PROPER-TIME FORMALISM IN A CONSTANT MAGNETIC FIELD AT FINITE TEMPERATURE AND CHEMICAL POTENTIAL

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Abstract

We investigate scalar and spinor field theories in a constant magnetic field at finite temperature and chemical potential. In an external constant magnetic field the exact solution of the two-point Green functions are obtained by using the Fock-Schwinger proper-time formalism. We extend it to the thermal field theory and find the expressions of the Green functions exactly for the temperature, the chemical potential and the magnetic field. For practical calculations the contour of the proper-time integral is carefully selected. The physical contour is discussed in a constant magnetic field at finite temperature and chemical potential. As an example, behavior of the vacuum self-energy is numerically evaluated for the free scalar and spinor fields.

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1   Introduction

Some of the interesting cosmological and astrophysical situations are found in the state with high density, temperature and strong magnetic field. Neutron star is a dense object which has a large chemical potential. Recently it is observed that some of the neutron stars have extremely strong magnetic field.[1] A primordial magnetic field in the early universe is also interesting.[2,3] To understand the physics of such situation we consider the quantum field theory in an external magnetic field at finite chemical potential and temperature.

One of the fundamental objects in the quantum field theory is a two-point Green function. The exact form of the Green function is necessary to deal with a strong magnetic field. Much interest has been paid to obtain the Green function under external fields. Schwinger found the exact expression of the Green function in an external magnetic field by using the proper-time formalism in 1951.[4] The proper-time method is extended to deal with the thermal system in Ref. 5. Trace of the Green function corresponds to the vacuum self-energy of the free field. It is obtained by QED effective action and evaluated in a constant electromagnetic field at finite temperature[6,7] and at finite chemical potential.[8,9] Variety of approaches are used to discuss the contributions from both the temperature and the chemical potential in Refs. 10–13.

In the present paper the proper-time formalism is re-considered in the imaginary time form of the thermal field theory. We modify the formalism to introduce both the temperature and the chemical potential exactly. In most of previous analysis the proper-time integral was analytically performed by the Landau level expansion. Since results are analytically continued to the wide range of parameters in the Landau level approach, these results may contain some approximation for the combined effect of temperature, chemical potential and external magnetic field. To obtain physical results we carefully choose the contour of the proper-time integral.[14] Here the explicit form of the scalar and fermion Green functions is written down in a proper-time form to discuss the physical contour in a constant magnetic field H at finite temperature T and chemical potential μ. As an example, we numerically calculate the vacuum self-energy for the free scalar and fermion fields and discuss the contour dependence of the proper-time integral.

2 Scalar Two-point Function at Finite H, T and μ

First we study the Green function for a complex scalar field in the constant magnetic field, H, at finite temperature, $T = 1/\beta$, and chemical potential, μ. The chemical potential is defined for the global U(1) symmetry of the complex field.[16] In the constant magnetic field, the Green function, $G(x, y; m_s)$, for the scalar field obeys the Klein-Gordon equation

$$\left\{ (\partial_\mu + ieA_\mu)^2 + m_s^2 \right\} G(x, y; m_s) = \delta^4(x - y). \tag{1}$$

We introduce the temperature and the chemical potential to this equation. Following the standard procedure of the imaginary time formalism,[15,16] the thermal Green function is defined in Euclidean space-time by

$$\left\{ (i\partial_4 - i\mu + eA_4)^2 + (i\partial_j + eA_j)^2 + m_s^2 \right\} G(x, y; m_s) = \delta_E^4(x - y), \tag{2}$$

where $\delta_E^4(x - y) = \delta(x_4 - y_4)\delta^3(\boldsymbol{x} - \boldsymbol{y})$. The time direction, x_4, is restricted between 0 and β, i.e. $x_4 \in [0, \beta]$. Thus the thermal theory has no Lorentz invariance. It naturally follows that the thermal equilibrium is defined along a specific time direction.

The Klein-Gordon equation is simplified by expanding the time direction, x_4, in Fourier series as

$$G(x, y) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(x_4 - y_4)} \widetilde{G}_n(\boldsymbol{x}, \boldsymbol{y}), \tag{3}$$

where the Matsubara frequency ω_n is given by $\omega_n = 2\pi n/\beta$ for a scalar field.

Here we consider the constant magnetic field along the z-axis, $A_\mu = \delta_{\mu 2} x_1 H$, for simplicity. For the constant magnetic field the fourth component of A_μ vanishes, $A_4 = 0$. For a constant A_μ Eq.(2) reduces to

$$\left\{ (\omega_n - i\mu)^2 - (\partial_j - ieA_j)^2 + m_s^2 \right\} \widetilde{G}_n(\boldsymbol{x}, \boldsymbol{y}) = \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{4}$$

The induced Green function, $\widetilde{G}_n$, has similar form to the three dimensional Green function for the scalar field with mass $M = \sqrt{(\omega_n - i\mu)^2 + m_s^2}$ in the electromagnetic field. M develops an imaginary part only if both the temperature and the chemical potential have non-vanishing value. Because

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of this imaginary part we must modify the original Fock-Schwinger method as is shown below.

