

## Information and noise in quantum measurement

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Even though measurement results obtained in the real world are generally both noisy and continuous, quantum measurement theory tends to emphasize the ideal limit of perfect precision and quantized measurement results. In this article, a more general concept of noisy measurements is applied to investigate the role of quantum noise in the measurement process. In particular, it is shown that the effects of quantum noise can be separated from the effects of information obtained in the measurement. However, quantum noise is required to “cover up” negative probabilities arising as the quantum limit is approached. These negative probabilities represent fundamental quantum-mechanical correlations between the measured variable and the variables affected by quantum noise.

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### I. INTRODUCTION

The interpretation of quantum-mechanical measurements and the connection between the classical regime and the quantum regime still remains somewhat of a mystery, even after nearly a century of quantum physics [1]. When introducing quantum mechanics, most textbooks and university courses tend to throw out all classical physics and start with mathematical axioms instead. Although no one would doubt that the concepts of classical physics are still successful in describing most of our everyday experience, there seems to be no gradual approach that introduces quantum modifications to an otherwise unaltered classical world. In fact, recent controversies [2] suggest that many physicists are meanwhile prepared to consider the classical limit as a somewhat special case justified only as a crude approximation or even as a subjective illusion (e.g., in the popular many-worlds interpretation) [3]. At the heart of these controversies is the problem of “measurement.” Classical theories were able to ignore the problem, because one could always assume knowledge of all facts. In quantum mechanics, there is a disturbing separation between the continuous deterministic evolution of a system state and the random selection of quantized results in the measurement.

The practical problems of interpreting the physical meaning of quantum states are solved by the statistical interpretation provided through the measurement postulate. However, no such postulate exists in classical physics. What would then be the analogy between quantum measurement and classical measurement? In the early days of quantum physics, such considerations resulted in the formulation of the well-known uncertainty principle [4]. All measurements must introduce some noise into the system in order to preserve the uncertainty relations. If the precision of a measurement approaches the quantum limit, the noise introduced during the measurement process completely obscures the original values of all physical properties that are not eigenvalues of the observed state. If the precision of the measurement is far below the quantum limit, the noise introduced into the measured system may be negligibly small and the classical situation is reproduced in most respects. It should be noted, however, that the assumption of infinitely precise coordi-

nates cannot be recovered—classical physics is the physics of low precision and noisy observations, just as nonrelativistic physics is the physics of low velocities. The mathematical representation of classical coordinates in terms of real numbers is therefore an approximation of the quantum-mechanical reality of finite precision.

In order to illustrate the transition from the classical (noisy) world to the world of quantization, it seems to be desirable to include the precision of measurements in the measurement theory. Indeed, several discussions of measurements with limited precision have been presented, usually based on the standard measurement postulate and an intermediate system. Much of the focus has been on continuous measurements, which can be described by quantum trajectories [5–7]. Some rather remarkable properties obtained by combining weak (i.e., low-precision) measurements with standard projective measurements have been pointed out by Aharonov and co-workers [8,9]. In the following, the concept of weak measurement is taken one step further by the introduction of a generalized measurement postulate for both weak and strong measurements. This eliminates the requirement of distinguishing qualitatively between two situations and allows a more direct approach to the transition from classical low-precision measurements to quantum-mechanical precision. Moreover, the generalized measurement postulate can be applied directly to the quantum system considered, without the introduction of a measurement system.

The generalized measurement postulate can be interpreted entirely in terms of classical measurement processes. Therefore, the noise introduced into the system during the measurement interaction can be separated from the information obtained. If the precision of the measurement is sufficiently low to avoid a resolution of quantization, this separation allows a classical interpretation of the information obtained in the measurement. As precision increases, however, the noise effect is required to “cover up” negative probabilities predicted by the information-induced change in the system state. These negative probabilities can then be interpreted as nonclassical correlations between the measurement result and the quantum fluctuations in the observed system.

## II. THE GENERALIZED MEASUREMENT POSTULATE

### A. Limitations of precise measurements

The axiomatic introduction of quantum mechanics given, e.g., in von Neumann's *Mathematical Foundations of Quantum Mechanics* [10] usually emphasizes the beauty and simplicity of the mathematical structure. It is therefore not surprising that the measurement postulate does not include the possibility of uncertainty in the measurement result that is emphasized by Heisenberg in his *Physical Principles of Quantum Theory* [11]. Instead, von Neumann merely shows that the measurement postulate based on an infinitely precise observation of eigenvalues is consistent with equally precise indirect measurement performed by letting the system interact with a meter. However, the uncertainty relations require that infinitely precise knowledge of one variable can only be obtained by introducing an infinite amount of uncertainty in another variable. A precise measurement of position always requires an infinite uncertainty of momentum, a precise measurement of the intensity of a radio signal would require complete uncertainty of the phase, and a precise measurement of the angular momentum of the moon would require a complete delocalization of the moon itself. A large number of similar examples can be constructed, showing that any quantum-mechanically precise measurement on a macroscopic system would cause macroscopic uncertainties. Thus, the projective measurement postulate has no classical limit and consequently fails to describe some of the most typical classical measurements performed on macroscopic objects.

In order to amend this shortcoming and to illustrate the continuous transition from quantum mechanics to classical physics, it is therefore desirable to replace the abstract mathematical definition of measurement given by von Neumann with a formulation closer to everyday experience. Actually, this can be achieved without any change in the fundamental structure of quantum theory, since the choice of a projection on eigenstates was not motivated by physical observations, but rather by considerations of mathematical simplicity. The generalized formalism may appear to be less elegant, but it faithfully reproduces all physical results, including a more natural transition to the classical limit.

