EUROPHYSICS LETTERS 20 May 1997

*Europhys. Lett.*, **38** (6), pp. 417-422 (1997)

# Heterogeneity-induced order in globally coupled chaotic systems

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(received 23 August 1996; accepted in final form 10 April 1997)

PACS. 05.45+b – Theory and models of chaotic systems.
PACS. 05.90+m– Other topics in statistical physics and thermodynamics.
PACS. 87.10+e – General, theoretical, and mathematical biophysics (including logic of biosystems, quantum biology, and relevant aspects of thermodynamics, information theory, cybernetics, and bionics).

**Abstract.** – Collective behavior is studied in globally coupled maps with distributed nonlinearity. It is shown that the heterogeneity enhances regularity in the collective dynamics. Low-dimensional quasi-periodic motion is often found for the mean field, even if each element shows chaotic dynamics. The mechanism of this order is due to the formation of an internal bifurcation structure, and the self-consistent dynamics between the structures and the mean field.

The dynamics of globally coupled systems has been extensively and intensively studied [1]-[10]. Such problems naturally appear in physical and biological systems. Coupled Josephson junction array [2] and nonlinear optics with multi-mode excitation [3] give such examples, while relevance to neural and cellular networks has been discussed [4]. Among others, the study of globally coupled chaotic systems has revealed novel concepts such as clustering, chaotic itinerancy, and partial order. In particular, the study of collective dynamics has gathered much attention [8]-[13]. Even if the elements are desynchronized, some kinds of collective dynamics, ranging from low-dimensional torus to high-dimensional chaos, are observed [6], [8].

In these recent studies, elements are homogeneous. In other words, identical elements are coupled. However, in many systems elements are heterogeneous. In a Josephson junction array, each unit is not identical. In an optical system, the gain of each mode depends on its wavenumber. In a biological system, each unit such as a neuron or a cell is heterogeneous. So far the study of a coupled system with distributed parameters is restricted to synchronization of oscillators [9], [10]. Thus it is important to check how the notions constructed in globally coupled dynamical systems can be applicable to a heterogeneous case. In the present letter we demonstrate that the collective order emerges in a heterogeneous system through self-consistent dynamics between the mean-field and internal differentiation of dynamics. Here we adopt a globally coupled map with a distributed parameter:

$$x_{n+1}(i) = (1-\epsilon) f_i(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f_j(x_n(j)) \quad (i = 1, 2, 3, \ldots, N),$$

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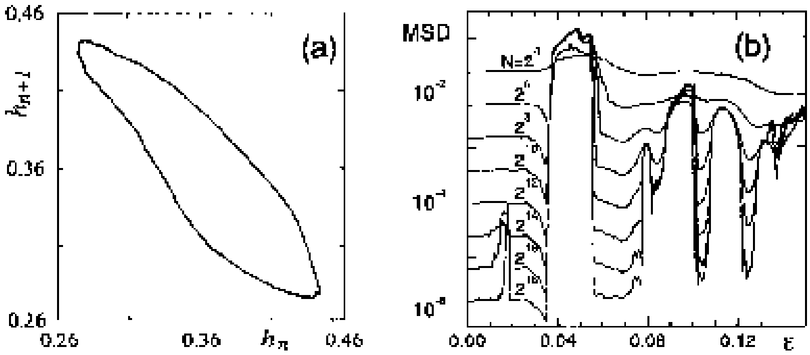


Fig. 1. – *a*) Return map of the mean-field $h$. $a_0 = 1.9$, $\Delta a = 0.05$, $\epsilon = 0.11$, $N = 8 \times 10^6$. *b*) The mean-square deviation of the mean field is plotted *vs.* coupling strength $\epsilon$. $a = 1.9$, $\Delta a = 0.05$. The number of elements is varied from $N = 2^4$ to $2^{18}$. Collective motions are clearly seen in some parameter regimes such as $\epsilon \approx 0.018$, $0.038 < \epsilon < 0.058$, $0.083 < \epsilon < 0.1$, $0.105 < \epsilon < 0.12$, and $0.13 < \epsilon < 0.15$.

where $x_n(i)$ is the variable of the $i$-th element at discrete time $n$, and $f_i(x(i))$ is the internal dynamics of each element. For the dynamics we choose the logistic map $f_i(x) = 1 - a(i)x^2$, where the parameter $a(i)$ for the nonlinearity is distributed between $[a_0 - \frac{\Delta a}{2}, a_0 + \frac{\Delta a}{2}]$ as $a(i) = a_0 + \frac{\Delta a(2i-N)}{2N}$. We note that the essentially same behavior is found when $a(i)$ is randomly distributed in an interval or the coupling $\epsilon(i)$ is distributed instead of $a$.

