MAPPING PROPERTIES OF NONLINEAR INTEGRAL OPERATORS AND PRE-SCHWARZIAN DERIVATIVES

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ABSTRACT. In this paper, nonlinear integral operators on normalized analytic functions in the unit disk are investigated in connection with the pre-Schwarzian derivative and the Hornich operation. In particular, several non-trivial relations between these operators and the class of strongly starlike functions will be deduced.

1. INTRODUCTION

H. Hornich [13] defined an unusual operation (often called the Hornich operation) on the set of locally univalent (analytic) functions in the unit disk (or, more generally, a convex domain). Without essential loss of generality, we may restrict ourselves to the set $\mathcal{A}$ of analytic functions f in the unit disk $\mathbb{D} = \{|z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Let f and g be locally univalent functions in $\mathcal{A}$ and α be a complex number. Then the Hornich operation is defined by

$$f \oplus g(z) = \int_0^z f'(w)g'(w)dw, \quad \text{and}$$

$$\alpha \star f(z) = \int_0^z \{f'(w)\}^\alpha dw,$$

where the branch of $(f')^\alpha = \exp(\alpha \log f')$ is taken so that $(f')^\alpha(0) = 1$. Thus, this manipulation gives a structure of vector space to the set $\mathcal{LU}$ of locally univalent functions in $\mathcal{A}$, namely, $\mathcal{LU} = \{f \in \mathcal{A} : f'(z) \neq 0 \ \forall z \in \mathbb{D}\}$.

Hornich [13] also introduced a norm to a subset of $\mathcal{LU}$ which makes it a separable real Banach space with the above operation. After then, J. A. Cima and J. A. Pfaltzgraff [9] gave another separable real Banach space structure to a slightly different set from Hornich's one. These spaces, however, both do not contain the whole set $\mathcal{S}$ of normalized univalent functions. On the other hand, D. M. Campbell, Cima and Pfaltzgraff [8] considered a complex Banach space structure on the set of locally univalent functions of

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finite order, where the order of a function f is defined by

$$\mathrm{ord}(f) = \sup_{z\in\mathbb{D}} \left| -\bar{z} + \frac{1}{2}(1-|z|^2)\frac{f''(z)}{f'(z)} \right|,$$

see [24]. This space is non-separable, but has the advantage that it contains the whole $\mathcal{S}$ as a closed subset with non-empty interior as we shall explain later. It is quite easy to see that $\mathrm{ord}(f) < \infty$ if and only if the norm

$$\|f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)|T_f(z)|$$

is finite, where $T_f = f''/f'$ denotes the pre-Schwarzian derivative $T_f = f''/f'$ of f. Note that the Schwarzian derivative of f is defined as $S_f = (f''/f')' - (f''/f')^2/2 = T_f' - (T_f)^2/2$. It should also be noted that the norm $\|f\|$ is nothing but the Bloch semi-norm of the function $\log f'$. The above norm is not same as, but equivalent to, that considered in [8].

S. Yamashita observed in [29] that $f \in \mathcal{LU}$ is of finite order if and only if f is *uniformly locally univalent* in $\mathbb{D}$, namely, there is a positive constant ρ for which f is univalent in every hyperbolic disk in $\mathbb{D}$ of radius ρ. Earlier than this, Ch. Pommerenke obtained in [24, Satz 2.6] an explicit estimate for the radius ρ of the univalent hyperbolic disk for f in terms of $\mathrm{ord}(f)$.

We can now view the Hornich operation more naturally through the pre-Schwarzian derivative. Indeed, since $T_{f\oplus g} = T_f + T_g$ and $T_{\alpha\star f} = \alpha T_f$, it is just the transformation of the usual linear operation under the inverse of taking pre-Schwarzian derivative. This simple fact is, however, a source of ideas developed in the present paper. Note that this point of view was used by Yamashita [28] in a more general context.

The complex Banach space $\mathcal{B} = \{f \in \mathcal{LU} : \|f\| < \infty\}$ with the Hornich operation and the norm $\|f\|$ is thus a natural object to investigate. It is well known that $\mathcal{S}$ is closed in $\mathcal{B}$ and contained in $\{f \in \mathcal{B} : \|f\| \le 6\}$. The Koebe function is an example so that $\|f\| = 6$. Also, sufficient conditions for univalence and boundedness for f are known.

Theorem A.

(i) *If* $\|f\| \le 1$, *then* f *is univalent in* $\mathbb{D}$ *and* $\|f\| \le k < 1$, *then* f *has a* K*-quasiconformal extension to* $\mathbb{C}$, *where* $K = (1+k)/(1-k)$.

(ii) *If* $\|f\| < 2$, *then* f *is bounded in* $\mathbb{D}$.

The bounds 1 *and* 2 *are sharp.*

The first assertion was proved by J. Becker [4]. The sharpness of the constant 1 is due to [5]. The second assertion is immediate. The reader can find a proof of it in [15] with examples showing the sharpness of the bound.

The set $\mathcal{T} \subset \mathcal{S}$ consisting of those functions f which have quasiconformal extensions to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is known to be an open set in $\mathcal{B}$ and considered as a model of the universal Teichmüller space. See [2] and [31]. One of the most interesting features of $\mathcal{T}$ is the following theorem due to I. V. Zhuravlev [31].

Theorem B. *The space $\mathcal{T}$ decomposes into the uncountably many connected components $\mathcal{T}_0$ and $\mathcal{T}_\zeta$, $\zeta \in \partial\mathbb{D}$, where*

$$\mathcal{T}_0 = \{f \in \mathcal{T} : f(\mathbb{D}) \text{ is bounded } \} \quad \text{and} \quad \mathcal{T}_\zeta = \{f \in \mathcal{T} : f(z) \to \infty \text{ as } z \to \zeta\}.$$

Moreover, $\{f \in \mathcal{B} : \|f - L_\zeta\| < 1\} \subset \mathcal{T}_\zeta$ holds for each $\zeta \in \partial\mathbb{D}$, where $L_\zeta(z) = z/(1 - \bar{\zeta}z)$.