### B. The generalized measurement operator

If a careful experimentalist obtains the result that a variable  $A$  corresponding to a Hermitian operator  $\hat{A}$  in quantum theory has a value of  $\bar{A} \pm \delta A$ , then we need not assume that the experimentalist performed a projective measurement on  $A$  and failed to read out the correct result. Instead, the uncertainty of  $\delta A$  may be a consequence of the experimentalist's attempts to minimize the noise introduced into the system according to the uncertainty principle. Typical examples of such measurements are optical back action evasion quantum nondemolition measurements of photon number [12,13] and of quadrature components of the light field [14,15]. In all these experimental realizations, the measurement resolution was finite and the coherence between eigenstates was reduced but not lost due to the measurement interaction [16,17].

In order to obtain a measurement value in a back action evasion measurement, the observed system must be coupled to a measurement apparatus. Ideally, a meter variable  $\hat{x}$  of the measurement device is shifted by an amount equal to the system variable  $\hat{A}$ . This interaction is described by the unitary operator  $\hat{S}$  which transforms the eigenstates  $|x;A\rangle$  of the pointer variable  $\hat{x}$  and the system variable  $\hat{A}$  according to

$$\hat{S}|x;A\rangle = |x+A;A\rangle. \quad (1)$$

The shift in  $\hat{x}$  may also be expressed using the conjugate meter variable  $\hat{p}$ , which is defined by the operator property

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i. \quad (2)$$

The interaction operator then reads

$$\hat{S} = \exp(-i\hat{p}\hat{A}). \quad (3)$$

The meter variable  $\hat{p}$  thus acts on all system variables that do not commute with  $\hat{A}$ , causing noise in the system. In order to perform a measurement of  $\hat{A}$  with a resolution of  $\delta A$ , it is necessary to prepare the initial meter state in a Gaussian wave packet with a variance of  $\delta A^2$  in  $\hat{x}$ . According to the uncertainty principle, this requires a variance of  $1/(2\delta A)^2$  in  $\hat{p}$ . In order to obtain information about the system, it is necessary to read out only  $\hat{x}$ , thereby eliminating all information about the noise variable  $\hat{p}$ .

For an initial Gaussian meter state and an arbitrary system state  $|\Phi_S\rangle$ , the entangled state of meter and system after the measurement interaction reads

$$|\psi_f\rangle = \int dx \sum_A (2\pi\delta A^2)^{-1/4} \exp\left(-\frac{(A-x)^2}{4\delta A^2}\right) \times \langle A|\Phi_S\rangle |x;A\rangle. \quad (4)$$

A measurement readout of the meter variable  $\hat{x}$  selects a subspace of the total Hilbert space. Within this subspace, the system state corresponding to a measurement readout of  $|x=\bar{A}\rangle$  is given by

$$\hat{P}_{\delta A}(\bar{A})|\Phi_S\rangle = \sum_A (2\pi\delta A^2)^{-1/4} \exp\left(-\frac{(A-\bar{A})^2}{4\delta A^2}\right) \times \langle A|\Phi_S\rangle |A\rangle. \quad (5)$$

The effect of a measurement of  $\hat{A}$  with finite resolution  $\delta A$  is therefore described by a set of generalized measurement operators  $\hat{P}_{\delta A}(\bar{A})$  corresponding to the possible measurement results  $\bar{A}$ ,

$$\hat{P}_{\delta A}(\bar{A}) = (2\pi\delta A^2)^{-1/4} \exp\left(-\frac{(\hat{A}-\bar{A})^2}{4\delta A^2}\right). \quad (6)$$

Instead of projecting the system state into an eigenstate of  $\hat{A}$ , this operator modifies the statistical weight of each eigenstate, while preserving as much coherence as possible [18].

The probability distribution  $p(\bar{A})$  over possible measurement results  $\bar{A}$  corresponding to a given initial state  $|\psi_i\rangle$  is given by

$$\begin{aligned} p(\bar{A}) &= \langle \psi_i | \hat{P}_{\delta A}^\dagger(\bar{A}) \hat{P}_{\delta A}(\bar{A}) | \psi_i \rangle \\ &= \langle \psi_i | [\hat{P}_{\delta A}(\bar{A})]^2 | \psi_i \rangle. \end{aligned} \quad (7)$$

This probability distribution may be characterized by the averages  $\langle \bar{A} \rangle$  and  $\langle \bar{A}^2 \rangle$ . These averages are related to the quantum-mechanical expectation values by

$$\begin{aligned} \langle \bar{A} \rangle &= \langle \hat{A} \rangle, \\ \langle \bar{A}^2 \rangle &= \langle \hat{A}^2 \rangle + \delta A^2. \end{aligned} \quad (8)$$

Thus the average results of the general measurement postulate correspond to the expectation value and the total variance is equal to the sum of the variance in the system and the squared uncertainty of the measurement.

It is important to understand that the additional fluctuations in the measurement result do not correspond to additional noise in the system. Instead, increasing the noise in the measurement result decreases the noise introduced in the system according to the uncertainty principle. After the measurement, the system state will have changed to

$$|\psi_f(\bar{A})\rangle = \frac{1}{\sqrt{p(\bar{A})}} \hat{P}_{\delta A}(\bar{A}) |\psi_i\rangle. \quad (9)$$

This is still a pure state. The extent to which it differs from the initial state is determined both by the decomposition of the initial state  $|\psi_i\rangle$  in terms of eigenstates of  $\hat{A}$ , and by the uncertainty  $\delta A^2$  of the measurement. For very large uncertainties, the changes in the system state are only weak, illustrating the low level of noise in the measurement interaction.

