Test for Parameter Change in ARIMA Models

Sangyeol Lee[1] Siyun Park[2] Koichi Maekawa[3] and Ken-ichi Kawai[4]

Abstract

In this paper we consider the problem of testing for parameter changes in ARIMA models based on the cusum test. The proposed test procedure is applicable to testing for the change from stationary models to nonstationary models, and vice versa. The idea is to transform the time series via differencing to make the whole time series as a combination of stationary subseries. For this task, we propose a graphical method to identify the right order of differencing. Then the cusum test statistic proposed by Lee et al. (2003) is constructed based the differenced time series. Simulation study and real data analysis are provided for illustration.

Key words : Test for parameter changes, cusum test, ARIMA model, graphical method, autocovariance function, and Brownian bridge.

1 Introduction

The problem of testing for parameter changes in time series models has been an important issue among statisticians and econometricians. There are a large number of articles as to the change point analysis in iid samples, linear models and time series models. See, for example, Brown, Durbin and Evans (1975), Wichern, Miller and Hsu (1976), Picard (1985), Inclán and Tiao (1994), Bai (1994), Csörgő and Horváth (1997), and Lee and Park (2001), and the papers cited therein. Recently, Lee, Ha, Na and Na (2003) proposed a cusum test aimed at testing for a parameter change in time series models. The cusum test not only deals with the classical mean and variance change problem, but also covers a more general parameter case, such as the coefficients in RCA and ARCH models. The cusum method turns out to perform adequately in a large class of time series models and to be useful for allocating the locations of changes (cf. Lee, Na and Na (2004), Lee and Lee (2004), and Lee and Na (2004), and Lee, Toktsu and Maekawa (2004)). However, despite its wide applicability, attention was only paid to stationary time series models. This motivates us

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to consider the change point problem for nonstationary models, particularly, the most well known ARIMA models.

In handling the problem in stationary ARMA models, it is natural to employ a test based on the ACF (autocovariance function) since the ACF characterizes ARMA models. For instance, one can use the cusum test introduced by Lee et al. (2003). However, if time series are nonstationary, their method is no longer applicable since it is based on the stationary assumption. Therefore, to apply the method to nonstationary ARIMA models, one has to transform the time series data to make a combination of stationary subseries. A simple way is to do differencing repeatedly until all subseries have stationary properties. In this case, however, one encounters the question as to determining the right order of the differencing to ensure the stationarity. Of course, if we deal with this task for ARIMA models with no changes, it is nothing but an ordinary model selection problem. However, in the presence of changes, it is not an easy task to design a suitable method in a formal manner. Thus, we propose a graphical method to determine the right order of differecing.

The basic idea is to examine the plot of the averaged partial sum of squares of observations. For example, if time series are stationary, the averaged partial sum converges to its second moment by a law of large numbers. Furthermore, if the time series are random walk, the partial sums exhibit a hyperbolic trend. Therefore, the partial sum at lag t divided by t^2 lies in a certain boundary. Similar reasoning is applicable to other ARIMA processes, and even to the time series with structural changes. Once the order is determined, one can conduct the cusum test based on the differenced time series immediately.

The organization of this paper is as follows. In Section 2, we present the cusum test for ACF. In Section 3, we explain the visual method to determine the differecing order. In Section 4, we perform a simulation study to examine whether the proposed method works properly or not. In Section 5, we apply our method to 3-month Euroyen interest rate data. Finally, we provide concluding remarks in Section 6.

2 Test for parameter change

Let $\{X_t\}$ be an ARIMA time series, and suppose that one wishes to test the following hypotheses:

$H_0 : X_t,\ t = 1, \ldots, n$, follow an ARIMA$(p, d, q)$ model vs.
$H_1 : X_t,\ t = 1, \ldots, l$, $1 \le l < n$, follow the ARIMA(p, d, q) model
and $X_t,\ t = l + 1, \ldots, n$, follow another ARIMA$(p', d', q')$ model.

If the orders d and d' are known, one can test H_0 vs. H_1 applying Lee et al.'s (2003) method to $(1-B)^D X_t$, where D denotes the maximum of d and d'. In the following we describe the test procedure.