We can see, in particular, that $\{f \in \mathcal{B} : \|f\| < 1\} \subset \mathcal{T}_0$ and $\{f \in \mathcal{B} : \|f - L_\zeta\| < 1\} \subset \mathcal{T}_\zeta \subset \{f \in \mathcal{B} : 2 < \|f\| < 6\}$ for $\zeta \in \partial\mathbb{D}$. The non-separability of $\mathcal{B}$ follows also from the above result (cf. [8]). K. Astala and F. W. Gehring showed in [2] that the interior of $\mathcal{S}$ in $\mathcal{B}$ coincides with $\mathcal{T}$ but the closure of $\mathcal{T}$ does not coincide with $\mathcal{S}$.

The above aspects can be used to consider problems about some integral operators. Let $I_\alpha[f] = \alpha \star f$ for $f \in \mathcal{LU}$ and $J_\alpha[f] := I_\alpha[J[f]]$ for $f \in \mathcal{ZF} = \{f \in \mathcal{A} : f(z) \neq 0 \text{ for all } z \in \mathbb{D}\setminus\{0\}\}$, where $\alpha \in \mathbb{C}$ and J stands for the Alexander transformation. More explicitly,

$$I_\alpha[f](z) = \int_0^z \{f'(w)\}^\alpha dw \quad \text{and} \quad J_\alpha[f](z) = \int_0^z \left\{\frac{f(w)}{w}\right\}^\alpha dw.$$

Note that the map $J = J_1 : \mathcal{ZF} \to \mathcal{LU}$ is bijective and that the inverse of J can be represented by $J^{-1}[f](z) = zf'(z)$. It is also well known that $J(\mathcal{S}^*) = \mathcal{K}$, where $\mathcal{S}^*$ and $\mathcal{K}$ stand for the classes of starlike and convex functions, respectively (see [1]). These operators have been studied by many authors. We refer the reader to [10, § 8.5] and [12, Chapter 15] for basic information about these operators, and standard terminology in the theory of univalent functions, as well.

Problems concerning the operator J_α can reduce to ones concerning I_α instead, if we once know about the image under the Alexander transformation J because of the relation $J_\alpha = I_\alpha \circ J$. For example, we know that $J(\mathcal{S}^*) = \mathcal{K}$, and therefore, $J_\alpha(\mathcal{S}^*) = I_\alpha(\mathcal{K})$. Though one can regard J as a smoothing operator, the behaviour of J is not so simple as it is known that $J(\mathcal{S})$ is not contained in $\mathcal{S}$. On the other hand, we certainly have a better estimate for functions in $J(\mathcal{S})$ than the estimate $\|f\| \le 6$ for $f \in \mathcal{S}$. In Section 2, we give a proof of the following result as well as some remarks. Compare with the inequality $\|I_\alpha[f]\| = |\alpha|\|f\| \le 6|\alpha|$ for $f \in \mathcal{S}$.

Theorem 1.1. *The inequality $\|J_\alpha[f]\| \le 4|\alpha|$ holds for every $f \in \mathcal{S}$ and every complex number α. The bound is sharp.*

Pfaltzgraff [23] showed that $I_\alpha(\mathcal{S}) \subset \mathcal{S}$ if $|\alpha| \le 1/4$. On the other hand, W. C. Royster [25] proved that, for each number α other than 1 with $|\alpha| > 1/3$, there is a function f in $\mathcal{S}$ with $I_\alpha(f) \notin \mathcal{S}$. Up to now, nothing better has been obtained in this general situation.

The problem to find the sharp constant now reduces to find the supremum of the ratio of the outer and the inner radii of the set

$$U(f) = \{\alpha \in \mathbb{C} : \alpha \star f \in \mathcal{S}\} \tag{1.1}$$

around the origin when f runs all over the set $\mathcal{S}$. Indeed, Royster's observation was essentially that $U(f_1) = \{\alpha \in \mathbb{C} : |\alpha| \le 1 \text{ or } |\alpha - 2| \le 1\}$, where $f_1(z) = \log(1+z)$. Note that $\|f_1\| = 2$ and that $2 \star f_1 = L_{-1}$. Especially, it may be interesting to observe that $\{\alpha \in \mathbb{C} : \alpha \star f_1 \in \mathcal{T}_0\} = \{|\alpha| < 1\}$ and $\{\alpha \in \mathbb{C} : \alpha \star f_1 \in \mathcal{T}_{-1}\} = \{|\alpha - 2| < 1\}$ (see Theorem B). The difficulty of determining the set $\{\alpha \in \mathbb{C} : I_\alpha(\mathcal{S}) \subset \mathcal{S}\}$ seems to come from the fact that we have only very few functions f for which the shapes of $U(f)$ are completely determined.

By virtue of the relation $J(\mathcal{S}^*) = \mathcal{K}$, Theorem 1 of E. P. Merkes [19] can read as $I_\alpha(\mathcal{K}) \subset \mathcal{S}$ for $|\alpha| \le 1/2$, where the constant $1/2$ is sharp. Relating this result, he gave in [19] the conjecture that $I_\alpha(\mathcal{K}) \subset \mathcal{S}$ for $|\alpha - 1| \le 1/2$. We will use the above aspects to settle the conjecture by giving a complete characterization of α satisfying $I_\alpha(\mathcal{K}) \subset \mathcal{S}$.

Theorem 1.2. *The set* $M = \{\alpha \in \mathbb{C} : I_\alpha(\mathcal{K}) \subset \mathcal{S}\} = \{\alpha \in \mathbb{C} : J_\alpha(\mathcal{S}^*) \subset \mathcal{S}\}$ *equals the union of the closed disk* $|\alpha| \le 1/2$ *and the line segment* $[1/2, 3/2]$.

