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A New Series of μ-Resolvable BIB Designs

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A method of constructing a series of μ-resolvable BIB designs from mutually orthogonal mates of BIB designs is provided. The resolvable designs obtainable through this method are new and involve lesser number of blocks for a fixed number of treatments than those obtained by Kageyama and Mohan (1983) and Subramani (1990).

Keywords: μ-Resolvability, BIB design, mutually orthogonal mates.

1. Introduction

The concept of resolvable and affine resolvable balanced incomplete block (BIB) designs was introduced by Bose [1]. The concept was generalized to μ-resolvable and affine μ-resolvable BIB designs by Shrikhande and Raghavarao [4]. The constructions of μ-resolvable BIB designs with their combinatorial properties have been discussed by several authors. Kageyama and Mohan [2] developed a method for construction of μ-resolvable BIB designs and an extensive list of the designs developed by the application of the same was presented with $v \leq 125, b \leq 250, \lambda \leq 100$ and $3 \leq k \leq v-3$ for $\mu \geq 2$. Furthermore, Subramani [5] developed a method of constructing μ-resolvable BIB designs using a set of mutually orthogonal Latin squares. The developed designs were also tabulated for $v \leq 20, b \leq 350, \lambda \leq 250$ and $3 \leq k \leq v-3$ with $\mu \geq 3$.

In this paper a new method of constructing μ-resolvable BIB designs is presented. The method has been developed from a set of mutually orthogonal mates (MOM's) of BIB designs. The concept of orthogonal mates (OM's) of BIB designs was given in Pal and Goswami [3]. The definitions of OM and MOM of BIB designs are presented below.

Definition 1.1. A BIB design with parameters v, b, r, k, λ is said to be μ-resolvable if the blocks can be grouped into t sets, $S_1, S_2, ..., S_t$, each of m blocks, such that in each set every treatment is replicated μ times. The following parametric relations hold in the design:

$$v\mu = mk, \; b = mt, \; r = \mu t.$$

Definition 1.2. Two BIB designs with the same parameters v, b, r, k, λ and with the same treatments are said to be orthogonal mates (OM's) of each other if, when they are superimposed on one another, every ordered pair of distinct treatments occurs equal number of times.

Definition 1.3. If in a set of BIB designs, every pair of designs is an OM of each other, then the set of BIB designs is called a set of mutually orthogonal mates (MOM's) of BIB designs.

Definition 1.4. Consider the Galois field $GF(v)$ with v elements, where v is a prime or a prime power. Let $x \in GF(v)$ and $a_{ij} \in GF(v)$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. When

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$A = (a_{ij})$ is a matrix of size $m \times n$, the symbolic inner product of A and x is defined by an $m \times n$ matrix B being represented by $B = A \odot x = (a_{ij}x)$, i.e., any element of B is the product of x and the corresponding element of A, the product being reduced under $GF(v)$.

The designs developed by using Definitions 1.3 and 1.4 are tabulated in Table 2.1 and these designs require lesser number of blocks than those of Kageyama and Mohan [2] and Subramani [5]. Recently, Vasiga, Furino and Ling [6] showed the existence of all the μ-resolvable BIB designs with $k = 4$, under the exception of $(\mu, v, \lambda) = (2,10,2)$.

2. Method of Construction

When v is a prime or a prime power, the procedure of construction of a set of MOM's of BIB designs with v treatments is described below. Let the treatments be elements from $GF(v)$.

Let B_1 be a BIB design with parameters $v, b = pv, r = pk, k, \lambda$, with p initial blocks, each block containing k treatments. From the design B_1, let us construct a set of $q (q \leq p)$ MOM's of BIB designs B in the following way.

The content of the design B_1 is written by placing side by side the p initial blocks columnwise and then by developing the p initial blocks cyclically columnwise and there by generating p sets, each set containing v blocks. Next, develop another BIB design B_2 with the same parameters as B_1 in the following way.

Let B_{11} represent the $k \times p$ matrix of elements containing p initial blocks of B_1. Construct $B_{21} = B_{11} \odot x$ of size $k \times p$, where x is a primitive element of $GF(v)$. Thus a new matrix of p initial blocks is obtained. These p new initial blocks are then developed cyclically columnwise as was done in the case of B_1. The whole design so developed constitutes another BIB design B_2 which is shown to be an orthogonal mate of B_1. It can be noted that B_2 also contains p sets, each set containing v blocks after development. The sets of B_2 are automatically placed below the corresponding sets of B_1.

The designs, $B_3, B_4, ..., B_q$ $(q \leq p)$ can be obtained by following the procedure described above, noting that $B_{i1} = B_{11} \odot x^{i-1}$, $i = 3, 4, ..., q$. It follows that the designs, $B_1, B_2, B_3, B_4, ..., B_q$, form a set of $q (\leq p)$ MOM's.