For infinitely precise measurements, the properties of the generalized measurement operator correspond to the properties of the projection operator. The projective measurement postulate is thus reproduced by  $\delta A \rightarrow 0$ . Since most actual measurements have a finite resolution, however, the generalized measurement operator  $\hat{P}_{\delta A}(\bar{A})$  provides a more realistic description of measurements than the original projection postulate. Discussions of quantum-mechanical phenomena may be simplified considerably by accepting the validity of the generalized measurement postulate without requiring the formal derivation given above. In particular, the relationship between the measurement information obtained and the noise introduced in the measurement can be investigated in far greater detail, directly revealing fundamental properties of the operator formalism in the measurement data. Note that in the following, the generalized measurement postulate will be applied without further reference to the actual physical interaction by which the measurement result  $\bar{A}$  is obtained. The

measurement resolution  $\delta A$  is considered to be a property characterizing the measurement effect on the system rather than a property of the external measurement setup. It is then possible to focus on the system properties as the source of the measurement statistics.

### C. Density matrix formulation

It is a straightforward matter to apply the generalized measurement postulate to an initial mixed state density matrix  $\hat{\rho}_i$ . The probability and the effect of the measurement result  $\bar{A}$  on the density matrix read

$$\begin{aligned} p(\bar{A}) &= \text{Tr}\{\hat{P}_{\delta A}(\bar{A}) \hat{\rho}_i \hat{P}_{\delta A}^\dagger(\bar{A})\} \\ &= \text{Tr}\{[\hat{P}_{\delta A}(\bar{A})]^2 \hat{\rho}_i\}, \\ \hat{\rho}_f(\bar{A}) &= \frac{1}{p(\bar{A})} \hat{P}_{\delta A}(\bar{A}) \hat{\rho}_i \hat{P}_{\delta A}^\dagger(\bar{A}). \end{aligned} \quad (10)$$

Using the density matrix formulation, the physical effect of the measurement interaction on the system may be determined by mixing the final state density matrices  $\hat{\rho}_f(\bar{A})$  according to their respective statistical weights  $p(\bar{A})$ ,

$$\begin{aligned} \hat{\rho}_f(\text{total}) &= \int d\bar{A} p(\bar{A}) \hat{\rho}_f(\bar{A}) \\ &= \int d\bar{A} \hat{P}_{\delta A}(\bar{A}) \hat{\rho}_i \hat{P}_{\delta A}^\dagger(\bar{A}). \end{aligned} \quad (11)$$

This density matrix represents the system between the measurement interaction and the readout of the measurement result. It therefore describes the (average) noise effect caused by the measurement interaction. In particular, the elements of the density matrix that describe coherence between eigenstates of  $\hat{A}$  are reduced by

$$\langle A_1 | \hat{\rho}_f(\text{total}) | A_2 \rangle = \exp\left(-\frac{(A_1 - A_2)^2}{8\delta A^2}\right) \langle A_1 | \hat{\rho}_i | A_2 \rangle, \quad (12)$$

where  $|A_1\rangle$  and  $|A_2\rangle$  are eigenstates of  $\hat{A}$  with eigenvalues of  $A_1$  and  $A_2$ , respectively. Thus there is a gradual decrease of coherence depending only on the separation of the eigenvalues  $A_1$  and  $A_2$ . The Gaussian dependence of the suppression factor on the difference of the eigenvalues indicates that the decoherence effect is extremely sensitive to the relationship between the eigenvalue difference  $|A_1 - A_2|$  and the resolution  $\delta A$  of the measurement. Indeed, the decoherence factor is greater than 0.88 for  $|A_1 - A_2| < \delta A$  and lower than 0.14 for  $|A_1 - A_2| > 4\delta A$ . This rapid transition from almost no decoherence to almost complete decoherence corresponds to the notion that the ability to distinguish the eigenvalues  $A_1$  and  $A_2$  requires decoherence between the corresponding eigenstates. If the separation of eigenvalues  $|A_1 - A_2|$  is large, even a very weak measurement that otherwise preserves microscopic coherences will destroy the coherence

between the eigenstates of  $A_1$  and  $A_2$ . It is therefore much more difficult to preserve the quantum coherence between states with quantitatively different physical properties than to preserve coherence between quantitatively similar states [19].

Since Eq. (11) makes no reference to the measurement result actually obtained, it represents only the effects of physical interaction involved in the measurement. Thus it is equivalent to a description of decoherence in open systems interacting with an unknown environment. However, in the case of a quantum measurement, the meter takes the place of the environment, and the meter information is recovered in the measurement. Consequently, it is not possible to average over the meter state, and interpretational problems related to the entanglement of system and meter can arise as soon as the actual information obtained in the measurement is considered. Specifically, the measurement readout requires an interpretation of  $\hat{\rho}_f(\text{total})$  as a mixture corresponding to the different possible measurement results, while the simpler alternative of assuming random phase noise in the coherence between eigenstates of  $\hat{A}$  cannot be recovered.