Concerning the Hornich operation, the linear structure of typical classes of univalent functions has been investigated. For convenience, we denote by $[f, g]$ the closed line segment joining f and g, namely, $[f, g] = \{(1-t) \star f \oplus t \star g : 0 \le t \le 1\}$. It is shown in [9] that the class $\mathcal{K}$ of convex functions is convex, namely, $[f, g] \subset \mathcal{K}$ for any pair of functions f and g in $\mathcal{K}$. Y. J. Kim and Merkes [17] proved that the class $\mathcal{C}$ of close-to-convex functions is also convex. In contrast, we will see that the class $\mathcal{S}^*$ of starlike functions is not convex in Section 3. At least, however, it is reasonable to pose the following problem:

Problem 1.3. *Is the class* $\mathcal{S}^*$ *starlike with respect to the origin concerning the Hornich operation?*

In other words, is it true that $[\mathrm{id}, f] \subset \mathcal{S}^*$ for each $f \in \mathcal{S}^*$? We consider this problem and give some partial solutions to it in Section 3.

Let α be a non-negative number. A function $f \in \mathcal{A}$ is called *strongly starlike of order* α if $|\arg(zf'(z)/f(z))| \le \alpha\pi/2$ for $z \in \mathbb{D}$. (Here and hereafter, for a zero-free analytic function p on $\mathbb{D}$ with $p(0) = 1$, $\arg p(z)$ is thought of a single-valued harmonic function in $\mathbb{D}$ with normalization $\arg p(0) = 0$.) We denote by $\mathcal{SS}(\alpha)$ the set of strongly starlike functions in $\mathcal{A}$ of order α. Note that $\mathcal{SS}(\alpha) \subset \mathcal{SS}(1) = \mathcal{S}^*$ for $0 \le \alpha \le 1$. As is well known, for $\alpha \in (0,1)$, each function $f \in \mathcal{SS}(\alpha)$ is bounded (see [7]) and has a $K(\alpha)$-quasiconformal extension to $\mathbb{C}$, where $K(\alpha) = (1 + \sin(\alpha\pi/2))/(1 - \sin(\alpha\pi/2))$ (see [11]). In particular, $\mathcal{SS}(\alpha) \subset \mathcal{T}_0$ for $\alpha \in (0,1)$. For further properties of strongly starlike functions, see, in addition, [26] and [27].

We will next consider the following problem.

Problem 1.4. *Find a sufficient condition on $f \in \mathcal{A}$ under which $I_\alpha[f] \in \mathcal{SS}(\alpha)$ for all $\alpha \in [0,1]$.*

Note that $f \in \mathcal{S}^* \cap \overline{\mathcal{T}_0}$ is a necessary condition for the above property. In particular, a function f in $\mathcal{S}^* \cap \mathcal{T}_\zeta$ (such as $f = L_\zeta$) for some $\zeta \in \partial\mathbb{D}$ does not have the above property. On the other hand, the condition $I_\alpha[f] \in \mathcal{SS}(\alpha)$ for all $\alpha \in [0,1]$ clearly implies $[\mathrm{id}, f] \subset \mathcal{S}^*$.

One may suspect that the conclusion in this problem is too strong. However, a mild condition can guarantee the positive answer to the problem as we see in the next couple of results, which will be proved in Section 4 in a slightly more general form.

Theorem 1.5. *Let $f \in \mathcal{A}$. Suppose that $\operatorname{Re} f' > 0$ in $\mathbb{D}$ and that $\arg f'(tz)$ lies between 0 and $\arg f'(z)$ for each $t \in [0,1]$ and $z \in \mathbb{D}$. Then $I_\alpha[f] \in \mathcal{SS}[\alpha]$ for $\alpha \in [0,2]$. In particular, $f \in \mathcal{S}^*$.*

As is well known, J. W. Alexander [1] showed that if $f \in \mathcal{A}$ satisfies $\operatorname{Re} f' > 0$ in $\mathbb{D}$ then $f \in \mathcal{S}$. However, without any additional condition, $\operatorname{Re} f' > 0$ does not imply $f \in \mathcal{S}^*$. The authors learned from J. Stankiewicz that this fact was first pointed out by J. Krzyż [18]. For convenience of the reader, we will give a simplified example as well as related results in Section 4.

We give a different kind of sufficient condition for Problem 1.4:

Theorem 1.6. *Suppose that $f \in \mathcal{A}$ satisfies the condition*

$$\left| \operatorname{Im} \frac{zf''(z)}{f'(z)} \right| \le \frac{\pi^2}{8G}, \quad z \in \mathbb{D},$$

where G is Catalan's constant. Then $I_\alpha[f] \in \mathcal{SS}[\alpha]$ for each $\alpha \in [0,1]$. In particular, $f \in \mathcal{S}^*$.

The proof of the theorem will be given also in Section 4. We recall that Catalan's constant G is given by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594\ldots. \tag{1.2}$$

For interesting formulae involving Catalan's constant, see [6, Chapter 9]. Note also that $\pi^2/8G \approx 1.346885252$.

2. Proof of Theorems 1.1 and 1.2

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $T_{J_\alpha[f]} = \alpha T_{J[f]}$, it is enough to show the inequality $\|J[f]\| \le 4$ for $f \in \mathcal{S}$. For a function f in $\mathcal{S}$, the inequality due to Grunsky

$$\left| \log \frac{zf'(z)}{f(z)} \right| \le \log \frac{1+|z|}{1-|z|}$$

holds (see [10, p. 126]). Set $w = \log(zf'(z)/f(z))$. Then we compute

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = |e^w - 1| \le \sum_{k=1}^{\infty} \frac{|w|^k}{k!} = e^{|w|} - 1 \le \frac{1+|z|}{1-|z|} - 1 = \frac{2|z|}{1-|z|}.$$

Therefore, we come to the estimate

$$(1-|z|^2)|T_{J[f]}(z)| = (1-|z|^2)\left|\frac{f'(z)}{f(z)} - \frac{1}{z}\right| \le (1-|z|^2)\,\frac{2}{1-|z|} = 2(1+|z|).$$

Hence, $\|J[f]\| \le 4$. The sharpness of the bound can be seen by the Koebe function. □

The idea of using the Grunsky inequality was suggested by Vladimir Gutlyanskiĭ to one of the authors. They would like to express their sincere thanks to him.

