Necessary Conditions for Attaining a Crystalline Beam

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Molecular dynamics is employed to study the nature and magnitude of beam cooling that is required in order to achieve a crystalline beam. Analysis is presented of a number of cooling systems now in use, or whose use is contemplated, none of which has been shown to be able to achieve the crystalline state. However, for an adequately strong cooling system that produces on average a constant *angular* velocity among the particles, a crystalline state can be achieved. In this paper, we present numerical values for a particular example under typical experimental conditions; these values appear to be achievable in practice. [S0031-9007(98)05644-0]

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For the last decade there has been interest in and experimental effort to achieve crystalline beams. These beams are sufficiently cold in the beam rest frame, so that the particles making up the beam "lock into" a position where the repelling Coulomb force just balances the external focusing force. Seen from the laboratory, the whole ordered structure circulates at great speed. The interest, besides intrinsically on this new state of matter, is primarily on the possibility of studying the physics of completely space-charge dominated beams, the possibility of studying Wigner crystal, and the possibility of obtaining high luminosity in colliders.

The ground state of a crystalline beam was proposed by Dikanskiĭ and Pestrikov [1] based on an experimental anomaly observed on an electron-cooled proton beam at the storage ring NAP-M, and was first studied using the molecular dynamics (MD) method by Schiffer and coworkers [2]. At the same time, experimental efforts have succeeded in achieving very low beam temperatures, but not yet a crystalline state [3].

In a long series of papers we explored the conditions for a crystalline beam ground state in a real storage ring [4–7]. Despite the large amount of work and the many publications in this field, there has not previously been a careful study of the nature and magnitude of the cooling force required to reach a crystalline state (see the papers in Ref. [6]). We have undertaken such a study and report on the results in this Letter.

Particle motion can be described by a Hamiltonian [4,6] in the rest frame (x, y, z, t) of a circulating reference particle in which the orientation of the axes is rotating so that the axes are constantly aligned to the radial (x), vertical (y), and tangential (z) direction. Consider in general a system of multispecies of particles under Coulomb interaction and external fields. Define a reference particle with electric charge $Z_0 e$ and atomic mass M_0, and define for the ith species of particles with charge $Z_i e$ and mass M_i, $\overline{Z}_i \equiv Z_i/Z_0$, and $m_i \equiv M_i/M_0$. Measure dimensions in units of the characteristic distance ξ with $\xi^3 = r_0\rho^2/\beta^2\gamma^2$, time in units of $\rho/\beta\gamma c$, and energy in units of $\beta^2\gamma^2 Z_0^2 e^2/\xi$, where $r_0 = Z_0^2 e^2/M_0 c^2$ is the classical radius, βc and $\gamma M_0 c^2$ are the velocity and energy of the reference particle, and ρ is the radius of curvature in bending regions of magnetic field B_0. In a bending region with pure dipole magnetic field, the Hamiltonian for particles of the ith species is

$$H_i = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) + \frac{1}{2}\left[x^2 + \left(\frac{\overline{Z}_i}{m_i} - 1\right)\gamma^2 z^2\right] - \left(2 - \frac{\overline{Z}_i}{m_i}\right)\gamma x P_z + \left(\frac{\overline{Z}_i}{m_i} - 1\right)\frac{\rho}{\xi}x + V_{Ci}, \quad (1)$$

where the Coulomb potential is

$$V_{Ci} = \sum_j \frac{\overline{Z}_i \overline{Z}_j}{m_i\sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2}}, \quad (2)$$

and the summation j is over all the other particles and their image charges [4]. In a nonbending region with longitudinal electric field and nondipole magnetic fields, the Hamiltonian is

$$H_i = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) - \frac{n_1}{2}\frac{\overline{Z}_i}{m_i}(x^2 - y^2) - n_{1s}\frac{\overline{Z}_i}{m_i}xy - \frac{n_2\xi}{6}\frac{\overline{Z}_i}{m_i}(x^3 - 3xy^2) + V_{Ci} + U_s, \quad (3)$$

where the quadrupole, skew quadrupole, and sextupole strengths are represented by

$$n_1 = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial x}, \quad n_{1s} = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial y}, \quad n_2 = -\frac{\rho}{B_0}\frac{\partial^2 B_y}{\partial x^2},$$

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respectively. The electrical force F_s can be expressed in terms of electrical field E_s measured in the laboratory frame,

$$F_s \equiv -\frac{\partial U_s}{\partial z} = \frac{\overline{Z}_i}{m_i}\frac{Z_0 e E_s \xi}{M_0 c^2}\left(\frac{\rho}{\xi\beta\gamma}\right)^2.$$

In the usual case of a single species, $\overline{Z}_i = m_i = 1$.

