2次元拡散過程の再帰性を保存する変形

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はしがき

本研究は下記分担者の協力を得て２次元拡散過程の再帰性を保存する変形に関して遂行したものであり、本冊子はその研究内容の報告である。

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研究課題　２次元拡散過程の再帰性を保存する変形

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1 An Example

Given $a > 0$ consider the surface $x_3 = \frac{1}{2}a((x_1)^2 + (x_2)^2)$ in $\mathbb{R}^3$. The standard Euclidean structure $dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3$ induces a Riemannian structure g on the surface. We are interested in a complex C^1-parameter ζ such that

$$g = \mathrm{Re}(\overline{\lambda d\zeta} \otimes \lambda d\zeta) \text{ for some } \mathbb{C}^\times\text{-valued function } \lambda. \tag{1.1}$$

This means the metric g is Hermitian with respect to the parameter ζ. Such a parameter is called *isothermal* with respect to the metric g.

To discuss an equation for isothermal parameters to satisfy, we fix an known complex parameter z, in this case $z = x_1 + \sqrt{-1}x_2$ is a natural one. The quantity $|\partial_z\zeta|^2 - |\partial_{\bar{z}}\zeta|^2$ is exactly the Jacobian $J[\zeta : z]$ where $\partial_z = \frac{\partial}{\partial z}$ and $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$. Without loss of generality we may restrict ourself to the situation $J[\zeta : z] > 0$. The right hand side of (1.1) reads

$$(|\alpha|^2 + |\beta|^2)\,\mathrm{Re}(d\bar{z} \otimes dz) + 2\,\mathrm{Re}(\overline{\alpha}\beta d\bar{z} \otimes d\bar{z}). \tag{1.2}$$

Here we write $\alpha = \lambda\partial_z\zeta$ and $\beta = \lambda\partial_{\bar{z}}\zeta$. Note that $|\lambda\partial_z\zeta|^2 - |\lambda\partial_{\bar{z}}\zeta|^2$ is positive. Actually any Riemannian metric have such a representation for some complex valued functions α and β with $|\alpha|^2 - |\beta|^2 > 0$. We now pay attention to the gauge invariance.

1.3 Lemma. *Two pairs (α, β) and (α', β') determine the same Riemannian metric by (1.2) if and only if there exists a $U(1)$-valued function u such that $\alpha' = u\alpha$ and $\beta' = u\beta$.*

We note that if $|\alpha|^2 - |\beta|^2 > 0$ then the ratio $\mu := \frac{\beta}{\alpha}$ is well defined and $|\mu| < 1$.

1.4 Definition. The gauge invariant quantity μ is called the Beltrami coefficient, denoted by $b[g; z]$, of the Riemannian metric g with respect to the parameter z.

We now determine the Beltrami coefficient of g with respect to the parameter

$$z := x_1 + \sqrt{-1}x_2.$$

Observe that $dx_1 \otimes dx_1 + dx_2 \otimes dx_2 = \mathrm{Re}(d\bar{z} \otimes dz)$. While

$$dx_3 \otimes dx_3 = \tfrac{a}{2}(\bar{z}dz + zd\bar{z}) \otimes \tfrac{a}{2}(\bar{z}dz + zd\bar{z}).$$

Expanding the right hand side we get

$$g = (1 + \frac{1}{2}a^2|z|^2)\,\mathrm{Re}(d\bar{z} \otimes dz) + \frac{1}{2}a^2\,\mathrm{Re}(z^2 d\bar{z} \otimes d\bar{z}). \tag{1.5}$$

Comparing (1.2) and (1.5), we infer that

$$|\alpha|^2(1 + |b[g; z]|^2) = 1 + \frac{1}{2}a^2|z|^2 \text{ and } 2|\alpha|^2 b[g; z] = \frac{1}{2}a^2 z^2.$$

We see that $2|\alpha| = 1 + \sqrt{1 + a^2|z|^2}$ (α is determined up to $U(1)$-factor due to the gauge symmetry) and hence

$$b[g; z] = a^2z^2/(1 + \sqrt{1 + a^2|z|^2})^2. \tag{1.6}$$

Once the Beltrami coefficient μ is determined, the equation to be solved is

$$\lambda d\zeta = \alpha(dz + b[g;z]d\bar{z}).$$

We see that the function λ is an integral factor, i.e., the 1-form $\frac{\alpha}{\lambda}(dz + b[g;z]d\bar{z})$ is closed. To eliminate the gauge factor, we extract the ratio of the coefficients of dz and $d\bar{z}$. We thus reach the following equation with $b[g;z]$ given in (1.6):

$$\frac{\partial \zeta}{\partial \bar{z}} = b[g;z]\frac{\partial \zeta}{\partial z} \tag{1.7}$$

called the Beltrami equation associated with g with respect to the parameter z.

1.8 Lemma. *Let ζ be a complex C^1-parameter on a surface equipped with a Riemannian structure. Then ζ is isothermal if and only if it solves the associated Beltrami equation.*

An complex parameter introduces a complex structure on the surface. A complex structure τ and a Riemannian metric g are compatible if and only if all holomorphic parameters are isothermal with respect to g. In the next section we discuss the subject from the view point of complex structures.

2 Some general theory

Let S be a Riemann surface and let g be a Riemannian metric on S. We do *not* assume that the complex structure and g are compatible.

Suppose (U,ζ) and (V,ξ) are isothermal C^1-parameters with respect to g. If $U \cap V \neq \emptyset$ then there exists a nonnegative function c such that $\mathrm{Re}(d\bar{\zeta} \otimes d\zeta) = c\,\mathrm{Re}(d\bar{\xi} \otimes d\xi)$ on $U \cap V$. If both ζ and ξ are positively oriented, this means the mapping $\zeta \circ \xi^{-1} : \xi(U \cap V) \to \mathbb{C}$ is conformal. Thus an atlas consisting of positively oriented isothermal C^1-parameters with respect to g introduces a complex structure compatible with g.

The way to find isothermal parameters with respect to g is as follows:

Step 1 Given a holomorphic parameter (U,z), determine the Beltrami coefficient $b[g;z]$. That is to find a function μ on U such that $|\mu| < 1$ and g is a conformal change of $(1 + |\mu|^2)\,\mathrm{Re}(\overline{dz} \otimes dz) + 2\,\mathrm{Re}(\mu d\bar{z} \otimes d\bar{z})$.

Step 2 Find a C^1-parameter solving the associate Beltrami equation $\partial_{\bar{z}}\zeta = b[g;z]\partial_z\zeta$. If necessary choose U sufficiently small so that the equation is solvable on U.

2.1 Remark. Suppose (V,w) is another holomorphic parameter with $U \cap V \neq \emptyset$. Then $b[g;w] = \frac{\lambda}{\bar{\lambda}}b[g;z]$ on $U \cap V$ where $dw = \lambda dz$.

Thus the isothermal parameter problem leads us to the the existence problem of (locally) diffeomorphic solutions to the associated Beltrami equation. To develop a deeper theory it is necessary to accept coefficients which may not originate from Riemannian metrics. We even relax their continuity. In what follows D and E denote domains in $\mathbb{C}$.

2.2 Definition. $\mu : D \to \mathbb{C}$ is called a generalized Beltrami coefficient if it is measurable and $\sup_A |\mu| < 1$ for all compact subsets A. $\mathrm{Blt}(D)$ denotes the space of all generalized Beltrami coefficients on D. Let $\mu \in \mathrm{Blt}(D)$. We say a function $f : D \to \mathbb{C}$ solves the Beltrami equation with coefficient μ if it is partially ACL and measurable and the weak partial derivatives are locally integrable and satisfy $\partial_{\bar{z}}f = \mu\partial_z f$ a.e.

If μ vanishes then Beltrami equation reduces to Cauchy-Riemann equation in the sense of distribution, whose solution is necessarily holomorphic.

2.3 Lemma. *Let $\mu \in \mathrm{Blt}(D)$. Suppose that a function $f : D \to \mathbb{C}$ solves the Beltrami equation with coefficient μ.*
(i) *If $f(D) \subset E$ and $g : E \to \mathbb{C}$ is a holomorphic function then $g \circ f : D \to \mathbb{C}$ solves the Beltrami equation with coefficient μ.*
(ii) *If $w : D \to \mathbb{C}$ is a holomorphic parameter then $f \circ w^{-1} : w(D) \to \mathbb{C}$ solves the Beltrami equation with coefficient $\left(\mu \frac{\overline{w'}}{w'}\right) \circ w^{-1}$.*

The argument on the existence problem is quite deep.

2.4 Lemma. *Given $\mu \in \mathrm{Blt}(\mathbb{C})$ with compact support, there exists a self-homeomorphism $\zeta : \mathbb{C} \to \mathbb{C}$ which solves the Beltrami equation with coefficient μ.*

Any self-homeomorphism of $\mathbb{C}$ continues to a self-homeomorphism of $\mathbb{C} \cup \{\infty\}$. Thus, making use of Möbius transformations, we deduce the next from Lemma 2.3 and Lemma 2.4.

2.5 Corollary. *If $\mu \in \mathrm{Blt}(\mathbb{C})$ vanishes on a nonvoid open subset and $\|\mu\|_\infty < 1$ then there is a homeomorphism $\zeta : \mathbb{C} \to \mathbb{C}$ solving the Beltrami equation with coefficient μ.*

2.6 Theorem. *Let D be the unit disk and $\mu \in \mathrm{Blt}(D)$. If $\|\mu\|_\infty < 1$ then there exists a homeomorphism $\zeta : D \to D$ solving the Beltrami equation with coefficient μ.*

Proof. According to Corollary 2.5 there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ solving the Beltrami equation with coefficient $\mu 1_D$. Clearly $f(D)$ is simply connected and omit some neighbourhood of ∞. We infer by Riemann's mapping theorem that there exists a biholomorphic mapping $g : f(D) \to D$. The composition $g \circ f$ is a desired one. □

To relax the support condition we introduce the notion of quasi-conformal mappings.

