THE ALEXANDER TRANSFORM OF A SPIRALLIKE FUNCTION

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Abstract. We show that the Alexander transform of a β-spirallike function is univalent when $\cos\beta \le 1/2$, which settles the problem posed by Robertson. We also solved a problem considered by Y. J. Kim and Merkes.

1. Introduction

Let $\mathscr{A}$ denote the class of analytic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ and let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of univalent functions. A function $f \in \mathscr{S}$ is called *starlike* (resp. *convex*) if the image $f(\mathbb{D})$ is starlike with respect to 0 (resp. convex). Alexander is the first who observed that the integral transformation J, defined by

$$J[f](z) = \int_0^z \frac{f(\zeta)}{\zeta} \mathrm{d}\zeta$$

and now called the *Alexander transformation*, maps the class $\mathscr{S}^*$ of starlike functions onto the class $\mathscr{K}$ of convex functions in a one-to-one fashion. Biernacki conjectured that $J(\mathscr{S}) \subset \mathscr{S}$ in 1960. In 1963, Krzyż and Lewandowski disproved it by giving the example $f(z) = z(1-\mathrm{i}z)^{\mathrm{i}-1}$, which is $\pi/4$-spirallike but is transformed to a non-univalent function by J. Here, a function f in $\mathscr{A}$ is called β-spirallike and known to be univalent if

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\beta}\frac{zf'(z)}{f(z)}\right) > 0, \quad |z| < 1,$$

for a real number β with $-\pi/2 < \beta < \pi/2$. We denote by $\mathscr{S}_{\mathrm{p}}(\beta)$ the class of β-spirallike functions in $\mathscr{A}$ and set $\mathscr{S}_{\mathrm{p}} = \bigcup_\beta \mathscr{S}_{\mathrm{p}}(\beta)$. See, for instance, [2] as a general reference and, especially, §8.4 in the book for the above topic. Note that $\mathscr{S}_{\mathrm{p}}(0)$ is nothing but the class $\mathscr{S}^*$ of starlike functions.

Robertson [7] studied the Alexander transform of a β-spirallike function. He showed that $J(\mathscr{S}_{\mathrm{p}}(\beta)) \subset \mathscr{S}$ when $\cos\beta \le x_0$, where $x_0 = 0.2034\cdots$ is the unique positive root of the equation $16x^3 + 16x^2 + x - 1 = 0$. (It seems that the paper [7] contains an error in a numerical evaluation of x_0.) Soon later, Libera and Ziegler [6] replaced x_0 by $x_1 = 0.2564\cdots$, where x_1 is the unique positive root of the equation $9x^3 + 9x^2 + x - 1 = 0$.

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Robertson [7] also observed that x_0 cannot be replaced by any number greater than $1/2$ and asked about the best value for this. We settle this problem in this short note. Note also that $J(\mathscr{S}_{\mathrm{p}}(0)) = \mathscr{K} \subset \mathscr{S}$.

Theorem 1. *The inclusion relation* $J(\mathscr{S}_{\mathrm{p}}(\beta)) \subset \mathscr{S}$ *holds precisely if either* $\cos\beta \le 1/2$ *or* $\beta = 0$.

In the literature, many papers have been devoted to the integral operator $I_\alpha,\ \alpha \in \mathbb{C}$, defined by

$$I_\alpha[f](z) = \int_0^z \{f'(\zeta)\}^\alpha \,\mathrm{d}\zeta = z\int_0^1 \{f'(tz)\}^\alpha \,\mathrm{d}t$$

for a locally univalent function f in $\mathscr{A}$ (see, for instance, [2, §8.5], [3] and references therein). Note that $I_\alpha \circ I_{\alpha'} = I_{\alpha\alpha'}$. Y. J. Kim and Merkes [5] considered essentially the problem of determining (or estimating) the set

$$A(\mathscr{F}) = \{\alpha \in \mathbb{C} : I_\alpha(\mathscr{F}) \subset \mathscr{S}\}$$

for a class $\mathscr{F}$ of locally univalent functions in $\mathscr{A}$. For example, Theorem 1 is equivalent to the assertion that $1 \in A(J(\mathscr{S}_{\mathrm{p}}(\beta)))$. They proved in [5] the relations

$$\{|\alpha| \le 1/4\} \subset A(J(\mathscr{S})) \subset A(J(\mathscr{S}_{\mathrm{p}})) \subset \{|\alpha| \le 1/2\}.$$

The second result in this note is the following.

Theorem 2.

$$A(J(\mathscr{S}_{\mathrm{p}})) = \{|\alpha| \le 1/2\}.$$

As we will see later, these two theorems are deduced from the concrete description of the set $A(J(\mathscr{S}_{\mathrm{p}}(\beta)))$. In what follows, $[z, w]$ denotes the closed line segment with endpoints z and w for $z, w \in \mathbb{C}$.

Theorem 3. *For* $-\pi/2 < \beta < \pi/2$,

$$A(J(\mathscr{S}_{\mathrm{p}}(\beta))) = \left\{|\alpha| \le \tfrac{1}{2\cos\beta}\right\} \cup \left[\tfrac{e^{i\beta}}{2\cos\beta}, \tfrac{3e^{i\beta}}{2\cos\beta}\right].$$

2. Proof of the results.

Libera and Ziegler observed in [6] that a function $f \in \mathscr{A}$ is in $\mathscr{S}_{\mathrm{p}}(\beta)$ if and only if $f(z)/z = (s(z)/z)^\alpha$ for some starlike function $s \in \mathscr{S}^*$, where $\alpha = e^{-i\beta}\cos\beta$. This result can be interpreted as the following lemma, which is crucial in the proof of our theorems. For convenience of the reader, we give an outline of the proof.