We have done both analytic and numerical calculations using the equations derived from these Hamiltonians and the MD methods [4]. Starting with a study of the ground state, it has been shown that there are two necessary conditions for the formation and maintenance of a crystalline beam. They are as follows: (1) The storage ring must be alternating-gradient (AG) focusing, and the energy of the beam must be less than the transition energy of the ring; i.e., $\gamma < \gamma_T$. (2) The ring lattice periodicity is at least $2\sqrt{2}$ as high as the maximum betatron tune. Condition (1) arises from the criterion of stable kinematic motion under Coulomb interaction when particles are subject to bending in a storage ring. Condition (2) arises from the criterion that there is no linear resonance between the phonon modes of the crystalline structure and the machine lattice periodicity.

Existing storage rings upon which attempts have been made to obtain crystalline beams do not satisfy the conditions just stated, although with minor modifications they would. However, the TARN II lattice does satisfy them. Among a wide range of choice we choose the TARN II parameters [8] for the example of this Letter, as given in Table I. In Fig. 1, we show a set of three-dimensional (3D) stereoscopic pictures of a typical crystalline beam at its ground state, along with particle trajectories over a lattice period. The bunching in the longitudinal direction of this crystal ball is a result of the focusing force produced by a radio-frequency (rf) electric voltage $V\sin(h\omega_0 t)$ (expressed in the laboratory frame) varying sinusoidally with time. In the absence of this rf focusing, one obtains a crystalline state that extends over the full circumference of the ring as depicted in Fig. 1 of Ref. [4].

We turn next to the subject of behavior of a crystal at nonzero temperature. The crystal temperature is defined to be proportional to the deviations of momentum from their ground-state values, then squared and averaged over many lattice locations and periods and over all the particles, i.e., $T = \beta^2\gamma^2\xi^2 M_0 c^2 \sum_{i=x,y,z}\langle(\Delta P_i)^2\rangle/2k_B\rho^2$, where k_B is the Boltzmann constant. Such an averaging is necessary because, without doing so, the quantities in the definition of temperature are functions of location and time. A more expansive discussion of the temperature is presented in Ref. [9]. A crystal in its ground state ($T = 0$), despite its transverse breathing (due to the AG focusing) and its longitudinal shear motion (due to straight sections and bend regions), will remain forever in the ground state. That is, it will *not* take up energy from the lattice. However, for any nonzero temperature the crystal will continually absorb energy and so heat up. In the high temperature limit this is called intrabeam scattering, and it becomes ever larger as the temperature decreases simply because coulomb scattering increases as beam-occupied

TABLE I. Parameters of TARN II.

Quantity	Value
Ring circumference, $2\pi R$	77.7 m
Number of lattice periods per turn	6
Horizontal and vertical tunes, ν_x, ν_y	1.68, 1.85
Transition energy, γ_T	1.87
Skew quad integral strength per period	0.02 m^{-1}
Dipole bending radius, ρ	4.01 m
Ion species	$^{24}Mg^+$
Kinetic energy	1 MeV
RF voltage, V per period	33.33 V
RF harmonic number, h	1000
Synchrotron tune, ν_s	0.107

