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Varieties with non-linear Gauss fibers

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Abstract For any given projective variety Y, we construct a projective variety $X \subset \mathbf{P}^N$ whose general fiber of the Gauss map with reduced scheme structure is isomorphic to Y when the characteristic > 0.

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1 Introduction

Let $X \subset \mathbf{P}^N$ be a projective variety over an algebraically closed field K. The Gauss map γ from X to the Grassmannian $\mathbf{G}(\dim X, N)$ is the rational map, defined by $\gamma(p) = \mathbf{T}_p X$ for a smooth point $p \in X$, and $\mathbf{T}_p X \in \mathbf{G}(\dim X, N)$ is the projective tangent space.

If the characteristic of K is 0, then it is classically known that a general fiber of the Gauss map is a linear subspace of dimension $\dim X - \dim \gamma(X)$ (see, for example, [9]). If the characteristic of K is positive, it is no longer true. Wallace ([8, Section 7]) pointed out that there exists a plane curve which has infinitely many multiple tangents, or equivalently, whose Gauss map has separable degree > 1 onto its image (see also [5, I-3]). Kaji ([3, Example 4.1],[4]) and Rathmann ([7, Example 2.13]) gave smooth curves with infinitely many multiple tangents. Noma constructed smooth or normal projective varieties whose Gauss maps have separable degree > 1 onto its image ([6]). In these cases, Gauss fibers are finite number of points. In [1], an example of a surface whose Gauss fibers are smooth conics is found. In the author's best knowledge, this is the first example that Gauss fibers are *not* finite unions of linear subspaces. We are naturally led to the following question:

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Question 1 What kind of variety appears in the general fiber of the Gauss map?

Our answer in this paper is: “Any projective varieties can appear”. To be more precise, we will prove the following:

Theorem 1 *Let* $\mathrm{char}K > 0$. *For any positive integers* $k \geq 2$, $r \leq k$ *and* $N \geq k+r$ *and any projective variety* $Y \subset \mathbf{P}^k$ *of codimension* r, *there exists a closed variety* $X \subset \mathbf{P}^N$ *of dimension* k, *not contained in any hyperplane, such that for a general point* $p \in X$, Y *is the fiber of the Gauss map* $\gamma^{-1}(\gamma(p)) \subset \mathbf{T}_p X \cong \mathbf{P}^k$ *set-theoretically (up to choices of coordinates).*

Notation Unless otherwise stated, the base field K is an algebraically closed field of characteristic $p > 0$. $\mathbf{G}(k, N)$ is the Grassmannian of k-dimensional linear subspaces of $\mathbf{P}^N$. Varieties are integral algebraic schemes. Points mean closed points. $[v] \in \mathbf{P}^N$ denotes the point of $\mathbf{P}^N$ corresponding to the equivalence class of $v \in \mathbf{A}^{N+1} \setminus 0$. Given a linear subspace $V \subset \mathbf{A}^{N+1}$, $\mathbf{P}(V) \subset \mathbf{P}^N$ means the linear subspace of $\mathbf{P}^N$ corresponding to V.

2 Construction

Let $k \geq 2$, $r \leq k$, $N > k$ be positive integers and let $Y \subset \mathbf{P}^k$ be a closed variety of codimension r. Let $B \subset \mathbf{G}(k-r, N)$ be a closed variety of dimension r, $I_B = \{(x, E) \in \mathbf{P}^N \times B | x \in E\}$ and let $f : I_B \to \mathbf{P}^N$, $g : I_B \to B$ be the natural projections. Let $u_1, \dots, u_r$ be a local parameter system of B, and let $\rho_0, \dots, \rho_{k-r} : U \to \mathbf{A}^{N+1}$ be a system of morphisms on some open set $U \subset B$ such that $\mathbf{P}(\langle \rho_0(s), \dots, \rho_{k-r}(s) \rangle)$ is equal to the $(k-r)$-dimensional linear subspace given by s for all points $s \in U$. We assume that

(1) f is generically étale onto its image,
(2) $\dim \left\langle \left\{ \frac{\partial \rho_i}{\partial u_j} | i, j \right\} \right\rangle = r$ and $\tau_1, \dots, \tau_r$ form its base, and
(3) $\frac{\partial}{\partial u_j}(\frac{\partial \rho_l}{\partial u_i}) = 0$ for any i, j, l.