Lemma 4. *For* $-\pi/2 < \beta < \pi/2$,

$$J(\mathscr{S}_{\mathrm{p}}(\beta)) = I_{e^{-i\beta}\cos\beta}(\mathscr{K}).$$

Proof. Set $\alpha = \mathrm{e}^{-\mathrm{i}\beta}\cos\beta$. For $f \in J(\mathscr{S}_{\mathrm{p}}(\beta))$, we write

$$\mathrm{e}^{\mathrm{i}\beta}\left(\frac{zf''(z)}{f'(z)}+1\right) = p(z)\cos\beta + \mathrm{i}\sin\beta,$$

where p is analytic, has positive real part in $\mathbb{D}$ and satisfies $p(0) = 1$. If we take $k \in \mathscr{K}$ such that $1 + zk''(z)/k'(z) = p(z)$, then $f''(z)/f'(z) = \alpha k''(z)/k'(z)$. Integrating both sides, we obtain the relation $f' = (k')^{\alpha}$, namely, $f = I_{\alpha}[k]$. Since we can trace back the above procedure, we obtain the assertion. □

The next result was proved by Aksent'ev and Nezhmetdinov [1] (see also [3]).

Lemma 5.

$$A(\mathscr{K}) = \{|\alpha| \le 1/2\} \cup [1/2, 3/2].$$

Proof of Theorem 3. By Lemma 4, we have

$$I_{\alpha}(J(\mathscr{S}_{\mathrm{p}}(\beta))) = I_{\alpha}(I_{\mathrm{e}^{-\mathrm{i}\beta}\cos\beta}(\mathscr{K})) = I_{\alpha\mathrm{e}^{-\mathrm{i}\beta}\cos\beta}(\mathscr{K}).$$

Therefore, $\alpha \in A(J(\mathscr{S}_{\mathrm{p}}(\beta)))$ if and only if $\alpha\mathrm{e}^{-\mathrm{i}\beta}\cos\beta \in A(\mathscr{K})$. Lemma 5 now yields the required relation. □

Proof of Theorem 1. We assume that $\beta \ne 0$. As we remarked above, $J(\mathscr{S}_{\mathrm{p}}(\beta)) \subset \mathscr{S}$ if and only if $1 \in A(\mathscr{S}_{\mathrm{p}}(\beta))$. Therefore, by Theorem 3, this is equivalent to $1 \le 1/(2\cos\beta)$, namely, $\cos\beta \le 1/2$. Thus the proof is now complete. □

Proof of Theorem 2. By definition,

$$A(\mathscr{S}_{\mathrm{p}}) = \bigcap_{\beta} A(\mathscr{S}_{\mathrm{p}}(\beta)).$$

With the aid of Theorem 3, a simple observation gives $A(\mathscr{S}_{\mathrm{p}}) = \{|\alpha| \le 1/2\}$. □

We finally mention the norm estimate of pre-Schwarzian derivatives. The hyperbolic norm of the pre-Schwarzian derivative $T_f = f''/f'$ of $f \in \mathscr{A}$ is defined to be

$$\|f\| = \sup_{|z|<1}(1 - |z|^2)|T_f(z)|.$$

It is known (cf. [4]) that f is bounded if $\|f\| < 2$ and the bound depends only on the value of $\|f\|$. Since $\|I_{\alpha}[f]\| = |\alpha|\|f\|$, we obtain the following result.

Proposition 6. *For each $\beta \in (-\pi/2, \pi/2)$, the sharp inequality $\|f\| \le 4\cos\beta$ holds for $f \in J(\mathscr{S}_{\mathrm{p}}(\beta))$. Moreover, if $\cos\beta < 1/2$, then a function in $J(\mathscr{S}_{\mathrm{p}}(\beta))$ is bounded by a constant depending only on β.*

Proof. For each $f \in J(\mathscr{S}_{\mathrm{p}}(\beta))$, by Lemma 4, there is a function $k \in \mathscr{K}$ such that $f = I_\alpha[k]$, where $\alpha = \mathrm{e}^{-\mathrm{i}\beta} \cos\beta$. Noting that $\|k\| \le 4$ (cf. [8]), we obtain

$$\|f\| = |\alpha|\|k\| \le 4|\alpha| = 4\cos\beta.$$

Since the inequality $\|k\| \le 4$ is sharp, the above inequality is also sharp. If $\cos\beta < 1/2$, the above inequality implies $\|f\| \le 4\cos\beta < 2$, from which the latter assertion follows. $\square$

In the last theorem, the bound $1/2$ cannot be replaced by any number greater than $1/\sqrt{2}$. Indeed, the function $f_\beta(z) = [(1-z)^{1-2\alpha} - 1]/(2\alpha - 1) \in J(\mathscr{S}_{\mathrm{p}}(\beta))$, where $\alpha = \mathrm{e}^{-\mathrm{i}\beta}\cos\beta$, is unbounded when $\cos\beta > 1/\sqrt{2}$.

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