### III. SEPARATION OF INFORMATION AND NOISE

#### A. Formal separation

The total effect of a measurement result of  $\bar{A} \pm \delta A$  on the density matrix element  $\langle A_1 | \hat{\rho}_i | A_2 \rangle$  is given by Eq. (10). In terms of matrix elements of the eigenstates of  $\hat{A}$  it reads

$$\begin{aligned} \langle A_1 | \hat{\rho}_f(\bar{A}) | A_2 \rangle &= \frac{1}{\sqrt{2\pi\delta A^2} p(\bar{A})} \\ &\times \exp\left(-\frac{[(A_1 + A_2)/2 - \bar{A}]^2}{2\delta A^2}\right) \\ &\times \exp\left(-\frac{(A_1 - A_2)^2}{8\delta A^2}\right) \langle A_1 | \hat{\rho}_i | A_2 \rangle \\ &= \frac{1}{\sqrt{2\pi\delta A^2} p(\bar{A})} \exp\left(-\frac{[(A_1 + A_2)/2 - \bar{A}]^2}{2\delta A^2}\right) \\ &\times \langle A_1 | \hat{\rho}_f(\text{total}) | A_2 \rangle. \end{aligned} \quad (13)$$

Since the effect of decoherence given in Eq. (11) can be identified with the Gaussian factor changing the matrix element of the initial density matrix  $\hat{\rho}_i$  to the corresponding matrix element of  $\hat{\rho}_f(\text{total})$ , the remaining factor should describe the effect of a noise-free measurement. This factorization of the decoherence factor and the factor associated with the measurement result obtained allows an unambiguous separation of information and noise in quantum measurements, even though these two aspects are connected by the requirements of the uncertainty relations. It is therefore possible to overcome the uncertainty limitations and to examine the structure of quantum-mechanical reality which is hidden beneath the noise.

The measurement may be interpreted as a two-step process. In the first step, decoherence is caused by the physical interaction between the system and the measurement setup, changing the density matrix from  $\hat{\rho}_i$  to  $\hat{\rho}_f(\text{total})$ . In the second step, information about the system is obtained without any (additional) interaction. This step may actually occur far away from the system. While the first step involves well-defined physical processes, the second step relates to a change in the probabilistic expression of the system state due to information gained about the system. Equation (11) shows that the total density matrix  $\hat{\rho}_f(\text{total})$  can be interpreted as a mixture of all possible final density matrices  $\hat{\rho}_f(\bar{A})$ , so it is possible to consider the change from  $\hat{\rho}_f(\text{total})$  to  $\hat{\rho}_f(\bar{A})$  as a selection of a reality that existed before the information was obtained. It therefore seems that the classical separation of information and physical interaction has been preserved.

However, the properties of  $\hat{\rho}_f(\text{total})$  have been modified by the decoherence in step 1, and it is impossible to remove this step without violating the uncertainty principle. In particular, the entanglement between system and meter ensures that the noise introduced into the system can never be compensated once a measurement readout is obtained.

#### B. Simulation of noise-free measurements

The only difference between the classical measurement situation and the quantum-mechanical situation is the uncertainty in the measurement interaction. Except for the required relationship between decoherence and measurement resolution, the two measurement steps can be separated. In a classical situation, the noise added in the measurement interaction is both undesirable and avoidable. It is assumed that the information obtained in the measurement refers to a reality of  $A$  that exists as an element of reality regardless of the measurement. By examining the quantum-mechanical version of a noise-free measurement, it is possible to find out ‘‘what is wrong with these classical elements of reality’’ [20].

The procedure for selecting a subensemble density matrix  $\hat{\rho}_f(\bar{A})$  from the total density matrix  $\hat{\rho}(\text{total})$  described by Eq. (11) can be applied directly to the initial density matrix  $\hat{\rho}_i$ . It is then possible to reverse the actual sequence of steps in the measurement process in order to investigate the changes in the system state caused by the information obtained before quantum noise is added. In a noise-free measurement, the initial density matrix  $\hat{\rho}_i$  is decomposed in analogy with Eq. (11),

$$\hat{\rho}_i = \int d\bar{A} p(\bar{A}) \hat{\rho}_m(\bar{A}). \quad (14)$$

The matrix elements of the density matrix  $\hat{\rho}_m(\bar{A})$  describing the effects of measurement without noise then read

$$\begin{aligned} \langle A_1 | \hat{\rho}_m(\bar{A}) | A_2 \rangle &= \frac{1}{\sqrt{2\pi\delta A^2} p(\bar{A})} \\ &\times \exp\left(-\frac{\left[\frac{1}{2}(A_1+A_2)-\bar{A}\right]^2}{2\delta A^2}\right) \\ &\times \langle A_1 | \hat{\rho}_i | A_2 \rangle. \end{aligned} \quad (15)$$

Indeed, the statistical weight factor modifying the density matrix looks harmless enough. The matrix element is enhanced or suppressed depending on the closeness of the average quantum number  $(A_1+A_2)/2$  of the matrix element to the measurement result  $\bar{A}$ . This effect would correspond to the classically expected modification if  $(A_1+A_2)/2$  were somehow to represent the (classical) value of  $\hat{A}$  associated with the matrix element. However, the matrix element indicates a coherent superposition of two different eigenvalues  $A_1$  and  $A_2$  of  $\hat{A}$ . Therefore the modification of its statistical weight should be represented by separate contributions from  $A_1$  and  $A_2$ . Since this is not so, however, a serious problem arises concerning the relation

$$\langle A_1 | \hat{\rho}_m | A_2 \rangle \langle A_2 | \hat{\rho}_m | A_1 \rangle \leq \langle A_1 | \hat{\rho}_m | A_1 \rangle \langle A_2 | \hat{\rho}_m | A_2 \rangle, \quad (16)$$

which guarantees that all probabilities obtained as expectation values of the density matrix  $\hat{\rho}_m$  are positive. Condition (16) is fulfilled only if

$$\frac{\langle A_1 | \hat{\rho}_i | A_2 \rangle \langle A_2 | \hat{\rho}_i | A_1 \rangle}{\langle A_1 | \hat{\rho}_i | A_1 \rangle \langle A_2 | \hat{\rho}_i | A_2 \rangle} \leq \exp\left(-\frac{(A_1-A_2)^2}{4\delta A^2}\right). \quad (17)$$