When elements are identical with $\Delta a = 0$, the present model reduces to a globally coupled map (GCM) studied extensively. In this case the mean field $h_n = \frac{1}{N}\sum_{j=1}^{N} f(x_n(j))$ is not stationary and a kind of collective dynamics is observed, when the chaotic dynamics of $x_n(i)$ are mutually desynchronized. Since the mean-field dynamics is generally complicated, the amplitude of oscillations is not easily measured as in the case for simple quasi-periodic dynamics. In this case, as a measure of the amplitude of the variation, we use the mean square deviation (MSD) of the mean-field distribution, given by $\langle(\delta h)^2\rangle = \langle(h - \langle h\rangle)^2\rangle$.

When elements are not identical, one might expect that collective dynamics would be destroyed and the mean field becomes stationary. On the contrary, the MSD remains finite even in the large-size limit. This implies the existence of some structure and coherence in the mean-field dynamics. In the present letter we clarify the form and the origin of such collective order in a heterogeneous system.

Figure 1 *a*) gives an example of the return map of the mean field. Here the width of scattered points along the one-dimensional curve decreases with $N$. Hence the figure clearly shows that the mean-field dynamics is on a 2-dimensional torus. The power spectrum of the mean-field time series also supports that the motion is quasi-periodic.

Figure 1 *b*) shows the MSD plotted as a function of the coupling strength $\epsilon$ with the increase of the system size. The MSD stays finite within a wide parameter region in which the power spectrum has delta peaks. Such collective behavior is rather general in our heterogeneous system and is observed more clearly than in the homogeneous GCM.

With the change of $a_0$, $\Delta a$, or $\epsilon$, the mean-field dynamics shows the bifurcation from torus to chaos accompanied by phase lockings. Further bifurcation proceeds to higher-dimensional chaos (while some structure is still kept). Several routes to chaos [12] are observed, including that through the doubling of torus (fig. 2). There are two cases for the collective motion, although for both cases each element oscillates chaotically without mutual synchronization. In one case (given in fig. 2) there are negative Lyapunov exponents (78 for $b$), for $N = 500$; whose

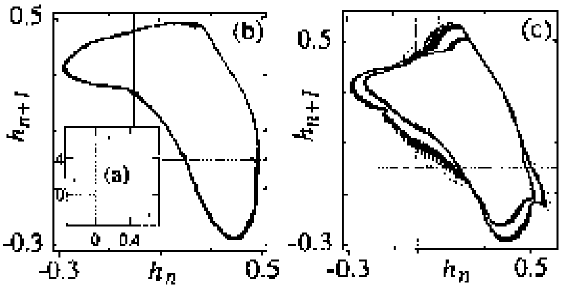


Fig. 2

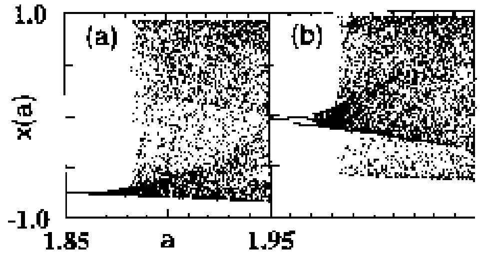


Fig. 3

Fig. 2. – Return map of the mean field $h$. $a_0 = 1.9$, $\Delta a = 0.1$. *a*) $\epsilon = 0.055$, $N = 2^{16}$. *b*) $\epsilon = 0.053$, $N = 2^{18}$. *c*) $\epsilon = 0.05$, $N = 2^{21}$.