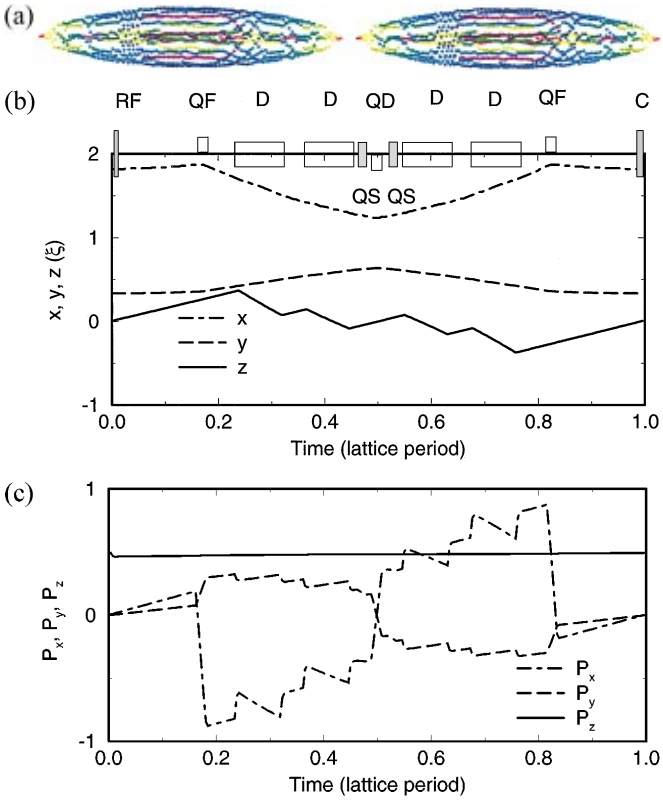

FIG. 1. A set of stereoscopic pictures of a crystal ball in TARN II at its ground state (a) (color), obtained by MD calculation with 1000 particles (total $N = 10^6$ in the ring) and displayed at the cooling location. (The stereoscopic effect can best be seen by holding the paper close to your nose.) The color code is associated with the scaled radial distance from the axis. The object is finite in all three dimensions due to the transverse focusing forces of the external magnets and the longitudinal rf force. The physical dimensions in the horizontal, vertical, and longitudinal directions are approximately 0.3, 0.2, and 8.0 mm, respectively. A typical distance between ions is 30 μm ($\xi = 23$ μm). The trajectory over a lattice period of a typical particle is shown in (b) and (c). Lattice components in each period are displayed on the figure: D is a bending dipole section, QF, QD, and QS are focusing, defocusing, and skew quadrupoles, RF is the bunching rf cavity, and C is the cooling section.

phase space area is reduced. On the other hand, at very low temperatures particles are frozen into a crystal and no longer move by each other. Thus the heating rate has a maximum near the temperature at which ordering starts to occur. Using the MD method, we have studied the heating rate as a function of the temperature. Figure 2 shows the results for the crystal ball case previously displayed in Fig. 1, and also compares the heating behavior of both bunched and coasting crystals at different density. For simplicity without loss of generality [10], we assume that the beam temperature is isotropic. The heating behavior of bunched and coasting crystals are found to be similar, and the peak heating rate is proportional to the average linear density over the crystal. A cooling system must be adequately powerful as to take the particles over the peak heating value and on down to a crystal.

We are now ready to explore the nature of beam cooling required in order to obtain a crystalline structure. In order to attain a crystalline state, the temperature must be low in all three directions. The laser cooling, which is the preferred method of reaching very low temperatures, provides adequate longitudinal cooling—for it works on the Doppler principle, but it does not cool in the transverse direction. There have been a number of suggestions, and even experimental efforts, to provide transverse cooling of the stored ion beams. Intrabeam scattering ("sympathetic cooling") does give some cooling in this direction and experimental observations of this effect have been reported [11]. Coupling cavities [12], or regular rf cavities in a region where there is dispersion [13], could provide significant energy transfer from the longitudinal to the transverse phase space; this has been proposed, but not yet experimentally attempted.

Neither of these approaches, sympathetic cooling or coupling cavities, has been shown to reach a crystalline beam (here approximately $T < 0.02$ K). Sympathetic cooling is too slow, and the heat exchange becomes ineffective (Fig. 2) as the beam approaches an ordered state [6]. Coupling cavities usually cease working before crystallization, as we have recently demonstrated using the MD method. For a single particle the coupling cavity scheme is effective, but as ordering starts to appear the Coulomb interaction between the particles (or "tune shift") becomes strong and destroys the synchrobetatron coupling mechanism.