2.7 Lemma. *Let $\alpha, \beta \in \mathbb{C}$ and $K \geq 1$. Then the following six conditions are equivalent:*
(i) $\min\{|\alpha|, |\beta|\} \leq \frac{K-1}{K+1} \max\{|\alpha|, |\beta|\}$.
(ii) $\max_{v \in U(1)} |\alpha v + \beta \bar{v}| \leq K \min_{v \in U(1)} |\alpha v + \beta \bar{v}|$.
(iii) $4|\alpha||\beta| \leq (K - \frac{1}{K})\big||\alpha|^2 - |\beta|^2\big|$.
(iv) $2(|\alpha|^2 + |\beta|^2) \leq (K + \frac{1}{K})\big||\alpha|^2 - |\beta|^2\big|$.
(v) $|u\alpha + v\bar{\beta}|^2 + |u\beta + v\bar{\alpha}|^2 \leq K(|u|^2 + |v|^2)\big||\alpha|^2 - |\beta|^2\big|$ *for all $u, v \in \mathbb{C}$.*
(vi) $|u\alpha + v\bar{\beta}|^2 + |u\beta + v\bar{\alpha}|^2 \geq \frac{1}{K}(|u|^2 + |v|^2)\big||\alpha|^2 - |\beta|^2\big|$ *for all $u, v \in \mathbb{C}$.*

2.8 Definition. A mapping $\zeta : D \to \mathbb{C}$ is called quasi-conformal if it is locally homeomorphic, partially ACL and, for each compact set A in D, there exists $K \geq 1$ such that

$$|\partial_z \zeta| + |\partial_{\bar{z}} \zeta| \leq K\big(|\partial_z \zeta| - |\partial_{\bar{z}} \zeta|\big) \text{ a.e. on } A.$$

If the bound K can be chosen independent of compact sets A then ζ is called K-quasi-conformal.

2.9 Remark. A local homeomorphism solving a Beltrami equation is quasi-conformal.

We are going to show that every quasi-conformal mapping satisfies a Beltrami equation.

2.10 Definition. $W_{\text{loc}}(D)$ denotes the space of all partially ACL measurable functions on D whose weak partial derivatives are locally square integrable. For $f \in W_{\text{loc}}(D)$ and a Borel subset A we write $\text{Enrg}(f; A) := \int_A \{|\partial_z f|^2 + |\partial_{\bar{z}} f|^2\}\text{vol}$ (energy integral).

2.11 Lemma. *Let $\zeta : D \to \mathbb{C}$ be quasi-conformal. Then it is totally differentiable a.e. and belongs to $W_{\text{loc}}(D)$. If ζ is injective then*

$$\zeta^*\text{vol} = \int_{\cdot} J[\zeta : z]\, \text{vol}$$

and, provided it is K-quasi-conformal on A,

$$\text{vol}(\zeta(A)) \leq \text{Enrg}(\zeta; A) \leq \tfrac{1}{2}(K + \tfrac{1}{K})\text{vol}(\zeta(A)).$$

Proof. The continuity and the partial ACL property imply that ζ is partially differentiable a.e. As for continuous open mappings from a 2-dimensional domain into $\mathbb{C}$, the a.e. partial differentiability automatically implies the a.e. total differentiability (c.f. Gehring and Lehto). On the other hand if a local homeomorphism is totally differentiable a.e. then its Jacobian is locally integrable. Observe that on each compact subset

$$|\partial_z\zeta|^2 + |\partial_{\bar{z}}\zeta|^2 \leq \frac{1}{2}(K + \tfrac{1}{K})\{|\partial_z\zeta|^2 - |\partial_{\bar{z}}\zeta|^2\} = \frac{1}{2}(K + \tfrac{1}{K})J[\zeta : z]$$

holds a.e. for some $K \geq 1$. Therefore the partial derivatives are locally square integrable. The dimension being 2, the change of variable formula is valid (?!). □

2.12 Corollary. 1-*quasi-conformal mappings are nothing but conformal mappings.*

Proof. Let $\zeta : D \to \mathbb{C}$ be a 1-quasi-conformal. Then $\partial_{\bar{z}}\zeta = 0$ a.e. Since $\zeta \in W_{\text{loc}}(D)$, it satisfies Cauchy-Riemann equation in the sense of distribution. □

2.13 Theorem. *Let $K \geq 1$ and $\zeta : D \to \mathbb{C}$ be a parameter. If it is orientation preserving, $\zeta \in W_{\text{loc}}(D)$ and there exists an open base $\mathfrak{D}$ such that*

$$\text{Enrg}(\zeta; A) \leq \tfrac{1}{2}(K + \tfrac{1}{K})\text{vol}(\zeta(A)) \textit{ for all } A \in \mathfrak{D}$$

then ζ is K-quasi-conformal.

Proof. Since ζ is injective and continuous, and $\zeta \in W_{\text{loc}}(D)$, the change of variable formula is valid. It therefore follows that

$$|\partial_z\zeta|^2 + |\partial_{\bar{z}}\zeta|^2 \leq \tfrac{1}{2}(K + \tfrac{1}{K})|J[\zeta : z]| \text{ a.e.}$$

Thus we infer by Lemma 2.7 that ζ is K-quasi-conformal. □

To prove the next statement the so-called geometric view point is indispensable.

2.14 Theorem. *Let $\zeta : D \to \mathbb{C}$ and $\xi : E \to \mathbb{C}$ be quasi-conformal mappings with $\zeta(D) \subset E$. Then $\xi \circ \zeta$ is quasi-conformal. If ζ is injective then $\zeta^{-1} : \zeta(D) \to \mathbb{C}$ is quasi-conformal.*

2.15 Lemma. *Let $\zeta : D \to \mathbb{C}$ be a finitely covered quasi-conformal mapping. Then the image measure $\zeta_*\text{vol}$ is absolutely continuous and its density function is given by the summation of $\frac{1}{J[\zeta:z]}$ over each fiber of ζ.*

Proof. We may assume that ζ is injective by localizing ζ. Applying Lemma 2.11 to ζ^{-1} we see that $\zeta_*\mathrm{vol} = (\zeta^{-1})^*\mathrm{vol}$ is absolutely continuous and its density function is given by $J[\zeta^{-1} : z]$. On the other hand applying Lemma 2.11 to ζ we see that $\zeta^*\mathrm{vol}$ is absolutely continuous, which means ζ-image of a null set also has zero-measure. Combining with the a.etotal differentiabilty of ζ and ζ^{-1}, we obtain that $J[\zeta : z] \circ \zeta^{-1} J[\zeta^{-1} : z] = 1$ a.e. □

2.16 Corollary. *Let $\zeta : D \to \mathbb{C}$ be a quasi-conformal mapping. Then there exists a Borel set of full measure on which ζ is totally differentiable and $|\partial_{\bar{z}}\zeta| < |\partial_z\zeta|$.*

Proof. Due to Lemma 2.11 and Lemma 2.15, the set

$$\{x \in D : \zeta \text{ is differentiable}, J[\zeta : z] > 0\}$$

is of full measure. Up to null sets the above set coincides with

$$\{x \in D : \zeta \text{ is differentiable}, \partial_z\zeta \neq 0\},$$

which is thus of full measure. Finally note that $\frac{K-1}{K+1} < 1$ for $K \geq 1$. □

2.17 Definition. Let $\zeta : D \to \mathbb{C}$ be a quasi-conformal mapping. We set

$$\mathrm{dlt}[\zeta] := \begin{cases} \partial_{\bar{z}}\zeta / \partial_z\zeta & \text{on } \{x \in D : \zeta \text{ is differentiable}, |\partial_{\bar{z}}\zeta| < |\partial_z\zeta|\} \\ 0 & \text{elsewhere} \end{cases}.$$

The measurable function $\mathrm{dlt}[\zeta]$ is called the complex dilatation of ζ.

Clearly the complex dilatations are generalized Beltrami coefficients.

2.18 Theorem. *If $\zeta : D \to \mathbb{C}$ is quasi-conformal then $\zeta \in W_{\mathrm{loc}}(D)$, $\partial_{\bar{z}}\zeta = \mathrm{dlt}[\zeta]\partial_z\zeta$ a.e.*

We investigate the transformation rule of Beltrami equations.

2.19 Definition. Let $\zeta : D \to \mathbb{C}$ be a quasi-conformal mapping with $\zeta(D) \subset E$. Given $\mu \in \mathrm{Blt}(E)$ define a generalized Beltrami coefficient on D by

$$\zeta^*\mu := \frac{\mathrm{dlt}[\zeta] + \mu \circ \zeta \, \frac{\overline{\partial_z\zeta}}{\partial_z\zeta}}{1 + \overline{\mathrm{dlt}[\zeta]}\, \mu \circ \zeta \, \frac{\overline{\partial_z\zeta}}{\partial_z\zeta}} \text{ on } \{x \in D : \zeta \text{ is differentiable}, |\partial_{\bar{z}}\zeta| < |\partial_z\zeta|\}$$

and $\zeta^*\mu := 0$ elsewhere. $\zeta^*\mu$ is called the pull-back of μ by ζ.

Since quasi-conformal mappings preserves the notion of measure zero, it follows that if $\mu = \nu$ a.e. then $\zeta^*\mu = \zeta^*\nu$ a.e.