Fig. 3. – Snapshot pattern of $x_n(i)$ plotted *vs.* $a(i)$. Here the mean field is locked to period 3 (to be more precise period 6). $a_0 = 1.9$, $\Delta a = 0.1$, $\epsilon = 0.055$. Time step 5000 (*a*), and 5001 (*b*). At the next iterate, the coherent structure of $x_n(a)$ for $a < 1.88$ moves to $x \approx 1$, while another iterate leads to the structure of $a$.

number decreases with $\epsilon$). In the other case (given in fig. 1*a*)), all exponents are positive.

Hereafter we show how this heterogeneity-induced order is possible (mainly for the former case). The scenario to be presented consists of two parts. First, we demonstrate the formation of "internal bifurcation structure", made possible by the distribution of parameters, which leads to the self-consistent relation between each dynamics and the mean field. Second, it is shown that the repetitive change of internal bifurcation structure makes it possible to form the low-dimensional collective motion in the mean field. The organization of the low-dimensional bifurcation in sub-systems from a high-dimensional system is a key concept for the collective dynamics.

First we study the formation of the internal bifurcation structure. In our system nonlinear parameters are distributed over elements. The dynamics of the $i$-th element depends on the parameter $a(i)$. Hence it is relevant to draw the motion *vs.* the parameter $a$. Figure 3 gives snapshot patterns of $x_n(a)$ for the period-3 locking [14] in the mean field (fig. 2 *a*)). It looks like an ordinary bifurcation diagram plotted against the change of external parameters, but the patterns of fig. 3 are just snapshot representations of one system consisting of $N$ elements. Still, successive plots of the pattern show that the dynamics of elements changes with $a(i)$. In fig. 3 elements with $1.85 < a(i) < 1.887$ show period-3 oscillation with almost synchronization. With the increase of $a(i)$, tangent bifurcation occurs at one point [15] around $a \approx 1.85$. For larger $a(i)$, period-doubling and crisis are observed in the snapshot pattern, if it is viewed as the transition of attractor with the change of control parameters. As the parameter $a$ is larger, successive bifurcation occurs beyond which elements fall into a fully chaotic state. Hence we call the structure as internal bifurcation. We also note that the clustered motion (at small $a(i)$) and fully desynchronized motions coexist.

If the mean field were an external parameter for each element, those bifurcations would occur in each sub-system according to their nonlinear parameters. The mean-field dynamics, in our case, is organized self-consistently from each element dynamics according to the internal bifurcation. The internal structure, consisting of period-3 synchronized motion and desynchronized motion, forms the period-3 oscillation of the mean field. On the other hand, the period-3 mean-field motion plays the role of external forcing which causes the period-3 clustered motion