We consider the proper-time Hamiltonian H_n;[4,17]

$$H_n = -\sum_{j=1}^{3}(\partial_j - ieA_j)^2 + (\omega_n - i\mu)^2 + m_s^2. \tag{5}$$

Evolution of the system is described by the proper-time, τ. Then the induced Green function $\widetilde{G}_n$ in Eq.(4) satisfies

$$H_n\widetilde{G}_n(\boldsymbol{x}, \boldsymbol{y}) = \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{6}$$

To solve this equation we introduce the unitary evolution operators U_n^1 and U_n^2 which are defined by

$$i\frac{\partial}{\partial\tau}U_n^\alpha(\boldsymbol{x}, \boldsymbol{y}; \tau) = H_n U_n^\alpha(\boldsymbol{x}, \boldsymbol{y}; \tau), \qquad (\alpha = 1, 2), \tag{7}$$

with the boundary conditions

$$\lim_{\tau\to-\infty} U_n^1(\boldsymbol{x}, \boldsymbol{y}; \tau) = \lim_{\tau\to\infty} U_n^2(\boldsymbol{x}, \boldsymbol{y}; \tau) = 0, \tag{8}$$

$$\lim_{\tau\to-0} U_n^1(\boldsymbol{x}, \boldsymbol{y}; \tau) = \lim_{\tau\to+0} U_n^2(\boldsymbol{x}, \boldsymbol{y}; \tau) = \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{9}$$

In the case of vanishing temperature or vanishing chemical potential the induced Green function, $\widetilde{G}_n$, can be described by only one evolution operator, U_n^1. However, two types of the evolution operators with different boundary conditions are necessary to obtain a finite $\widetilde{G}_n$ under non-vanishing temperature and chemical potential. In the previous works contributions from the evolution operator U_n^2 is not completely considered. The induced Green functions $\widetilde{G}_n(\boldsymbol{x}, \boldsymbol{y})$ are expressed by U_n^α as

$$\widetilde{G}_n(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} -i\displaystyle\int_{-\infty}^{-0} d\tau\, U_n^1(\boldsymbol{x}, \boldsymbol{y}; \tau), & (\text{ n} < 0\ ), \\ i\displaystyle\int_{+0}^{\infty} d\tau\, U_n^2(\boldsymbol{x}, \boldsymbol{y}; \tau), & (\text{ n} > 0\ ). \end{cases} \tag{10}$$

After some straightforward calculations, see for example Refs. 17 and 18, we

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obtain the evolution operators

$$U_n^\alpha(\boldsymbol{x}, \boldsymbol{y}; \tau) = \frac{a^\alpha}{(4\pi)^{3/2} |\tau|^{3/2}} \frac{eH\tau}{\sin(eH\tau)} \exp\left\{ ie \int_{\boldsymbol{y}}^{\boldsymbol{x}} d\xi \cdot A(\xi) \right\}$$

$$\times \exp\left[\frac{i}{4}(x-y)_i eF_{ij}[\coth(eF\tau)]_{jk}(x-y)_k \right.$$

$$\left. -i\tau\left\{ (\omega_n - i\mu)^2 + m_s^2 \right\} \right], \quad (11)$$

where F is the field strength and H is the magnetic field. a^α is defined by

$$a^\alpha = \begin{cases} e^{+3\pi i/4}, & (\ \alpha = 1\ ), \\ e^{-3\pi i/4}, & (\ \alpha = 2\ ). \end{cases} \quad (12)$$

The evolution operators, U_n^α, are exponentially suppressed at the limit $\tau \to -\infty$ for $n > 0$ and $\tau \to \infty$ for $n < 0$. It is clearly seen in Eq.(11) that the evolution operators obey the boundary conditions (8) and (9). Substituting Eqs.(10) and (11) in Eq.(3) we get the explicit expression of the Green function,

$$G(x, y; m_s) = \frac{-i}{\beta}\left[\sum_{n=-1}^{-\infty} e^{-i\omega_n(x_4-y_4)} \int_{-\infty}^{0} d\tau\, U_n^1(\tau) \right.$$

$$\left. - \sum_{n=1}^{\infty} e^{-i\omega_n(x_4-y_4)} \int_{0}^{\infty} d\tau\, U_n^2(\tau) \right]$$