Thus the decoherence factor of Eqs. (12) and (13) reappears in a requirement that can be fulfilled only if the coherence of the density matrix  $\hat{\rho}_i$  is sufficiently low. If the coherence of  $\hat{\rho}_i$  is high, however, condition (16) is violated and consequently negative probabilities are obtained for some of the possible coherent superpositions of the eigenstates  $|A_1\rangle$  and  $|A_2\rangle$ .

The difference between quantum mechanics and classical physics thus emerges as the measurement information obtained at low resolution is not only information on eigenvalues of  $\hat{A}$ , but also on the average values of off-diagonal matrix elements. Negative probabilities arise naturally, because there may be no possible eigenvalues corresponding to  $(A_1+A_2)/2$ . Nevertheless, measurement results close to  $(A_1+A_2)/2$  indicate that the coherent contribution of the corresponding off-diagonal matrix element is greater than that of the associated diagonal elements. In classical physics, the reality of  $A$  would be well defined. In the quantum formalism, however, the eigenvalues of  $\hat{A}$  represent only an incomplete description of the reality of the operator variable  $\hat{A}$ .

### C. Illustration of negative probabilities in a two-level system

At this point, a specific example should help to illustrate the case of negative probabilities after the measurement. If the system concerned is a spin- $\frac{1}{2}$  system described by the two orthogonal eigenstates of the  $\hat{s}_z$  component,  $|+Z\rangle$  and  $|-Z\rangle$ , then all physical properties can be described in terms of the operators of the spin components,

$$\hat{s}_x = \frac{1}{2}(|+Z\rangle\langle -Z| + |-Z\rangle\langle +Z|),$$

$$\hat{s}_y = -\frac{i}{2}(|+Z\rangle\langle -Z| - |-Z\rangle\langle +Z|),$$

$$\hat{s}_z = \frac{1}{2}(|+Z\rangle\langle +Z| - |-Z\rangle\langle -Z|). \quad (18)$$

If the initial state is given by the eigenstate of  $\hat{s}_x$  with the eigenvalue  $+\frac{1}{2}$ ,  $|+X\rangle$ , then the initial density matrix reads

$$\hat{\rho}_i = \frac{1}{2}(|+Z\rangle\langle +Z| + |-Z\rangle\langle -Z| + |+Z\rangle\langle -Z| + |-Z\rangle\langle +Z|). \quad (19)$$

The probability  $p(\bar{s}_z)$  of obtaining a measurement result of  $\bar{s}_z \pm \delta s_z$  can be determined according to Eq. (7) using the corresponding generalized measurement operator. It reads

$$\begin{aligned} p(\bar{s}_z) &= \frac{1}{\sqrt{2\pi\delta s_z^2}} \left[ \frac{1}{2} \exp\left(-\frac{(\bar{s}_z-1/2)^2}{2\delta s_z^2}\right) \right. \\ &\quad \left. + \frac{1}{2} \exp\left(-\frac{(\bar{s}_z+1/2)^2}{2\delta s_z^2}\right) \right] \\ &= \frac{1}{\sqrt{2\pi\delta s_z^2}} \exp\left(-\frac{\bar{s}_z^2+1/4}{2\delta s_z^2}\right) \cosh\left(\frac{\bar{s}_z}{2\delta s_z^2}\right). \end{aligned} \quad (20)$$

Spin quantization clearly emerges in this probability distribution if  $\delta s_z$  is smaller than  $\frac{1}{2}$ . The noise-free part of the measurement changes the density matrix to

$$\begin{aligned} \hat{\rho}_m(\bar{s}_z) &= \frac{1}{2 \cosh(\bar{s}_z/2\delta s_z^2)} \left[ \exp\left(\frac{\bar{s}_z}{2\delta s_z^2}\right) |+Z\rangle\langle +Z| \right. \\ &\quad \left. + \exp\left(-\frac{\bar{s}_z}{2\delta s_z^2}\right) |-Z\rangle\langle -Z| \right. \\ &\quad \left. + \exp\left(\frac{1}{8\delta s_z^2}\right) |+Z\rangle\langle -Z| \right. \\ &\quad \left. + \exp\left(\frac{1}{8\delta s_z^2}\right) |-Z\rangle\langle +Z| \right]. \end{aligned} \quad (21)$$

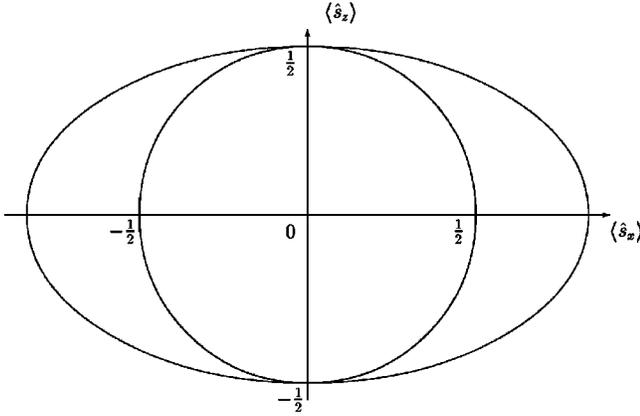


FIG. 1. Illustration of the expectation values of the spin- $\frac{1}{2}$  system after the noise-free measurement (ellipse) and after the complete measurement (circle) for a measurement uncertainty of  $\delta s_z = \frac{1}{2}$ .

