be valid. It is therefore not sufficient to extend the range of photon number values. Instead, the statistics of physical properties should be based on the measurement results obtained in a specific measurement setup. The ambiguity in the formalism can then be resolved by applying the appropriate generalized measurement postulate.

It seems that the physical property of light field intensity given by the photon number can not be attributed to any measurement independent elements of reality. Possibly, it might be a useful compromise to regard the measurement results n_m as elements of a fundamentally noisy reality, while acknowledging the qualitative dependence of the measurement result on the resolution δn. In the classical limit, the identification of n_m with the actual light field intensity is usually not problematic. Therefore, our classical concept of reality survives on the macroscopic level, even though it has to be abondoned in the microscopic regime. In the quantum limit, n_m can again be identified with the eigenvalues of the operator $\hat{n}$. In this manner, a continuous transition between our classical concept of reality and the mysterious properties of the quantum regime can be described.

B. Experimental possibilities

The measurement statistics described here should be obtainable by carefully evaluating the data obtained in any quantum nondemolition measurement followed by a measurement of field coherence, e.g., by homodyne detection. It is important, however, to keep track of the correlation between the measurement result n_m and the corresponding average results of the field measurements $\langle \hat{a} \rangle_f(n_m)$. This requires some amount of time resolution, for example in the form of light field pulses or perhaps of solitons in fibers [2]. Unfortunately, it is extremely difficult to realize quantum nondemolition measurements of high resolution in the optical regime. The experimental results cited here [1,2] are still well in the classical regime of $\delta n > 1$. Possibly, a realization based on the interaction of single atoms with a microwave mode [3,4] might be more promising. In particular, the use of a variable number of single probe atom passed through the cavity should allow a particularly reliable variation of the photon number resolution parameter δn.

The challenge presented by the aspects of quantum theory discussed above is to obtain sufficient control of quantum coherence to explore the properties at the very limit of quantum mechanical uncertainty. The effects observed in this regime should then help to illustrate the quantum mechanical properties utilized for quantum computation, quantum communication, and other aspects of quantum information [20]. The continuous transition from the classical aspects of optical coherence to the quantum properties of the light field can also serve as a tool to pinpoint the technological requirements for more complex implementations of quantum optical devices.

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