Put $x_t{}' = (1-B)^d X_t$. Under H_0, we assume that

$$\phi(B)x_t{}' = \theta(B)\epsilon_t,$$

where ϵ_t are iid random variables with mean 0, variance σ_ϵ^2, $E|\epsilon_1|^{4\lambda} < \infty$ for some $\lambda > 1$, and $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and $\theta(B) = 1 + \theta_1(B) + \cdots + \theta_q B^q$. Set $x_t = (1-B)^D X_t$. For $|h| < n$, define

$$\hat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}(x_t - \bar{x}_n)(x_{t+|h|} - \bar{x}_n), \quad \bar{x}_n = \frac{1}{n}\sum_{t=1}^{n} x_t,$$

and let $\{h_n\}$ be a sequence of positive integers, such that as $n \to \infty$,

$$h_n \to \infty \text{ and } h_n = O(n^\beta) \text{ for some } \beta \in (0, (\lambda-1)/2\lambda).$$

Let $\hat{\kappa}_4$ be a consistent estimator of the kurtosis κ_4 of ϵ_1, which, for instance, can be obtained by fitting a long AR(q) model to data and calculating the residuals (cf. Lee and Wei (1999)). Set

$$\hat{\Gamma}_{ij} = \hat{\kappa}_4 \hat{\gamma}_n(i)\hat{\gamma}_n(j) + \sum_{r=-h_n}^{h_n}\left(\hat{\gamma}_n(i+r)\hat{\gamma}_n(j+r) + \hat{\gamma}_n(i-r)\hat{\gamma}_n(j+r)\right), \quad i,j = 0,\ldots,m.$$

Theorem 4.2 of Lee et al. (2003) shows that if we put

$$\mathcal{S}_n(s) = \left(\frac{[ns]}{\sqrt{n}}\left(\hat{\gamma}_{[ns]}(0) - \hat{\gamma}_n(0)\right), \ldots, \frac{[ns]}{\sqrt{n}}\left(\hat{\gamma}_{[ns]}(m) - \hat{\gamma}_n(m)\right)\right)', \quad 0 \le s \le 1,$$

then under H_0,

$$\mathcal{S}_n'(s)\hat{\Gamma}^{-1}\mathcal{S}_n(s) \xrightarrow{w} ||\mathbf{W}_{m+1}^\circ(s)||^2,$$

where $\hat{\Gamma}$ denotes the $(m+1)\times(m+1)$ matrix whose (i,j)-th component is $\hat{\Gamma}_{ij}$, and $\mathbf{W}_{m+1}^\circ$ denotes an $(m+1)$-dimensional standard Brownian bridge. As a result,

$$T_n := \sup_{0<s<1} \mathcal{S}_n'(s)\hat{\Gamma}_n^{-1}\mathcal{S}_n(s) \xrightarrow{w} T := \sup_{0<s<1} ||\mathbf{W}_{m+1}^\circ(s)||^2.$$

We reject H_0 if T_n is large. The critical values are presented in Lee et al. (2003). Recall that one can detect multiple change points following the D_k plot method in Iclán and Tiao (see also Section 5). In Section 4, we will see through a simulation study that the test statistic performs adequately.

3 Graphical method to identify D

In this section we consider the case that d and d' are unknown. As mentioned earlier, if the time series has parameter changes, it is not easy to identify the correct orders. Therefore, here we develop a graphical method to estimate them. Suppose that δ_t are iid random variables with zero mean and unit variance. Denote

$$y_j(1) = \sum_{i=1}^{j} \delta_i \quad \text{and} \quad y_j(k) = \sum_{i=1}^{j} y_i(k-1), \quad k \geq 2.$$

Let

$$W_n(u) = n^{-1/2} \sum_{i=1}^{[nu]} \delta_i, \quad 0 \leq u \leq 1,$$

and let $W(u)$ denote a standard Brownian motion. Define

$$W^{(2)}(u) = \int_0^u W(u)du \quad \text{and} \quad W^{(k)} = \int_0^u W^{(k-1)}(u)du, \quad k \geq 3.$$

From Donsker's invariance principle (cf. Billingsley (1968)), we may write that

$$y_j(1) = n^{1/2} W_n(j/n) \overset{d}{\simeq} n^{1/2} W(j/n)$$

for large n, and

$$\begin{aligned} y_j(2) &= n^{3/2}\{\sum_{i=1}^{j} W_n(i/n)/n\} \simeq n^{3/2} \int_0^{j/n} W_n(u)du \\ &\overset{d}{\simeq} n^{3/2} \int_0^{j/n} W(u)du = n^{3/2} W^{(2)}(j/n). \end{aligned}$$