This density matrix violates condition (17) and predicts negative probabilities for several spin directions. The negative probabilities can be illustrated by the expectation values of the spin components,

$$\begin{aligned} \langle \hat{s}_x \rangle_m &= \frac{\exp(1/8\delta s_z^2)}{2 \cosh(\bar{s}_z/2\delta s_z^2)} \\ &= \frac{1}{2} \exp\left(\frac{1}{8\delta s_z^2}\right) \sqrt{1 - \tanh^2\left(\frac{\bar{s}_z}{2\delta s_z^2}\right)}, \\ \langle \hat{s}_y \rangle_m &= 0, \\ \langle \hat{s}_z \rangle_m &= \frac{1}{2} \tanh\left(\frac{\bar{s}_z}{2\delta s_z^2}\right). \end{aligned} \quad (22)$$

Figure 1 shows the expectation values in the  $xz$  plane for a measurement uncertainty of  $\delta s_z^2 = \frac{1}{4}$ . Except at  $\langle \hat{s}_x \rangle = 0$ , the length of the average spin vector is larger than  $\frac{1}{2}$ , indicating negative probabilities. In particular, a result of  $\bar{s}_z = 0$  increases the expectation value of  $\frac{1}{2}$  in  $\hat{s}_x$  to  $\frac{1}{2} \exp[1/(8\delta s_z^2)]$ . This corresponds to a probability greater than 1 for  $|+X\rangle$  and a corresponding negative probability for  $|-X\rangle$ . Note that this result seems to be related to the observation of spin components larger than the permitted eigenvalue limit reported in [8], even though actual measurements of the spin component  $\hat{s}_x$  are not considered in the present context.

Of course, negative probabilities cannot be observed in a measurement. They represent a statistical tool which is connected with the unavoidable presence of quantum noise. In order to recover the final density matrix  $\hat{\rho}_f(\bar{s}_z)$  after the measurement, quantum noise must be added. This reduces the coherence to normal levels, resulting in the pure state density matrix

$$\begin{aligned} \hat{\rho}_f(\bar{s}_z) &= \frac{1}{2 \cosh(\bar{s}_z/2\delta s_z^2)} \left[ \exp\left(\frac{\bar{s}_z}{2\delta s_z^2}\right) | +Z \rangle \langle +Z | \right. \\ &\quad + \exp\left(-\frac{\bar{s}_z}{2\delta s_z^2}\right) | -Z \rangle \langle -Z | + | +Z \rangle \langle -Z | \\ &\quad \left. + | -Z \rangle \langle +Z | \right]. \end{aligned} \quad (23)$$

The expectation values given by this density matrix read

$$\begin{aligned} \langle \hat{s}_x \rangle_f &= \frac{1}{2} \sqrt{1 - \tanh^2\left(\frac{\bar{s}_z}{2\delta s_z^2}\right)}, \\ \langle \hat{s}_y \rangle_f &= 0, \\ \langle \hat{s}_z \rangle_f &= \frac{1}{2} \tanh\left(\frac{\bar{s}_z}{2\delta s_z^2}\right), \end{aligned} \quad (24)$$

as shown in Fig. 1. In particular, the expectation value of  $\hat{s}_x$  for  $\bar{s}_z = 0$  is reduced to  $\frac{1}{2}$ . Thus, quantum noise is necessary in order to ‘‘cover up’’ any negative probabilities and any excessive expectation values arising in the noise-free formalism.

#### D. Interpretation of negative probabilities

The example above shows that negative probabilities can compensate the decoherence caused by the noisy measurement interaction. It could therefore be said that negative probabilities represent nonclassical information beyond the limits of uncertainty. Indeed, the density matrix  $\hat{\rho}_m$  may be ‘‘purer’’ than a pure state. The trace of the square of a pure state density matrix is 1. For a mixed state, it is smaller than 1. For  $\hat{\rho}_m$ , however, it may indeed be greater than 1. Thus, with the purity  $P_m$  of  $\hat{\rho}_m$  defined as

$$P_m = \text{tr}\{\hat{\rho}_m^2\}, \quad (25)$$

the density matrix  $\hat{\rho}_m(\bar{s}_z = 0)$  given in the example of Sec. III C above has a purity of

$$P_m(\bar{s}_z = 0) = \frac{1}{2} + \frac{1}{2} \exp\left(\frac{1}{4\delta s_z^2}\right), \quad (26)$$

which is greater than 1 in all cases. Note that there is no upper limit to  $P_m(0)$ . However, its increase does depend on the decrease in the likelihood of observing  $\bar{s}_z = 0$ .

Note that the ‘‘superpurity’’ of the noise-free density matrix  $\hat{\rho}_m$  corresponds to the classical notion that any new information gained about a physical system should reduce the uncertainty of the state. Therefore, obtaining information about any state has to increase the purity of this state. Negative probabilities allow such an increase in purity even for a pure state. While ‘‘superpure’’ states are of course unphysi-

cal by themselves, they can be used to provide a local interpretation of entanglement. In particular, entangled states can always be interpreted as a mixture of ‘‘superpure’’ product states. Classical probability theory then explains why entanglement cannot be utilized to transfer information instantaneously without physical interaction. Thus, the generalized measurement operator may explain the physical nature of entanglement in a far more intuitive way than the conventional formalism.