In this procedure we can obtain the following.

Theorem 2.1. The existence of a set of $q(\leq p)$ MOM's of a BIB design with parameters $v, b = pv, r = pk, k, \lambda$, having p initial blocks, implies the existence of a μ-resolvable BIB design with parameters $v^* = v, b^* = bk, r^* = qr, k^* = q$, $\lambda^* = rq(q-1)/(v-1)$, $\mu = q, m = v, t = pk$, where v is a prime or a prime power.

Proof (by construction). Let B_1 be a BIB design with parameters $v, b = pv, r = pk, k, \lambda$, having p initial blocks. The set of q MOM's of BIB designs B_j's $(j = 1, 2, ..., q)$ are constructed from B_1, following the way mentioned above.

Let us superimpose all q designs, each involving v treatments. Let us construct the content of each cell obtained by superimposition as a block (see Illustration 2.1). Then the total number of blocks, each of size q, is bk. Thus we obtain the required design B^* with bk blocks, each of size q. These bk blocks in B^* can be divided into pk sets, each set containing v blocks, as there are p initial blocks in B_j for $j = 1, 2, ..., q$. It can easily be verified that the design B^* is a μ-resolvable BIB design with parameters $v^* = v, b^* = bk, r^* = qr, k^* = q, \lambda^* = rq(q-1)/(v-1), \mu = q, m = v, t = pk$. □

Illustration 2.1. Let B_1, B_2 and B_3 be a set of three MOM's of BIB designs with parameters $v = 7, b = 21, r = 6, k = 2, \lambda = 1$. Their solutions are given below. Here the columns show blocks in each design.

$$B_1 \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \qquad \begin{array}{ccccccc} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{ccccccc} 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$B_2 \quad \begin{array}{ccccccc} 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{array} \qquad \begin{array}{ccccccc} 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{array} \qquad \begin{array}{ccccccc} 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \end{array}$$

$$B_3 \quad \begin{array}{ccccccc} 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{array} \qquad \begin{array}{ccccccc} 6 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \end{array} \qquad \begin{array}{ccccccc} 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \end{array}$$

Let the designs B_1, B_2 and B_3 be superimposed to construct a μ-resolvable BIB design B^* with parameters $v^* = 7, b^* = 42, r^* = 18, k^* = 3, \lambda^* = 6, \mu = 3, m = 7, t = 6$, whose solution is given by

(1,3,2), (2,4,3), (3,5,4), (4,6,5), (5,0,6), (6,1,0), (0,2,1);
(3,2,6), (4,3,0), (5,4,1), (6,5,2), (0,6,3), (1,0,4), (2,1,5);
(2,6,4), (3,0,5), (4,1,6), (5,2,0), (6,3,1), (0,4,2), (1,5,3);
(6,4,5), (0,5,6), (1,6,0), (2,0,1), (3,1,2), (4,2,3), (5,3,4);
(4,5,1), (5,6,2), (6,0,3), (0,1,4), (1,2,5), (2,3,6), (3,4,0);
(5,1,3), (6,2,4), (0,3,5), (1,4,6), (2,5,0), (3,6,1), (4,0,2).

Elements in the parentheses denote the contents of the blocks. Each row above shows a 3-resolution set (i.e., $\mu = 3$).

Remark 2.1. The number of blocks in the μ-resolvable BIB design constructed in Theorem 2.1 can be reduced further by adopting the following procedure. Let us consider s $(1 \le s \le k)$ rows of the design B_1 and denote it by B_1^*. Obtain $B_2^*, ..., B_q^*$ following the method of construction described earlier. By superimposing the designs $B_1^*, B_2^*, ..., B_q^*$ (procedure of construction being described in the proof of Theorem 2.1), we get a μ-resolvable BIB design B^{**} with parameters $v^{**} = v, b^{**} = sb, r^{**} = spq = sqb/v, k^{**} = q, \lambda^{**} = srq(q-1)/[k(v-1)], \mu = q, m = v, t = sp$, in which s is an integer such that λ^{**} is an integer. In fact, the constancy of λ^{**} can be proved in the following way: Consider pairs of type (i, j) for $i, j (i < j) = 1, 2, ..., v$. The total number of such pairs from each block is equal to $q(q-1)/2$ and hence the total number of such pairs from the design is $[q(q-1)/2](sb)$. On the other hand, the total number of the above type of pairs from the design is also equal to $\lambda^{**} v(v-1)/2$, because λ^{**} is the number of times a particular pair occurs in the design. Thus the expression of λ^{**} can be obtained by equating the two quantities, and by using the relations $p = b/v$ or $p = r/k$. Note that the value s should be chosen such that λ^{**} becomes an integer.