Similarly, we obtain $y_j(k) \overset{d}{\simeq} n^{k-1/2} W^{(k)}(j/n)$, and thus

$$n^{-2k} \sum_{j=1}^{t} (y_j(k))^2 \overset{d}{\simeq} n^{-1} \sum_{j=1}^{t} (W^{(k)}(j/n))^2 \simeq \int_0^{t/n} (W^{(k)}(u))^2 du,$$

which implies that for t close to n,

$$t^{-2k} \sum_{j=1}^{t} (y_j(k))^2 \overset{d}{\simeq} \int_0^1 (W^k(u))^2 du = O_P(1).$$

The above argument indicates that one can estimate the order d in $X_t = (1-B)^d \delta_t$ via examining the shape of the function $g_1 : t \rightarrow t^{-1} \sum_{i=1}^{t} X_i^2$ and $g_{2k} : t \rightarrow t^{-2k} \sum_{i=1}^{t} X_i^2$, $k \geq 1$. For example, if $d = 2$, it is anticipated that g_1 and g_2 explode fast, $g_4(t)$ are within some boundary, and $g_6(t)$ have the values close to 0. From the same reasoning, if g_{2k}, $k < d$, explode, $g_{2d}(t)$ lie in some boundary, and $g_{2(d+1)}(t)$ have values close to 0, then one can select d as the correct order. In fact, this reasoning is still true for δ_t in a class of linear processes including ARMA processes and strong mixing processes such as GARCH(1,1) processes (cf. Carrasco and Chen (2002)). Moreover, the graphical method is still valid for determining the correct order D even for time series with structural changes in ARIMA models. Figures 1-4 are concerned with the change from an ARIMA(1,1,1) model to an ARMA(1,1) model. From those figures, one can easily reason that D is equal to 1. Meanwhile, Figures 5-8 deal with the change from an ARMA(1,1) model to an ARIMA(1,1,1) model. Similarly, we can easily see that $D = 1$.

As mentioned earlier, it should be emphasized that designing a formal decision rule is not feasible since one has to take account of all possible cases including both stationary and nonstationary processes with parameter changes. Since the graphical method is not rigorous in terms of mathematics, one might be able to claim that the selected order is only a candidate. Thus, here we discuss on the issue of checking the correctness of the selected order.

Suppose that D is chosen by the graphical method. Letting $x_t = (1-B)^D X_t$, where X_t denote original data, we follow the testing procedure in Section 2 with x_t's to find change points. If the test detects the change points, say, t_i, $i = 1, \ldots, k$, we perform a unit root test for all the subseries $x_{t_{i-1}+1}, \ldots, x_{t_i}$, $i = 1, \ldots, k+1$, where $t_0 = 0$ and $t_{k+1} =$ the number of x_t's. On the other hand, if there are no change points, we conduct the unit root test for the whole series $\{x_t\}$. Firstly, if unit roots exist at least in one of those subseries, we decide that the D is not the correct order. In this case, we completely ignore the obtained result, including the change points, since nonstationary processes are involved. Then we repeat the same procedure with the updated order $D + 1$. By continuing until we do not find any unit roots, we can finally determine the correct order. Secondly, if no unit roots are detected with D, one may speculate that D might be overestimated. Note that the overestimation does not affect the locations of change points since our test is based on the structure of autocorrelations. If the test shows the presence of change points, we perform a unit root test based on $(1-B)^{(D-1)} X_t$ for all the subseries; otherwise we do it for the whole series. If the unit root is detected at least for one of those series, we conclude that D is the right order. Otherwise, we repeat the same procedure with $D - 1$. Here, we can keep the obtained change points unlike before. In general, if a unit root is detected for $(1-B)^{D-l} X_t$ for some $1 \le l \le D$, we determine the right order to be $D - l + 1$. Otherwise, it is determined to be 0. Following this way, we can eventually determine the correct order.