2.20 Lemma. *Suppose $\zeta : D \to \mathbb{C}$ is quasi-conformal, $\zeta(D) \subset E$ and $f : E \to \mathbb{C}$ is a.etotally differentiable and measurable. Then $f \circ \zeta$ is totally differentiable a.e.*
(i) $\partial_z(f \circ \zeta) = (\partial_z f) \circ \zeta \partial_z\zeta + (\partial_{\bar{z}} f) \circ \zeta \overline{\partial_{\bar{z}}\zeta}$ *a.e.*, $\partial_{\bar{z}}(f \circ \zeta) = (\partial_z f) \circ \zeta \partial_{\bar{z}}\zeta + (\partial_{\bar{z}} f) \circ \zeta \overline{\partial_z\zeta}$ *a.e.*
(ii) *If $\partial_{\bar{z}} f = \mu \partial_z f$ a.efor some $\mu \in \mathrm{Blt}(E)$ then $\partial_{\bar{z}}(f \circ \zeta) = \zeta^*\mu \, \partial_z(f \circ \zeta)$ a.e.*
(iii) *If $\partial_z f$ and $\partial_{\bar{z}} f$ are locally square integrable then so are $\partial_{\bar{z}}(f \circ \zeta)$ and $\partial_z(f \circ \zeta)$.*

Proof. Let N be the set of non totally differentiable points for f. Then it is of measure zero and hence so is $\zeta^{-1}(N)$. We next choose a subset A of D with full measure such that $A \cap \zeta^{-1}(N) = \emptyset$, ζ is totally differentiable on A and $|\partial_{\bar{z}}\zeta| < |\partial_z\zeta|$ on A. Consequently $f \circ \zeta$ is totally differentiable on A. Observe that

$$d(f \circ \zeta) = (\partial_z f) \circ \zeta d\zeta + (\partial_{\bar{z}} f) \circ \zeta d\bar{\zeta} \quad \text{on } A.$$

Extracting the coefficients of dz and $d\bar{z}$ respectively, we get the chain rule (i). Since

$$\partial_{\bar{z}}\zeta + \mu \circ \zeta\, \overline{\partial_z\zeta} = \zeta^*\mu(\partial_z\zeta + \mu \circ \zeta\, \overline{\partial_{\bar{z}}\zeta}) \quad \text{on } A$$

and $\partial_{\bar{z}} f = \mu \partial_z f$ a.e., the equation $\partial_{\bar{z}}(f \circ \zeta) = \zeta^*\mu\, \partial_z(f \circ \zeta)$ a.e. follows. We finally prove the square integrability of derivatives. Let B be a compact subset of D. Then there exists $K \geq 1$ such that $|\partial_z\zeta| + |\partial_{\bar{z}}\zeta| \leq K(|\partial_z\zeta| - |\partial_{\bar{z}}\zeta|)$ a.e. on B. Applying (v) and (vi) of Lemma 2.7 for $\alpha = \partial_z\zeta$, $\beta = \partial_{\bar{z}}\zeta$, $u = (\partial_z f) \circ \zeta$ and $v = (\partial_{\bar{z}} f) \circ \zeta$ we get

$$|\partial_z(f \circ \zeta)|^2 + |\partial_{\bar{z}}(f \circ \zeta)|^2 \leq K(|(\partial_z f) \circ \zeta|^2 + |(\partial_{\bar{z}} f) \circ \zeta|^2) J[\zeta : z],$$
$$|\partial_z(f \circ \zeta)|^2 + |\partial_{\bar{z}}(f \circ \zeta)|^2 \geq \tfrac{1}{K}(|(\partial_z f) \circ \zeta|^2 + |(\partial_{\bar{z}} f) \circ \zeta|^2) J[\zeta : z],$$

both of which hold a.e. on B. Consequently we have that

$$\int_B \{|\partial_z(f \circ \zeta)|^2 + |\partial_{\bar{z}}(f \circ \zeta)|^2\} \operatorname{vol} \leq K \int_{\zeta(B)} \{|\partial_z f|^2 + |\partial_{\bar{z}} f|^2\} \operatorname{vol}, \tag{2.21}$$

$$\int_B \{|\partial_z(f \circ \zeta)|^2 + |\partial_{\bar{z}}(f \circ \zeta)|^2\} \operatorname{vol} \geq \frac{1}{K} \int_{\zeta(B)} \{|\partial_z f|^2 + |\partial_{\bar{z}} f|^2\} \operatorname{vol} \tag{2.22}$$

by Lemma 2.11. □

2.23 Lemma. *If $\zeta : D \to \mathbb{C}$ and $\xi : E \to \mathbb{C}$ are quasi-conformal mappings with $\zeta(D) \subset E$ and $\nu \in \mathrm{Blt}(\xi(E))$ then*

$$\mathrm{dlt}[\xi \circ \zeta] = \zeta^*\mathrm{dlt}[\xi] \ \text{a.e. and } (\xi \circ \zeta)^*\nu = \zeta^*(\xi^*\nu) \ \text{a.e.}$$

If $\zeta : D \to \mathbb{C}$ is an injective quasi-conformal mapping and $\mu \in \mathrm{Blt}(D)$ then

$$(\zeta^{-1})^*\mu = \Big(\frac{\mu - \mathrm{dlt}[\zeta]}{1 - \overline{\mathrm{dlt}[\zeta]}\mu} \frac{\partial_z\zeta}{\overline{\partial_z\zeta}}\Big) \circ \zeta^{-1} \ \text{a.e.}$$

Proof. We see by Lemma 2.15, Theorem 2.18 and Lemma 2.20 that

$$\partial_{\bar{z}}(\xi \circ \zeta) = \zeta^*\mathrm{dlt}[\xi]\partial_z(\xi \circ \zeta) \text{ a.e.}$$

Since $\xi \circ \zeta$ is also quasi-conformal, we get the first relation. We may localize the problem to discuss the second relation. Given a relatively compact open subset U of $\xi(E)$ there exists a quasi-conformal mapping $g : U \to \mathbb{C}$ such that

$$\mathrm{dlt}[g] = \nu \text{ a.e.}$$

due to Lemma 2.4. It then follows that

$$\mathrm{dlt}[g \circ \xi] = \xi^*\nu \text{ a.e. and hence } \mathrm{dlt}[(g \circ \xi) \circ \zeta] = \zeta^*(\xi^*\nu) \text{ a.e.}$$

On the other hand we also have that

$$\mathrm{dlt}[g \circ (\xi \circ \zeta)] = (\xi \circ \zeta)^* \nu.$$

Consequently weget the second relation. We next discuss the last claim. Since $\zeta^*\mathrm{dlt}[\zeta^{-1}] = 0$ a.e., we see that

$$\mathrm{dlt}[\zeta]\frac{\overline{\partial_z \zeta}}{\partial_z \zeta} = -\mathrm{dlt}[\zeta^{-1}] \circ \zeta \text{ a.e.}$$

On the other hand we have that

$$(1,0) = ((\partial_z \zeta^{-1}) \circ \zeta, (\partial_{\bar z} \zeta^{-1}) \circ \zeta) \begin{pmatrix} \partial_z \zeta & \partial_{\bar z} \zeta \\ \overline{\partial_{\bar z} \zeta} & \overline{\partial_z \zeta} \end{pmatrix} \text{ a.e.}$$

Inverting the matrix we get

$$(|\partial_z \zeta|^2 - |\partial_{\bar z} \zeta|^2)(\partial_z \zeta^{-1}) \circ \zeta = \overline{\partial_z \zeta} \text{ a.e.,}$$

which implies the desired relation. □

2.24 Lemma. *Let $\mu \in \mathrm{Blt}(\mathbb{C})$. If $\|\mu\|_\infty < 1$ then there exists a quasi-conformal homeomorphism $\zeta : \mathbb{C} \to \mathbb{C}$ with $\mathrm{dlt}[\zeta] = \mu$ a.e., in other words, there exists a self-homeomorphism of $\mathbb{C}$ solving the Beltrami equation with coefficient μ.*

Proof. According to Corollary 2.5 there exists a homeomorphism $g : \mathbb{C} \to \mathbb{C}$ solving the Beltrami equation with coefficient $\mu 1_{B^{\mathrm{ext}}}$. Note that g is quasi-conformal and $\mathrm{dlt}[g] = \mu 1_{B^{\mathrm{ext}}}$ a.e. Define $\nu \in \mathrm{Blt}(\mathbb{C})$ by $\left(\mu \frac{\partial_z g}{\overline{\partial_z g}}\right) \circ g^{-1} 1_{g(\overline{B})}$ so that we have

$$g^*\nu = \frac{\mathrm{dlt}[g] + \nu \circ g \, \frac{\overline{\partial_z g}}{\partial_z g}}{1 + \overline{\mathrm{dlt}[g]}\, \nu \circ g \, \frac{\overline{\partial_z g}}{\partial_z g}} = \frac{\mu 1_{B^{\mathrm{ext}}} + \mu 1_{\overline{B}}}{1 + \overline{\mu 1_{B^{\mathrm{ext}}}}\, \mu 1_{\overline{B}}} = \mu \text{ a.e.}$$

The support being compact, there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ solving the Beltrami equation with coefficient ν. Then f is quasi-conformal and $\mathrm{dlt}[f] = \nu$ a.e. We see that $f \circ g$ is also quasi-conformal and, by Lemma 2.23, $\mathrm{dlt}[f \circ g] = \mu$ a.e. □

2.25 Theorem. *Suppose that both D and E are bounded Jordan domains. Let $\varphi : D \to E$ be a quasi-conformal homeomorphism with $\|\mathrm{dlt}[\varphi]\|_\infty < 1$. Then φ extends to a homeomorphism $\overline{D} \to \overline{E}$ (Caratheodory).*

Proof. Choose a homeomorphism $\zeta : \mathbb{C} \to \mathbb{C}$ solving the Beltrami equation with coefficient $\mathrm{dlt}[\varphi]1_D$. Note that $\zeta(D)$ is also a bounded Jordan domain and $\zeta \circ \varphi^{-1} : E \to \zeta(D)$ is a conformal isomorphism by Corollary 2.12 and Lemma 2.23. The composition extends to a homeomorphism $\psi : \overline{E} \to \overline{\zeta(D)}$ by Caratheodory's theorem. Since $\overline{\zeta(D)} = \zeta(\overline{D})$, we see that $\psi^{-1} \circ \zeta|_{\overline{D}}$ is a desired homeomorphism. □

3 More on quasi-conformal mappings

Recall that Lipschitz continuous functions are totally differentiable a.e.