In particular, the small design can be obtained when $s = 1$. In this case, the resulting μ-resolvable BIB design has the parameters $v^{**} = v, b^{**} = b, r^{**} = pq, k^{**} = q$, $\lambda^{**} = rq(q-1)/[k(v-1)], \mu = q, m = v, t = p$. In fact, all the designs given in Table 2.1 are constructed for $s = 1$ except for the design of No. 14 (the first source) in which $s = 2$.

Illustration 2.2. In Illustration 2.1, let us include in B_1^* the first row of B_1 only, i.e., $s = 1$. We get a μ-resolvable BIB design with parameters $v = 7, b = 21, r = 9, k = 3, \lambda = 3$, $\mu = 3, m = 7, t = 3$, whose solution is given by

(1,3,2), (2,4,3), (3,5,4), (4,6,5), (5,0,6), (6,1,0), (0,2,1);
(3,2,6), (4,3,0), (5,4,1), (6,5,2), (0,6,3), (1,0,4), (2,1,5);

(2,6,4), (3,0,5), (4,1,6), (5,2,0), (6,3,1), (0,4,2), (1,5,3).

A list of the available μ-resolvable BIB designs are presented in Table 2.1 with $3 \leq k \leq 10$ and $7 \leq v \leq 19$ which require fewer blocks than the designs given in Table 1 of Subramani [5] who listed 230 designs. The designs in the table are constructed by use of an idea of Remark 2.1 after the construction by Theorem 2.1. Note that the present table is not exhaustive. In the Source of the table a basic BIB design, BIB$(v, b = pv, r, k, \lambda)$, is given with the values of q being the number of MOM's of the BIB design and p as $[q, p]$.

Table 2.1: μ-Resolvable BIB designs

No.	v	b	r	k	λ	μ	m	t	Source with $[q,p]$
1	7	21	9	3	3	3	7	3	$BIB(7,21,6,2,1)[3,3]$
2	9	36	12	3	3	3	9	4	$BIB(9,36,8,2,1)[3,4]$
3	9	36	16	4	6	4	9	4	$BIB(9,36,8,2,1)[4,4]$
4	11	55	15	3	3	3	11	5	$BIB(11,55,10,2,1)[3,5]$
5	11	55	20	4	6	4	11	5	$BIB(11,55,10,2,1)[4,5]$
6	11	55	25	5	10	5	11	5	$BIB(11,55,10,2,1)[5,5]$
7	13	52	12	3	2	3	13	4	$BIB(13,52,12,3,2)[3,4]$
8	13	52	16	4	4	4	13	4	$BIB(13,52,12,3,2)[4,4]$
9	13	78	18	3	3	3	13	6	$BIB(13,78,12,2,1)[3,6]$
10	13	78	24	4	6	4	13	6	$BIB(13,78,12,2,1)[4,6]$
11	13	78	30	5	10	5	13	6	$BIB(13,78,12,2,1)[5,6]$
12	13	78	36	6	15	6	13	6	$BIB(13,78,12,2,1)[6,6]$
13	17	68	16	4	3	4	17	4	$BIB(17,68,16,4,3)[4,4]$
14	17	136	24	3	3	3	17	8	$BIB(17,68,16,4,3)[3,4]$ or $BIB(17,136,16,2,1)[3,8]$
15	17	136	32	4	6	4	17	8	$BIB(17,136,16,2,1)[4,8]$
16	17	136	40	5	10	5	17	8	$BIB(17,136,16,2,1)[5,8]$
17	17	136	48	6	15	6	17	8	$BIB(17,136,16,2,1)[6,8]$
18	17	136	56	7	21	7	17	8	$BIB(17,136,16,2,1)[7,8]$
19	17	136	64	8	28	8	17	8	$BIB(17,136,16,2,1)[8,8]$
20	19	57	9	3	1	3	19	3	$BIB(19,57,18,6,5)[3,3]$
21	19	114	18	3	2	3	19	6	$BIB(19,114,18,3,2)[3,6]$
22	19	114	24	4	4	4	19	6	$BIB(19,114,18,3,2)[4,6]$
23	19	114	36	6	10	6	19	6	$BIB(19,114,18,3,2)[6,6]$
24	19	171	27	3	3	3	19	9	$BIB(19,171,18,2,1)[3,9]$
25	19	171	36	4	6	4	19	9	$BIB(19,171,18,2,1)[4,9]$
26	19	171	45	5	10	5	19	9	$BIB(19,171,18,2,1)[5,9]$
27	19	171	54	6	15	6	19	9	$BIB(19,171,18,2,1)[6,9]$
28	19	171	63	7	21	7	19	9	$BIB(19,171,18,2,1)[7,9]$
29	19	171	72	8	28	8	19	9	$BIB(19,171,18,2,1)[8,9]$
30	19	171	81	9	36	9	19	9	$BIB(19,171,18,2,1)[9,9]$

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