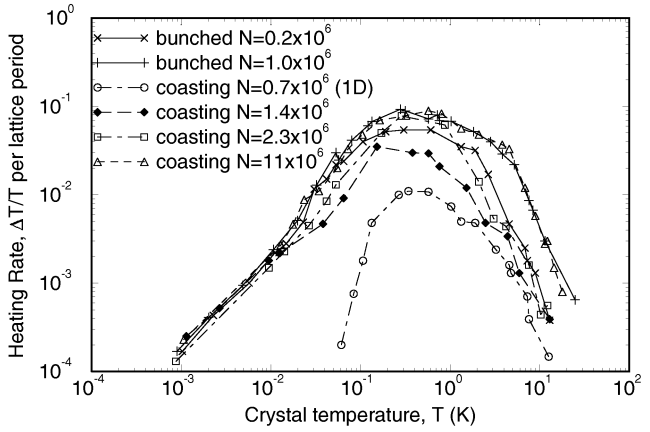

FIG. 2. Heating rate curves for bunched beams (including the one shown in Fig. 1) and coasting beams at different linear density. Except for the indicated 1D case, all the rest are 3D structures. The heating rate is significantly less for a 1D crystal.

We have found that longitudinal cooling, when it results in a state where the momentum is a function of the transverse displacement, is effective in reaching a crystalline state, i.e., a state in which particles of different radial position all have the same average angular velocity. We shall call this method "tapered cooling." The effect of cooling is described by a reduction in P_z in the cooling region by

$$\Delta P_z = -f_z(P_z - C_{xz}x), \tag{4}$$

where the coefficient describing the strength of cooling is f_z, and the coefficient describing the extent of tapering is C_{xz}. With proper choice of parameters purely longitudinal cooling also results in transverse cooling (and therefore a crystalline state): The coupling between z and x is provided by the dispersion at the cooling location, while coupling between x and y is provided by the skew quadrupoles. With tapered cooling and activated skew quadrupoles, temperature in the horizontal, vertical, and longitudinal directions can be reduced simultaneously. The acceptable strength of the skew quadrupoles is a function of the difference in transverse tunes.

In the low-density limit where Coulomb interaction between the particles is negligible, one can calculate the cooling rate by evaluating the dispersion relation from the one-turn transfer matrix. In the case of weak cooling, $f_z \ll 1$, and near a transverse coupling resonance, the transverse and longitudinal cooling rates are $\tau_{x,y}^{-1} = GC_{xz}$ and $\tau_z^{-1} = f_z\omega_0/2\pi - 4GC_{xz}$, where $G \equiv \gamma\eta_c f_z\omega_0/8\pi\rho$, and η_c is the dispersion at the location of the cooling device. The single-particle stability limit for the tapering coefficient is $C_{xz} \leq \rho/\gamma\eta_c$. When the cooling rate is not small compared to 1, the dispersion relation can be solved numerically.

The value of C_{xz} necessary to obtain a zero-temperature state is given simply by the requirement that the average angular velocity of the particles all are equal, i.e., $\langle\dot{z}\rangle = 0$, or $\langle P_z\rangle = \gamma\rho\langle x\rangle/R$, where $2\pi R$ is the machine circumference. Consequently, $\langle C_{xz}\rangle = \gamma\rho/R$, where $\langle C_{xz}\rangle$ is the average over one lattice period of C_{xz} defined in Eq. (4). In the laboratory frame, this argument is equivalent to writing $\langle x\rangle = (R/\gamma^2)\langle\Delta p/p\rangle$, where $\langle\Delta p/p\rangle$ is the average fractional momentum, which may be compared with the single-particle relation of $\langle x\rangle = (R/\gamma_T^2)\Delta p/p$. The value of C_{xz} at a cooling section differs from the average $\langle C_{xz}\rangle$ due to ground-state crystal breathing. Approximately, this difference is given by the azimuthal variation of the single-particle dispersion. The optimum value of C_{xz} is independent of the density, the bunching of the beam, and the cooling strength f_z. It can be precisely determined by a numerical evaluation of the desired crystalline state.