3.1 Lemma. *Suppose $\zeta : D \to \mathbb{C}$ is a quasi-conformal mapping and $f : E \to \mathbb{C}$ is a Lipschitz continuous function with $\zeta(D) \subset E$. Then $f \circ \zeta \in W_{\mathrm{loc}}(D)$. If ζ is injective and K-quasi-conformal on A then $\frac{1}{K}\mathrm{Enrg}(f; \zeta(A)) \le \mathrm{Enrg}(f \circ \zeta; A) \le K\mathrm{Enrg}(f; \zeta(A))$.*

Proof. Due the Lipschitz continuity of f the partial ACL property holds for the composition $f \circ \zeta$. Thus we get the result by Lemma 2.20 and the inequalities (2.21). □

To extend Lemma 3.1 to the statement for $f \in W_{\mathrm{loc}}(E)$ we need several results from Sobolev space theory.

3.2 Lemma. *If $f \in W_{\mathrm{loc}}(E)$ then there exists a sequence $\{f_k\}$ in $C_0^\infty(E)$ such that f_k converges to f a.e. and $\lim_{k,l\to\infty} \mathrm{Enrg}(f_k - f_l; B) = 0$ for all compact subsets B.*

The proof of this Lemma is easy. But the next one is much hard to prove.

3.3 Lemma. *Let $g : D \to \mathbb{C}$ be a Lebesgue measurable function and $\{g_k\}$ be a sequence in $W_{\mathrm{loc}}(D)$. If $\lim_{k,l\to\infty} \mathrm{Enrg}(g_k - g_l; A) = 0$ for all compact subsets A and g_k converges to g a.e. then $g \in W_{\mathrm{loc}}(D)$ and $\lim_{k\to\infty} \mathrm{Enrg}(g_k - g; A) = 0$ for all compact subsets A.*

3.4 Theorem. *Suppose $f \in W_{\mathrm{loc}}(E)$, $\zeta : D \to \mathbb{C}$ is quasi-conformal and $\zeta(D) \subset E$. Then $f \circ \zeta \in W_{\mathrm{loc}}(D)$ and the claims (i) and (ii) in Lemma 2.20 are valid. If ζ is injective and K-quasi-conformal on A then $\frac{1}{K}\mathrm{Enrg}(f; \zeta(A)) \le \mathrm{Enrg}(f \circ \zeta; A) \le K\mathrm{Enrg}(f; \zeta(A))$.*

Proof. Given $f \in W_{\mathrm{loc}}(E)$, by virtue of Lemma 3.2, we can find a sequence $\{f_k\}$ in $C_0^\infty(E)$ such that f_k converges to f a.e. and $\lim_{k,l\to\infty} \mathrm{Enrg}(f_k - f_l; B) = 0$ for all compact subsets B in E. Let A be a compact set in D. Then there exists $K \ge 1$ such that ζ is K-quasi-conformal on A. According to Lemma 3.1, $f_k \circ \zeta \in W_{\mathrm{loc}}(D)$ and

$$\mathrm{Enrg}(f_k \circ \zeta - f_l \circ \zeta; A) \le K\mathrm{Enrg}(f_k - f_l; \zeta(A)).$$

$\zeta(A)$ being compact, the latter converges to 0 as k and l tend to ∞. On the other hand by Lemma 2.15 that $f_k \circ \zeta$ converges to f a.e. Invoking Lemma 3.3 we infer that $f \circ \zeta \in W_{\mathrm{loc}}(D)$ and $\lim_{k\to\infty} \mathrm{Enrg}(f_k \circ \zeta - f \circ \zeta; A) = 0$ for all compact subsets A in D. To prove the chain rule we choose a subsequence so that $\partial_z(f_k \circ \zeta)$ converges to $\partial_z(f \circ \zeta)$ a.e., $\partial_{\bar z}(f_k \circ \zeta)$ converges to $\partial_{\bar z}(f \circ \zeta)$ a.e., $\partial_z f_k$ converges to $\partial_z f$ a.eand $\partial_{\bar z} f_k$ converges to $\partial_{\bar z} f$ a.e. Taking Lemma 2.15 into account we infer that $(\partial_z f_k) \circ \zeta$ converges to $(\partial_z f) \circ \zeta$ a.e. and $(\partial_{\bar z} f_k) \circ \zeta$ converges to $(\partial_{\bar z} f) \circ \zeta$ a.e. Thus obtain the chain rule for the limit. Consequently we get the rest of the statement. □

3.5 Remark. The energy integral is a conformal invariant. Using the energy integral we later introduce the capacity, which is another conformal invariant.

By a parameter we will mean a continuous injection from a surface into $\mathbb{C}$.

3.6 Corollary. *Let $\zeta : D \to \mathbb{C}$ be a quasi-conformal parameter and $f : D \to \mathbb{C}$ be a Lebesgue measurable function. Then $f \circ \zeta^{-1} : \zeta(D) \to \mathbb{C}$ has a holomorphic modification if and only if $f \in W_{\mathrm{loc}}(D)$ and $\partial_{\bar z} f = \mathrm{dlt}[\zeta]\partial_z f$ a.e.*

Proof. The implication $\Rightarrow$ was discussed in Lemma 2.3. Since $(\zeta^{-1})^*\mathrm{dlt}[\zeta] = 0$ a.eby Lemma 2.23, we can deduce the converse implication $\Leftarrow$ from Theorem 3.4. □

3.7 Definition. Given $\mu \in \mathrm{Blt}(D)$ set $\mathcal{O}_\mu(U) := \{f \in W_{\mathrm{loc}}(U) : \partial_{\bar z} f = \mu \partial_z f$ a.e.$\}$ where U runs through all open subsets. We call $\mathcal{O}_\mu(\cdot)$ the sheaf of the solutions of the Beltrami equation with coefficient μ.

3.8 Theorem. *Given $\mu \in \mathrm{Blt}(D)$ there exists a unique complex structure on D whose structure sheaf is the sheaf of the solutions of the Beltrami equation with coefficient μ.*

Proof. It follows from Theorem 2.6 that, for each $x \in D$, there exists a quasi-conformal parameter (U, ζ) at x with $\mathrm{dlt}[\zeta] = \mu$ a.e. We see by Lemma 2.23 that the family of such parameters (U, ζ) consistently defines a holomorphic atlas on D. Thus a complex structure is introduced on D. Corollary 3.6 tells us that its structure sheaf is exactly $\mathcal{O}_\mu(\cdot)$. □

3.9 Definition. Each $\mu \in \mathrm{Blt}(D)$ shall also denote the complex structure whose structure sheaf is the sheaf of the solutions of the Beltrami equation with coefficient μ.

We still agree that domains in $\mathbb{C}$ are equipped with the standard complex structure unless otherwise stated.

3.10 Theorem. *Let D be a simply connected domain in $\mathbb{C}$ and let $\mu \in \mathrm{Blt}(D)$. Then there exists a parameter $D \to \mathbb{C}$ solving the Beltrami equation with coefficient μ, in other words, there exists a quasi-conformal parameter $\zeta : D \to \mathbb{C}$ with $\mathrm{dlt}[\zeta] = \mu$ a.e.*

Proof. D being simply connected and non-compact, there exists a global holomorphic parameter $\zeta : (D, \mu) \to \mathbb{C}$ due to the uniformization theorem. We see by Theorem 3.8 that ζ in fact solves the Beltrami equation with coefficient μ. □

3.11 Corollary. *Let D be a domain in $\mathbb{C}$ and $\mu \in \mathrm{Blt}(D)$. Suppose there exists a simply connected domain E in $\mathbb{C}$ such that $D \subset E$ and $\sup_{D \cap A} |\mu| < 1$ for all compact subsets A of E. Then there exists a quasi-conformal parameter $\zeta : D \to \mathbb{C}$ with $\mathrm{dlt}[\zeta] = \mu$ a.e.*

Proof. We define $\nu : E \to \mathbb{C}$ by $\nu = 1_D \mu$. Then it follows that $\nu \in \mathrm{Blt}(E)$. Thus the present claim derives from Theorem 3.10. □

We may rephrase Theorem 2.6 and Theorem 2.24 as follows: If D is simply connected and $\|\mu\|_\infty < 1$ then the complex structure μ is isomorphic with the standard complex structure. We mention a bit about moduli problem of complex structures.

3.12 Theorem. *Let $\varphi : D \to E$ be a homeomorphism, $\mu \in \mathrm{Blt}(D)$ and $\nu \in \mathrm{Blt}(E)$. Then φ is holomorphic relative to μ and ν if and only if it is quasi-conformal and $\varphi^* \nu = \mu$ a.e.*

Proof. We first suppose that φ is holomorphic relative to μ and ν. Let (U, ζ) be a holomorphic parameter for (E, ν). Then $\zeta \circ \varphi : (\varphi^{-1}(U), \mu) \to \mathbb{C}$ is holomorphic, which means it solves the Beltrami equation with coefficient μ. Being injective, $\zeta \circ \varphi$ is quasi-conformal and $\mathrm{dlt}[\zeta \circ \varphi] = \mu$. Since ζ is quasi-conformal and $\varphi = \zeta^{-1} \circ \zeta \circ \varphi$, we infer that φ is quasi-conformal on $\varphi^{-1}(U)$. Thus, by Lemma 2.23,

$$\varphi^* \nu = \varphi^*(\mathrm{dlt}[\zeta]) = \mathrm{dlt}[\zeta \circ \varphi] = \mu.$$

Conversely suppose that φ is quasi-conformal and $\varphi^* \nu = \mu$. Let (U, ζ) be a holomorphic parameter for (E, ν). Then $\zeta \circ \varphi : \varphi^{-1}(U) \to \mathbb{C}$ is quasi-conformal and

$$\mathrm{dlt}[\zeta \circ \varphi] = \varphi^*(\mathrm{dlt}[\zeta]) = \varphi^* \nu = \mu.$$

This implies that $\zeta \circ \varphi : (\varphi^{-1}(U), \mu) \to \mathbb{C}$ is holomorphic and hence φ is holomorphic on $\varphi^{-1}(U)$ relative to μ and ν. □

We may rephrase Theorem 3.12 as follows: Let $\mu \in \mathrm{Blt}(D)$ and $\nu \in \mathrm{Blt}(E)$. Then two Riemann surfaces (D, μ) and (E, ν) are conformally equivalent if and only if there exists a quasi-conformal homeomorphism $\varphi : D \to E$ such that $\varphi^* \nu = \mu$ a.e.