The conditions (1) and (2) imply that $\dim \langle \rho_0, \dots, \rho_{k-r}, \tau_1, \dots, \tau_r \rangle = k+1$. Let $\eta : U \times \mathbf{P}^k \to \mathbf{P}^N$;

$$(s) \times (Y_0 : \cdots : Y_k) \mapsto [Y_0 \rho_0(s) + \cdots + Y_{k-r} \rho_{k-r}(s) + Y_{k-r+1} \tau_1(s) + \cdots + Y_k \tau_r(s)],$$

and let X be the closure of $\eta(U \times Y)$. Then, X is the closed subvariety in $\mathbf{P}^N$ of dimension $\leq k$. Let $\tau := \eta|_{(U \times Y)} : U \times Y \to X$.

Let $\widehat{Y} \subset \mathbf{A}^{k+1}$ be the affine cone of $Y \subset \mathbf{P}^k$. By changing the coordinate system if necessary, we may assume that $Y_0 - y_0, \dots, Y_{k-r} - y_{k-r}$ are a local parameter system of $\widehat{Y}$ at a smooth point $(y_0, \dots, y_k) \in \widehat{Y}$.

Proposition 1 τ *is generically étale, and* $\mathbf{T}_{\tau(s,y)} X = \eta(s \times \mathbf{P}^k)$ *for a general point* $s \in B$ *and a general point* $y \in Y$.

Proof Let $\hat{\tau} : U \times \widehat{Y} \to \mathbf{A}^{N+1}$ be the affine lifting of τ. By the assumption (3) and easy computation, we have

$$\begin{aligned}
\frac{\partial \hat{\tau}}{\partial u_1} &= Y_0 \frac{\partial \rho_0}{\partial u_1} + \ldots + Y_{k-r} \frac{\partial \rho_{k-r}}{\partial u_1} \\
&\vdots \\
\frac{\partial \hat{\tau}}{\partial u_r} &= Y_0 \frac{\partial \rho_0}{\partial u_r} + \ldots + Y_{k-r} \frac{\partial \rho_{k-r}}{\partial u_r} \\
\frac{\partial \hat{\tau}}{\partial Y_0} &= \rho_0 + \frac{\partial Y_{k-r+1}}{\partial Y_0} \tau_1 + \ldots + \frac{\partial Y_k}{\partial Y_0} \tau_r \\
&\vdots \\
\frac{\partial \hat{\tau}}{\partial Y_{k-r}} &= \rho_{k-r} + \frac{\partial Y_{k-r+1}}{\partial Y_{k-r}} \tau_1 + \ldots + \frac{\partial Y_k}{\partial Y_{k-r}} \tau_r .
\end{aligned}$$

By the assumptions (1) and (2), for a general point $(s, y) \in B \times \widehat{Y}$,

$$\operatorname{Im} d_{(s,y)} \hat{\tau} = \langle \tau_1(s), \ldots, \tau_r(s), \rho_0(s), \ldots, \rho_{k-r}(s) \rangle .$$

This implies our assertion.

We have the following theorem:

Theorem 2 *Let $k \geq 2$, $r \leq k$, $N > k$ be positive integers and let $Y \subset \mathbf{P}^k$ be a closed variety of codimension r. We assume that there exists an r-dimensional closed variety $B \subset \mathbf{G}(k - r, N)$ with a system of rational maps satisfying the conditions (1), (2) and (3) as above, such that $\bigcup_{E \in B} E \subset \mathbf{P}^N$ is not contained in any hyperplane and $\mathbf{P}(\langle \rho_0(s), \ldots, \rho_{k-r}(s), \tau_1(s), \ldots, \tau_r(s) \rangle) = \mathbf{P}(\langle \rho_0(s'), \ldots, \rho_{k-r}(s'), \tau_1(s'), \ldots, \tau_r(s') \rangle)$ implies $s = s'$ for general points $s, s' \in B$.*

Then, there exists a closed variety $X \subset \mathbf{P}^N$ of dimension k, not contained in any hyperplane, such that for a general point $p \in X$, Y is the fiber of the Gauss map $\gamma^{-1}(\gamma(p)) \subset \mathbf{T}_p X \cong \mathbf{P}^k$ set-theoretically.