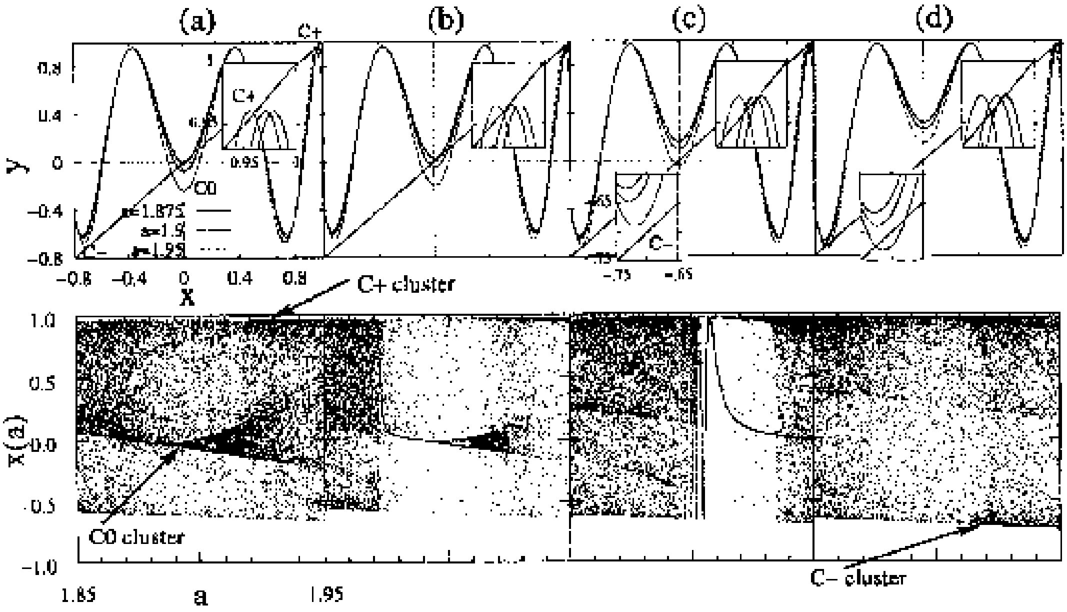


Fig. 4. – Dynamics of the internal bifurcation structure. $a_0 = 1.9$, $\Delta a = 0.1$, and $\epsilon = 0.053$. In the lower column snapshot patterns are plotted at time step 1080 ($a$), 1161 ($b$), 1206 ($c$), 1251 ($d$). In the upper column, the effective maps, eq. (2), are plotted for $a = 1.875, 1.9, 1.95$ at the corresponding time steps. These steps (with unequal intervals) are chosen so that the formation and collapse of clusters are seen, in correspondence with fig. 5.

and desynchronized motion according to the parameter $a(i)$ of each element.

When the coupling strength $\epsilon$ is smaller (fig. 2 $b$)), another tangent bifurcation occurs. This bifurcation leads to the formation of the 2nd clustered motion. As the 2nd clustered motion is formed, the mean field is varied, which changes the effective map for each element. Then the period-3 locking in the mean field collapses, when we need the second scenario for the self-consistent dynamics between the internal bifurcation structure and the mean field [16].

As the simplest example, we discuss the quasi-periodic case given by fig. 2 $b$). Figure 4 shows the snapshots of $x(a)$ corresponding to this case. We note that two clusters of coherent motions are formed for some parameter values of $a(i)$ and collapse successively, along with the quasi-periodic change of the mean field.

To see our scenario here, we need to clarify $a$) how the internal structure determines the mean field and $b$) how the mean field modifies the stability of this internal bifurcation structure.

The step $a$) is rather simple. The mean field is determined by

$$h_{n-1} = \frac{1}{N}\sum_{i=1}^{N} f_i(x_{n-1}(i)) = \frac{1}{N}\sum_{i=1}^{N} x_n(i). \tag{1}$$

On the other hand, the stability of clustered motion is determined by the effective map for each element:

$$x_{3n} = F_{h_{3n-1}}(F_{h_{3n-2}}(F_{h_{3n-3}}(x))), \tag{2}$$

which is the 3rd iterate of $F_{h_n}(x) = (1-\epsilon)(1-ax^2) + \epsilon h_n$. Change of the mean field modifies the shape of eq. (2), which induces the change in the stability of each clustered motion. Then

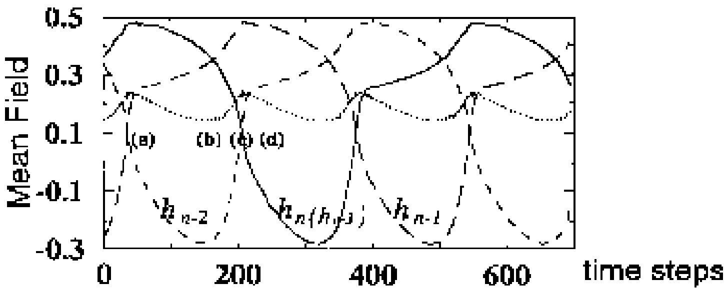


Fig. 5. – Time series of the mean fields $h_n(h_{n-3})$ (solid line), $h_{n-1}$ (long dashed line), $h_{n-2}$ (dashed line) are plotted every 3 steps ($n = 3m$). The average of the 3 steps $\frac{h_n+h_{n-1}+h_{n-2}}{3}$ (dotted line) is also plotted. The steps shown as *a*), *b*), *c*), *d*) correspond to those in fig. 4, respectively.

the internal bifurcation structure is modified. Thus the process *b*) is obtained. To sum up *a*) and *b*), the following feedback process exists:

1: Change in the Internal Bifurcation Structure

$\Downarrow$ $\Uparrow$

2: Change of the Mean Field | 4: Change of the stability in the structure (3)

$\Downarrow$ $\Uparrow$

3: Change in the Map for each element

(The 2nd, 3rd and 4th processes occur simultaneously, which determine the dynamics in each element.)