3.13 Definition. $\mathrm{qcAut}(D)$ denotes the group of all quasi-conformal automorphisms of D and $\mathrm{qcAut}_0(D) := \{\varphi \in \mathrm{qcAut}(D) :$ homotopic to the identity mapping$\}$. Given $\mu \in \mathrm{Blt}(D)$ write the group of all holomorphic automorphisms of (D, μ) by $\mathrm{cAut}(D; \mu)$.

The group $\mathrm{qcAut}(D)$ acts on the space $\mathrm{Blt}(D)$ by

$$\mathrm{Blt}(D) \times \mathrm{qcAut}(D) \to \mathrm{Blt}(D), \quad (\mu, \varphi) \mapsto \varphi^* \mu.$$

$\mathrm{qcAut}_0(D)$ is a normal subgroup of $\mathrm{qcAut}(D)$ and $\mathrm{Stab}(\mathrm{qcAut}(D), \mu) = \mathrm{cAut}(D; \mu)$ for each $\mu \in \mathrm{Blt}(D)$. The orbit space $\mathrm{Blt}(D)/\mathrm{qcAut}(D)$ is called the moduli space of complex structures while $\mathrm{Blt}(D)/\mathrm{qcAut}_0(D)$ is called the Teichmüler space.

3.14 Lemma. *Let $\mu \in \mathrm{Blt}(D)$. Then the $\mathrm{qcAut}_0(D)$-orbit of μ is given by $\{\varphi^* \mu\}$ where $\varphi \in \mathrm{qcAut}(D)$ runs through those elements homotopic to an element in $\mathrm{cAut}(D; \mu)$.*

Proof. $\mathrm{cAut}(D; \mu)\mathrm{qcAut}_0(D) = \{\varphi \in \mathrm{qcAut}(D) : \mathrm{cAut}(D; \mu) \cap (\mathrm{qcAut}_0(D)\varphi) \neq \emptyset\}$. □

4 The capacity as conformal invariant.

The locally convex topology on $C_0^\infty(D)$ induced by the semi-norm $\mathrm{Enrg}(\cdot; D)^{1/2}$ is Hausdorff. We denote its completion by $\iota : C_0^\infty(D) \to \overline{C_0^\infty(D)}^{\mathrm{Enrg}}$. Let U be an open subset of D. We always regard $C_0^\infty(U)$ as a subspace of $C_0^\infty(D)$. Then the mapping $C_0^\infty(U) \to \overline{\iota(C_0^\infty(U))}$ induced by ι realizes a completion relative to the norm $\mathrm{Enrg}(\cdot; U)^{1/2}$.

4.1 Definition. Let $W_0(U)$ be the set of all functions f on D such that there exists an $\mathrm{Enrg}(\cdot; D)^{1/2}$-Cauchy sequence $\{f_k\}$ in $C_0^\infty(U)$ which converges to f a.e.

We immediately deduce the next by the molifying method.

4.2 Lemma. *Let $f \in W_{\mathrm{loc}}(U)$. If its support is compact then $f \in W_0(U)$.*

4.3 Corollary. *Suppose $\zeta : D \to \mathbb{C}$ is a parameter and it is K-quasi-conformal on an open subset U. Let f be a function on $\zeta(U)$. Then $f \in W_0(\zeta(U))$ if and only if $f \circ \zeta \in W_0(U)$*

Proof. Combine Lemma 4.2 and the argument in the proof of Theorem 3.4. □

Thanks to Lemma 3.2 and Lemma 3.3 there exists a unique linear mapping Φ which preserves the semi-norm $\mathrm{Enrg}(\cdot; D)^{1/2}$ and makes the next diagram commutative:

$$\begin{array}{ccccc}
W_0(U) & \xrightarrow{\subset} & W_0(D) & \xrightarrow{\Phi} & \overline{C_0^\infty(D)}^{\mathrm{Enrg}} \\
\cup\uparrow & & \cup\uparrow & & \uparrow\iota \\
C_0^\infty(U) & \xrightarrow{\subset} & C_0^\infty(D) & = & C_0^\infty(D)
\end{array}$$

4.4 Lemma. *The canonical mapping Φ is bijective if and only if the space $W_0(D)$ equipped with a seminorm $\mathrm{Enrg}(\cdot; D)^{1/2}$ is a Hilbert space.*

4.5 Definition. We say the domain D transient if the space $W_0(D)$ equipped with a semi-norm $\mathrm{Enrg}(\cdot;D)^{1/2}$ is a Hilbert space.

4.6 Lemma. *Let U be a relatively compact open subset of D. Then Poincaré's inequality holds:* $\int_D |f|^2 \,\mathrm{vol} \le (\mathrm{diam}\, U)^2 \mathrm{Enrg}(f;D)$ *for all* $f \in C_0^\infty(U)$.

Consequently if D is bounded then it is transient. In general we have the next criterion.

4.7 Theorem. *D is transient if and only if there exists a positive bounded integrable function ρ such that* $\int_D |f|\, \rho \mathrm{vol} \le \mathrm{Enrg}(f;D)^{1/2}$ *for all* $f \in C_0^\infty(D)$.

In the sequel we assume that the domains D and E are *transient.*

4.8 Definition. Let A be a subset of D and U be an open subset of D with $A \subset U$.

$$\mathrm{Cap}(A;U) := \inf\{2\mathrm{Enrg}(f;D)\,;\, f \in W_0(U), f = 1 \text{ a.e.on some open set} \supset A\}.$$

We immediately see that $\mathrm{Cap}(A;U) = \inf\{\mathrm{Cap}(B;U)\,;\, B \text{ open}, A \subset B \subset U\}$ by the very definition. Therefore the next derives from Theorem 3.4 and Corollary 4.3.

4.9 Theorem. *Let $\zeta : D \to \mathbb{C}$ be a parameter. If it is K-quasi-conformal on an open subset U then* $\frac{1}{K}\mathrm{Cap}(A;U) \le \mathrm{Cap}(\zeta(A);\zeta(U)) \le K\mathrm{Cap}(A;U)$ *for all* $A \subset U$.

We note that sets of zero capacity always have zero measure. This is easily seen by the following lemma (a consequence of Lemma 4.6 and Theorem 4.7).

4.10 Lemma. *There exists a positive bounded integrable function ρ such that* $\int_A \rho \mathrm{vol} \le \mathrm{Cap}(A;D)^{\frac{1}{2}}$ *for all Borel sets A. If D is bounded then* $\rho = \frac{1}{(\mathrm{diam}\, D)^2}$ *is a choice.*

However lots of nullsets have positive capacity (see Lemma 4.12).

4.11 Lemma. (i) *Suppose A is a compact subset of U. Then* $\mathrm{Cap}(A;U) \le 2\mathrm{Enrg}(f;D)$ *for all* $f \in W_0(D) \cap C(U)$ *with* $f \ge 1$ *on A. Moreover* $\mathrm{Cap}(A;U)$ *is realized by*

$$\begin{aligned}\mathrm{Cap}(A;U) &= \inf\{2\mathrm{Enrg}(f;D)\,;\, f \in W_0(U) \cap C(D), f = 1 \text{ on } A\}\\ &= \inf\{2\mathrm{Enrg}(f;D)\,;\, f \in C_0^\infty(U), f \ge 1 \text{ on } A\}.\end{aligned}$$

(ii) *If A is a compact subset of D then* $\mathrm{Cap}(A;D) = \inf\{\mathrm{Cap}(A;U)\,;\, U \text{ open}, \supset A, \overline{U} \subset D\}$.

The relation in (i) is very useful to estimate capacities. As an example we show the next.

4.12 Lemma. $\mathrm{Cap}(I \times \{\frac{1}{2}\}; (0,1)^{\times 2}) \ge 4\mathrm{lgth}(I)$ *for each compact interval I in* $(0,1)$.

Proof. Suppose that $f \in C_0^\infty((0,1)^{\times 2})$ and $f \ge 1$ on $I \times \{1/2\}$. Then we have that

$$\int_{(0,1/2)} \partial_2 f(s,\cdot)\,\mathrm{lgth} = f(s,1/2) \ge 1 \text{ for } s \in I.$$

Just as in the proof of Poincaré's inequality we infer that $\int_{I\times(0,1/2)} |\nabla f|^2 \,\mathrm{vol} \ge 2\mathrm{lgth}(I)$ and similarly $\int_{I\times(1/2,1)} |\nabla f|^2 \,\mathrm{vol} \ge 2\mathrm{lgth}(I)$. Consequently $\mathrm{Enrg}(f;(0,1)^{\times 2}) \ge 2\mathrm{lgth}(I)$. Invoking Lemma 4.11 we reach the result. □

We list few important properties of the capacity and related notions.

4.13 Lemma. (i) *Suppose* $f \in W_{\mathrm{loc}}(D)$ *and* g *is a measurable function on* D. *If there exists a constant* C *such that* $|g(x)-g(y)| \le C|f(x)-f(y)|$ *for all* $x, y \in D$ *then* $g \in W_{\mathrm{loc}}(D)$ *and* $\mathrm{Enrg}(g; A) \le C^2 \mathrm{Enrg}(f; A)$ *for all compact subsets* A.
(ii) *Suppose* $f \in W_0(D)$ *and* g *is a measurable function on* D. *If there is a constant* C *such that* $|g(x)-g(y)| \le C|f(x)-f(y)|$ $\forall x, y$ *and* $|g(x)| \le C|f(x)|$ $\forall x$ *then* $g \in W_0(D)$.