As a consequence of the inequality $\|J[f]\| \le 4$ for $f \in \mathcal{S}$, it follows that $J[f]$ is uniformly locally univalent in $\mathbb{D}$. Note that the same is no longer true for functions f in $\mathcal{B}$ because $f(z)/z$ may have zeros.

Combining Theorem 1.1 with Theorem A, we obtain the assertion that $J_\alpha(\mathcal{S}) \subset \mathcal{S}$ for each α with $|\alpha| \le 1/4$. Note that Y. J. Kim and Merkes [16] first showed this relation by using the weaker inequality $(1-|z|^2)|zT_{J[f]}(z)| \le 4$ for $f \in \mathcal{S}$.

We next show Theorem 1.2.

Proof of Theorem 1.2. The part $\{|\alpha| \le 1/2\} \subset M$ was proved by Merkes [19]. On the other hand, the inclusion relation $[0, 3/2] \subset M$ is due to M. Nunokawa [22]. We show that any other point does not belong to M. By definition, the set M can be described by

$$M = \bigcap_{f \in \mathcal{K}} U(f),$$

where $U(f)$ is the set given by (1.1). First, we consider the function $f_1(z) = \log(1+z)$ and set $f_\alpha = I_\alpha[f_1]$ for $\alpha \in \mathbb{C}$. As we noted in Introduction, Royster [25] showed that $U(f_1) = \{|\alpha| \le 1 \text{ or } |\alpha - 2| \le 1\}$. We observe here the simple relation $U(f_\alpha) = \{\beta : \alpha\beta \in U(f_1)\}$ for $\alpha \in \mathbb{C}$. By the relation $1 + zf_\alpha''/f_\alpha' = 1 - \alpha z/(1+z)$ and by the fact that the image of $\mathbb{D}$ under $z/(1+z)$ is the half-plane $\mathrm{Re}\, w < 1/2$, we see that $\{\alpha \in \mathbb{C} : f_\alpha \in \mathcal{K}\} = [0, 2]$. Since $f_2 \in \mathcal{K}$, we get first $M \subset U(f_2) = \{|\alpha| \le 1/2 \text{ or } |\alpha - 1| \le 1/2\}$. This is the reason why Merkes came to the aforementioned conjecture. In particular, we obtain $M \subset \{|\alpha| \le 3/2\}$. Secondly, we take any number r_0 from $(1/2, 3/2]$. Then $f_{1/r} \in \mathcal{K}$, and thus, $M \subset U(f_{1/r})$ for each $r \in (1/2, r_0)$. In particular,

$$\{\alpha \in M : |\alpha| = r_0\} \subset \bigcap_{1/2 < r < r_0} \{\alpha \in U(f_{1/r}) : |\alpha| = r_0\} = \{r_0\}.$$

It is now shown that $\{\alpha \in M : |\alpha| > 1/2\} \subset (1/2, 3/2]$. □

3. The class of starlike functions

In this section, we consider the familiar class $\mathcal{S}^*$ of starlike functions. First, we show the next result by giving an example of two starlike functions f and g so that $h = (1/2)\star(f \oplus g)$ is not starlike.

Theorem 3.1. *The class $\mathcal{S}^*$ of starlike functions is not convex concerning the Hornich operation.*

We remark on the subtlety of this fact. Since $\mathcal{S}^* \subset \mathcal{C}$ and $\mathcal{C}$ is convex [17], we can see that the segment $[f, g]$ is contained entirely in $\mathcal{C}$ for $f, g \in \mathcal{S}^*$. Therefore, when we try to construct such an example as above, we cannot choose f and g so that the midpoint h is not univalent.

Proof. Putting $\gamma = e^{\pi i/4}$, we now define the functions f and g in $\mathcal{S}^*$ by $f'(z) = (1 + \gamma z)/(1 - \gamma z)^3$ and $g'(z) = (1 + \bar{\gamma} z)/(1 - \bar{\gamma} z)^3$. Note that f and g both are rotations of the Koebe function $z/(1 - z)^2$, and therefore, starlike functions. Then the midpoint h of f and g can be expressed in the form

$$h(z) = \int_0^z \frac{(1 + \gamma w)^{1/2}(1 + \bar{\gamma} w)^{1/2}}{(1 - \gamma w)^{3/2}(1 - \bar{\gamma} w)^{3/2}} dw = \int_0^z \sqrt{\frac{1 + \sqrt{2}w + w^2}{(1 - \sqrt{2}w + w^2)^3}} dw,$$

which is a variant of the Schwarz-Christoffel transformation. Indeed, on the boundary of the unit disk, we can write

$$h(e^{i\theta}) = \begin{cases} h(1) + \dfrac{i}{2} \displaystyle\int_0^\theta \sqrt{\frac{\cos t + 1/\sqrt{2}}{(\cos t - 1/\sqrt{2})^3}} dt, & |\theta| < \pi/4, \\ h(-1) - \dfrac{i}{2} \displaystyle\int_0^{\theta - \pi} \sqrt{\frac{\cos t - 1/\sqrt{2}}{(\cos t + 1/\sqrt{2})^3}} dt, & |\theta - \pi| < 3\pi/4. \end{cases}$$

Therefore, we see that $h(\mathbb{D}) = \{z \in \mathbb{C} : \operatorname{Re} z < h(1)\} \setminus E$, where E is the closed half parallel strip given by $\{z \in \mathbb{C} : \operatorname{Re} z \le h(-1) \text{ and } |\operatorname{Im} z| \le a\}$ and a is the positive number defined by

$$a = \frac{1}{2} \int_0^{\pi/4} \sqrt{\frac{\cos t - 1/\sqrt{2}}{(\cos t + 1/\sqrt{2})^3}} dt.$$

In particular, the image $h(\mathbb{D})$ is not starlike with respect to the origin. □

We next consider Problem 1.3. So far, we have no complete solution to it. We have yet the following sufficient condition for a function $f \in \mathcal{S}^*$ to satisfy $[\mathrm{id}, f] \subset \mathcal{S}^*$.