It should be noted that the time scale of the change of the structure is longer than the oscillation of each element (see fig. 5). For the above case, the oscillation of each element is near period-3, while the modulation (to change the phase of period-3 oscillations) takes $170 \times 3$ steps. For the case of fig. 1 *a*) the distribution of elements has a 7-banded structure (roughly speaking), where the modulation to change the phase has a longer time scale ($25 \times 7$ steps). Such slow modulation of the mean-field dynamics is formed by the above feedback diagram. This separation of time scales is necessary to have a low-dimensional collective order; otherwise high-dimensional chaotic dynamics remains [6].

Let us consider the above scenario in detail along fig. 4, which are shown through $3n$ steps, because the clustered motion has period 3.

Two-clustered motion is formed at $x = 0$ (denoted by $c0$) and at $x = 1$ (denoted by $c+$) (fig. 4 *a*)). The $c0$-clustered motion breaks down at $a \approx 1.918$ by crisis and the elements with larger $a(i)$ than this value leave the cluster. The 2nd tangent bifurcation occurs near $x = 1.0$ and $a = 1.92$, and this $c+$-clustered motion attracts elements which have left the $c0$-cluster. This makes the mean field $h_{n-1}$ increase due to eq. (1) (*a*) (*b*) in fig. 5). At the same time, $h_n$ and $h_{n-2}$ decrease, as is expected by successive mappings of $c0$ and $c+$ from step $n$. This change of the mean field modifies the map (2).

The modification of the map (2) destabilizes the $c0$-cluster and stabilizes the $c+$-cluster (fig. 4 *b*), *c*)). The $c0$-cluster starts to collapse from smaller values of $a$ successively, and with this process the $c+$-cluster grows from larger to smaller $a$. This makes the mean fields $h_{n-1}$ and $h_{n-2}$ increase, and $h_n$ decrease (*b*), *c*) in fig. 5). This change of the mean field modifies the map for each element, and a new clustered motion is formed near $x = -0.7$ (denoted as $c-$), besides the cluster at $x = 1$ ($c+$). The same process as above continues, by changing the roles of clusters until $c+$-cluster collapses and $c0$-cluster is formed (fig. 4 *d*)). In this way the feedback (3) is repeated.

We note that the internal bifurcation structure is clear, since the value of $a$ is nonidentical. The role of elements is differentiated as to the synchronization and desynchronization, which temporally changes as in the case for chaotic itinerancy [1], [13]. The extension of the present scenario to GCM with identical parameters will be given elsewhere, where the change of roles of elements is discussed only through a distribution.

Although we have explained the above scenario for the period-3 window case due to its simplicity, this mechanism is generally applied to our system, since each (logistic) dynamics contains a variety of windows and bifurcations. Although in some cases the window structures in the internal bifurcation are not clearly visible, the present scenario for collective motion generally holds when the roles of elements are differentiated according to the internal bifurcation, as is typical in a heterogeneous system (see fig. 1). Thus our scenario for the collective order can be observed in coupled systems such as Josephson junction arrays, and multi-mode laser systems, as well as in biological networks, where parameters are distributed by elements.

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The authors would like to thank N. Nakagawa for useful discussions. This work is partially supported by Grant-in-Aids for Scientific Research from the Ministry of Education, Science, and Culture of Japan.

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[14] The period can be $3 \times 2^k$, although this doubling is irrelevant to the following arguments. For example, the period is 6 for fig. 2 *a*).
[15] In the single logistic map, the 3rd iterate of the map $y = f(f(f(x)))$ is tangential to $y = x$ at 3 points corresponding to the periodic points. On the other hand, for the mapping $f_{\delta_n}(x) = 1 - ax^2 + \delta_n$ with a time-dependent parameter $\delta_n$ as in the present case, the 3rd iterate $y = f_{\delta 3}(f_{\delta 2}(f_{\delta 1}(x)))$ is tangential to $y = x$ only at one point, unless there is a certain restriction to the external field $\delta$. In other words, one specific phase of the period-3 oscillation is selected in accordance with the external parameter.
[16] Within a small parameter region between locking state (fig. 2 *a*)) and quasi-periodic state (fig. 2 *b*)), the two attractors coexist depending on the initial conditions.