This means that the normal contraction operates on the spaces $W_{\mathrm{loc}}(D)$ and $W_0(D)$.

4.14 Lemma. (i) $\mathrm{Cap}(A; D) \le \mathrm{Cap}(B; D)$ *if* $A \subset B$. $\mathrm{Cap}(A; D) \ge \mathrm{Cap}(A; E)$ *if* $D \subset E$.
(ii) *If* A_k *are compact and* $A_k \supset A_{k+1}$ *then* $\mathrm{Cap}(\bigcap_{k=1}^{\infty} A_k; D) = \inf_{k=1,2,\dots} \mathrm{Cap}(A_k; D)$.
(iii) $\mathrm{Cap}(A \cup B; D) + \mathrm{Cap}(A \cap B; D) \le \mathrm{Cap}(A; D) + \mathrm{Cap}(B; D)$.
(iv) *If* A_k *is nondecreasing then* $\mathrm{Cap}(\bigcup_{k=1}^{\infty} A_k; D) = \sup_{k=1,2,\dots} \mathrm{Cap}(A_k; D)$.
In general the countable subadditivity holds: $\mathrm{Cap}(\bigcup_{k=1}^{\infty} A_k; D) \le \sum_{k=1}^{\infty} \mathrm{Cap}(A_k; D)$.
(v) *If* A *is a Borel set then* $\mathrm{Cap}(A; D) = \sup\{\mathrm{Cap}(B; D)\,; B$ *compact,* $\subset A\}$.

We see by (i),(ii) and (iv) that the set function $\mathrm{Cap}(\cdot; D)$ is a Choquet capacity.

4.15 Lemma. *Let* ζ *be a parameter on* D *such that* $f \circ \zeta \in W_{\mathrm{loc}}(D)$ *for all* $f \in C_0^{\infty}(\zeta(D))$ *and* $K := \sup\{\mathrm{Enrg}(f \circ \zeta; D)\,; f \in C_0^{\infty}(\zeta(D)), \mathrm{Enrg}(f; \zeta(D)) \le 1\} < +\infty$.
(i) $\mathrm{Cap}(A; U) \le K\mathrm{Cap}(\zeta(A); \zeta(U))$ *for any subset* A *and open subset* U *with* $A \subset U$.
(ii) *If* $\{f \circ \zeta : f \in C_0^{\infty}(\zeta(D))\}$ *is dense in* $W_0(D)$ *and* $\mathrm{Enrg}(f; \zeta(D)) \le K\mathrm{Enrg}(f \circ \zeta; D)$ *for all* $f \in C_0^{\infty}(\zeta(D))$ *then* $\mathrm{Cap}(\zeta(A); \zeta(U)) \le K\mathrm{Cap}(A; U)$ *for any* A *and* U *with* $A \subset U$.

Proof. By virtue of Lemma 4.14(v) it suffices to prove the statement when A is compact. Note that $\zeta(A)$ is compact and $\zeta(U)$ is open. Given $\varepsilon > 0$ there exists $f \in C_0^{\infty}(\zeta(D))$ such that $\mathrm{supp}\, f \subset \zeta(U)$, $f \ge 1$ on $\zeta(A)$ and

$$2\mathrm{Enrg}(f; \zeta(D)) \le \mathrm{Cap}(\zeta(A); \zeta(U)) + \varepsilon/K.$$

We see that $f \circ \zeta \in W_{\mathrm{loc}}(D)$ and

$$2\mathrm{Enrg}(f \circ \zeta; D) \le K\mathrm{Cap}(\zeta(A); \zeta(U)) + \varepsilon.$$

On the other hand $\mathrm{supp}\, f \circ \zeta = \zeta^{-1}(\mathrm{supp}\, f)$ being compact and contained in U, it follows by Lemma 4.2 that $f \circ \zeta \in W_0(U)$. Clearly $f \circ \zeta$ is continuous and $f \circ \zeta \ge 1$ on A. Thus we infer by Lemma 4.11(i) that

$$\mathrm{Cap}(A; U) \le 2\mathrm{Enrg}(f \circ \zeta; D).$$

Tending ε to 0 we get the desired inequality. We can prove (ii) by making use of that $\{f \circ \zeta : f \in C_0^{\infty}(\zeta(D))\}$ is a special standard core. □

Under the assumption of Lemma 4.15 there exists a unique continuous linear mapping $W_0(\zeta(D)) \to W_0(D)$ whose restriction on $C_0^{\infty}(\zeta(D))$ coincides with $f \mapsto f \circ \zeta$. However it is *not* yet verified whether $f \circ \zeta \in W_{\mathrm{loc}}(D)$ hold for generic $f \in W_0(\zeta(D))$. Concerning this point see Lemma 4.23.

4.16 Definition. Let $A \subset D$ and $f : A \to \mathbb{R}$. f is called quasi-continuous relative to Cap if given $\varepsilon > 0$ there is an open set V such that $\mathrm{Cap}(V; D) < \varepsilon$ and f is continuous off V.

4.17 Lemma. *Let* $f : U \to \mathbb{R}$ *be a function defined on an open subset of* D. *If* f *is quasi-continuous relative to* Cap *and* $f \ge 0$ *a.e. then* $\{x \in U : f(x) < 0\}$ *is of capacity zero.*

4.18 Lemma. *Let f be a function on D. Then $f \in W_{\mathrm{loc}}(D)$ if and only if for each $x \in D$ there exist $\widetilde{f} \in W_0(D)$ such that $f = \widetilde{f}$ a.e. in some neighbourhood of x.*

4.19 Theorem. *Every function in $W_{\mathrm{loc}}(D)$ admits a quasi-continuous modification relative to* Cap. *Two quasi-continuous modifications coincide up to sets of capacity zero.*

4.20 Remark. In view of Lemma 4.18 it suffices to prove Theorem 4.19 for functions in $W_0(D)$. The key fact is the estimate $\mathrm{Cap}(\{|f| > \lambda\}; D) \leq \frac{2}{\lambda^2}\mathrm{Enrg}(f; D)$ for $f \in W_0(D) \cap C(D)$. It turns out that this is also valid for quasi-continuous functions belonging to $W_0(D)$.

4.21 Theorem. *Suppose that $\{f_k\}$ is a sequence of quasi-continuous functions in $W_0(D)$. If $\lim_{k,l\to\infty} \mathrm{Enrg}(f_k - f_l; D) = 0$ then there exists a subsequence which converges uniformly off an open set with arbitrary small capacity.*

4.22 Corollary. *Let $\{f_k\}$ be a sequence of quasi-continuous functions in $W_0(D)$. If f_k is* $\mathrm{Enrg}(\cdot; D)^{1/2}$*-Cauchy and converges q.e. then the limit function is quasi-continuous.*

4.23 Lemma. *Let ζ be a parameter on D such that $f \circ \zeta \in W_{\mathrm{loc}}(D)$ for all $f \in C_0^\infty(\zeta(D))$ and $K := \sup\{\mathrm{Enrg}(f \circ \zeta; D)\,; f \in C_0^\infty(\zeta(D)), \mathrm{Enrg}(f; \zeta(D)) \leq 1\} < +\infty$. Then for every quasi-continuous $f \in W_0(\zeta(D))$ the composition $f \circ \zeta$ is a quasi-continuous element of $W_0(D)$ and* $\mathrm{Enrg}(f \circ \zeta; D) \leq K\mathrm{Enrg}(f; \zeta(D))$.

Proof. Fix a quasi-continuous $f \in W_0(\zeta(D))$. There exists a sequence $\{f_k\}$ in $C_0^\infty(\zeta(D))$ and a subset A in D such that $\lim_{k,l\to\infty} \mathrm{Enrg}(f_k - f_l; \zeta(D)) = 0$ and f_k converges to f on $\zeta(A)$ and $\mathrm{Cap}(\zeta(D \setminus A); \zeta(D)) = 0$ according to Theorem 4.21 and Corollary 4.22. It follows by Lemma 4.15(i) that $\mathrm{Cap}(D \setminus A; D) = 0$. Hence $f_k \circ \zeta$ converges to $f \circ \zeta$ q.e. On the other hand, supports being compact, $f_k \circ \zeta \in W_0(D)$ for all k. Thus $\{f_k \circ \zeta\}$ is a Cauchy sequence in $W_0(D)$. Consequently we infer by Corollary 4.22 that $f \circ \zeta$ is quasi continuous and belongs to $W_0(D)$. □

It is a proper place to mention the spectral synthesis.

4.24 Theorem. *Let U be a relatively compact open set in D and $f : D \to \mathbb{R}$. Then $f \in W_0(U) \Leftrightarrow f \in W_{\mathrm{loc}}(D)$ and a quasi-continuous modification vanishes q.e. off U.*

We next discuss equilibrium potentials.

4.25 Theorem. *Suppose that* $\mathrm{Cap}(A; D) < +\infty$. *Then* $\mathrm{Cap}(A; D) \leq 2\mathrm{Enrg}(f; D)$ *for all $f \in W_0(D)$ which are quasi-continuous and $f \geq 1$ q.e. on A and moreover the equality is attained by a unique element.*

4.26 Definition. We set $\mathrm{eqlb}\phi[A; D] := \bigwedge\{f\,; \text{excessive on } D, f = 1 \text{ q.e. on } A\}$, which is called the equilibrium potential of A relative to D.