$$= -\frac{ie^{3\pi i/4}}{(4\pi)^{3/2}\beta} \sum_{n=1}^{\infty} e^{i\omega_n(x_4-y_4)}$$

$$\times \int_{0}^{\infty} d\tau \frac{eH}{\tau^{1/2}\sin(eH\tau)} \exp\left\{ ie \int_{\boldsymbol{y}}^{\boldsymbol{x}} d\xi \cdot A(\xi) \right\}$$

$$\times \exp\left[-\frac{i}{4}(x-y)_i eF_{ij}[\coth(eF\tau)]_{jk}(x-y)_k \right.$$

$$\left. +i\tau\left\{ (\omega_n + i\mu)^2 + m_s^2 \right\} \right] + (c.c.). \quad (13)$$

The first term on the final result in Eq.(13) comes from the integration of the evolution operator, U_n^1. The complex conjugate of this, $(c.c.)$, appears from U_n^2 which is introduced to deal with the non-vanishing temperature and chemical potential.

3 Spinor Two-point Function at Finite H, T and μ

Next we consider the Green function for a fermion field in an external constant magnetic field at finite temperature and chemical potential. The Green function, $S(x, y; m_f)$, is defined by the Dirac equation;

$$(i\slashed{\partial} + e\slashed{A} - i\mu\gamma_4 - m_f) S(x, y; m_f) = \delta_E^4(x - y), \tag{14}$$

where m_f is the mass of the fermion field.

To calculate the analytical form of $S(x, y; m_f)$ it is more convenient to introduce the bi-spinor function $G_f(x, y)$,

$$S(x, y; m_f) = (i\slashed{\partial} + e\slashed{A} - i\mu\gamma_4 + m_f) G_f(x, y). \tag{15}$$

The explicit form of $S(x, y; m_f)$ is determined by solving the following equation for $G_f(x, y)$,

$$\left\{ D^2 + \frac{i}{2}eH(\gamma_1\gamma_2 - \gamma_2\gamma_1) - 2\mu\partial_4 + \mu^2 - m_f^2 \right\} G_f(x, y) = \delta_E^4(x - y), \tag{16}$$

where $D_j = \partial_j - ieA_j$. Here we choose the direction of the constant magnetic field along the z-axis, $A_\mu = \delta_{\mu 2} x_1 H$. For a constant magnetic field the function, $G_f(x, y)$, is expanded in Fourier series

$$G_f(x, y) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} e^{-i\omega_n(x_4 - y_4)} \widetilde{G}_n^f(\boldsymbol{x}, \boldsymbol{y}), \tag{17}$$

and Eq.(16) reads

$$\left\{ \sum_{j=1}^{3} D_j^2 + \frac{i}{2}eH(\gamma_1\gamma_2 - \gamma_2\gamma_1) - (\omega_n - i\mu)^2 - m_f^2 \right\} \widetilde{G}_n^f(\boldsymbol{x}, \boldsymbol{y}) = \delta^3(\boldsymbol{x} - \boldsymbol{y}). \tag{18}$$

As in the scalar case, the induced function, $\widetilde{G}_n^f(x, y)$, is calculated by introducing the proper-time Hamiltonian,

$$H_n^f = \sum_{j=1}^{3} D_j^2 + \frac{i}{2}eH(\gamma_1\gamma_2 - \gamma_2\gamma_1) - (\omega_n - i\mu)^2 - m_f^2. \tag{19}$$

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According to the similar way in the scalar field we can find the bi-spinor function, $G_f(x, y)$. It is described by two types of the unitary evolution operators U_n^1 and U_n^2 which satisfy the boundary conditions (8) and (9). Therefore the explicit expression of $G_f(x, y)$ is obtained by

$$
G_f(x, y) = \frac{-i}{\beta} \left[\sum_{n=0}^{\infty} e^{-i\omega_n(x_4 - y_4)} \int_{-\infty}^{-0} d\tau \, U_n^1(\tau) \right.
$$

$$
\left. - \sum_{n=-1}^{-\infty} e^{-i\omega_n(x_4 - y_4)} \int_{+0}^{\infty} d\tau \, U_n^2(\tau) \right]. \quad (20)
$$