4 Simulation results

In this section, we evaluate the performance of the test statistics T_n in Section 2 through a simulation study. The empirical sizes and powers are calculated at a nominal level of 0.1. Here $m = 1$, $h_n = n^{1/4}$ and $q = [(\log n)^2]$ are used for T_n, and the critical value is 2.054. In order to examine the performance of T_n, we consider the ARIMA(1,d,1) process $(1-B)^d(1-\phi B)X_t = (1+\theta B)\varepsilon_t$, where ε_t are iid standard normal r.v.'s, and $X_0 = 0$. The empirical sizes and powers are calculated with sets of 300, 500 and 800 observations generated from an ARIMA(1,d,1) model. Tables 1-5 summarize the empirical sizes and powers for the following alternative hypothesis.

$$
\begin{aligned}
H_1 \;\; : \;\; & (1-B)^d(1-\phi B)X_t = (1+\theta B)\varepsilon_t, \quad t = 1, \cdots [n/2], \\
& (1-B)^{d'}(1-\phi' B)X_t = (1+\theta B)\varepsilon_t, \quad t = [n/2]+1, \cdots n,
\end{aligned}
$$

where $\theta = 0.2$ and ϕ and ϕ' are assumed to vary, taking values of 0.2, 0.5 and 0.8. Table 1 shows that the empirical sizes and powers are reasonably good unless ϕ is close to 1. Actually, it is well known that high correlation damages statistical inferences. In actual practice, however, highly correlated time series can be regarded to form a unit root process, so that this case can be classified into the category that d and d' are equal to 1. Tables 2-5 also exhibit that the procedure based on the time series $\{(1-B)^D X_t\}$, where D is obtained through the graphical method in Section 3, performs adequately.

5 Real data analysis

In this section we analyze a real data set and demonstrate that our method presented in the previous sections is properly applicable. For this task, we analyze the 3-month Euroyen interest rate data set obtained from International Financial Statistics over the period from 07/1989 to 12/2002: the time series $\{X_t\}$, $t = 1, \ldots, 162$, is plotted in Figure 9. First, we apply the graphical method in Section 3 to determine D. Figures 10 and 11 manifestly suggest that we can choose $D = 1$. Now, for testing for parameter changes, we utilize the test statistic T_n with $m = 1$ for the differenced time series $x_t = (1-B)X_t$. At the nominal level of 0.1, the critical value is 2.054 (cf. Lee et al. (2003)). As a consequence, it appears that there is one parameter change. The change point can be selected by examining the D_k plot, where $D_k = \mathcal{S}_n'(k/n)\hat{\Gamma}^{-1}\mathcal{S}_n(k/n)$. Since D_k is maximized at $k = 50$, we can see that the parameter change occurs at the lag 50: the vertical lines in Figure 9 and 12 indicate the location of the change point. Now, as we described in the Remark of Section 3, we perform Dickey-Fuller's unit root test for the two subseries of $\{(1-B)X_t\}$. Since the result indicates that there are no unit roots, we perform the unit root test for the original $\{X_t\}$. The result shows that the first subseries has a unit root while the second has no unit roots. Threfore, we conclude that D should be equal to 1. By fitting ARIMA(p, d, q) models, $d = 0, 1$ and $p, q \leq 2$, to the two original subseries (using AIC), we obtain that the first subseries $\{X_{1t}\}$ of $\{X_t\}$ follows the ARIMA(1,1,1) model and the second subseries $\{X_{2t}\}$ follows the ARMA(1,2) model as follows:

$$(1 - B)(1 - 0.927B)X_{1t} = (1 - 0.731B)\epsilon_t, \; t = 1, 2, \cdots, 50,$$

and

$$(1 - 0.961B)X_{2t} = (1 + 0.289B + 0.351B^2)\epsilon_t, \; t = 51, 52, \cdots, 162.$$

6 Concluding remarks

In this paper, we proposed a method for detecting parameter change points in ARIMA models based on the cusum test in Lee et al. (2003) and the graphical method introduced in Section 3. The graphical method was designed to determine the correct order of differencing, based on which we transform the time series data to form a combination of stationary subseries. The simulation study in Section 4 demonstrated that the graphical method and the cusum test performs appropriately. This method was applied to a real data set, the 3-month Euroyen interest rate data. As a result, we could detect one change point: it turned out that the first subseries before the point follows an ARIMA model and the second subseries after it follows an ARMA model. This result strongly advocates the validity of our method in actual practice. Our method, however, should not be used to every data set. In particular, if data has high volatility and jumps, our method will be likely to lead to a wrong conclusion. Therefore, in advance of using it, one should carefully check whether or not a given time series data can be handled within the framework of ARIMA models. However, insofar as the data is generated from ARIMA models, our method is feasibly applicable. Overall, we conclude that our method can be a functional tool to detect change points in ARIMA models.