Theorem 3.2. *Let $f \in \mathcal{A}$. Suppose that the function $zf''(z)/f'(z)$ takes no values in the set $E_0 = \{x + yi : x \le -1$ and $|y| \ge -\sqrt{3}x\}$. Then $f_\alpha = I_\alpha[f] \in \mathcal{S}^*$ for $0 \le \alpha \le 1$.*

This result can be obtained as an immediate consequence of the following special case of the Open Door Lemma due to P. Mocanu [21]. Here, E_1 denotes the closed subset $\{-1+yi : |y| \geq \sqrt{3}\}$ of $\mathbb{C}$ consisting of two rays.

Lemma 3.3 (Open Door Lemma). *Let f be a function in $\mathcal{A}$. If $zf''(z)/f'(z) \in \mathbb{C} \setminus E_1$ for every $z \in \mathbb{D}$, then f is starlike.*

Proof of Theorem 3.2. For $\alpha \in [0,1]$, $zf_\alpha''(z)/f_\alpha'(z) = \alpha zf''(z)/f'(z) \in \{\alpha w : w \in E_0\} \subset \mathbb{C} \setminus E_1$. Now Lemma 3.3 yields that $f_\alpha \in \mathcal{S}^*$. □

Note that, for the Koebe function $K(z) = z/(1-z)^2$, the function $zK''(z)/K'(z) = 2z(2+z)/(1-z^2)$ is known to map the unit disk conformally onto the slit domain $\mathbb{C} \setminus E_1$. Lemma 3.3 means exactly that if $zf''(z)/f'(z)$ is subordinate to $zK''(z)/K'(z)$ then $zf'(z)/f(z)$ is subordinate to $zK'(z)/K(z) = (1+z)/(1-z)$ for a function $f \in \mathcal{A}$. We also see that the Koebe function does not satisfy the hypothesis of Theorem 3.2 because the set E_0 is larger than E_1. Though Theorem 3.2 does not imply $[\mathrm{id}, K] \subset \mathcal{S}^*$, this claim itself can be proved directly (see Proposition 4.5 in the next section). Since the Koebe function is extremal in various aspects, the validity of the statement $[\mathrm{id}, K] \subset \mathcal{S}^*$ may be thought as supporting evidence for the affirmative answer to Problem 1.3. For sufficient conditions of different types for $f \in \mathcal{S}^*$ to satisfy $[\mathrm{id}, f] \subset \mathcal{S}^*$, see also results concerning Problem 1.4.

4. Strongly starlike functions

In this section, we concentrate on Problem 1.4. We first prove Theorem 1.5 by showing a slightly more general result. For $\theta_0, \theta_1 \in \mathbb{R}$, we set

$$\Gamma[\theta_0, \theta_1] = \{z \in \mathbb{C} : z \neq 0, \arg z = (1-t)\theta_0 + t\theta_1 \text{ for some } t \in [0,1]\}$$

and

$$\Gamma(\theta_0, \theta_1) = \{z \in \mathbb{C} : z \neq 0, \arg z = (1-t)\theta_0 + t\theta_1 \text{ for some } t \in (0,1)\}.$$

It is important in the sequel to note that the set $\Gamma[\theta_0, \theta_1]$ is convex if $|\theta_1 - \theta_0| < \pi$ and that $\Gamma(\theta_0, \theta_1)$ is convex if $|\theta_1 - \theta_0| \leq \pi$. Then one can prove the following result, from which Theorem 1.5 follows.

Theorem 4.1. *Let β be a positive constant and $f \in \mathcal{A}$. If $|\arg f'(z)| < \beta\pi/2$ and if $f'(tz) \in \Gamma[0, \arg f'(z)]$ for each $t \in [0,1]$ and $z \in \mathbb{D}$, then the function $f_\alpha = I_\alpha[f]$ is strongly starlike of order $\alpha\beta$ provided that $\alpha\beta \leq 2$.*

Proof. Let $\alpha \leq 2/\beta$. By assumption, $f_\alpha'(tz) = (f'(tz))^\alpha \in \Gamma[0, \alpha \arg f'(z)]$ for $t \in [0,1]$ and $z \in \mathbb{D}$. Therefore, the average

$$\frac{f_\alpha(z)}{z} = \int_0^1 f_\alpha'(tz)dt \tag{4.1}$$

of $f'_\alpha(tz)$ belongs to the convex set $\Gamma[0, \alpha \arg f'(z)]$ for each $z \in \mathbb{D}$. Hence, $zf'_\alpha(z)/f_\alpha(z) \in \Gamma[0, \alpha \arg f'(z)] \subset \Gamma[-\alpha\beta\pi/2, \alpha\beta\pi/2]$. Now the conclusion follows. □

Without the additional assumption about $f'(tz)$ in Theorem 4.1, we would only have the conclusion that $f_\alpha(z)/z \in \Gamma[-\alpha\beta\pi/2, \alpha\beta\pi/2]$ for $\alpha \in [0, 1/\beta]$ merely from the above argument. Therefore, we still have the inequality $|\arg(zf'_\alpha(z)/f_\alpha(z))| \le \alpha\beta\pi$ for $\alpha \in [0, 1/\beta]$. We record it as a proposition for a future reference.

Proposition 4.2. *Suppose that $f \in \mathcal{A}$ satisfies $|\arg f'(z)| \le \beta\pi/2$ in $\mathbb{D}$. Then $f_\alpha \in \mathcal{SS}(2\alpha\beta)$ for $\alpha \in [0, 1/\beta]$. In particular, if f' has positive real part, $|\arg(zf'_\alpha(z)/f_\alpha(z))| \le \alpha\pi$ for $z \in \mathbb{D}$ and $\alpha \in [0, 1]$.*

Example 4.3. We consider the case when $f(z) = -2\log(1-z) - z$. This function clearly satisfies the hypothesis in Theorem 1.5. Therefore, the function F_α defined by

$$F_\alpha(z) = I_\alpha[f](z) = \int_0^z \left(\frac{1+w}{1-w}\right)^\alpha dw \tag{4.2}$$

is strongly starlike of order α for $\alpha \in [0, 2]$, and hence, univalent in $\mathbb{D}$ for $\alpha \in [0, 1]$. On the other hand, F_α is not univalent in $\mathbb{D}$ for $\alpha > 1$, see [15].