The Matsubara frequency for the fermion field is given by $\omega_n = (2n + 1)\pi/\beta$. The evolution operators $U_n^\alpha(\tau)$ are found to be

$$
U_n^\alpha(\tau) = \frac{b^\alpha}{(4\pi)^{3/2}|\tau|^{3/2}} \frac{eH\tau}{\sin(eH\tau)} \exp\left\{ ie \int_y^x d\xi \cdot A(\xi) \right\}
$$

$$
\times \exp\left[-\frac{i}{4}(x - y)_i eF_{ij}[\coth(eF\tau)]_{jk}(x - y)_k \right.
$$

$$
\left. -i\tau\left\{ \frac{1}{2}eF_{jk}\sigma_{jk} - (\omega_n - i\mu)^2 - m_f^2 \right\} \right], \quad (21)
$$

where $\sigma_{jk} = \dfrac{i}{2}[\gamma_j, \gamma_k]$, F is the field strength and b^α is

$$
b^\alpha = \begin{cases} e^{-3\pi i/4}, & (\ \alpha = 1\ ), \\ e^{+3\pi i/4}, & (\ \alpha = 2\ ). \end{cases} \quad (22)
$$

Inserting Eq.(21) into Eq.(20), one easily derive the two-point Green function,

$$
S(x, y; m_f) =
$$

$$
(i\slashed{\partial} + e\slashed{A} - i\mu\gamma_4 + m_f)\left(-\frac{ie^{-3\pi i/4}}{(4\pi)^{3/2}\beta} \right) \sum_{n=0}^{\infty} e^{-i\omega_n(x_4 - y_4)}
$$

$$
\times \int_0^{\infty} d\tau \frac{eH}{\tau^{1/2}\sin(eH\tau)} \exp\left\{ ie \int_y^x d\xi \cdot A(\xi) \right\}
$$

$$
\times \exp\left[\frac{i}{4}(x - y)_i eF_{ij}[\coth(eF\tau)]_{jk}(x - y)_k \right.
$$

$$
\left. +i\tau\left\{ \frac{1}{2}eF_{jk}\sigma_{jk} - (\omega_n - i\mu)^2 - m_f^2 \right\} \right] + (c.c.). \quad (23)
$$

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The complex conjugate term, (c.c.), in Eq.(23) comes from U_n^2 which is introduced to deal with effects of both the temperature and the chemical potential.

4   Behavior of the Vacuum Self-energy

At the one loop level the vacuum self energy for a free field is given by the trace of the Green function. Here we calculate it for a free scalar and a free fermion fields at finite H, T and μ.

For a free scalar field with mass m_s the trace of the Green function (13) becomes

$$\frac{1}{\beta V} \text{Tr} G(x,x) = \frac{e^{\pi i/4}}{(4\pi)^{3/2}\beta} \sum_{n=1}^{\infty} \int_0^{\infty} d\tau \frac{eH}{\tau^{1/2} \sin(eH\tau)}$$
$$\times \exp\left[i\tau\left\{(\omega_n + i\mu)^2 + m_s^2\right\}\right] + (c.c.), \quad (24)$$

where V is the 3-dimensional volume. If it is larger than m_s^2, naive perturbation loses validity to evaluate the radiative correction in a scalar theory with interactions. To get rid of this difficulty we must use a resumed propagator known as the ring diagram resummation in the thermal field theory.[16]

Carrying out the trace of the Green function (23) with respect to space-time and spinor legs, we find the vacuum self-energy for a free fermion field,

$$\frac{1}{\beta V} \text{Tr} S(x,x) = \frac{4e^{3\pi i/4}m_f}{(4\pi)^{3/2}\beta} \sum_{n=0}^{\infty} \int_0^{\infty} d\tau \frac{eH \cot(eH\tau)}{\tau^{1/2}}$$
$$\times \exp\left[-i\tau\left\{(\omega_n - i\mu)^2 + m_f^2\right\}\right] + (c.c.). \quad (25)$$

For the vanishing temperature and/or chemical potential the contribution from the evolution operator U_n^2 is coincide with the one from U_{-n}^1 and contour dependence of the proper-time integral disappears. Therefore our result must be coincide with the one found in the previous work. For $\mu \to 0$ Eq.(25) exactly agrees with the one obtained in Ref. 6. At the limit, $T \to 0$ ($\beta \to \infty$), Eq.(25) reproduce the results obtained in Refs. 8 and 9.