The change in the density matrix caused by a quantum measurement can now be compared with the changes caused by information obtained about a classical probability distribution. A classical noise-free measurement can change the statistics of physical properties only if the corresponding properties are correlated with the measured variable. In the example given in Sec. III C, however, the statistics of  $\hat{s}_x$  are changed by the measurement even though the initial state is an eigenstate of  $\hat{s}_x$ . Classically, a well-defined variable cannot be correlated with any other variable. By introducing negative probabilities, however, this situation is changed. In the case above, the measurement reduces the expectation value of  $\hat{s}_x$  to a very low value if a value of  $\bar{s}_z$  close to the quantized values of  $\pm\frac{1}{2}$  is observed. On the other hand, negative probabilities appear if  $\bar{s}_z$  is close to the average between the two quantized values. The original pure state is retained if one averages over all measurement results, as shown by Eq. (14). Thus, there is a statistical correlation between the spin component  $\hat{s}_x$  and the spin component  $\hat{s}_z$ , which may be expressed by averages over the measurement results  $\bar{s}_z$  and the expectation values  $\langle\hat{s}_x\rangle$  after the measurement as

$$\overline{\bar{s}_z^2\langle\hat{s}_x\rangle} - \overline{\bar{s}_z\langle\hat{s}_x\rangle} = -\frac{1}{4}\langle\hat{s}_x\rangle. \quad (27)$$

In words, the measurement of a quantized value of  $\hat{s}_z$  is correlated with the nonvanishing possibility of a negative value of  $\hat{s}_x$ , while the measurement of a value of  $\hat{s}_z$  between the quantized values is correlated with a negative probability for  $\hat{s}_x < 0$ , indicating that such negative values are ‘‘more than impossible.’’

### E. Nonclassical correlations as fundamental operator properties

Usually, entanglement is analyzed in terms of operator properties. In particular, nonclassical features of spin- $\frac{1}{2}$  statistics can often be traced to the anticommutation of the spin components. Using the generalized projection operator  $\hat{P}_{\delta A}(\bar{A})$ , it is possible to derive an analytical expression for the correlation of  $\bar{s}_z^2$  and  $\langle\hat{s}_x\rangle$  given in Eq. (27). In general, the correlation between the squared measurement result  $\bar{A}^2$  and a variable  $\hat{B}$  after the measurement is given by

$$\begin{aligned} C(\bar{A}^2; \langle\hat{B}\rangle) &= \overline{\bar{A}^2\langle\hat{B}\rangle} - \overline{\bar{A}}^2\langle\hat{B}\rangle \\ &= \int d\bar{A} p(\bar{A}) \text{Tr}\{\rho_f(\bar{A})\hat{B}\}\bar{A}^2 \\ &\quad - \text{Tr}\{\rho_f(\text{total})\hat{B}\}[\delta A^2 + \text{Tr}\{\rho_f(\text{total})\hat{A}^2\}]. \end{aligned} \quad (28)$$

By solving the integral over  $\bar{A}$ , the correlation may be expressed entirely in terms of expectation values of the final state density matrix  $\hat{\rho}_f(\text{total})$ . It reads

$$\begin{aligned} C(\bar{A}^2; \langle\hat{B}\rangle) &= \frac{1}{4}\langle\hat{A}^2\hat{B} + 2\hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2\rangle_f(\text{total}) \\ &\quad - \langle\hat{A}^2\rangle_f(\text{total})\langle\hat{B}\rangle_f(\text{total}). \end{aligned} \quad (29)$$

Since  $\hat{B}$  does not necessarily commute with  $\hat{A}$ , this correlation can be nonzero even if the system is in an eigenstate of  $\hat{A}$ . In the case of the spin- $\frac{1}{2}$  system discussed above, the anticommutation of  $\hat{s}_x$  and  $\hat{s}_z$  is responsible for the result that

$$C(\bar{s}_z^2; \langle\hat{s}_x\rangle) = -\langle\hat{s}_z^2\rangle_f(\text{total})\langle\hat{s}_x\rangle_f(\text{total}). \quad (30)$$

In the case of  $\delta s_z \rightarrow \infty$ , or when removing the noise effect from the measurement, the final density matrix can be replaced with the initial state. It is then possible to obtain nonzero correlations between  $\hat{s}_x$  and  $\hat{s}_z$  even for the eigenstates of  $\hat{s}_x$ . This result suggests that the physically relevant value of  $\hat{s}_x$  is not even well defined for eigenstates of  $\hat{s}_x$ . In the words of the famous Einstein-Podolsky-Rosen (EPR) paper [21], the fact that the outcome of an  $\hat{s}_x$  measurement can be predicted with certainty does not mean that there exists an element of reality corresponding to this *potential* measurement result, unless the measurement is *actually* performed. The  $\hat{s}_x$  fluctuations of an eigenstate of  $\hat{s}_x$  are revealed in the noise-induced changes of  $\hat{s}_x$  if any property other than  $\hat{s}_x$  interacts with the environment.