We next show the following assertion, from which Theorem 1.6 follows as a corollary.

Theorem 4.4. *Suppose that a function $f \in \mathcal{A}$ satisfies the inequality*

$$\left|\mathrm{Im}\,\frac{zf''(z)}{f'(z)}\right| \le \frac{\pi\beta}{2}$$

in $|z| < 1$, where β is a positive constant Then $f_\alpha = I_\alpha[f] \in \mathcal{SS}(4G\alpha\beta/\pi)$ holds as long as $4G\alpha\beta \le \pi$, where G is Catalan's constant given in (1.2).

Proof. Set $v(z) = \mathrm{Im}\,(zf''(z)/f'(z))$. Then v is a bounded harmonic function in $\mathbb{D}$ with $|v| \le \pi\beta/2$. Since $v(0) = 0$, the harmonic Schwarz lemma (cf. [3, Chapter 6]) yields the inequality

$$|v(z)| \le 2\beta \arctan|z|, \quad z \in \mathbb{D}. \tag{4.3}$$

Next we observe the formula

$$\frac{d}{ds}\log f'(sz) = \frac{zf''(sz)}{f'(sz)}$$

for $z \in \mathbb{D}$ and positive parameter $s \le 1$. Taking the imaginary part of the both sides and integrating it in s over the interval $[t, 1]$, we obtain

$$\arg\frac{f'(z)}{f'(tz)} = \int_t^1 \mathrm{Im}\left(\frac{zf''(sz)}{f'(sz)}\right) ds = \int_t^1 \frac{v(sz)}{s} ds.$$

By (4.3), we have

$$\left|\arg\frac{f'(z)}{f'(tz)}\right| \le 2\beta\int_t^1 \frac{\arctan(s|z|)}{s}ds$$
$$< 2\beta\int_0^1 \frac{\arctan s}{s}ds$$

for each $t \in [0,1]$. Since $\arctan x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$, we have the well-known relation

$$\int_0^1 \frac{\arctan x}{x}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = G.$$

We now conclude that the value $f'(z)/f'(tz)$ belongs to the set $\Gamma(-2G\beta, 2G\beta)$. This implies that $(f'(tz)/f'(z))^\alpha$ lies in $\Gamma(-2G\alpha\beta, 2G\alpha\beta)$. Therefore, the average

$$\frac{f_\alpha(z)}{zf_\alpha'(z)} = \int_0^1\left(\frac{f'(tz)}{f'(z)}\right)^\alpha dt$$

belongs to the convex set $\Gamma(-2G\alpha\beta, 2G\alpha\beta)$ as long as $2G\alpha\beta \le \pi/2$. This means that $f_\alpha = I_\alpha[f] \in \mathcal{SS}(4G\alpha\beta/\pi)$ when $4G\alpha\beta \le \pi$. □

We are now ready to show the fact that $[\mathrm{id}, K] \subset \mathcal{S}^*$ by giving a more refined relation.

Proposition 4.5. *Let* $K(z) = z/(1-z)^2$ *be the Koebe function. Then* $K_\alpha = I_\alpha[K] \in \mathcal{SS}(\min\{1, 3\alpha\})$ *for* $\alpha \in [0,1]$.

Proof. We first note that $\arg K'(tz) \in \Gamma[0, \arg K'(z)]$ for $t \in [0,1]$. Indeed, when $\operatorname{Im} z \ge 0$, we have $0 \le \arg((1+tz)/(1-tz)) \le \arg((1+z)/(1-z))$ and $0 \le \arg(1/(1-tz)) \le \arg(1/(1-z))$. Since $\arg K'(z) = \arg((1+z)/(1-z)) + 2\arg(1/(1-z))$, the required claim follows when $\operatorname{Im} z \ge 0$. When $\operatorname{Im} z \le 0$, we show it in the same way as above.

Since $|\arg K'(z)| < 3\pi/2$ in $\mathbb{D}$, we conclude that $K_\alpha \in \mathcal{SS}(3\alpha)$ for $0 \le \alpha \le 1/3$ from Theorem 4.1.

Next, we consider the case when $1/3 < \alpha < 1$. Then the desired conclusion is that $p_\alpha(z) = zK_\alpha'(z)/K_\alpha(z)$ has positive real part in $\mathbb{D}$. First note that p_α extends to a holomorphic function in some open neighbourhood of $\overline{\mathbb{D}} \setminus \{1, -1\}$. We now examine the behaviour of $p_\alpha(z)$ around $z = 1$. Since $(1+z)^\alpha = 2^\alpha(1-(1-z)/2)^\alpha = 2^\alpha[1 - \alpha(1-z)/2 + \alpha(\alpha-1)(1-z)^2/8 + \cdots]$ as $z \to 1$, one obtains the asymptotic expansion of K_α' near to $z = 1$ in $\mathbb{D}$:

$$K_\alpha'(z) = 2^\alpha(1-z)^{-3\alpha} - \alpha 2^{\alpha-1}(1-z)^{1-3\alpha} + \alpha(\alpha-1)2^{\alpha-3}(1-z)^{2-3\alpha} + O(1).$$

Integrating the above gives us the expansion

$$K_\alpha(z) = \frac{2^\alpha}{3\alpha-1}(1-z)^{1-3\alpha} - \frac{\alpha 2^{\alpha-1}}{3\alpha-2}(1-z)^{2-3\alpha} + O(1)$$

as $z \to 1$ in $\mathbb{D}$, where the second term should be eliminated when $2 - 3\alpha \geq 0$. Therefore,

$$p_\alpha(z) = (3\alpha - 1)\frac{z}{1-z}\Big(1 + O(|1-z|)\Big) = c\,\frac{1+z}{1-z} + O(1)$$

as $z \to 1$ in $\mathbb{D}$, where $c = (3\alpha - 1)/2 > 0$.