Performing the proper-time integral and the summation in Eq.(24) and Eq.(25) numerically, we evaluate the vacuum self-energy for the free scalar and the free fermion fields. We choose the contour of proper-time integration in the first term of the right hand side of Eq.(24) slightly above the real axis. As is known, this contour gives physical results at the limit $\mu \to 0$ and $T \to 0$.[4] It is natural to take the same contour at finite μ and T.
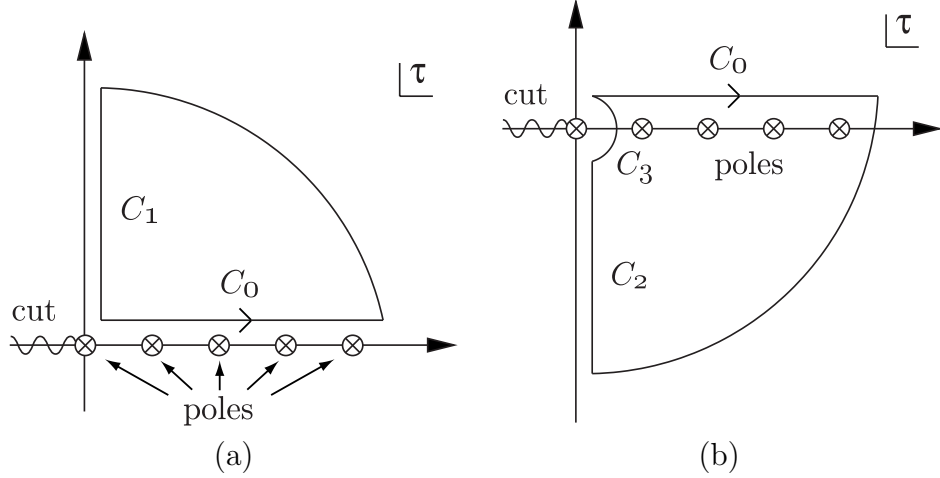
Figure 1: Contour on the complex τ plane.

For $\omega_1^2 - \mu^2 + m_s^2 > 0$ the integrand is exponentially suppressed at the infinity above the real axis. We close the contour as is shown in Fig. 4 (a) and perform the proper-time integral along the path C_1. The complex conjugate of the result gives the second term in the right hand side in Eq.(24). Thus the $\mathrm{Tr}G(x,x)$ is found to be

$$\frac{1}{\beta V}\mathrm{Tr}G(x,x) = \frac{1}{4\pi^{3/2}\beta}\sum_{n=1}^{\infty}\int_{1/\Lambda^2}^{\infty}d\tau\, f_s(\tau, n),\tag{26}$$

where $f_s(\tau, n)$ is

$$f_s(\tau, n) = \frac{eH}{\sqrt{\tau}}\frac{\cos(2\omega_n\mu\tau)}{\sinh(eH\tau)}e^{-\tau(\omega_n^2-\mu^2+m_s^2)}.\tag{27}$$

Here we introduce the proper-time cut-off Λ to regularize the theory.

In the case $\omega_1^2 - \mu^2 + m_s^2 < 0$ we must consider two kinds of paths in Fig. 4 (a) and (b). For a positive $\omega_n^2 - \mu^2 + m_s^2$ the integrand drops at the infinity above the real axis. If $\omega_n^2 - \mu^2 + m_s^2$ is negative, the integrand is suppressed at the infinity below the real axis. We calculate the proper-time integral along the paths C_1, C_2, C_3 and add the contribution from the poles

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on the real axis. After some calculations $\mathrm{Tr}G(x,x)$ is obtained by

$$
\frac{1}{\beta V}\mathrm{Tr}G(x,x) = \frac{1}{4\pi^{3/2}\beta}\sum_{n>[N]}^{\infty}\int_{1/\Lambda^2}^{\infty}d\tau f_s(\tau,n)
$$

$$
+\frac{1}{4\pi^{3/2}\beta}\sum_{n=1}^{[N]}\left[h_{s0}(n)+h_{sj}(n)-\int_{1/\Lambda^2}^{\infty}d\tau g_s(\tau,n)\right],(28)
$$

where $N = \beta\sqrt{\mu^2-m_s^2}/(2\pi)$, $[N]$ is the Gauss notation and

$$
g_s(\tau,n) = \frac{eH}{\sqrt{\tau}}\frac{\sin(2\omega_n\mu\tau)}{\sinh(eH\tau)}e^{-\tau(\mu^2-\omega_n^2-m_s^2)}, \tag{29}
$$

$$
h_{s0}(n) = \frac{e^{-\pi i/4}}{2}\int_{-\pi/2}^{\pi/2}d\theta\frac{eH}{\Lambda}\frac{e^{i\theta/2}}{\sin(eHe^{i\theta}/\Lambda^2)}
$$

$$
\times\exp[i\{(\omega_n+i\mu)^2+m_s^2\}e^{i\theta}/\Lambda^2]+(c.c.), \tag{30}
$$

$$
h_{sj}(n) = 2\sqrt{\pi}\sum_{l=1}^{\infty}(-1)^l\left(\frac{eH}{l}\right)^{1/2}e^{-2\pi l\omega_n\mu/(eH)}
$$

$$
\times\cos\left[\frac{\pi l}{eH}(\mu^2-\omega_n^2-m_s^2)+\frac{\pi}{4}\right]. \tag{31}
$$

$h_{s0}(n)$ is the contribution from the path C_3 around the pole at $\tau = 0$ and $h_{sj}(n)$ is sum of residues at $eH\tau = \pi l$.