For the bunched crystal ball shown in Fig. 1, we find that C_{xz} must be near 0.26, f_z larger than about 0.3 (at the peak of the heating curve), and skew quadrupole parameters as given in Table I. Since there is no qualitative difference between a bunched and a coasting beam in both heating (Fig. 2) and cooling behavior, we, for simplicity, present results for tapered cooling of a coasting beam (parameters in Table I above the dividing line). In Fig. 3 we display the acceptable range (for $T \leq 0.02$ K) of tapering, C_{xz}, as a function of beam linear density. The difference of C_{xz} from its optimum value determines the lowest reachable crystal temperature, and the acceptable range is not dependent on f_z as long as f_z is large enough to overcome heating (Fig. 3). For 1D low density string structures, the acceptable ranges for both C_{xz} and f_z are drastically increased, but yet a tapering is still needed to achieve a crystalline state. Numerical evaluation of the single-particle dispersion relation yields a stable range of $0 < C_{xz} \leq 0.97$.

In practice, in order to achieve a cooling system that provides a constant angular velocity it is necessary to have dispersion in the cooling electrons (for electron cooling) or in the laser beam (for Doppler laser cooling). In either case the dispersion that is required is very small and can easily be provided by a bending magnet or a light prism. For the TARN II example with the laser cooling, at optimum tapering we would need the laser wavelength to vary linearly by 0.01% (say, from 280.0 to 280.3 nm) horizontally across a beam of 1.5 mm. With the electron cooling we would need the electron velocity to vary by $(\Delta v/v) = 0.01\%$ across the same distance. Although the degree of dispersion is small, it is, as we have emphasized in this Letter, essential for obtaining a crystalline beam.

In summary, the powerful numerical methods of molecular dynamics allow one to determine the desired crystalline state and the parameters necessary for achieving it, as we have shown by an explicit example in this Letter. We have demonstrated that with a properly designed ring (alternating gradient and high lattice periodicity), operating below the transition energy and with a sufficiently powerful cooling system that produces a constant angular velocity, one can achieve a crystalline beam. It appears that these criterion can be met in practice.

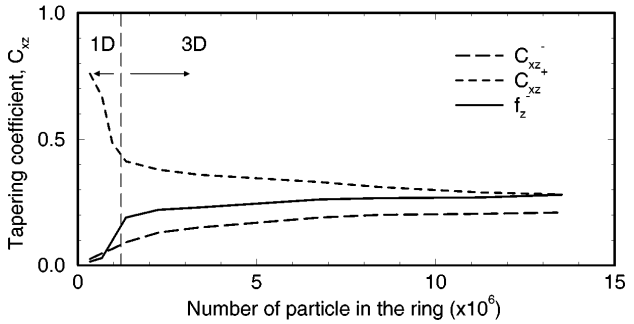

FIG. 3. Acceptable range of tapering coefficients (C_{xz}^-, C_{xz}^+) and minimum cooling strength f_z^-, in order to achieve a crystalline state, as a function of the number of particles in the storage ring (i.e., beam linear density). Numerical evaluation of the single-particle dispersion relation described in the text yields C_{xz}^+ of 0.97.

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[9] At high temperature, $T \gg 1$ K, the crystal temperature T is equivalent to the conventionally defined beam temperature T_B, given by the relation

$$[T_{Bx}, T_{By}, T_{Bz}] = \frac{\beta^2\gamma^2 M_0 c^2}{2k_B}\left[\frac{\epsilon_x}{\langle\beta_x\rangle}, \frac{\epsilon_y}{\langle\beta_y\rangle}, \frac{1}{\gamma^2}\left(\frac{\Delta p}{p}\right)^2\right],$$

where $\epsilon_{x,y}$ are the unnormalized rms beam emittance, $\Delta p/p$ is the rms momentum spread, and $\langle\beta_{x,y}\rangle$ are the averaged betatron amplitude functions. However, when the beam is cooled longitudinally so that $\Delta p/p$ and T_{B_z} both approach zero, the longitudinal crystal temperature $T_z = \beta^2\gamma^2\xi^2 M_0 c^2\langle(\Delta P_z)^2\rangle/2k_B\rho^2$ can still be high since the z motion is coupled with the x motion through dispersion, and that transverse temperature is still high.

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