On the other hand, $p_\alpha(z)$ is bounded around $z = -1$. Therefore, the real part of $p_\alpha(z)$ can be written in the form $h(z) + c(1 - |z|^2)/|1-z|^2$, where h is a bounded harmonic function in $\mathbb{D}$. Now it remains to show that $h > 0$ in $\mathbb{D}$. To this end, it suffices to see that $P_\alpha(\theta) := 1/p_\alpha(-e^{-i\theta})$ has non-negative real part for each $\theta \in (0, \pi)$. We have used here the symmetric property $P_\alpha(-\theta) = P_\alpha(\theta)$ of P_α.

Since

$$K_\alpha'(-e^{-i\theta}) = \frac{(1-e^{-i\theta})^\alpha}{(1+e^{-i\theta})^{3\alpha}} = 2^{-2\alpha}e^{i\alpha(\theta+\pi/2)}\sin^\alpha(\theta/2)\cos^{-3\alpha}(\theta/2), \quad \theta \in (0,\pi),$$

we compute

$$\begin{aligned} K_\alpha(-e^{-i\theta}) &= \left(\int_0^{-1} + \int_{-1}^{-e^{-i\theta}}\right)\frac{(1+w)^\alpha}{(1-w)^{3\alpha}}dw \\ &= -\int_0^1 \frac{(1-x)^\alpha}{(1+x)^{3\alpha}}dx + ie^{i\alpha\pi/2}2^{-2\alpha}\int_0^\theta e^{i(\alpha-1)t}\sin^\alpha(t/2)\cos^{-3\alpha}(t/2)dt \end{aligned}$$

for $\theta \in (0, \pi)$. Therefore, we obtain

$$\frac{\cos^{3\alpha}(\theta/2)}{\sin^\alpha(\theta/2)}\,P_\alpha(\theta) = 2^{2\alpha}e^{i\alpha(\theta+\pi/2)-i\theta}C_\alpha + i\int_0^\theta e^{i(1-\alpha)(\theta-t)}\sin^\alpha(t/2)\cos^{-3\alpha}(t/2)dt,$$

where $C_\alpha = \int_0^1 (1-x)^\alpha(1+x)^{-3\alpha}dx > 0$, and thus, the real part of $P_\alpha(\theta)$ has the same signature as

$$2^{2\alpha}C_\alpha\cos\big\{(1-\alpha)\theta - \alpha\pi/2\big\} + \int_0^\theta \sin(1-\alpha)(\theta - t)\sin^\alpha(t/2)\cos^{-3\alpha}(t/2)dt.$$

It is clear that the integrand of the second term is nonnegative. On the other hand, since $(1-\alpha)\theta - \alpha\pi/2 \leq (1-\alpha)\pi - \alpha\pi/2 = \pi/2 + (1-3\alpha)\pi/2 \leq \pi/2$, the first term is also nonnegative. The proof has been completed. □

Remark. It is easy to see that $K_\alpha = I_\alpha[K]$ is unbounded when $\alpha \geq 1/3$. Recalling the fact that strongly starlike functions of order < 1 are bounded, we see that $f = K_\alpha$ does not satisfy the conclusion in Problem 1.4 for $\alpha > 1/3$. On the other hand, the above proposition asserts that the function $f = K_{1/3}$ does!

As we noted in Introduction, the condition $\operatorname{Re} f' > 0$ does not imply starlikeness of f. J. Krzyż [18] constructed a counterexample in a clever but a little complicated way. Since the reference [18] may be difficult to access for the reader, we give a somewhat simplified example for convenience.

Example 4.6. Let Ω be the Jordan domain $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \mathrm{Im}\, z > -1\}$ and $p : \mathbb{D} \to \Omega$ be the conformal homeomorphism determined by $p(0) = 1$ and $p(1) = 0$. We can give an explicit form of p by $p(z) = (1+i)\sqrt{(\alpha - z)/(\alpha + z)} - i$, where $\alpha = (3+4i)/5$, although it does not matter below. Then $f(z) = \int_0^z p(\zeta)d\zeta$ is such a function, namely, $\mathrm{Re}\, f' > 0$ while $zf'(z)/f(z)$ takes a value with negative real part for some $z \in \mathbb{D}$. Note that p analytically extends to the point $z = 1$. We now use the following claim. We can see it directly but we prefer to give a geometric proof below in order to clarify what conditions are essential.

Claim. $\mathrm{Im}\, p(x) > 0$ *for* $0 < x < 1$.

Set $q(z) = f(z)/z$. Since $q(1)$ can be written as the average $\int_0^1 p(x)dx$ of $p(x)$ over the interval $[0, 1]$, the above claim implies $\mathrm{Im}\, q(1) > 0$. By continuity, $\mathrm{Im}\, q(e^{i\theta}) > 0$ holds for sufficiently small $\theta > 0$. On the other hand, $p(e^{i\theta})$ takes the form $i\,P(\theta)$, where $P(\theta) < 0$ for sufficiently small $\theta > 0$. Therefore, $\mathrm{Re}\,(zf'(z)/f(z)) = \mathrm{Re}\,(p(z)/q(z)) = P(\theta)|q(e^{i\theta})|^{-2}\mathrm{Im}\, q(e^{i\theta}) < 0$ for $z = e^{i\theta}$ with $\theta > 0$ small enough.

Proof of Claim. Note that the image of the segment $(0, 1)$ under p is the hyperbolic geodesic in Ω joining 1 and 0. Therefore, the claim follows from the inequality $\rho_\Omega(\bar{z}) < \rho_\Omega(z)$ for $z \in \Omega^- = \{z \in \Omega : \mathrm{Im}\, z < 0\}$, where ρ_Ω denotes the hyperbolic (or Poincaré) density of the domain Ω. The last inequality is implied by the reflection principle of hyperbolic metric due to D. Minda [20, Theorem 3]. □

We end this section with a small remark on Problem 1.4. If $I_\alpha[f] \in \mathcal{SS}(\alpha)$ for some $\alpha < 1$, then the function $I_\alpha[f]$ is necessarily univalent and bounded in $\mathbb{D}$. This conclusion itself can be deduced only from the assumption $\mathrm{Re}\, f' > 0$. Indeed, we have a stronger result as in the following.