In the case of the fermion field the above contour gives the physical result at $\mu \to 0$ and $T \to 0$ for the second term "$(c.c.)$" of right hand side of Eq.(25). According to the similar analysis with the scalar field, we calculate the $\mathrm{Tr}S(x,x)$. For a positive $\omega_0^2 - \mu^2 + m_s^2$ the proper-time integral is performed along the path C_1,

$$
\frac{1}{\beta V}\mathrm{Tr}S(x,x) = -\frac{m_f}{\pi^{3/2}\beta}\sum_{n=0}^{\infty}\int_{1/\Lambda^2}^{\infty}d\tau f(\tau,n), \tag{32}
$$

where $f(\tau,n)$ is

$$
f(\tau,n) = \frac{eH}{\sqrt{\tau}}\coth(eH\tau)\cos(2\omega_n\mu\tau)e^{-\tau(\omega_n^2-\mu^2+m_f^2)}. \tag{33}
$$

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For a negative $\omega_0^2 - \mu^2 + m_s^2$ we evaluate the proper-time integral along the path C_1, C_2, C_3 and consider the influence from pole on the real axis.

$$\frac{1}{\beta V}\mathrm{Tr}S(x,x) = -\frac{m_f}{\pi^{3/2}\beta}\sum_{n>[N]}\int_{1/\Lambda^2}^{\infty}d\tau f(\tau,n)$$

$$+\frac{m_f}{\pi^{3/2}\beta}\sum_{n=0}^{[N]}\left[h_0(n) + h_j(n) + \int_{1/\Lambda^2}^{\infty}d\tau g(\tau,n)\right], \quad (34)$$

where $N = \beta\sqrt{\mu^2 - m_f^2}/(2\pi) - 1/2$, $g(\tau,n)$, $h_0(n)$ and $h_j(n)$ are given by

$$g(\tau,n) = \frac{eH}{\sqrt{\tau}}\coth(eH\tau)\sin(2\omega_n\mu\tau)e^{-\tau(\mu^2-\omega_n^2-m_f^2)}, \quad (35)$$

$$h_0(n) = \frac{e^{3\pi i/4}}{2}\int_{-\pi/2}^{\pi/2}d\theta\frac{eH}{\Lambda}e^{i\theta/2}\cot(eHe^{i\theta}/\Lambda^2)$$

$$\times \exp[i\{(\omega_n + i\mu)^2 + m_f^2\}e^{i\theta}/\Lambda^2] + (c.c.), \quad (36)$$

$$h_j(n) = 2\sqrt{\pi}\sum_{l=1}^{\infty}\left(\frac{eH}{l}\right)^{1/2}e^{-2\pi l\omega_n\mu/(eH)}$$

$$\times \sin\left[\frac{\pi l}{eH}(\mu^2 - \omega_n^2 - m_f^2) - \frac{\pi}{4}\right]. \quad (37)$$

h_0 corresponds to the contribution from the path C_3 and h_j is sum of residues at $eH\tau = \pi l$.