Acknowledgements. The first author wishes to acknowledge that this research was supported by Korea Research Foundation Grant 2003-070-C00008. The third author acknowledges the support from Grant-in Aid for Scientific Research 14330005.

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Table 1. ARIMA(1,0,1); $\phi \to \phi'$; $\theta = 0.5$

ϕ		0.2			0.5			0.8		
ϕ'		0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
n	300	.080	.672	1.00	.424	.164	.896	.926	.660	.612
	500	.124	.836	1.00	.774	.134	.992	1.00	.910	.512
	800	.082	.954	1.00	.938	.128	.998	1.00	.990	.426

Table 2. ARIMA(1,0,1)$\to$ ARIMA(1,1,1); $\phi \to \phi'$; $\theta = 0.5$

ϕ		0.2			0.5			0.8		
ϕ'		0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
n	300	1.00	1.00	1.00	.998	1.00	1.00	.722	.998	1.00
	500	1.00	1.00	1.00	1.00	1.00	1.00	.914	1.00	1.00
	800	1.00	1.00	1.00	1.00	1.00	1.00	.986	1.00	1.00

Table 3. ARIMA(1,1,1)$\to$ ARIMA(1,0,1); $\phi \to \phi'$; $\theta = 0.5$

ϕ		0.2			0.5			0.8		
ϕ'		0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
n	300	.950	.994	.742	.998	1.00	.996	1.00	1.00	1.00
	500	1.00	1.00	.932	1.00	1.00	1.00	1.00	1.00	1.00
	800	1.00	1.00	.992	1.00	1.00	1.00	1.00	1.00	1.00

Table 4. ARIMA(1,1,1)$\to$ ARIMA(1,2,1); $\phi \to \phi'$; $\theta = 0.5$

ϕ		0.2			0.5			0.8		
ϕ'		0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
n	300	1.00	1.00	1.00	.996	1.00	1.00	.726	.996	1.00
	500	1.00	1.00	1.00	.998	1.00	1.00	.918	1.00	1.00
	800	1.00	1.00	1.00	1.00	1.00	1.00	.990	1.00	1.00

Table 5. ARIMA(1,2,1)→ ARIMA(1,1,1); $\phi \to \phi'$; $\theta = 0.5$

ϕ		0.2			0.5			0.8		
ϕ'		0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
n	300	.960	.992	.766	1.00	1.00	1.00	1.00	1.00	1.00
	500	1.00	1.00	.914	1.00	1.00	1.00	1.00	1.00	1.00
	800	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Figure 1: Change from $(1-B)(1-0.5B)X_t = (1+0.5B)\varepsilon_t$ to $(1-0.5B)X_t = (1+0.5B)\varepsilon_t$