4.27 Theorem. $\mathrm{eqlb}\phi[A; D]$ *is excessive. If* $\mathrm{Cap}(A; D) < +\infty$ *then* $\mathrm{eqlb}\phi[A; D]$ *is quasi-continuous and belongs to $W_0(D)$ and moreover it is the unique function solving the variation problem in Theorem 4.25.*

4.28 Corollary. (i) $\mathrm{Cap}(A; D) = 0$ *if and only if* $\mathrm{eqlb}\phi[A; D] = 0$.
(ii) *Suppose $D \subset E$. Then* $\mathrm{Cap}(A; D) = 0$ *if and only if* $\mathrm{Cap}(A; E) = 0$.

Proof. (ii) derives form $\mathrm{Cap}(A; D) \geq \mathrm{Cap}(A; E)$ and $\mathrm{eqlb}\phi[A; D] \leq \mathrm{eqlb}\phi[A; E]|_D$. □

4.29 Remark. Suppose D and E are bounded and $D \subset E$. Then there exists a constant C such that $\mathrm{dist}(A, \partial D)^2\mathrm{Cap}(A; D) \leq C(\mathrm{diam}\, E)^2\mathrm{Cap}(A; E)$ for all $A \subset D$.

5 Variation problems and modules of quadrilaterals.

For the time being let D be a Jordan domain in $\mathbb{C}$, Δ be a non-empty compact subset of ∂D such that each of its connected component has at least two points. We note that both Δ and $\partial D \setminus \Delta$ have finitely many connected components.

5.1 Definition. Let $g \in C(\Delta)$. We set

$$C^1(D, \Delta, g) := \{f \in C(D \cup \Delta) : C^1 \text{ in } D, f = g \text{ on } \Delta, \mathrm{Enrg}(f; D) < +\infty\}.$$

Given an open and closed subset I of Δ we define the quantity $\mathrm{mdl}(D, \Delta, I)$, called the module of the triplet, by $\inf\{2\mathrm{Enrg}(f; D) \, ; f \in C^1(D, \Delta, 1_I)\}$.

Recall that any quasi-conformal homeomorphism $\phi : D \to E$ between Jordan domains with $\|\mathrm{dlt}[\phi]\|_\infty < 1$ automatically extends to a homeomorphism $\overline{D} \to \overline{E}$, called the canonical extension (see Theorem 2.25).

5.2 Lemma. *Let ϕ be a bi-holomorphic mapping from D to a Jordan domain. Then* $\mathrm{mdl}(\phi(D), \phi(\Delta), \phi(I)) = \mathrm{mdl}(D, \Delta, I)$ *where ϕ is canonically extended to $\overline{D}$.*

Proof. Let $f : D \cup \Delta \to \mathbb{R}$. Then clearly $f \in C^1(\phi(D), \phi(\Delta), 1_{\phi(I)})$ if and only if $f \circ \phi \in C^1(D, \Delta, 1_I)$. Thus the conformal invariance derives from that of the energy integral. □

5.3 Lemma. *If $g : \Delta \to \mathbb{R}$ is locally constant then $C^1(D, \Delta, g) \neq \emptyset$.*

Proof. We may regard D is a unit disk by constructing a suitable bi-holomorphic mapping. We see by the assumption on Δ and g that g extends to a C^1-class function $\tilde{g}$ on ∂D. The unique harmonic function on D with boundary data $\tilde{g}$ clearly belongs to $C^1(D, \Delta, g)$. □

In Theorem 5.4 and Corollary 5.5 we assume that Δ consists of two connected components and I is one of the connected components.

5.4 Theorem. $2\mathrm{Enrg}(f; D) \geq \mathrm{mdl}(D, \Delta, I)$ *for $f \in C(D \cup \Delta)$; partially ACL, $f|_\Delta = 1_I$. There exists $h \in C(\overline{D})$ which is harmonic in D, coincides with 1_I on Δ and whose energy integral attains $\frac{1}{2}\mathrm{mdl}(D, \Delta, I)$. h is the unique function minimizing the energy integral.*

Proof. We may assume that $D = (0, a) \times (0, b)$, $\Delta = ([0, a] \times \{0\}) \cup ([0, a] \times \{b\})$ and $I = [0, a] \times \{b\}$. Just as in the proof of Lemma 4.12 we infer that $b \int_D |\nabla f|^2 \mathrm{vol} \geq a$. Clearly $f = \frac{1}{b} \mathrm{Im}\, z$ attains the lower bound. □

5.5 Corollary. $\frac{1}{K}\mathrm{mdl}(D, \Delta, I) \leq \mathrm{mdl}(\zeta(D, \Delta, I)) \leq K\mathrm{mdl}(D, \Delta, I)$ *for each K-quasi-conformal parameter $\zeta : D \to \mathbb{C}$ whose image is also a Jordan domain.*

The quantity $\mathrm{mdl}(D, \Delta, I)$ coincides with the module M of the quadrilateral (D, Δ). In this case the imaginary part of a bi-holomorphic mapping $D \to (0, M) \times (0, 1)$ solves the variation problem. Let z be the parameter on D induced by this mapping. Then the function $e^{\sqrt{-1}2\pi z/M}$ maps D to the annulus $\{x \in \mathbb{C} : e^{-2\pi/M} < |x| < 1\}$.

5.6 Remark. Is an analogous picture still in force? If the number of connected components for Δ is $2n$ or $2n+1$ then a certain solution for the variation problem constructs a holomorphic covering mapping from D to a disk from which $(n-1)$-disks are removed?

We return to the general setting, i.e., no restriction on the number of connected components of Δ.

5.7 Definition. $\mathcal{C}(D,\Delta,g) := \{f \in C(D\cup\Delta) : \text{partialy ACL}, \nabla f \in L^2(D), f = g \text{ on } \Delta\}$ for $g \in C(\Delta)$. Let $W(D,\Delta,g)$ be the set of all functions f on $D\cup\Delta$ such that there exists an $\mathrm{Enrg}(\cdot;D)^{1/2}$-Cauchy sequence $\{f_k\}$ in $\mathcal{C}(D,\Delta,g)$ which converges to f a.e.

5.8 Lemma. *$W(D,\Delta,g)$ is an affine space with translation vector space $W(D,\Delta,0)$. $cW(D,\Delta,g) = W(D,\Delta,cg)$ for $c\neq 0$ and $W(D,\Delta,g_1)+W(D,\Delta,g_2) = W(D,\Delta,g_1+g_2)$.*

Proof. $\mathcal{C}(D,\Delta,g)$ is an affine space with translation vector space $\mathcal{C}(D,\Delta,0)$. □

Clearly $\mathcal{C}(D,\Delta,g)$ contains $C^1(D,\Delta,g)$.

5.9 Lemma. *Suppose that $D := (0,a)\times(0,1)$ and $\Delta \supset [0,a]\times\{0\}$.*
(i) $\int_{(0,a)\times(0,\varepsilon)} |f|^2 \operatorname{vol} \leq \varepsilon^2 \mathrm{Enrg}(f;(0,a)\times(0,\varepsilon))$ *for all $f\in W(D,\Delta,0)$ and $\varepsilon\in(0,1)$.*
(ii) *Let $\chi : \overline{D} \to \mathbb{R}$ be Lipschitz continuous. Then $\chi f \in W(D,\Delta,0)$ for all $f\in W(D,\Delta,0)$ and there exists C such that* $\mathrm{Enrg}(\chi f;D) \leq C\mathrm{Enrg}(f;D)$ *for all $f\in W(D,\Delta,0)$.*
(iii) *Let $\chi : \overline{D} \to \mathbb{R}$ be Lipschitz continuous and $\varepsilon\in\mathbb{R}_{(0,1)}$. If $0\leq\chi\leq 1$, $\chi = 1$ on $[0,a]\times[\varepsilon,1]$ and there exists C such that $|\chi(x)-\chi(y)|\leq C|x-y|$ for all $x,y\in\overline{D}$ then* $\mathrm{Enrg}(\chi f - f;D) \leq (2+C^2\varepsilon^2)\mathrm{Enrg}(f;(0,a)\times(0,\varepsilon))$ *for all $f\in W(D,\Delta,0)$.*

Proof. The procedure to obtain (i) for $f\in\mathcal{C}(D,\Delta,0)$ is exactly the same as in the proof of Poincaré's inequality. Invoking Fatou's lemma we can transfer the inequality to the limit of approximating sequence for $f\in W(D,\Delta,0)$. We next show (ii). Suppose $f\in\mathcal{C}(D,\Delta,0)$. Then χf is clearly continuous and vanishes on Δ. The product χf is also partially ACL and its partial derivative reads $\partial_i(\chi f) = (\partial_i\chi)f + \chi(\partial_i f)$. Both χ and its partial derivatives are bounded. Applying (i) we get the square integrability of $\nabla(\chi f)$. Consequently $\chi f\in\mathcal{C}(D,\Delta,0)$. Moreover we can find a constant C such that $\mathrm{Enrg}(\chi f;D)\leq C\mathrm{Enrg}(f;D)$ for all $f\in\mathcal{C}(D,\Delta,0)$. Through the limit procedure we get the complete statement (ii). □

5.10 Corollary. *The space $W(D,\Delta,0)$ with the norm $\mathrm{Enrg}(\cdot;D)^{\frac{1}{2}}$ is a Hilbert space.*

Proof. We can find $a>0$ and a bi-holomorphic mapping $\phi : D\to(0,a)\times(0,1)$ such that $\phi(\Delta)\supset[0,a]\times\{0\}$. Thus we can apply Lemma 5.9(i). □

5.11 Lemma. *$\{f\in C(D\cup\Delta) : C^\infty$ in $D, \operatorname{supp} f\cap\Delta=\emptyset, \mathrm{Enrg}(f;D)<+\infty\}$ is dense in $W(D,\Delta,0)$. In general $C^1(D,\Delta,g)$ is dense in $W(D,\Delta,g)$.*