Performing the integration over τ and the summation numerically, we obtain the behaviors of the trace of the Green functions, i.e. the vacuum self energy at the one loop level. In Fig. 2 we show behaviors of the vacuum self-energy as a function of the external magnetic field H with T and μ fixed. For a case of neutron star many interest has been payed to the QCD phase structure. Thus we suppose the proper-time cut-off is more than QCD scale, $\Lambda_{\mathrm{QCD}} < \Lambda \sim 2\mathrm{GeV}$. The scalar and fermion mass is taken to be the pion mass scale, $m_s \sim m_f \sim 0.1\mathrm{GeV}$. The present upper limit for the magnetic field in the neutron star is of the order, $eH \sim O(0.01\mathrm{GeV}^2)$.[19] An oscillating mode is observed for both the scalar and fermion field. The amplitude of the oscillation becomes larger as H increases and/or T decreases. The oscillation disappears for higher temperature. For the neutron star the upper limit of the magnetic field, eH, is of the order $O(0.01\mathrm{GeV}^{-2})$. It seems to be difficult
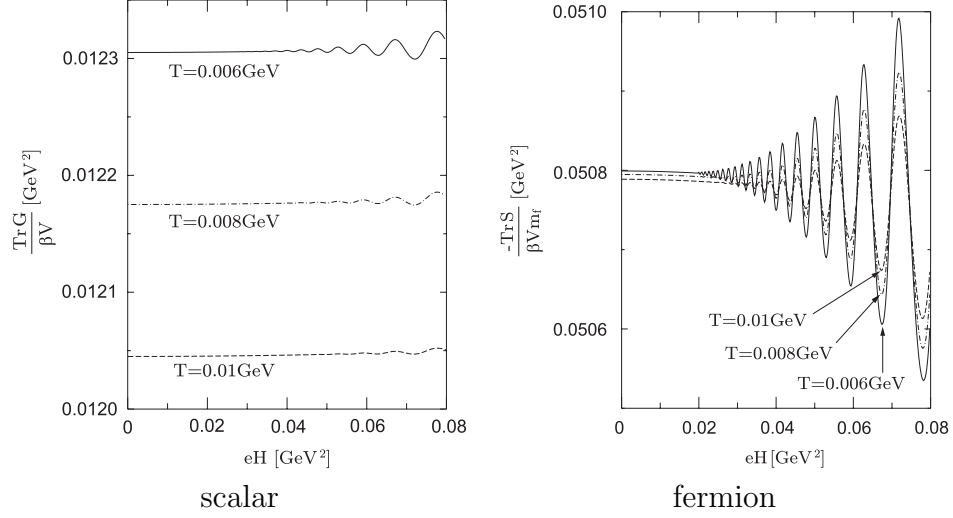
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Figure 2: Behaviors of the vacuum self energy as a function of H with T and μ fixed. We set $\Lambda = 2\text{GeV}$, $m_s = m_f = 0.1\text{GeV}$, $\mu = 1\text{GeV}$ and $T = 0.006\text{GeV}$, $T = 0.008\text{GeV}$, $T = 0.01\text{GeV}$.
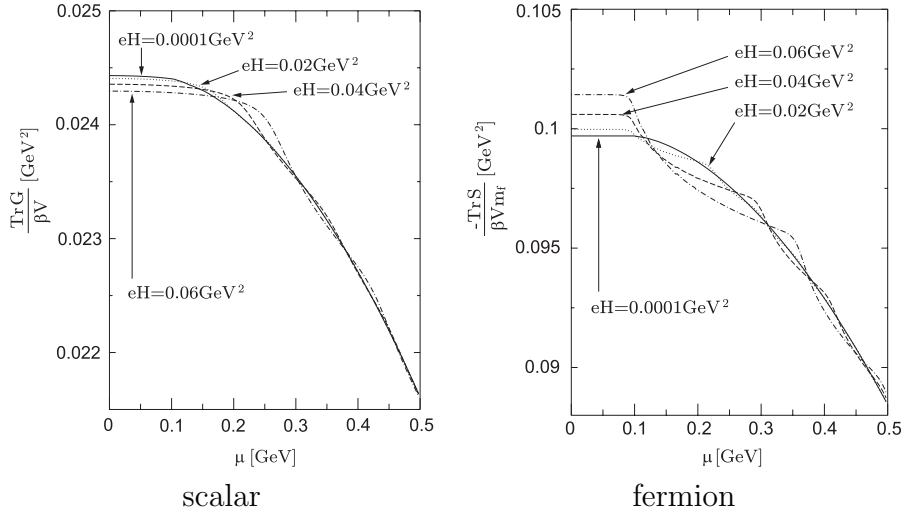

Figure 3: Behaviors of the vacuum self energy as a function of μ with H and T fixed. We set $\Lambda = 2\text{GeV}$, $m_s = m_f = 0.1\text{GeV}$, $T = 0.006\text{GeV}$ and $eH = 0.0001\text{GeV}^2$, $eH = 0.02\text{GeV}^2$, $eH = 0.04\text{GeV}^2$, $eH = 0.06\text{GeV}^2$.

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to see the magnetic oscillation appeared in Fig. 3 unless the neutron star is extremely cold. It agrees with the result obtained in Ref. 11.

Behaviors of the vacuum self-energy is illustrated as a function of μ with T and H fixed in Fig. 3. The trace of the two-point function goes down as μ increases for both the scalar and fermion field. We can see the oscillating mode for both cases. For the fermion field the mode is the origin of the van Alphen-de Haas magnetic oscillations as is shown in Ref. 11. Such a effect is not found in the scalar field theory, since the scalar field has no sharp Fermi surface.

As in known, the trace of the scalar two-point function contains a term which is proportional to T^2.[16] Indeed, Eq.(27) reduces to the well-known result,

$$\frac{\mathrm{Tr}G}{\beta V} = 2\frac{T^2}{24} + \mathrm{O}(T),\qquad(38)$$

at the limit μ, H and $m_s \to 0$. To obtain it we drop the surface term of the proper-time integral and use a formula $\zeta(-1) = -1/12$. T^2 behavior is not observed in Fig. 2, because the surface term is proportional to ΛT. The temperature considered here is too small compared with the cut-off scale Λ. T-dependence of the surface term is canceled out if we take the T-dependent cut-off, $\Lambda \propto 1/T$. Similar property is found in the momentum cut-off regularization for $H = 0$.