### F. Negative probabilities and quantization

One of the fundamental consequences of quantum mechanics is the replacement of continuous classical variables with discrete quantum numbers. In particular, the components of angular momentum have eigenvalues equal to multiples of  $\hbar$  and the energies of harmonic oscillators or wave modes have eigenvalues equal to multiples of  $\hbar\omega$ . One of the strangest features of quantum mechanics is the inconsistency this introduces into classical arguments. For example, it should be necessary to conclude that, if  $\hat{s}_x$  is equal to  $\pm\frac{1}{2}$  and  $\hat{s}_y$  is equal to  $\pm\frac{1}{2}$ ,  $\hat{s}_x + \hat{s}_y$  should be equal to zero or  $\pm 1$ . However, the eigenvalues of  $\hat{s}_x + \hat{s}_y$  are  $\pm 1/\sqrt{2}$ . It is this contradiction of classical arguments based on quantized values that is exploited in the formulation of Bell’s inequalities [22]. Usually, one tries to escape the dilemma by arguing that the classical meaning of the quantized observable is lost completely. Alternatively, however, one could assume that eigenvalues do *not* represent all physical values of the ob-

servable. The observation of quantized values in precise measurements could instead be explained as a fundamental statistical effect based on the presence of negative probabilities and quantum correlations.

In the example above, the eigenvalues of  $\hat{s}_z$  are  $\pm 1$ . However, there is a correlation between  $\vec{s}_z^2$  and  $\hat{s}_x$  which suggests some measure of physical reality for fluctuations in  $\vec{s}_z^2$ , especially for the possibility of nonquantized values near  $\vec{s}_z = 0$ . In the case of light field quantization, the same principles can be applied to quantum nondemolition measurements of photon number [23], revealing high phase coherence at half-integer photon numbers and low phase coherence at integer photon numbers. The measurement process can now be analyzed in far greater depth. If separate quantum states are not resolved, negative probabilities and quantum correlations are hidden by the remaining uncertainty in the observed variable. If quantization is resolved, quantum correlations modify the statistics of all variables that do not commute with the observed variable. Such correlations can only be interpreted in terms of negative probabilities. However, quantum noise “covers up” the negative probabilities. Nevertheless, negative probabilities can be observed indirectly in nonclassical correlations.

Recently, the question of what truly characterizes the differences between classical physics and quantum physics has been raised in a new context regarding the potential of quantum computers [24,25]. It seems that on the quite technical level represented, e.g., by NMR quantum computation, the statistical relationships between those operator variables actually utilized are far more important than observable independent concepts such as entanglement. Possibly, contemporary quantum theory has paid too little attention to the observable properties of quantum systems. By interpreting quantum mechanics in terms of nonclassically correlated observables, a smooth transition between the classical regime and the quantum regime is possible and the problem of suddenly having to change the vocabulary from physical properties to Hilbert spaces can be avoided. Quantum properties can then be explored within a framework similar to that of classical physics, with the main quantum correction originating from the nonclassical correlations possible due to the appearance of negative probabilities. It should then be possible to identify the correlations required for quantum computation and other applications of quantization effects.

#### IV. CONCLUSIONS

In conclusion, a physical interpretation of the measurement process based on a separation of information and noise is possible. This separation corresponds exactly to the classical notion of a reality unchanged by the measurement interaction. However, negative probabilities appear in the measurement decomposition of the initial density matrix  $\hat{\rho}_i$  into the conditioned density matrices  $\hat{\rho}_m$  given by Eq. (14). These negative probabilities represent a type of nonclassical information only available in quantum-mechanical systems. They are responsible for the failure of measurement-independent concepts of local reality such as the one proposed in the EPR paper [21], and it is likely that this type of nonclassical information is also responsible for the advantages of quantum computing as compared with classical computing. Moreover, the negative probabilities are directly related to quantization itself, since they arise from correlations that distinguish between the observation of quantized values and the observation of values between two quantized eigenvalues of the observable.

These results clearly show that there is much more to quantum measurements than the observation of eigenvalues. Possibly, the main interpretational problem in quantum measurement theory is the assumption that physical variables should be restricted to their eigenvalues. However, negative probabilities and the real physical consequences of measuring a value between two eigenvalues seem to indicate that the effective physical properties of a variable are not restricted to eigenvalues only. Instead, some measure of physical reality should be attributed to the continuum of values between and even beyond the eigenvalues. In particular, the off-diagonal matrix elements of the density matrix can be associated with the average of the two associated eigenvalues, even if this average does not correspond to any actual eigenvalue. Quantum coherence can then be understood in terms of nonclassical correlations between the physical properties of a system. Thus the generalized measurement postulate represents an opportunity for developing new interpretational concepts in quantum theory which may allow us to improve our intuitive understanding of the physical nature of quantum effects.

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- [1] See, for example, the Introduction of *Quantum Theory and Measurement*, edited by J.A. Wheeler and W.H. Zurek (Princeton University Press, Princeton, NJ, 1983).
- [2] See in particular the letters in Phys. Today **52** (2), 11 (1999), concerning the two-part article by S. Goldstein, *ibid.* **51** (3), 42 (1998); **51** (4), 38 (1998), and the letters in *ibid.* **52** (1), 15 (1999), concerning the article by M. Beller, *ibid.* **51** (9), 29 (1998).
- [3] See, e.g., N.D. Mermin, e-print quant-ph/9609013, p. 13: “So what do measurement, or classical domain, or knowledge have

to do with objective reality? Nothing—nothing whatever. They have to do with *us*.”

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- [19] This consideration is well known from the study of so-called cat states. For a discussion, see, e.g., S. Haroche, *Phys. Today* **51** (7), 36 (1998).
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