Proof. Given a connected component I of Δ we can find $a>0$ and a bi-holomorphic mapping $\phi : D\to(0,a)\times(0,1)$ such that $\phi(I) = [0,a]\times\{0\}$ and $\phi(\Delta\setminus I)\subset[0,a]\times\{1\}$. It follows from (ii) and (iii) of Lemma 5.9 that any element in $\mathcal{C}(\phi(D),\phi(\Delta),0)$ is approximated by a sequence in $\{f\in\mathcal{C}(\phi(D),\phi(\Delta),0) : \operatorname{supp} f\cap\phi(I)=\emptyset\}$. Due to the conformal invariance of the energy integral we see that $\{f\in\mathcal{C}(D,\Delta,0) : \operatorname{supp} f\cap I=\emptyset\}$ is dense in $W(D,\Delta,0)$. Repeating this argument we see that $\{f\in\mathcal{C}(D,\Delta,0) : \operatorname{supp} f\cap\Delta=\emptyset\}$ is dense in $W(D,\Delta,0)$. To complete the proof we fix a holomorphic parameter on D whose image is a rectangle. We thus regard D itself as rectangle. Suppose $f\in\mathcal{C}(D,\Delta,0)$ and $\operatorname{supp} f\cap\Delta=\emptyset$. By using reflection method and cutting off we can construct an extension $\widetilde{f}\in W_0(\mathbb{C})$ of f with $\operatorname{supp}\widetilde{f}\cap\Delta=\emptyset$. Then there exists a sequence $\{f_k\}$ in $C_0^\infty(\mathbb{C})$ such that $\operatorname{supp} f_k\cap\Delta=\emptyset$, $\mathrm{Enrg}(f_k-\widetilde{f};\mathbb{C})$ converges to 0 and f_k converges to $\widetilde{f}$ a.e. Clearly $\{f_k|_{D\cup\Delta}\}$ is a desired approximating sequence. □

5.12 Corollary. $\mathrm{mdl}(D, \Delta, I) \leq 2\mathrm{Enrg}(f; D)$ *hold for all* $f \in W(D, \Delta, 1_I)$. *There is* $h \in W(D, \Delta, 1_I)$ *which is harmonic in* D *and whose energy integral attains* $\frac{1}{2}\mathrm{mdl}(D, \Delta, I)$. h *is a unique function in* $W(D, \Delta, 1_I)$ *minimizing the energy integral.*

5.13 Theorem. *Let* $\zeta : D \to \mathbb{C}$ *be a* K*-quasi-conformal parameter whose image is also a Jordan domain. Then* $\frac{1}{K}\mathrm{mdl}(D, \Delta, I) \leq \mathrm{mdl}(\zeta(D, \Delta, I)) \leq K\mathrm{mdl}(D, \Delta, I)$.

Proof. Let f be a function on $\zeta(D \cup \Delta)$. Then, by Theorem 3.4, $f \in \mathcal{C}(\zeta(D), \zeta(\Delta), 1_{\zeta(I)})$ if and only if $f \circ \zeta \in \mathcal{C}(D, \Delta, 1_I)$. Combining with Corollary 5.12 we get the statement. □

5.14 Theorem. *Let* $f : D \cup \Delta \to \mathbb{R}$ *be Lebesgue measurable. Then* $f \in W(D, \Delta, g)$ *if and only if* f *is partially ACL on* D*, its weak partial derivatives square integrable on* D *and it admits a quasi-continuous modification coinciding with* g *q.e. on* Δ.

5.15 Corollary. *If* $g_1 \neq g_2$ *then* $W(D, \Delta, g_1) \cap W(D, \Delta, g_2) = \emptyset$.

6 The case $\|\mu\|_\infty = 1$.

Even if $\|\mu\|_\infty = 1$ the Beltrami equation may have a homeomorphic solution $\mathbb{C} \to \mathbb{C}$. A concrete example is given by the isothermal parameter problem on the surface

$$x_3 = \frac{1}{2}a((x_1)^2 + (x_2)^2)$$

in $\mathbb{R}^3$. The Beltarami coefficient $b[g; z]$ of the induced meric is given by (1.6). In fact we can solve the associated Beltrami equation explicitly.

6.1 Lemma. *There exists a diffeomorphism* $\zeta : \mathbb{C} \to \mathbb{C}$ *such that*

$$\mathrm{dlt}[\zeta] = a^2 z^2/(1 + \sqrt{1 + a^2|z|^2})^2.$$

Proof. The function $\zeta := \exp\{\frac{1}{2}\int_0^{a^2|z|^2} \frac{1}{1+\sqrt{1+t}}\, dt\}z$ is a desired one. □

6.2 Remark. Since $\zeta(\mathbb{C}) = \mathbb{C}$ in Lemma 6.1, the complex structure induced by the parameter ζ is equivalent to the standard complex structure on $\mathbb{C}$, which is recurrent, i.e., any positive superharmonic function is a constant function.

6.3 Lemma. *There exists an into diffeomorphism* $\zeta : \mathbb{C} \to \mathbb{C}$ *such that*

$$\mathrm{dlt}[\zeta] = -a^2 z^2/(1 + \sqrt{1 + a^2|z|^2})^2$$

and $\zeta(\mathbb{C})$ *is the unit disk.*

Proof. The function $\zeta := \frac{az}{1+\sqrt{1+a^2|z|^2}}$ is a desired one. □

6.4 Remark. Since $\zeta(\mathbb{C})$ is the unit disk in Lemma 6.3, the complex structure induced by the parameter ζ is equivalent to the standard complex structure on the unit disk, which is transient, i.e., Green's functions exist.

The situation in Lemma 6.3 is by no means exceptional.

6.5 Lemma. *Suppose* $c \in U(1)$ *and* $c \neq 1$. *Then there exists an into diffeomorphism* $\zeta : \mathbb{C} \to \mathbb{C}$ *such that*

$$\mathrm{dlt}[\zeta] = ca^2z^2/(1+\sqrt{1+a^2|z|^2})^2$$

and $\zeta(\mathbb{C})$ is the unit disk.

Taking these examples into account we may raise the following question:

6.6 Question. Let μ be a generalized Beltrami coefficient on $\mathbb{C}$. Then, by Theorem 3.10, there exist a quasi-conformal parameter $\zeta : \mathbb{C} \to \mathbb{C}$ with complex dilatation μ. When is the mapping ζ surjective, equivalently, when is the associated complex structure recurrent?

We give a partial answer.

6.7 Assumption. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a measurable function with $\sup_{s\in[0,t]} s|\rho(s)| < 1$ for all t. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be an absolutely continuous function such that $\frac{f'(s)}{1+sf'(s)} = \rho(s)$ for almost all s. In fact f is given by $f(t) = \int_0^t \frac{\rho(s)}{1-s\rho(s)}ds$ up to an integral constant.

If $\zeta = e^{f(|z|^2)}z$ then we see that $d\zeta = e^{f(|z|^2)}\{(1+zf'(|z|^2)\overline{z})dz + zf'(|z|^2)zd\overline{z}\}$. Therefore $\zeta = e^{f(|z|^2)}z$ is a global solution to Beltrami equation with coefficient $\mu = z^2\rho(|z|^2)$.

6.8 Lemma. (i) *$t \mapsto 2\,\mathrm{Re}\, f(t) + \log t$ is strictly increasing.*
(ii) $\lim_{t\to+\infty}(2\,\mathrm{Re}\, f(t) + \log t) = +\infty$ *if and only if* $\int_1^{+\infty} \frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2}\frac{dt}{t} = +\infty$.

Proof. $2\,\mathrm{Re}(f(t) - f(1)) + \log t = 2\mathrm{Re} \int_1^t \frac{\rho(s)}{1-s\rho(s)}ds + \int_1^t \frac{1}{s}ds = \int_1^t \frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2}\frac{dt}{t}$. □

6.9 Corollary. *The complex structure on $\mathbb{C}$ associated with the generalized Beltrami coefficient $\mu = z^2\rho(|z|^2)$ is recurrent if and only if $\int_1^{+\infty} \frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2}\frac{dt}{t} = +\infty$.*

Let $\delta \in \mathbb{R}_{(0,1)}$. Observe that $\max_{|x|\leq\delta} \frac{1-|x|^2}{|1-x|^2} = \frac{1+\delta}{1-\delta}$ and $\min_{|x|\leq\delta} \frac{1-|x|^2}{|1-x|^2} = \frac{1-\delta}{1+\delta}$. Suppose $t|\rho(t)| \leq \delta$ for almost all t. Then $(2\,\mathrm{Re}\, f(t) + \log t) \geq \frac{1-\delta}{1+\delta}\log t$, which diverges as $t \to +\infty$.

Let $\alpha \in U(1)$ and ε be a measurable function such that $\sup_{s\in[0,t]} |1-\varepsilon(s)| < 1$ ($\Leftrightarrow$ $|\varepsilon(t)|^2 < 2\,\mathrm{Re}\,\varepsilon(t)$) for all t and $\lim_{t\to+\infty}\varepsilon(t) = 0$. Suppose $t\rho(t) \sim \alpha(1-\varepsilon(t))$ as $t \to +\infty$. If $\alpha \neq 1$ then $\frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2} \sim (2\,\mathrm{Re}\,\varepsilon(t) - |\varepsilon(t)|^2)/|1-\alpha|^2$ as $t \to +\infty$ and hence

$$\int_1^{+\infty} \frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2}\frac{dt}{t} < +\infty \Leftrightarrow \int_1^{+\infty} (2\,\mathrm{Re}\,\varepsilon(t) - |\varepsilon(t)|^2)\frac{dt}{t} < +\infty.$$

While if $\alpha = 1$ then $\frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2} \sim (2\,\mathrm{Re}\,\varepsilon(t) - |\varepsilon(t)|^2)/|\varepsilon(t)|^2$ as $t \to +\infty$ and hence $\int_1^{+\infty} \frac{1-t^2|\rho(t)|^2}{|1-t\rho(t)|^2}\frac{dt}{t} = +\infty$.