Proof We construct $X \subset \mathbf{P}^N$ as above consideration. Let τ be as above. By Proposition 1, $\tau(s \times Y)$ is contained in $\gamma^{-1}(\gamma(\tau(s, y)))$ for each general point $(s, y) \in B \times Y$. If the tangential space $\mathbf{P}(\langle \rho_0(s), \ldots, \rho_{k-r}(s), \tau_1(s), \ldots, \tau_r(s) \rangle)$ is uniquely determined from a general point of B then τ is generically one-to-one, because a point contained in two distinct Ys is a singular point of X. This implies that $\tau(s \times Y)$ coincides with the fiber $\gamma^{-1}(\gamma(\tau(s, y)))$ for a general point $(s, y) \in B \times Y$.

The nondegeneration of X follows from the nondegeneration of the tangent variety Tan Z of the ruled variety $Z = \bigcup_{E \in B} E$, because Tan Z is also the tangent variety of X.

Theorem 1 is given as the corollary of Theorem 2.

Proof (Proof of Theorem 1) If $N \geq k + r$ then we can take B as the closure of the image of the morphism $\mathbf{A}^r \to \mathbf{G}(k - r, N)$; $s \mapsto \mathbf{P}(\langle v, p_1, \ldots, p_{k-r} \rangle)$, where v is the morphism from $\mathbf{A}^r$ to $\mathbf{A}^{N+1}$ given by

$$v = (1, 0, \ldots, 0, u_1, \ldots, u_r, u_1^p, \ldots, u_r^p, u_r^{p^2}, \ldots, u_r^{p^{N-k-r+1}})$$

and $p_i \in \mathbf{A}^{N+1}$ is the point whose l-th coordinate is 0 for any $l \neq i$ and 1 for $l = i$. We have the result by Theorem 2.

Remark 1 The conditions (1) and (2) force the ruling of $B \subset \mathbf{G}(k - r, N)$ to be developable, i.e. tangent spaces of the ruled variety $\bigcup_{E \in B} E \subset \mathbf{P}^N$ are constant on each linear subspace $E \in B$ ([1]).

3 Examples

Example 1 Let char$K = 2$. We consider the hypersurface X in $\mathbf{P}^3$ given by $F = X^6 + Y^6 + Z^6 + YZ^4W + Y^2Z^2W^2 + Y^3W^3$. For a general point $(x : y : z : w) \in X$, the tangent plane given by $wY + yW = 0$ because the Gauss map is given by $(\partial F/\partial X : \partial F/\partial Y : \partial F/\partial Z : \partial F/\partial W) = (0 : Z^4W + Y^2W^3 : 0 : YZ^4 + Y^3W^2) = (0 : W : 0 : Y)$. The intersection of X and this plane is the plane curve $wY + yW = \sqrt{y^3}X^3 + \sqrt{y^3 + w^3}Y^3 + \sqrt{y^3}Z^3 + \sqrt{y^2w}YZ^2 + \sqrt{yw^2}Y^2Z = 0$. This is the fiber of the Gauss map at $\mathbf{T}_{(x:y:z:w)}X \in \mathbf{P}^{3*}$. This curve is smooth and of degree 3 (for a general point $(x : y : z : w) \in X$), hence this is an elliptic curve.

The above surface X is given by our method in Section 2: B is the closure of the image of the morphism $\mathbf{A}^1 \to \mathbf{G}(1, 3) \subset \mathbf{P}^5; u \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (1 : u : u^2 : 0 : 0 : 0)$ where p_{ij} are Plücker coordinates, and $Y \subset \mathbf{P}^2$ is the hypersurface given by $Y_0^3 + Y_1^3 + Y_2^3 = 0$ where Y_0, Y_1, Y_2 are coordinates on $\mathbf{P}^2$.