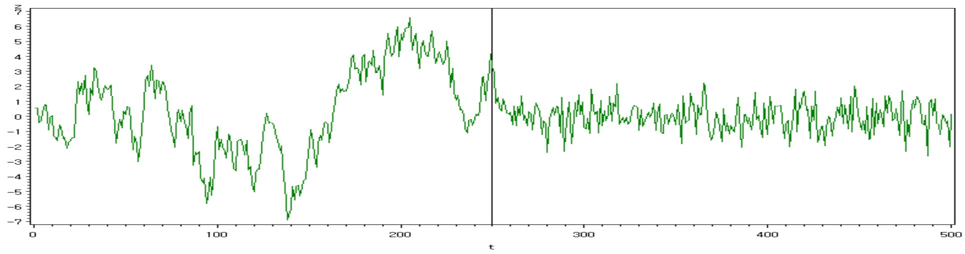

Figure 2: $\frac{1}{t}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 1

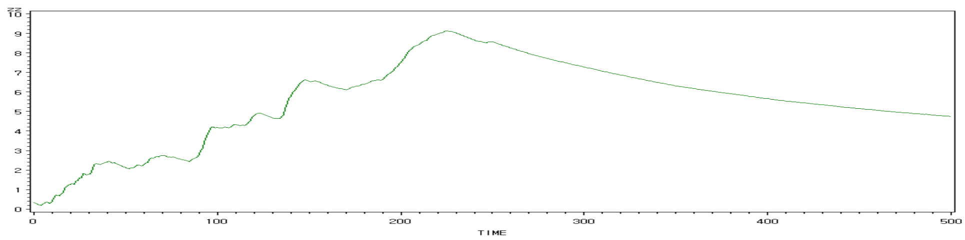

Figure 3: $\frac{1}{t^2}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 1

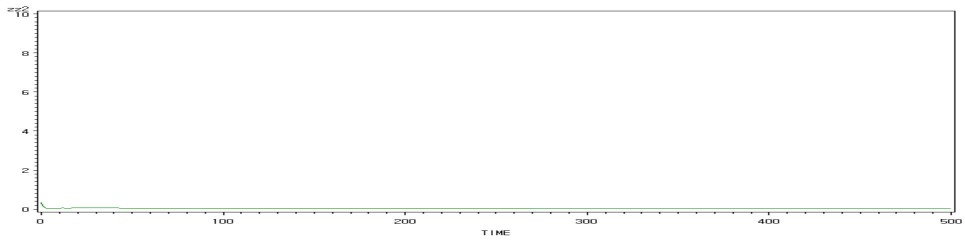

Figure 4: $\frac{1}{t^4}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 1

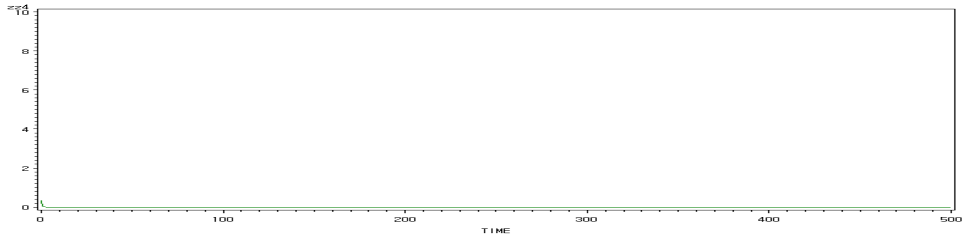

Figure 5: Change from $(1-0.5B)X_t = (1+0.5B)\varepsilon_t$ to $(1-B)(1-0.5B)X_t = (1+0.5B)\varepsilon_t$