Theorem 4.7. *Suppose that $e^{i\beta} f'$ has positive real part in $\mathbb{D}$ for some $f \in \mathcal{A}$ and a real constant β. Then, $I_\alpha[f] \in \mathcal{S}$ whenever $|\alpha| \le 1/2$ or $\alpha \in [0, 1]$ and $I_\alpha[f]$ is bounded whenever $|\alpha| < 1$. The latter bound is sharp.*

Theorem 4.7 follows immediately from the next more general theorem up to the assertion that $I_\alpha[f] \in \mathcal{S}$ for $\alpha \in [0, 1]$, which is, however, a direct consequence of the convexity of $\mathcal{C}$ because $f \in \mathcal{C}$ in this case.

A function p analytic in $\mathbb{D}$ with $p(0) = 1$ is called Gelfer if $p(z)+p(w) \neq 0$ for every pair of points z, $w \in \mathbb{D}$. In particular, if $e^{i\beta}p$ has positive real part for some real constant β, then p is Gelfer. We refer the reader to [30] for interesting properties of Gelfer functions.

Theorem 4.8. *Suppose that the derivative of a function $f \in \mathcal{A}$ is Gelfer. Then $I_\alpha[f] \in \mathcal{S}$ whenever $|\alpha| \le 1/2$ and $I_\alpha[f]$ is bounded whenever $|\alpha| < 1$. The latter bound is sharp.*

Proof. Gelfer's theorem implies that $\|f\| = \sup(1-|z|^2)|f''(z)/f'(z)| \le 2$ (see [14] for a simple proof). Therefore, $\|I_\alpha[f]\| = |\alpha|\|f\| \le 1$ for $|\alpha| \le 1/2$. Becker's theorem (Theorem A) now yields the univalence of $I_\alpha[f]$. The boundedness follows from the inequality $\|I_\alpha[f]\| < 2$ for $|\alpha| < 1$. We see that this bound is sharp by considering the function $F_\alpha = I_\alpha[f]$ defined in (4.2). □

Since $J_\alpha = I_\alpha \circ J$, the above theorem is translated into the following equivalent form through the Alexander transformation J.

Corollary 4.9. *Suppose that the function $f(z)/z$ is Gelfer for some $f \in \mathcal{A}$. Then $J_\alpha[f] \in \mathcal{S}$ whenever $|\alpha| \le 1/2$ and $J_\alpha[f]$ is bounded whenever $|\alpha| < 1$. The latter bound is sharp.*

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References

1. J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12–22.
2. K. Astala and F. W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Analyse Math. **46** (1986), 16–57.
3. S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory, Second Edition*, Springer, 2001.
4. J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.
5. J. Becker and Ch. Pommerenke, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354** (1984), 74–94.
6. B. C. Berndt, *Ramanujan's Notebooks, Part 1*, Springer-Verlag, 1985.
7. D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2) **1** (1969), 431–443.
8. D. M. Campbell, J. A. Cima, and J. A. Pfaltzgraff, *Linear spaces and linear-invariant families of locally univalent analytic functions*, Manuscripta Math. **4** (1971), 1–30.
9. J. A. Cima and J. A. Pfaltzgraff, *A Banach space of locally univalent functions*, Michigan Math. J. **17** (1970), 321–334.
10. P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983.
11. M. Fait, J. G. Krzyż, and J. Zygmunt, *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. **51** (1976), 279–285.
12. A. W. Goodman, *Univalent Functions, 2 vols.*, Mariner Publishing Co. Inc., 1983.
13. H. Hornich, *Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen*, Monatsh. Math. **73** (1969), 36–45.
14. Y. C. Kim and T. Sugawa, *Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions*, Preprint.

15. ______, *Growth and coefficient estimates for uniformly locally univalent functions on the unit disk*, Rocky Mountain J. Math. **32** (2002), 179–200.
16. Y. J. Kim and E. P. Merkes, *On an integral of powers of a spirallike function*, Kyungpook Math. J. **12** (1972), 249–253.
17. ______, *On certain convex sets in the space of locally schlicht functions*, Trans. Amer. Math. Soc. **196** (1974), 217–224.
18. J. Krzyż, *A counterexample concerning univalent functions*, Folia Soc. Sci. Lublin. **2** (1962), 57–58.
19. E. P. Merkes, *Univalence of an integral transform*, Contemporary Math. **38** (1985), 113–119.
20. D. Minda, *A reflection principle for the hyperbolic metric and applications to geometric function theory*, Complex Variables Theory Appl. **8** (1987), 129–144.
21. P. T. Mocanu, *Some integral operators and starlike functions*, Rev. Roumaine Math. Pures Appl. **31** (1986), 231–235.
22. M. Nunokawa, *On the univalence of a certain integral*, Trans. Amer. Math. Soc. **146** (1969), 439–446.
23. J. A. Pfaltzgraff, *Univalence of the integral of* $f'(z)^\lambda$, Bull. London Math. Soc. **7** (1975), 254–256.
24. Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen* I, II, Math. Ann. **155** (1964), 108–154, ibid. **156**(1964), 226–262.
25. W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. **12** (1965), 385–387.
26. J. Stankiewicz, *Quelques problèmes extrémaux dans les classes des fonctions* α*-angulairement étoilées*, Ann. Univ. Mariae Curie-Skłodowska, Sectio A **20** (1966), 59–75.
27. T. Sugawa, *A self-duality of strong starlikeness*, Preprint.
28. S. Yamashita, *Banach spaces of locally schlicht functions with the Hornich operations*, Manuscripta Math. **16** (1975), 261–275.
29. ______, *Almost locally univalent functions*, Monatsh. Math. **81** (1976), 235–240.
30. ______, *Gelfer functions, integral means, bounded mean oscillation, and univalency*, Trans. Amer. Math. Soc. **321** (1990), 245–259.
31. I. V. Zhuravlev, *Model of the universal Teichmüller space*, Siberian Math. J. **27** (1986), 691–697.

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