Example 2 Let char$K = 3$. Let B be the closure of the image of the morphism $\mathbf{A}^1 \to \mathbf{G}(1, 3) \subset \mathbf{P}^5; u \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) = (1 : u : u^3 : 0 : 0 : 0)$, and $Y \subset \mathbf{P}^2$ be the hypersurface given by $Y_0^4 + Y_1^4 + Y_2^4 = 0$. Then we have the surface X given by $X^{12}+Y^{12}+Z^{12}+2Y^2Z^9W+2Y^6Z^3W^3+Y^8W^4$. By easy computation, we can check that the general fibers of the Gauss map of this surface are smooth curves of genus 3.

Example 3 Now, we give an example of a 3-fold $X \subset \mathbf{P}^4$ whose general fibers of the Gauss map are twisted cubic curves.

Let char$K = 2$. Let $B \subset \mathbf{G}(1, 4)$ be the closure of the image of the morphism $\mathbf{A}^2 \to \mathbf{G}(1, 4), (u, v) \mapsto \mathbf{P}(\langle \rho_0, \rho_1 \rangle)$ where

$$\begin{aligned} \rho_0 &= (1\ 0\ u\ 0\ u^2) \\ \rho_1 &= (0\ 1\ 0\ v\ v^2), \end{aligned}$$

and let $Y \subset \mathbf{P}^3$ be a curve given by $Y_0Y_2 - Y_1^2, Y_1Y_3 - Y_2^2, Y_0Y_3 - Y_1Y_2$. Then our construction in Section 2 gives the hypersurface $X \subset \mathbf{P}^4$ whose defining polynomial is $X_0X_1^5 + X_1^6 + X_0^3X_1X_2^2 + X_0^4X_3^2 + X_0^4X_1X_4$ (by using Groebner Basis).

Example 4 We give an example of a hypersurface $X \subset \mathbf{P}^9$ whose general Gauss fibers are abelian surfaces.

Let char$K = 2$. Let $B \subset \mathbf{G}(1, 4)$ be the closure of the image of the morphism $\mathbf{A}^6 \to \mathbf{G}(2, 9), (u_1, u_2, u_3, u_4, u_5, u_6) \mapsto \mathbf{P}(\langle \rho_0, \rho_1, \rho_2 \rangle)$ where

$$\begin{aligned} \rho_0 &= (1\ 0\ 0\ u_1\ \ 0\ \ 0\ \ u_2\ \ 0\ \ 0\ \ u_1^3 + u_2^3) \\ \rho_1 &= (0\ 1\ 0\ \ 0\ \ u_3\ \ 0\ \ 0\ \ u_4\ \ 0\ \ u_3^3 + u_4^3) \\ \rho_2 &= (0\ 0\ 1\ \ 0\ \ 0\ \ u_5\ \ 0\ \ 0\ \ u_6\ u_5^3 + u_6^3), \end{aligned}$$

and let Y be the surface $E \times E \subset \mathbf{P}^2 \times \mathbf{P}^2$ where E is the elliptic curve in $\mathbf{P}^2$ given by $x_0^3 + x_1^3 + x_2^3 = 0$. We embed Y to $\mathbf{P}^8$ by Segre embedding. We have the hypersurface $X \subset \mathbf{P}^9$ as the closure of the image of the morphism $U \times Y \to \mathbf{P}^9$ (where U is an open subset of B), $(s) \times (x_0 : x_1 : x_2) \times (y_0 :$

$y_1 : y_2) \mapsto [x_0y_0\rho_0(s) + x_1y_1\rho_1(s) + x_2y_2\rho_2(s) + x_0y_1\frac{\partial\rho_0}{\partial u_1}(s) + x_0y_2\frac{\partial\rho_0}{\partial u_2}(s) + x_1y_0\frac{\partial\rho_1}{\partial u_3}(s) + x_1y_2\frac{\partial\rho_1}{\partial u_4}(s) + x_2y_0\frac{\partial\rho_2}{\partial u_5}(s) + x_2y_1\frac{\partial\rho_2}{\partial u_6}(s)]$.

Recently, varieties with non-constant Gauss fibers are found, which result will be published in [2].

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