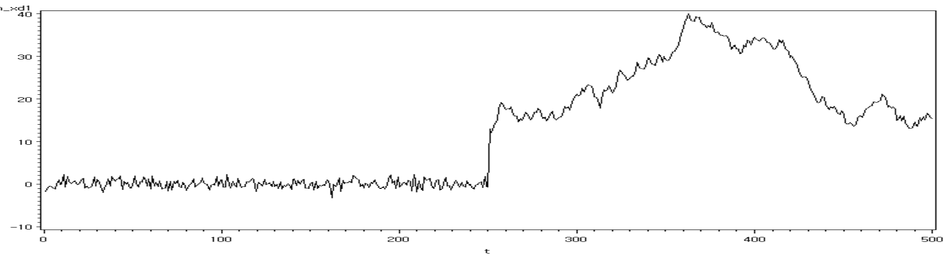

Figure 6: $\frac{1}{t}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 5

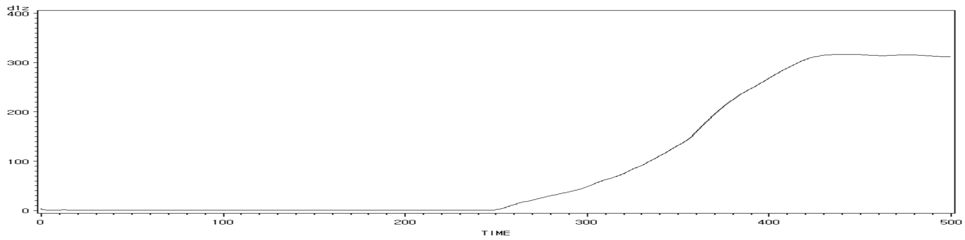

Figure 7: $\frac{1}{t^2}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 5

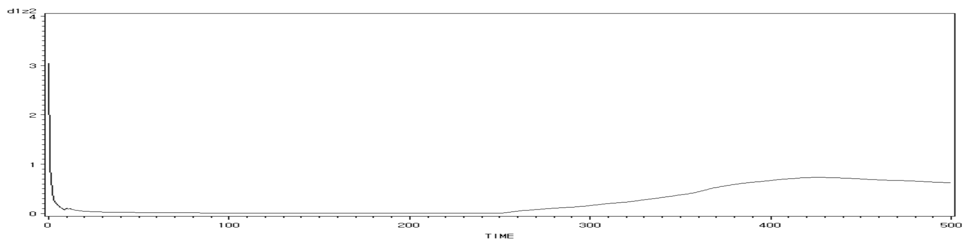

Figure 8: $\frac{1}{t^4}\sum_{i=1}^{t} X_i^2$ plot for the series in Figure 5

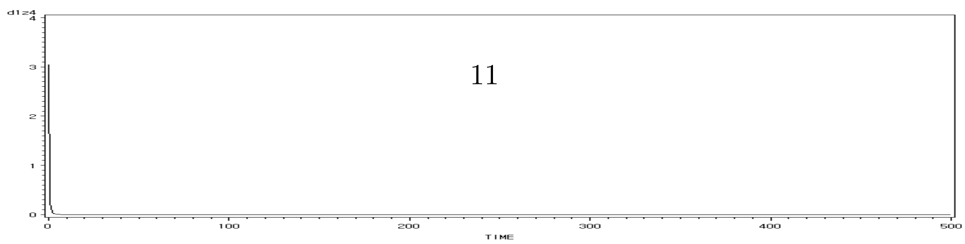

Figure 9: The plot of 3-month Euroyen interest rate

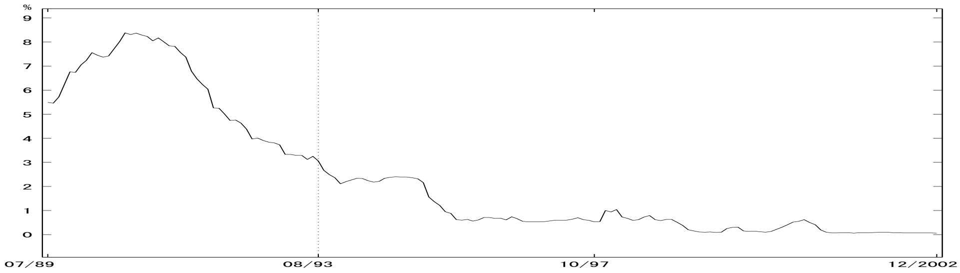

Figure 10: The plot of g_1 for 3-month Euroyen interest rate

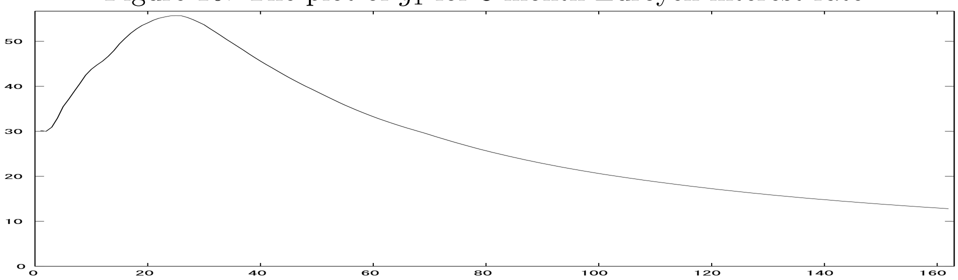

Figure 11: The plot of g_2 for 3-month Euroyen interest rate

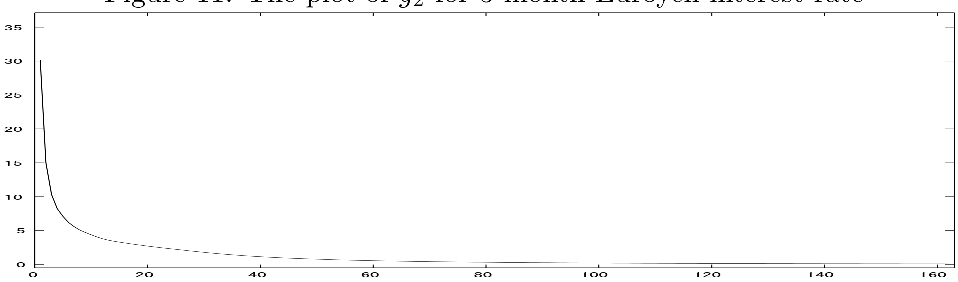

Figure 12: The plot of D_k for 3-month Euroyen interest rate