5   Conclusion

We have investigated the scalar and fermion field theories at finite temperature, chemical potential and constant magnetic field. The explicit expressions of the two-point Green functions are found by using the proper-time formalism. If both the temperature and the chemical potential exist, we must modify the Fock-Schwinger proper-time method by introducing two types of the evolution operators with different boundary conditions.

The proper-time integrations remain in our final expressions of the two-point Green functions $G(x, y; m_s)$ and $S(x, y; m_f)$. The remained integrand is exponentially suppressed at the limit $\tau \to \infty$. There are poles at $eH\tau = n\pi$ for any integer n. Because of these poles the naive Wick rotation has no validity. [a] We carefully take the contour of proper-time integration slightly

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[a]After the naive Wick rotation $\tau \to it$, we can perform the summations in Eq.(24) and

above the real axis to avoid these poles in the complex τ plane. In the case with large chemical potential we must consider two kinds of contour. One of the contour contains the poles at $eH\tau = n\pi$. On the other hand, the naive analytic continuation from the low chemical potential takes the contour below the real axis for $\omega_n - \mu^2 + m^2 < 0$. Thus it drops the contribution from the poles on the real axis. It gives only approximate results.

We apply our formalism to the vacuum self-energy at the one-loop level. At the limit $T \to 0$ and/or $\mu \to 0$ contributions from the poles at $eH\tau = n\pi$ disappear and our results coincide with the previous one. We numerically perform the proper-time integral and the Matsubara mode summation. In a strong magnetic field oscillating mode is observed for both the scalar and the fermion fields. Our results qualitatively agree with the previous analysis obtained by the naive Wick rotation[13] for $\mu < m_f$. Performing the naive Wick rotation, Eq.(25) reads

$$\frac{1}{\beta V} \mathrm{TrS} = -\frac{m_f}{2\pi^{3/2}\beta} \int_{1/\Lambda^2}^{\infty} d\tau \frac{eH}{\sqrt{\tau}} \coth(eH\tau) e^{\tau(\mu^2 - m_f^2)} \theta_2(2\mu\tau/\beta, 4\pi i\tau/\beta^2) \ . \tag{39}$$

In Fig. 4 we show the behaviors of the fermion vacuum self-energy (39). T and μ are fixed on the same value with the Fig. 3. We cut the proper-time integration at $10^4 \mathrm{GeV}^{-2}$ because it does not converge for $\mu > m_f$. As is shown in Fig. 4, the vacuum self-energy coincides with the results in Fig. 3 for $\mu < m_f = 0.1 \mathrm{GeV}$. But it is completely different for $\mu > 0.1 \mathrm{GeV}$. In Fig. 4 the vacuum self-energy is not analytic at some points.

There are some interesting applications of our work. Using the two-point functions obtained here we calculate the radiative correction of some QED process, radiative decay of axion, neutrino, and so on. As an example, we calculated the trace of the two-point functions. It is necessary to determine the phase structure of the theory at finite T, μ and H.[20] The influence of the external magnetic field will modify the decay rate and phase structure at finite T and μ. It may affect the evolution of our universe and/or neutron stars.

The present work is restricted to the analysis of the influence of the temperature, chemical potential and a constant magnetic field. There are some interesting objects in an external electromagnetic field. However it is

Eq.(25) and the vacuum self-energy is described by the elliptic theta function,[13] However, it is not valid in our formalism.
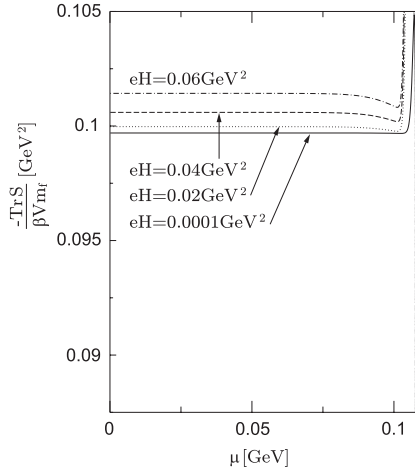
Figure 4: Behaviors of the vacuum self energy as a function of μ with H and T fixed. We set $\Lambda = 2\text{GeV}$, $m_s = m_f = 0.1\text{GeV}$, $T = 0.006\text{GeV}$ and $eH = 0.0001\text{GeV}^2$, $eH = 0.02\text{GeV}^2$, $eH = 0.04\text{GeV}^2$, $eH = 0.06\text{GeV}^2$. A vertical line of right side consists of all the value of eH.

not clear to extend our procedure to the state in an external electric field. It is also interesting to introduce the curvature effects in our analysis.[18]

Acknowledgements

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